Predictive Control of Timed Event Graphs with Specifications Defined by P-time Event Graphs

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I. ABSTRACT

The aim of this paper is the predictive control of Timed Event Graphs with specifications defined by P-time Event Graphs. We propose a fixed-point approach which leads to a pseudo-polynomial algorithm. As the performance of the algorithm is crucial in on-line control, we highlight an important case where the resolution of this first algorithm is efficient. The second technique is a space controller on a horizon leading to a strongly polynomial algorithm.

keywords: Timed Event Graphs, P-time Petri nets, (min, max, +) functions, fixed point, predictive control.

II. INTRODUCTION

In this paper, we focus on model predictive control of Timed Event Graphs with specifications defined by P-time Event Graphs. A classical problem is the control of a Timed Event Graph where some events are stated as controllable, meaning that the corresponding transitions (input) may be delayed from firing until some arbitrary time provided by a supervisor. The specifications are defined by a P-time Event Graph [11] [6] which describes the desired behavior of the interconnections of all the internal transitions. We wish to determine an input in order to obtain the desired behavior defined by the specifications.

Model predictive control is an on-line approach which needs *efficient algorithms*: a crucial point is that a calculation of the control that is too slow can postpone the application of the control at the calculated dates. With the aim of a strongly polynomial algorithm, one objective is the analysis of the state space.

Naturally, variations of the classical problem have been considered in the literature. A first class of approaches [7] considers extremal points of the state space and develops optimal control in order to keep trajectories close to a reference trajectory following additional constraints. Contrary to these approaches where the resolution uses conventional algebra, our approach describes every trajectory in the formalism of the (max, +) algebra. Moreover, we only use classical operations such as the Kleene star which can be determined by known efficient algorithms: They are polynomial *in the strong sense*, that is, the complexity depends only on the dimensions but not on the values of the parameters. We can recall that the classical algorithms of linear programming are not polynomial in the strong sense (the complexities of the ellipsoid algorithm of Khashiyan and the interior point algorithm of Karmarkar are respectively $O(n^4.L)$ and $O(n^{3.5}.L)$ where n is the number of variables and L is the number of bits necessary in the storage of the data [14]).

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Another class of approaches analyzes state space and develops controllers in order to keep trajectories inside a space deduced from a given specification. Without considering a desired output, the computation of the maximal set of the initial states is analyzed in [10]. Contrary to these approaches where the initial condition must be applied to the process, our approach considers that the current state is the result of an unknown evolution.

In this paper, we consider that each transition is observable: The event date of each transition firing is assumed to be available. No hypothesis is made on the structure of the Event Graphs which does not need to be strongly connected. The initial marking should only satisfy the classical liveness condition and the usual hypothesis of First In First Out (FIFO) places is used. Due to the lack of space, the presentation of the model of the P-time Event Graph is omitted; the reader can find the preliminaries and the presentation of the models in [6]. The principle of the model predictive control can be found in [7]. The consistency of the models is beyond the scope of this paper and we assume their consistency, that is, the existence of *finite* solutions [4] [6].

The plan is as follows: The problem is first rewritten under a fixed-point form. Deduced from the algorithm of Mc Millan and Dill[13], Algorithm 1 provides a way of determining the largest solution. The analysis of Algorithm 1 highlights an important case where the resolution of Algorithm 1 is efficient. Moreover, a restriction to a specific subspace leads to Algorithm 2. Finally, the results are illustrated by an example where the specifications are described by a P-time Event Graph: This Petri net naturally contains lower bounds (that is, with the expression $x(k-1)+T \leq x(k)$ with the notation given in Section IV) contrary to the additional constraints of the examples given in [1] and [10].

III. Preliminary remarks

A monoid is a pair $(S, \oplus)$ where the operation $\oplus$ is associative and presents a neutral element ε. A semi-ring S is a triple $(S, \oplus, \otimes)$ where $(S, \oplus)$ and $(S, \otimes)$ are monoids, $\oplus$ is commutative, $\otimes$ is distributive in relation to $\oplus$ and the zero element ε of $\oplus$ is the absorbing element of $\otimes$ $(\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon)$. A dioid $\mathcal{D}$ is an idempotent semi-ring (the operation $\oplus$ is idempotent, that is $a \oplus a = a$). The set $\mathbb{R} \cup \{-\infty\}$, provided with the maximum operation denoted $\oplus$ and the addition denoted $\otimes$ is an example of dioid denoted $\mathbb{R}_{\max}$ = $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$: so, $x \oplus y = max(x, y)$ and $x \otimes y = x + y$. The neutral elements of $\oplus$ and $\otimes$ are represented by $\varepsilon = -\infty$ and $e = 0$, respectively. The absorbing element of $\otimes$ is ε. The minimum operation is denoted $\wedge$. The partial order denoted $\leqslant$ is defined in $\mathbb{R}^n$ as follows: $x \leqslant y \Longleftrightarrow x \oplus y = y \Longleftrightarrow x \wedge y = x \Longleftrightarrow x_i \leqslant y_i$, for i from 1 to n. The notation $x < y$ means that $x \leqslant y$ and $x \neq y$. A dioid $\mathcal{D}$ is complete if it is closed for infinite sums, and the distributivity of the multiplication with respect to the addition applies to infinite sums. $(\forall\ c \in \mathcal{D}\ )\ (\forall\ A \subseteq \mathcal{D})\ c \otimes (\bigoplus_{x \in A} x) = \bigoplus_{x \in A} c \otimes x$. For example, $\overline{\mathbb{R}}_{\max}$ = $(\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, \oplus, \otimes)$ is complete. The operations $\oplus$ and $\otimes$ are extended to matrices as follows: If $\alpha \in \mathcal{D}$ and if P, $Q \in \mathcal{D}^{mxn}$ then $(\alpha \otimes P)_{i,j}$=$\alpha \otimes P_{i,j}$ and $(P \oplus Q)_{i,j}$=$P_{i,j} \oplus Q_{i,j}$ for all i, j; If $P \in \mathcal{D}^{mxp}$ and $Q \in \mathcal{D}^{pxn}$ then $(P \otimes Q)_{i,j}$=$\bigoplus_{k=1}^{p} P_{i,k} \otimes Q_{k,j}$ for all i, j. The identity matrix is denoted I_d: $(I_d)_{i,j} = e$ if $i = j$ and $(I_d)_{i,j} = \varepsilon$ if $i \neq j$. The zero matrix is only composed of the entries ε and is denoted ε. The dimensions of the matrices I_d and ε can easily be deduced from the context. The set of nxn matrices with entries in the complete dioid $\mathcal{D}$ including the two operations $\oplus$ and $\otimes$ is a complete dioid, which is denoted $\mathcal{D}^{nxn}$. We can deal with non-square matrices if we complete them with rows or columns provided the entries equal ε. The mapping

f is said to be residuated if for all $y \in \mathcal{D}$, the least upper bound of subset $\{x \in \mathcal{D} \mid f(x) \leq y\}$ exists and lies in this subset. The mapping $x \in (\overline{\mathbb{R}}_{\max})^n \mapsto A \otimes x$, defined over $\overline{\mathbb{R}}_{\max}$ is residuated (see [2]) and the left $\otimes-$residuation of B by A is denoted by $A \backslash B = \max\{x \in (\overline{\mathbb{R}}_{\max})^n$ such that $A \otimes x \leqslant B\}$. The maximum of this last set is denoted x^+. The notation $card(X)$ stands for the cardinality of the set X.

The following Theorem uses the Kleene star defined by: $A^* = \bigoplus_{i=0}^{+\infty} A^i$.

Theorem 1: (Theorem 4.75 Part 1 in [2]) Consider the equation $x = A \otimes x \oplus B$ and the inequality $x \geq A \otimes x \oplus B$ with A and B in a complete dioid $\mathcal{D}$. Then, $A^* \otimes B$ is the least solution to these two relations. ■

IV. Control problem

Let us consider the objective of this paper. Below, the variable $x_i(k)$ is the date of the k^{th} firing of the transition x_i and n is the dimension of $x(k)$.

A. Objective (Problem 1)

The objective of this paper is the determination of the greatest control u on an arbitrary horizon $[k_s + 1, k_f]$ with $h = k_f - k_s \in \mathbb{N}$ such that its application to the Timed Event Graph defined by [2]

$$\begin{cases} x(k+1) = A \otimes x(k) \oplus B \otimes u(k+1) \\ y(k) = C \otimes x(k) \end{cases} \tag{1}$$

for $k \geq k_s$, satisfies the following conditions:

a) $y \leq \underline{z}$ knowing the trajectory of the desired output $\underline{z}$;

b) The state trajectory follows the model of the P-time Event Graph defined by

$$\begin{pmatrix} x(k) \\ x(k+1) \end{pmatrix} \geq \begin{pmatrix} A^{=} & A^{+} \\ A^{-} & A^{=} \end{pmatrix} \otimes \begin{pmatrix} x(k) \\ x(k+1) \end{pmatrix} \; ; \tag{2}$$

c) The initial value of the state trajectory $x(k)$ for $k \geq k_s$ is finite and is a known vector denoted $\underline{x}(k_s)$. This " non-canonical " initial condition can be the result of a past evolution of a process. Since $\underline{x}(k_s)$ is finite, the trajectories considered in this paper are finite.

Underlined symbols like $\underline{x}(k_s)$ correspond to known data of the problem and $x(k)$ and $y(k)$ are estimated in the following resolutions.

Remark 1: Applications of P-time Event Graphs can be found in production systems, micro-circuit design [15], transportation systems [10], and the food industry [1]. A simple example is cooking a product [1]: The cooking time must not be too long, otherwise the product will be damaged; at the same time, the cooking time needs to be long enough.

The system (2) can always be obtained and corresponds to a P-time Event Graph where the initial marking of each place is equal at the most to one. When we consider the places having a unitary (respectively, null) initial marking, the lower bound a of the temporization of the place linking its ingoing transition x_j to its outgoing transition x_i generates the entry $A^{-}_{i,j} = a \geq 0$ (respectively, $A^{=}_{i,j} = a \geq 0$) and the upper bound b of the temporization of the place linking its ingoing transition x_i to its outgoing transition x_j generates the entry $A^{+}_{i,j} = -b \leq 0$ (respectively, $A^{=}_{i,j} = -b \leq 0$). More details can be found in [6].

B. Relations on horizon $[k_s, k_f]$

The relations of the Timed Event Graph can be rewritten under the following classical form on horizon $[k_s, k_f]$.

$$X = \Omega_h \otimes x(k_s) \oplus \Psi_h \otimes U \tag{3}$$

where $h = k_f - k_s$, $X = \begin{pmatrix} x(k_s+1)^t & x(k_s+2)^t & \cdots & x(k_f-1)^t & x(k_f)^t \end{pmatrix}^t$ (t: transposed), $U = \begin{pmatrix} u(k_s+1)^t & u(k_s+2)^t & \cdots & u(k_f-1)^t & u(k_f)^t \end{pmatrix}^t$, $\Omega_h =$ $\begin{pmatrix} A \\ A^2 \\ A^3 \\ \dots \\ A^h \end{pmatrix}$ and $\Psi_h = \begin{pmatrix} B & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ A \otimes B & B & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ A^2 \otimes B & A \otimes B & B & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A^{h-1} \otimes B & A^{h-2} \otimes B & A^{h-3} \otimes B & \cdots & A^2 \otimes B & A \otimes B & B \end{pmatrix}$.

Below we consider the additional constraints (2) for $k \geq k_s$ and an autonomous Timed Event Graph defined by the inequality $x(k) \geq A \otimes x(k-1)$ which is the relaxation of the earliest firing rule, starting from $x(k_s) = \underline{x}(k_s)$.

$$\begin{cases} \begin{pmatrix} x(k_s) \\ X \end{pmatrix} \geq D_h \otimes \begin{pmatrix} x(k_s) \\ X \end{pmatrix} \\ x(k_s) = \underline{x}(k_s) \end{cases} \tag{4}$$

where $D_h = \begin{pmatrix} A^= & A^+ & \varepsilon & \cdots & & \\ A \oplus A^- & A^= & A^+ & \cdots & & \\ \varepsilon & A \oplus A^- & A^= & \cdots & & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ & & & \cdots & A^= & A^+ \\ & & & \cdots & A \oplus A^- & A^= \end{pmatrix}$.

The matrix D_h presents a tridiagonal structure: This is a square matrix, composed of a main diagonal (square submatrix $A^=$ for rows $i \in [1, h+1]$), an upper diagonal (square submatrix A^+ for rows $i \in [1, h]$), a lower diagonal (square submatrix $A \oplus A^-$ for rows $i \in [2, h+1]$) and all other blocks being zero matrices (square submatrix ε). The matrix D_h is a $n.(h+1)$ x $n.(h+1)$ matrix where n is the dimension of x.

C. Fixed point form

We introduce the following extended state vector $\overline{x} = \begin{pmatrix} (x(k_s))^t & (X)^t \end{pmatrix}^t$ which expresses the complete state trajectory. Let $(\overline{x})^+$ be the greatest estimate of state trajectory and $F = \begin{pmatrix} \underline{x}(k_s)^t & (C\backslash \underline{z}(k_s+1))^t & (C\backslash \underline{z}(k_s+2))^t & \cdots & (C\backslash \underline{z}(k_f))^t \end{pmatrix}^t$.

***Theorem** 2:* The greatest state and control trajectory of the control problem is the greatest solution of the following fixed point inequality system

$$\begin{cases} \overline{x} \leq D_h \backslash \overline{x} \wedge F \\ U \leq \Psi_h \backslash X \\ X \leq \Omega_h \otimes x(k_s) \oplus \Psi_h \otimes U \end{cases} \tag{5}$$

with condition $\underline{x}(k_s) \leq x^+(k_s)$.

Proof: Using the previous description of the state and control trajectories (3) (4), we can easily rewrite the problem under a general, fixed-point formulation $x \leq f(x)$ which allows the control problem to be resolved. For instance, equality $X = \Omega_h \otimes x(k_s) \oplus \Psi_h \otimes U$ is equivalent to $X \leq \Omega_h \otimes x(k_s) \oplus \Psi_h \otimes U$ and $X \geq \Omega_h \otimes x(k_s) \oplus \Psi_h \otimes U$ (see the proof of Theorem 1 in [4] for more details). Function f is a (min, max, +) function which can be defined by the following grammar: $f = b, x_1, x_2, \ldots, x_n \mid f \otimes a \mid f \wedge f \mid f \oplus f$ where a, b are arbitrary real numbers ($a, b \in \mathbb{R}$). The existence of the greatest solution on complete lattices can be proven by using the famous fixed point theorem of Knaster-Tarski [16]. The conditions of the Knaster-Tarski theorem are satisfied: The general form of the problem is such that $x \leq f(x)$ where f is an isotone function defined on a complete lattice $\overline{\mathbb{R}}_{max}$=($\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, \leq$) and where x corresponds to $(x(k_s)^t, X^t, U^t)^t$. ■

D. Algorithm 1

The effective calculation of the greatest control can be made by the classical iterative algorithm of Mc Dillan and Dill [13]. The general resolution of $x \leq f(x)$ is given by the iterations of $x_{\langle i \rangle} \leftarrow x_{\langle i-1 \rangle} \wedge f(x_{\langle i-1 \rangle})$ if the finite starting point $x_{\langle 0 \rangle}$ is greater than the final solution. Here, number $\langle i \rangle$ represents the number of iterations and not the number of components of vector x.

Even if a solution exists, the sequence $\{x_{\langle i \rangle}\}$ in the algorithm of Mc Dillan and Dill [13] is required to be finite as a finite sequence guarantees the finite termination of the recursion. As algorithm [13] is known to be pseudo-polynomial (the complexity is here considered in the worst case), the algorithm has a finite termination. Another reasoning based on the following definition and limited to $\mathbb{Z}$ is as follows. A poset in which, for any chain $x_{\langle 0 \rangle} \geq x_{\langle 1 \rangle} \geq \ldots \geq x_{\langle i \rangle} \geq \ldots$ of elements there exists an integer $\eta \geq 0$ such that $x_{\langle i \rangle} = x_{\langle \eta \rangle}$ for all $i \geq \eta$ is said to satisfy the Descending Chain Condition (DCC).

***Property** 1:* The recursion of Mc Dillan and Dill's algorithm [13] has a finite termination over $\mathbb{Z}$ assuming the consistency of $x \leq f(x)$.

Proof: In Lemma 1 of [9] satisfaction of the DCC guarantees the finite termination of the recursion. Clearly, a finite lattice satisfies the DCC as the length of the longest chain is finite in a finite lattice [3]. Therefore, we can prove the finite convergence by defining a finite sublattice with the following three points. Firstly, we reduce the number of elements of the sublattice by considering $\mathbb{Z}$. Secondly, a finite upper bound is the finite starting point $x_{\langle 0 \rangle}$. Thirdly, a finite lower bound is the greatest solution to the problem: If we assume the consistency, that is, the existence of *finite* solutions [4], the greatest solution over $\mathbb{Z}$, has no component equal to $\varepsilon = -\infty$; It has no component equal to $+\infty$ as the elements of the set are lower than or equal to the finite starting point. Finally, the application of Lemma 1 in [9] to this finite sublattice guarantees the finite termination of the algorithm [13] and every derived algorithm [15]. ■

We now provide an algorithm specific to the determination of the greatest state and control. Since it follows the algorithm of Mc Dillan and Dill, this algorithm is also pseudo-polynomial. Starting from $x_{\langle 0 \rangle} = F$, the trajectory $\overline{x}$ is minimized in each iteration of the following algorithm where $(\overline{x})^1 = \begin{pmatrix} (x^1(k_s))^t & (X^1)^t \end{pmatrix}^t$ and $(\overline{x})^2 = \begin{pmatrix} (x^2(k_s))^t & (X^2)^t \end{pmatrix}^t$ correspond to useful intermediate values. Each iteration $\langle i \rangle$ with $i > 0$ considers the three steps 1, 2 and 3.

Algorithm 1

Step 0 (initialization): $\langle i \rangle \leftarrow \langle 0 \rangle$; $(\overline{x})^2 \leftarrow F$

Repeat

- $\langle i \rangle \leftarrow \langle i+1 \rangle$ (numbering of the iteration)
- Step 1: $(\overline{x})^1 \leftarrow D_h^* \backslash (\overline{x})^2$
- Step 2: $U \leftarrow \Psi_h \backslash X^1$
- Step 3: $(\overline{x})^2 \leftarrow (\overline{x})^1 \wedge \begin{pmatrix} +\infty \\ \Omega_h \otimes x^1(k_s) \oplus \Psi_h \otimes U \end{pmatrix}$

until $X^1 = X^2$. ■

Step 1 is deduced from the resolution of $\overline{x} \leq D_h \backslash \overline{x} \wedge (\overline{x})^2$ and the application of Theorem 4.73 in ([2]). The obtained solution $(\overline{x})^1$naturally satisfies $(\overline{x})^1 \leq D_h \backslash (\overline{x})^1$ which is equivalent to the first relation in (4). The rest of the algorithm checks that this calculated solution, also satisfies $X^1 = \Omega_h \otimes x^1(k_s) \oplus \Psi_h \otimes U$ with $U = \Psi_h \backslash X^1$. Let X' be a co-state such that $X' = \Omega_h \otimes x^1(k_s) \oplus \Psi_h \otimes U$. The co-state, also considered in the well-known "backward approach" (see part 5.6.2 in [2]), is now analyzed.

***Property** 2:* $X' \leq X^1$ and $X^2 = X'$ where $X' = \Omega_h \otimes x^1(k_s) \oplus \Psi_h \otimes U$.

Proof: First, $X^1 \geq (\Omega_h) \otimes x^1(k_s)$: the components of X^1 satisfy the constraint $x(k+1) \geq A \otimes x(k)$ for $k \geq k_s$ which is expressed in $(\overline{x})^1 \leq D_h \backslash (\overline{x})^1$. Secondly, $\Psi_h \otimes U = \Psi_h \otimes (\Psi_h \backslash X^1) \leq X^1$ by definition of the residuation. Finally, $X^2 = X^1 \wedge X' = X'$ as $X' \leq X^1$. ■

The previous property 2 shows that $X^1 = X'$ when $X^1 = X^2$. As a consequence, relations (3) and the first relation of (4) are satisfied when $X^1 = X^2$.

Remark 2: Let $x^0(k_s)$ be the state vector $x(k_s)$ generated by the calculation of $D_h^* \backslash (\overline{x})^2$. Note that we have $x^0(k_s) = x^1(k_s) = x^2(k_s)$ by construction and that the convergence test $x^1(k_s) = x^2(k_s)$ is always satisfied.

When convergence is obtained, a state $x^1(k_s) \leq \underline{x}(k_s)$ is generated and the expression $\Omega_h \otimes x^1(k_s) \oplus \Psi_h \otimes U$ describes a trajectory starting from $x^1(k_s)$: The consideration of the initial state generalizes Algorithm 1 in [4]. The solution of the control problem 1 is given when condition $x^1(k_s) = \underline{x}(k_s)$ is satisfied. The algorithm also proposes an initial state $x^1(k_s)$ if the condition $x^1(k_s) = \underline{x}(k_s)$ is not satisfied.

E. Space Analysis

Let us recall that Algorithm 1 is polynomial but not in the strong sense (pseudo-polynomial complexity). In order to improve the complexity, we highlight two important cases where its resolution is more efficient. The strategy is that the control U calculated in Step 2 must produce the exact state trajectory $(\overline{x})^1$ which is expected in Step 1 of the first iteration of Algorithm 1.

***Theorem** 3:* The trajectory $(\overline{x})^2$ satisfies the system composed of (3) and the first relation of (4) when $\begin{pmatrix} I & \varepsilon \\ \varepsilon & \Psi_h \end{pmatrix} \otimes \begin{pmatrix} x^0(k_s) \\ U \end{pmatrix} = (\overline{x})^1$. Moreover, $(\overline{x})^2 = (\overline{x})^1$.

Proof: Step 1 calculates a solution $(\overline{x})^1$ which clearly satisfies $(\overline{x})^1 \leq D_h \backslash (\overline{x})^1$ which is equivalent to $(\overline{x})^1 \geq D_h \otimes (\overline{x})^1$ (we can also take any non-optimal $(\overline{x})$ satisfying this last inequality in the algorithm). Moreover, Property 2 shows that $X^2 = X'$. Let us now prove that $X^1 = X'$. Indeed, Property 2 shows that $X^1 \geq X' = \Omega_h \otimes x^1(k_s) \oplus \Psi_h \otimes U$ and we have $X^1 = \Psi_h \otimes U$ by assumption. Finally, $X^2 = X' = X^1$ and the algorithm stops at the first iteration $\langle 1 \rangle$. Remark 2 completes the previous equalities and gives the final equality $(\overline{x})^2 = (\overline{x})^1$. ■

Theorem 3 highlights an important case where Algorithm 1 gives the final state trajectory at the first iteration $\langle 1 \rangle$: Algorithm 1 is strongly polynomial since the resolution is reduced to a unique iteration composed of the simple application of elementary operations $\oplus$, $\otimes$, $\wedge$ and $\backslash$. Rewritten with a simpler notation, the condition of Theorem 3 is now analyzed: The problem is to check the solution existence of $\overline{u} \in \mathbb{R}^{\overline{q}}$ in the equality

$$\overline{B} \otimes \overline{u} = \overline{x} \text{ for any } \overline{x} \in \mathbb{R}^{\overline{n}} \text{ satisfying } \overline{x} \geq \overline{A} \otimes \overline{x} \qquad (6)$$

with the following notation: $\overline{B} = \begin{pmatrix} I & \varepsilon \\ \varepsilon & \Psi_h \end{pmatrix}$, $\overline{u} = \begin{pmatrix} x(k_s) \\ U \end{pmatrix}$, $\overline{x} = \begin{pmatrix} x(k_s) \\ X \end{pmatrix}$ and $\overline{A} = D_h$.

Let $\mathcal{S}_{\overline{x}}$ be the maximal set of vectors $\overline{x}$ such that $\overline{B} \otimes \overline{u} = \overline{x}$ and $\overline{x} \geq \overline{A} \otimes \overline{x}$ has a solution $\overline{x}$. Let $\overline{n}$=card($\overline{x}$) and $\overline{q}$=$card(\overline{u})$. The characterization of $\mathcal{S}_{\overline{x}}$ is as follows. Note that matrix $\overline{B}$ has no null rows as $\overline{x}$ is finite. Without loss of generality, we assume that matrix $\overline{B}$ has no null columns so that $\overline{B} \setminus \overline{x}$ is finite. We assume that the associated graph of $\overline{A}$ does not contain circuits with strictly positive weight so that $\overline{A}^* \in \mathbb{R}_{\max}^{\overline{n}x\overline{n}}$.

***Property** 3:* $\mathcal{S}_{\overline{x}}$ =Im$\overline{B}$ $\cap$Im$\overline{A}^*$.

Proof: Indeed, { $\overline{x} \in \mathbb{R}^{\overline{n}}$ such that $\overline{B} \otimes \overline{u} = \overline{x}$ with $\overline{u} \in \mathbb{R}^{\overline{q}}$}=Im$\overline{B}$ by definition of the image (Im$\overline{B}$ ={$\overline{B} \otimes \overline{u}$ such that $\overline{u} \in \mathbb{R}^{\overline{q}}$}). Moreover, { $\overline{x} \in \mathbb{R}^{\overline{n}}$ such that $\overline{x} \geq \overline{A} \otimes \overline{x}$}=Im$\overline{A}^*$ (Lemma 4.77 page 191 in [2], [12], [10]). ■

Below we study the existence of a solution $\overline{u}$ in (6). Let $K = \{1, .., \overline{n}\}$ be the set of indices of columns of $\overline{A}^*$. Property 4 follows from known results.

***Property** 4:* The greatest vector $\overline{u}$=$\overline{B} \backslash \overline{x}$ satisfies the system (6) if and only if $\overline{B} \otimes (\overline{B} \backslash \overline{A}^*) = \overline{A}^*$.

Proof: Considering the space $\mathcal{S}_{\overline{x}}$, Corollary 3 in [8] gives $\overline{B} \otimes \overline{u} = \overline{x} \Leftrightarrow \overline{B} \otimes (\overline{B} \setminus \overline{x}) = \overline{x}$ where we take $\overline{x} = \overline{A}^* \otimes \lambda$ for any $\lambda \in \mathbb{R}^{\overline{n}}$. So, we can deduce that the equality $\overline{B} \otimes (\overline{B} \backslash \overline{x}) = \overline{x}$ is true for each vector $(\overline{A}^*)_{.,k}$ for $k \in K$: the equality $\overline{B} \otimes (\overline{B} \backslash \overline{A}^*) = \overline{A}^*$ is satisfied.

Conversely, if the equality $\overline{B} \otimes (\overline{B} \backslash \overline{x}) = \overline{x}$ is satisfied for each vector $(\overline{A}^*)_{.,k}$ for $k \in K$, we prove below that the greatest vector $\overline{u}$=$\overline{B} \backslash \overline{x}$ satisfies (6), that is, $\overline{B} \otimes \overline{u} = \overline{x}$ with $\overline{x} \in$Im$\overline{A}^*$ for $\overline{u}$=$\overline{B} \backslash \overline{x}$: So, we have $\overline{B} \otimes (\overline{B} \backslash \overline{x}) = \overline{B} \otimes [\overline{B} \backslash (\overline{A}^* \otimes \lambda)] = \overline{B} \otimes [\overline{B} \backslash (\bigoplus_{k \in K} \lambda_k \otimes (\overline{A}^*)_{.,k})] \geq$

$\overline{B} \otimes [\bigoplus_{k \in K} \otimes \overline{B} \backslash (\lambda_k \otimes (\overline{A}^*)_{.,k})]$ (Property f2 page 180 in [2]) $= \overline{B} \otimes [\bigoplus_{k \in K} \lambda_k \otimes \overline{B} \backslash (\overline{A}^*)_{.,k}] =$ $\bigoplus_{k \in K} \lambda_k \otimes \overline{B} \otimes [\overline{B} \backslash (\overline{A}^*)_{.,k}] = \bigoplus_{k \in K} \lambda_k \otimes (\overline{A}^*)_{.,k} = \overline{A}^* \otimes \lambda = \overline{x}$

Finally, as $\overline{B} \otimes (\overline{B} \backslash \overline{x}) \geq \overline{x}$ for any vector $\overline{x} \in$Im$\overline{A}^*$ and $\overline{B} \otimes [\overline{B} \backslash \overline{x}] \leq \overline{x}$ by definition of the residuation, the result is obtained. ■

In short, Property 4 provides new conditions such that Algorithm 1 is strongly polynomial in the case described by Theorem 3: this result gives a practical test which uses only the entries of $\overline{B}$ and $\overline{A}^*$ without calculating the state and the control.

We now analyze the case where $\overline{B} \otimes (\overline{B} \backslash \overline{A}^*) = \overline{A}^*$ is not satisfied for all columns $(\overline{A}^*)_{.,k}$ but for some columns denoted $(\overline{A}^*)^{=}_{.,k}$. Let $L \subset K$ be the set of indices of column vectors $(\overline{A}^*)_{.,k}$ satisfying the equality $\overline{B} \otimes (\overline{B} \backslash (\overline{A}^*)_{.,k}) = (\overline{A}^*)_{.,k}$.

***Property** 5:* The greatest vector $\overline{u}$=$\overline{B} \backslash \overline{x}$ satisfies the system (6) if $\overline{x} \in$Im$(\overline{A}^*)^{=}$.

Proof: The proof is almost identical to the converse of the previous Property 4 but Im$\overline{A}^*$ and K are replaced by Im$(\overline{A}^*)^{=}$ and L, respectively. ■

Therefore, Algorithm 1 can stop for any iteration when $\overline{x} \in$Im$(\overline{A}^*)^{=}$. Since this condition is not guaranteed, the following control approach reduces the state space to the subspace Im$(\overline{A}^*)^{=}$. Applying Property 5, Algorithm 2 below calculates the greatest control U by generating a vector $\overline{x}$ in the subspace defined by Im$(\overline{A}^*)^{=}$, that is, $\overline{x} = (\overline{A}^*)^{=} \otimes \lambda$, with the problem constraint $\overline{x} \leq F$.

Algorithm 2

- Step 1: $\overline{x} \leftarrow (\overline{A}^*)^{=} \otimes ((\overline{A}^*)^{=} \backslash F)$
- Step 2: $U \leftarrow \Psi_h \backslash X$ ■

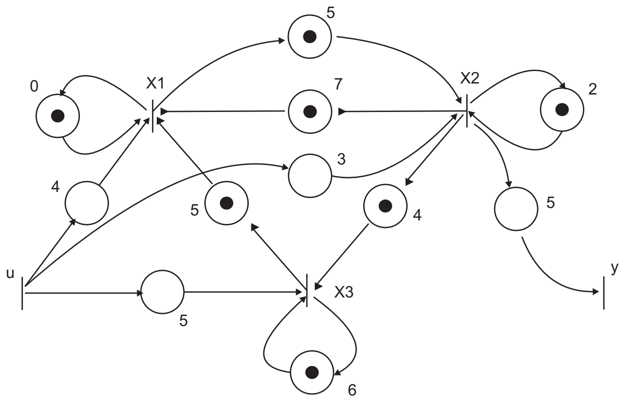

Fig. 1. Plant: Timed Event Graph

Under the condition $\mathrm{Im}(\overline{A}^{*})^{=} \neq \emptyset$ ($card(L) \neq 0$), Algorithm 2 compensates for the non-satisfaction of the condition $\overline{B} \otimes(\overline{B} \setminus \overline{A}^{*}) = \overline{A}^{*}$ by reducing the state space to $\mathrm{Im}(\overline{A}^{*})^{=}$. Algorithm 2 is strongly polynomial, contrary to Algorithm 1 that is considered in the general case.

F. Example

Timed Event Graph (Fig. 1): $A = \begin{pmatrix} 0 & 7 & 5 \\ 5 & 2 & \varepsilon \\ \varepsilon & 4 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$ and $C = \begin{pmatrix} \varepsilon & 5 & \varepsilon \end{pmatrix}$

P-time Event Graph (Fig. 2): $A^{=} = \begin{pmatrix} \varepsilon & \varepsilon & -11 \\ \varepsilon & \varepsilon & -11 \\ 1 & 1 & \varepsilon \end{pmatrix}$, $A^{-} = \begin{pmatrix} \varepsilon & 0 & 1 \\ 3 & \varepsilon & 4 \\ 1 & 2 & \varepsilon \end{pmatrix}$ and $A^{+} = \begin{pmatrix} \varepsilon & -5 & -9 \\ -8 & \varepsilon & -9 \\ -6 & -11 & \varepsilon \end{pmatrix}$

Taking $h = 3$, the desired output $z(k)$ and the initial condition $\underline{x}(k_s)$ are as follows:

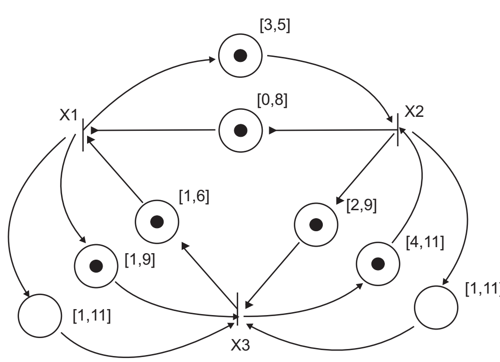

Fig. 2. Specifications: P-Time Event Graph

k	k_s+1	k_s+2	k_s+3
z	25	25	28

and $\underline{x}(k_s) = \begin{pmatrix} 2 & 0 & 3 \end{pmatrix}^t$. Needing three iterations, Algorithm 1 gives the following results:

k	k_s+1	k_s+2	k_s+3
u	4	10	16

,

k	k_s	k_s+1	k_s+2	k_s+3
x_1	2	8	14	20
x_2	0	7	13	19
x_3	3	9	15	21

and

k	k_s+1	k_s+2	k_s+3
y	12	18	24

.

Analysis. We have $\overline{n} = (h+1).n = 12$ and $\overline{q} = n + h.card(u)$= 6 as $n = 3$ and $h = 3$.

$$(\overline{B})^t = \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 & 3 & 5 & 10 & 9 & 11 & 16 & 15 & 17 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 4 & 3 & 5 & 10 & 9 & 11 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 4 & 3 & 5 \end{pmatrix}$$

$$\overline{A}^* = \begin{pmatrix} 0 & 0 & -1 & -7 & -5 & -8 & -13 & -12 & -14 & -20 & -18 & -21 \\ -2 & 0 & -3 & -8 & -7 & -9 & -15 & -13 & -16 & -21 & -20 & -22 \\ 1 & 1 & 0 & -6 & -4 & -7 & -12 & -11 & -13 & -19 & -17 & -20 \\ 6 & 7 & 5 & 0 & 1 & -1 & -7 & -5 & -8 & -13 & -12 & -14 \\ 5 & 5 & 4 & -2 & 0 & -3 & -8 & -7 & -9 & -15 & -13 & -16 \\ 7 & 8 & 6 & 1 & 2 & +0 & -6 & -4 & -7 & -12 & -11 & -13 \\ 12 & 13 & 11 & 6 & 7 & 5 & 0 & 1 & -1 & -7 & -5 & -8 \\ 11 & 12 & 10 & 5 & 6 & 4 & -2 & 0 & -3 & -8 & -7 & -9 \\ 13 & 14 & 12 & 7 & 8 & 6 & 1 & 2 & 0 & -6 & -4 & -7 \\ 18 & 19 & 17 & 12 & 13 & 11 & 6 & 7 & 5 & 0 & 1 & -2 \\ 17 & 18 & 16 & 11 & 12 & 10 & 5 & 6 & 4 & -2 & 0 & -3 \\ 19 & 20 & 18 & 13 & 14 & 12 & 7 & 8 & 6 & 1 & 2 & 0 \end{pmatrix}$$

As the equality $\overline{B} \otimes \overline{u} = (\overline{A}^*)_{.,k}$ does not hold for any $k \in K$ but for the subset of the columns $L = \{1, 3, 5\} \subset K = \{1, ..., 12\}$ which define the subspace Im$(\overline{A}^*)^=$, Property 4 cannot be applied and the convergence in one iteration of Algorithm 1 is not guaranteed: Several iterations are necessary here. Fortunately, Im$(\overline{A}^*)^= \neq \emptyset$ and Property 5 can be applied in the two following situations: Firstly, the numerical results in the above tables obtained at the convergence of Algorithm 1, show that $\overline{x} \in$Im$(\overline{A}^*)^=$ ($\overline{x} = 3 \otimes (\overline{A}^*)_{.,3}$) although this condition is not satisfied in the first iteration; secondly, Algorithm 2 based on the space Im$(\overline{A}^*)^=$ converges in one iteration.

V. Conclusion

In this paper, we propose a fixed point approach solving the control problem of Timed Event Graphs with specifications defined by P-time Event Graphs. Algorithm 1 makes it possible to determine the greatest state and control when the Timed Event Graph starts from an arbitrary initial condition. Since Algorithm 1 is pseudo-polynomial, we analyze the state space and highlight an important case where the resolution of Algorithm 1 is more efficient. The second technique leads to a reduction of the state space which leads to predictive control based on a space controller. Algorithm 1 under the condition $\overline{B} \otimes (\overline{B} \setminus \overline{A}^*) = \overline{A}^*$ and Algorithm 2 under the weak condition Im$(\overline{A}^*)^= \neq \emptyset$, are strongly polynomial contrary to many approaches for this subject which are only polynomial in the weak sense. More details about the CPU time of the Algorithms can be found in [5].

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