Two-staged Approach for Estimation of sequences in Partially Observable P-time Petri Nets on a sliding horizon with schedulability analysis

P. Declerck and P. Bonhomme

Abstract

In this paper, we consider the on-line estimation of current subsequences for Partially Observable P-time Petri Nets and their starting markings on a sliding horizon composed of steps defined by two successive occurrences of observable transition firings. We propose a general strategy composed of two phases: Phase 1 exploits a simplification of the P-time Petri net under the form of a Timed Petri net; considering a candidate count vector and the relevant starting marking proposed at Phase 1, Phase 2 makes a schedulability analysis by building a system of relations which can be represented by an acyclic conflict-free computation graph. The complete approach avoids the generation of sets which is generally time and space consuming, and provides an optimal solution for each subproblem by using efficient standard tools.

Keywords: P-timed Petri nets, P-time Petri nets, Partially Observable, Estimation, Sliding Horizon.

I. Introduction

A. Aim and motivation

The problem of estimating the state of a dynamic system is a fundamental issue in system theory. Indeed, it is not always possible to associate a sensor with each state

P. Declerck is with the University of Angers, Lisa-Laris, 62 avenue Notre-Dame du Lac, F-49000 Angers, France, philippe.declerck@univ-angers.fr

P. Bonhomme is with the University of Tours, INSA CVL, LIFAT EA 6300 CNRS, ROOT ERL CNRS 7002, patrice.bonhomme@insa-cvl.fr

due to the cost and the physical location in many processes. This characteristic can be found in many discrete event systems such as manufacturing systems, microcircuit design, transportation systems, and the food industry. The unknown data can be crucial for the control system which supervises the process. Another motivation is the fault diagnosis as the occurrence of faults can perturb or stop the production of the process.

As any process clearly follows a time evolution, the consideration of timing information is important for any problem as control and fault detection, and is indispensable to define the concept of production rate. In particular, the knowledge of the past events and their timestamps guarantees the order of these events and allows future actions to be determined by the control system. In Petri nets (PNs) which is a well-suited model for discrete event systems, temporal intervals can be associated with places or transitions for Time PNs, but the corresponding subclasses (P-Time PNs and T-Time PNs) are fundamentally different and must not be confused. In Time PNs [16], a temporal interval of firing is associated with each transition enabled by the marking while a temporal interval of availability is associated with each token which enters a place in P-Time PNs [2] [3] [8] [9]. In this paper, we focus on P-time PNs whose evolution can undergo token deaths which express the loss of resources or parts and failures to meet time specifications. Applications can be found in many fields as production systems, food industry and transportation systems [7] [1] [10]. Up to now, the consideration of the time factor as intervals in P-time PNs containing conflicts and synchronizations, remains an open problem in estimation as, at the best of our knowledge, only few papers have considered this type of general model: [2] [3] provide an estimation of marking sets consistent with an observed label sequence. Contrary to these studies, we focus on estimation of an optimal subsequence with respect to a general linear criterion in this paper. Particularly, the aim is the on-line estimation of current subsequences for Partially Observable P-time Petri Nets and its starting marking on a sliding horizon composed of steps defined by two successive occurrences of observable transition firings.

B. Contribution of the paper

Let us highlight the main advances of the paper before the technical treatment of the problem.

1) The complete approach avoids the generation of marking sets which is generally time and space consuming.

2) The proposed approach uses standard computer programs of Linear Programming and Integer Linear Programming.

3) The general linear criterion is not restrictive and can be adapted to many fields. This criterion can be a global price which is expressed with a function depending of the costs and the gains of the process [12] or expresses the presence and absence of faults in fault detection [5] [6].

4) The solution of approach can be represented by a simple form which is an acyclic conflict-free computation graph which clearly describes the connections between the events of the obtained sequence.

5) The approach gives a schedulability analysis of the obtained sequence particularly with respect to the P-time aspect.

6) The considered model is general (P-time PNs) while the assumptions taken in this paper are relatively not restrictive. The assumption of earliest functioning firing rule, boundedness of the marking and the hypothesis of acyclicity are not considered in this article contrary to many papers in this topic. The Petri net can contain self-loops without null time durations and any transition can have several firings at each time (particularly, the occurrences of observable events can be simultaneous). No firing rule (as a race policy [14]) is taken a priori.

C. Strategy and context

Many papers and books in "max-plus algebra" show that an algebraic model of Petri nets exploiting the "counter" form or the "dater" form facilitates the study and leads to powerful approaches. A large number of applications can be found in [15]. Allowing to describe the time sequences of complex Petri nets with weights on the arcs, the "counter" form represents the number of events at each moment while the "dater" form describes each event by a numbered date and can express complex synchronizations which can be found in P-time event graphs and time stream event graphs. However, the building of the algebraic model faces some specific difficulties [9]:

- The counter approach cannot be used to model P-time PNs and particularly the constraints connected to the upper bounds.

- The unique efficient way to model P-time PNs is the dater approach, but this technique is only adapted to the structure of event graph which cannot take account of conflict structures.

To summarize, neither the counter approach, nor dater approach can be used directly to completely describe the time trajectories of the algebraic model [9] for the class of P-time PNs. We can hope that ideally the scheduling problem should be solved by a unique optimization problem with respect to some criteria but an important conclusion is that no algebraic approach can solve directly the general estimation problem for any time model by giving an optimal solution. By reason of the inherent complexity of the problem, suboptimal approaches must be developed. To solve this difficulty, we propose a strategy which is a two-phased approach: the Phase 1 proposes sequences based on a simplification of the time model while the Phase 2 makes a schedulability of the candidate vector by building the relevant time schedule. The aims of the Phases 1 and 2 are now presented.

- The Phase 1 is to partially simplify the time aspect of the problem which is relaxed. The upper bounds of the time durations are neglected in the P-time PN which becomes a P-timed PN which can be described by an algebraic model using the counter technique and can be optimized by known tools [10]: the result for each step is a pair composed of a time trajectory which is a candidate solution to the global problem and also a starting marking. However, the simplification implies that these sequences do not correspond to the desired schedule in general: remember that a known result is that for P-time Event Graphs, the minimum trajectory also depends on the upper bounds of the time durations [8] (respectively, the maximum trajectory also depends on the lower bounds).

- The last Phase 2 is to adapt this first result to the P-time PN as the computed trajectory can be inconsistent with the constraints of the model. Therefore, the strategy is to consider the untimed sequence relevant to the time trajectory and to determine a trajectory coherent with the desired model, here a P-time PN. In other words, we make a consistency analysis also named a schedulability analysis. Generalizing [3], this analysis is possible if a variant of the dater technique is used.

In this paper, the aim is to treat P-time PNs by a simplification under the form of a P-timed PN while the paper [11] considers P-timed PNs by building an untimed PN and analyzing the candidate count vectors. The objective of [12] is also different as this study focuses on the estimation of sequences in untimed PNs. The proposed technique differs from [2] which develops a state observer of the form of an automaton whose nodes can be viewed as macro-markings and from [3] where the state observer for P-time labeled PNs with some indistinguishable events is built online in a decentralized context.

In this paper, we assume the following assumptions for the different PNs under investigation:

- Assumption $\mathcal{AS}-1$: the incidence matrices and the initial marking (denoted M^{init} below) are known.

- Assumption $\mathcal{AS}-2$: the Petri net is live.

- Assumption $\mathcal{AS}-3$: the observations are distinguishable, that is, the same label cannot be associated with more than one observable transition.

- Assumption $\mathcal{AS}-4$: the origin of firing count is absolute. In that case, the firing numbers (also named counters) of the successive observations can be defined with respect to this origin and are known.

- Assumption $\mathcal{AS}-5$: the time is discrete and is defined over $\mathbb{Z}$. Note that the time depends on the origin of time and can be negative.

- Assumption $\mathcal{AS}-6$: the time durations of the places of the Timed and P-time PNs are over $\mathbb{N}$ and are known.

To simplify the presentation and the notation, the step horizon is limited to two successive steps only.

The models considered in this paper are: the P-timed PN in section IV-D; the P-time PN in section V for the Phase 2.

The paper is organized as follows. In Section II, we first present a reminder of the basics of untimed PNs, notations for estimation and description of time trajectories. After presenting the aim of the two-phased strategy, the following section IV describes the Phase 1 where we present the procedure of the sequence estimation. Contrary to [12], we consider time models and focus on the adaptation of the estimation to a horizon composed of two successive steps. The generation of the relevant untimed sequences and the starting marking is discussed. Then, the Phase 2 in Section V is a schedulability analysis of the proposed sequence which must be consistent with a P-time PN: adapting the technique [3] to the current step with a two-staged systematic approach, the first stage is the generation of a token evolution based on the FIFO assumption while the second stage is the establishment of the time relations and the checking of their consistency. The approach is illustrated by different pedagogical examples where Example 2 is a running example.

II. Preliminary

A. Notations for Petri nets and models

The notation $|Z|$ is the cardinality of set Z and the notation A^Tcorresponds to the transpose of matrix A. The $i-th$ row (respectively $j-th$ column) of A is denoted $A_{i,.}$(respectively $A_{.,j}$) . A Place/Transition (P/TR) net is the structure $N = (P, TR, W^+, W^-)$, where P is a set of $|P|$ places and TR is a set of $|TR|$ transitions. The matrices W^+ and W^- are respectively the $|P| \times |TR|$ post and pre-incidence matrices over $\mathbb{N}$, where each row $l \in \{1, ..., |P|\}$ specifies the weight of the incoming and outgoing arcs of the place $p_l \in P$. The incidence matrix is $W = W^+ - W^-$. The preset and postset of the node $v \in P \bigcup TR$ are denoted by ${}^\bullet v$ and $v^\bullet$, respectively. The notation Ω^* represents the set of firing sequences, denoted σ, consisting of transitions of the set $\Omega \subset TR$. The vector $\overline{\sigma}$ of dimension $|TR|$ expresses the firing vector or count vector of the sequence $\sigma \in TR^*$, where the i-th component $\overline{\sigma_i}$ is the firing number of the transition $tr_i \in TR$ which is fired $\overline{\sigma_i}$ times in the sequence σ.

The marking of the set of places P is a vector $M \in \mathbb{N}^{|P|}$ that assigns to each place $p_i \in P$ a non-negative integer number of tokens M_i, represented by black dots. The i-th component M_i is also written $M(p_i)$. The marking M reached from the initial marking M^{init} (which replaces the usual notation M_0) by firing the sequence σ can be calculated by the fundamental relation: $M = M^{init} + W.\overline{\sigma}$. The transition tr is enabled at M if $M \geq W^-(.,tr)$ and may be fired yielding the marking $M' = M + W(.,tr)$. We write $M[\sigma \succ$ to denote that the sequence of transitions σ is enabled at M, and we write $M[\sigma \succ M'$ to denote that the firing of σ yields M'.

P-timed PNs allow the modeling of discrete event systems with time constraints of the tokens inside the places where each time duration denoted T_l over $\mathbb{N}$ describes a *minimum* sojourn time of a token in place $p_l \in P$.

Definition *1:* A P-timed PN is a triple (GR, M^{init}, f) where GR is a Petri net, M^{init} is the initial marking and the mapping f is defined by $p_l \mapsto T_l$ with $0 \leq T_l$ from P to $\mathbb{R}^+$.

Like the P-timed PNs, the P-time PNs allow the modeling of discrete event systems with time constraints for tokens to remain in place. A temporal interval $[T_l^-, T_l^+]$ defined in $\mathbb{R}^+ \times (\mathbb{R}^+ \cup \{+\infty\})$ where T_l^- is the lower bound and T_l^+ is the upper bound is associated with each place $p_l \in P$.

***Definition** 2:* A P-time PN is a triple (GR, M^{init}, f) where GR is a Petri net, M^{init} is the initial marking and the mapping f is defined by $p_l \mapsto [T_l^-, T_l^+]$ with $0 \leq T_l^- \leq T_l^+$ from P to $\mathbb{R}^+ \times (\mathbb{R}^+ \cup \{+\infty\})$.

The token must stay in this place during the *minimum* residence duration T_l^-. Before this duration, the token is in a state of unavailability for firing an output transition. The value $T_l^+$is a *maximum* residence duration after which the token must leave place p_l. If not, the system finds itself in a token-dead state. The token is therefore available to fire an output transition in the time interval $[T_l^-, T_l^+]$.

B. General notations for estimation

A labeling function $L : TR \to AL \cup \{\varepsilon\}$ assigns to each transition $tr \in TR$ either a symbol from a given alphabet AL or the empty string ε. In a partially observed PN, we assume that the set of transitions TR can be partitioned as $TR = TR_{obs} \bigcup TR_{un}$, where the set TR_{obs} (respectively, TR_{un}) is the set of observable transitions associated with a label of AL (respectively, the empty string ε). In this paper, we assume that the same label of AL cannot be associated with more than one transition of TR_{obs} (Assumption $\mathcal{AS}-3$).

The $TR_{un}-$induced subnet of the Petri net N is defined as the new net $N_{un} = (P,$ TR_{un}, W_{un}^+, $W_{un}^-)$, where W_{un}^+ and W_{un}^- (respectively, W_{obs}^+ and W_{obs}^-) are the restrictions of W^+ and W^- to $P \times TR_{un}$ (respectively, $P \times TR_{obs}$). Therefore, $W_{un} = W_{un}^+ - W_{un}^-$ (respectively, $W_{obs} = W_{obs}^+ - W_{obs}^-$). A reorganization of the columns with regards to TR_{un} and TR_{obs} yields $W = \begin{pmatrix} W_{un} & W_{obs} \end{pmatrix}$. Notation x_i expresses an unobservable transition, belonging to TR_{un} while an observable transition belonging to TR_{obs} is denoted y_i.

The notation of the count vectors is taken for $\overline{x}$ of dimension $|TR_{un}|$ and $\overline{y}$ of dimension $|TR_{obs}|$. The reorganization of the components of $\overline{\sigma}$ yields $\overline{\sigma} = \begin{pmatrix} \overline{x}^T & \overline{y}^T \end{pmatrix}^T$.

Starting from the marking $M^{<1>}$ which is the initial marking M^{init}, the estimation of the current unobservable sequence is based on the treatment of the data produced by the transitions observed successively in an on-line procedure at the dates $t^{<1>}, t^{<2>}, t^{<3>}, \ldots$ where: $t^{<k>}$ is the $k-th$ firing date of a set of observable transitions; moreover, $< k >$ is the $k-th$ step relevant to a time interval $[t^{<k-1>}, t^{<k>}]$ for $k \geq 1$. By construction, $t^{<k-1>} \neq t^{<k>}$ for $k \geq 1$ and $t^{<1>}$ is relevant to the first observations. The notation $\overline{y}^{<k>}$ represents the count vector of observable transitions firing at time $t^{<k>}$ exactly for step $< k >$. Notation $\overline{x}^{<k>}$ represents the count vector for the unobservable transitions TR_{un} for step $< k >$. From the marking $M^{<k>}$ at step $< k >$, the transition firings relevant to $\overline{x}^{<k>}$ and $\overline{y}^{<k>}$ allow the establishment of marking $M^{<k+1>}$: formally, $M^{<k>}[\sigma^{<k>} \succ M^{<k+1>}$ such that $\overline{\sigma}^{<k>} = \begin{pmatrix} (\overline{x}^{<k>})^T & (\overline{y}^{<k>})^T \end{pmatrix}^T$. As $M^{<1>}$ is the initial marking relevant to time $t^{<0>} = 0$, we assume that $\overline{x}^{<k>} = 0$ and $\overline{y}^{<k>} = 0$ for $k \leq 0$. So, the estimation must consider $M^{<1>}[x^{<1>}y^{<1>} \succ M^{<2>}$ for step $< 1 >$ where the time horizon is $[t^{<0>}, t^{<1>}]$ and $t^{<1>}$ is relevant to the first observation $y^{<1>}$, then $M^{<2>}[x^{<2>}y^{<2>} \succ M^{<3>}$ for step $< 2 >$ where the time horizon is $[t^{<1>}, t^{<2>}]$ and $t^{<2>}$ is relevant to the second observation $y^{<2>}$, and so on. So, the estimation of $x^{<k>}$ must treat $M^{<k>}[x^{<k>}y^{<k>} \succ M^{<k+1>}$ for step $< k >$ where the time horizon is $[t^{<k-1>}, t^{<k>}]$. Note that *these notations are not cumulative* as we can have $\overline{x}^{<3>} = 0$ but $\overline{x}^{<1>} \neq 0$ and $\overline{x}^{<2>} \neq 0$: the condition $\overline{x}^{<1>} \leq \overline{x}^{<2>} \leq \overline{x}^{<3>}$ does not hold.

III. Aim and principle of the two-phased approach

Let us consider a P-time PN where the incidence matrix W and the initial marking $M^{<1>}$ are known. We focus on the subsequences of the unobservable firing events of the transitions of TR_{un} for each step $< k >$. Given a sequence of the observed firing events of the transitions of TR_{obs} generated by the activity of the Petri net, we want to find the subsequences of the unobservable firing events of the transitions of TR_{un} for each step $< k >$ which are ideally optimal with respect to a general criterion.

Therefore, the framework of this paper is as follows.

- In Phase 1, the objective is to propose candidate untimed sequences denoted $\sigma^{<k>}$ starting from a starting marking $M^{<k>}$ for each step $< k >$. The model is a P-timed PN which is a simplified form of the P-time Petri Net with respect to the time aspect. Extracted from the computed trajectory by neglecting time, the untimed sequence σ is a data analyzed in the following phase. Precisely, the objective of Phase 1 is to find a pair $(M^{<k>}, \sigma^{<k>})$ composed of a starting marking $M^{<k>}$ leading to a sequence $\sigma^{<k>}$.

- In Phase 2, the model is a P-time PN. The objective is to analyze the time consistency of the pair $(M^{<k>}, \sigma^{<k>})$ by giving a time trajectory. Note that the determination of this trajectory which contains $t^{<k-1>}$ and $t^{<k>}$ implies the deduction of the corresponding horizon.

IV. Phase 1: simplified optimization problem

A. Problem of Phase 1

Considering a P-timed PN which is a simplified form of the P-time Petri Net, the objective is the estimation of a pair $(M^{<k>}, \sigma^{<k>})$ composed of a sequence $\sigma^{<k>}$ and a relevant starting marking $M^{<k>}$. These two components need the estimation of a time trajectory described in Section IV-D which is based on an Integer Linear Programming Problem (ILP problem), that is an optimization of a linear criterion where the constraint system is defined by (4).

We now present the necessary notations, the algebraic model of the P-timed PNs and the polyhedron developed on a time horizon which will be analyzed in the following section.

B. Counter notations for time trajectories

As the modelling of time must be made, the previous notations must be completed with the following notations which are *cumulative* from the starting of the Petri net. If we consider that the origin of time is at the starting of the Petri net, each transition x_i is associated with the *number of events* that happen *before or at time* θ and this number is denoted $\overline{x_i(\theta)}$. Assuming that the events can only occur at $\theta \geq 1$ with $\theta \in \mathbb{Z}$ (Assumption $\mathcal{AS}-5$), we have $x(\theta) = 0$ for $\theta \leq 0$.

The above notions are usual in max-plus algebra but must be adapted to the problem. Let $\overline{y(\theta)}$ (respectively, $\overline{x(\theta)}$) be the count subvector at time θ such that the relevant transitions belong to the set of observable transitions TR_{obs} (respectively, unobservable transitions TR_{un}).

The on-line diagnosis procedure is based on a sliding horizon which is relevant to a current step $< k >$ for $k \geq 1$ separating two successive dates of observations $t^{<k-1>}, t^{<k>}$ of the last successive observations $y^{<k-1>}$, $y^{<k>}$. Step $< 0 >$ corresponds to the initialization with $\overline{x}^{<0>} = 0$ and $\overline{y}^{<0>} = 0$ while the state computation starts at step $< 1 >$. A time horizon relevant to step $< k >$ can be defined

$$h^{<k>} = t^{<k>} - t^{<k-1>}$$

and can be calculated as the two dates at step $< k >$ are known. The notation $h^{<k>}$ clearly shows that the horizon depends on the step. For simplicity of the writing, $h^{<k>}$ and $t^{<k>}$ are denoted h and t respectively in this section. Considering that the time is discrete (Assumption $\mathcal{AS}-5$), the observations and the unknown time trajectory on the horizon h are defined as follows:

$\overline{\gamma}^{<k>} = \begin{pmatrix} (\overline{y(t-h)})^T & (\overline{y(t-h+1)})^T & \ldots & (\overline{y(t-1)})^T & (\overline{y(t)})^T \end{pmatrix}^T$ and

$\overline{\varkappa}^{<k>} = \begin{pmatrix} (\overline{x(t-h)})^T & (\overline{x(t-h+1)})^T & \ldots & (\overline{x(t-1)})^T & (\overline{x(t)})^T \end{pmatrix}^T$.

Contrary to $\overline{y}^{<k>}$ introduced at the previous section, $\overline{y(\theta)}$ is cumulative on all past iterations, and so, $\overline{y(t)} = \sum_{k'=0,\ldots,k} \overline{y}^{<k'>}$. The introduction of the notation $\overline{y}^{<0>\rightarrow<k>} = \sum_{k'=0,\ldots,k} \overline{y}^{<k'>}$ allows to give short expressions of the known data for the iteration $< k >$ which are:

$$\begin{aligned} \overline{y(t-h)} &= \overline{y}^{<0>\rightarrow<k-1>} \\ \overline{y(t)} &= \overline{y}^{<0>\rightarrow<k>} \end{aligned} \tag{1}$$

and $\overline{y}^{<k>} = \overline{y(t)} - \overline{y(t-h)}$ with $t = t^{<k>}$ and $t - h = t^{<k-1>}$. By construction, there is no firing of an observable transition in step $< k >$ for $\theta \in \{t-h+1, t-h+2, \ldots, t-1\}$ and formally, $\overline{y(\theta)} = \overline{y(t-h)} = \overline{y}^{<0>\rightarrow<k-1>}$ for $t-h+1 \leq \theta \leq t-1$.

Similarly, the notation $\overline{x(\theta)}$ is cumulative on all past iterations contrary to $\overline{x}^{<k>}$ and particularly $\overline{x(t)} = \sum_{k'=0,\ldots,k} \overline{x}^{<k'>}$. The notation $\overline{x}^{<0>\rightarrow<k>} = \sum_{k'=0,\ldots,k} \overline{x}^{<k'>}$ allows to write

$$\begin{aligned} \overline{x(t-h)} &= \overline{x}^{<0>\rightarrow<k-1>} \\ \overline{x(t)} &= \overline{x}^{<0>\rightarrow<k>} \end{aligned} \tag{2}$$

and $\overline{x}^{<k>} = \overline{x(t)} - \overline{x(t-h)}$.

In this paper, we consider that the firing numbers of observations are known (Assumption $\mathcal{AS}-4$) and can be defined with respect to a unique origin of count corresponding to the initial marking where no event has still occurred. Practically, the event counter starts when the Petri net starts.

C. Generation of the untimed sequence and starting marking

The analysis of the time trajectory $\overline{\varkappa}^{<k>}$ for step $< k >$ (and particularly its variations of the count components) allows to simply provide the untimed sequence $\sigma^{<k>}$.

***Property** 1:* For $\theta \in \{t-h+1, t-h+2, \ldots, t\}$, $\overline{x_i(\theta)} - \overline{x_i(\theta-1)}$ gives the number of firings of unobservable transition $i \in \{1, \ldots, |TR_{un}|\}$ at θ. ■

Proof. This result is the direct application of the counter notation. ■

Example 1.

Let $h = 2$, $|TR_{un}| = 2$ and $\overline{\varkappa}^{<k>} = \left(\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 \end{pmatrix} \right)^T$ (in order to ease the understanding of the vector, the subvectors are presented). The change $\begin{pmatrix} 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 2 \end{pmatrix}$ corresponds to the firing of x_1 at $t-1$ while the non-evolution $\begin{pmatrix} 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 2 \end{pmatrix}$ expresses no firing. ■

Following the increasing time, the relevant events can be numbered. If two events or more occur at the same time, an arbitrary order can be fixed.

Starting marking

To complete the useful data, we need to obtain $M^{<k>}$ which is made with the following property.

Property *2:* Knowing $\overline{\varkappa}^{<k>}$, the estimate $M^{<k>}$ is deduced with $M^{<k>} = M^{<1>} + W.\overline{\sigma(t^{<k-1>})}$

Proof.

As the notations $\overline{x(\theta)}$ and $\overline{y(\theta)})$ are cumulative on all past iterations, the marking at θ can directly be deduced with $M = M^{<1>} + W.\overline{\sigma(\theta)}$ with $\overline{\sigma(\theta)} = \begin{pmatrix} (\overline{x(\theta)})^T & (\overline{y(\theta)})^T \end{pmatrix}^T$. ■

Therefore, if we desire to make the estimation of a pair $(M^{<k>}, \sigma^{<k>})$, we must estimate the marking at step $< k-1 >$ at least: to simplify the presentation, we assume that the general estimation of the pair $(M^{<k>}, \sigma^{<k>})$ considers the step horizon $\{< k-1 >, < k >\}$ which corresponds to the time horizon $\{t^{<k-2>}, t^{<k-2>}+1, \dots, t^{<k-1>}-1, t^{<k-1>}, t^{<k-1>}+1, \dots, t^{<k>}-1, t^{<k>}\}$ with

$h^{<k>} = t^{<k>} - t^{<k-1>}$ and $h^{<k-1>} = t^{<k-1>} - t^{<k-2>}$.

So, $t^{<k-1>} = t^{<k>} - h^{<k>}$, $t^{<k-2>} = t^{<k>} - h^{<k>} - h^{<k-1>}$ and the horizon for the step horizon $\{< k-1 >, < k >\}$ is

$$h^{<k-1> \rightarrow <k>} = h^{<k>} + h^{<k-1>}.$$

This point implies a modification of $\overline{\varkappa}^{<k>}$ and $\overline{\gamma}^{<k>}$ to a larger step horizon. The new notations relevant to two successive steps are as follows. We take $\overline{\gamma}^{<k-1> \rightarrow <k>} = \begin{pmatrix} \overline{y(t^{<k-2>})}^T & \overline{y(t^{<k-2>}+1)}^T & \dots & \overline{y(t^{<k-1>}-1)}^T & \overline{y(t^{<k-1>})}^T & \overline{y(t^{<k-1>}+1)}^T & \dots & \overline{y(t^{<k>}-1)}^T & \overline{y(t^{<k>})}^T \end{pmatrix}^T$ and $\overline{\varkappa}^{<k-1> \rightarrow <k>} = \begin{pmatrix} \overline{x(t^{<k-2>})}^T & \overline{x(t^{<k-2>}+1)}^T & \dots & \overline{x(t^{<k-1>}-1)}^T & \overline{x(t^{<k-1>})}^T & \overline{x(t^{<k-1>}+1)}^T & \dots & \overline{x(t^{<k>}-1)}^T & \overline{x(t^{<k>})}^T \end{pmatrix}^T$

D. P-Timed Petri net and trajectories

Let us remember the following algebraic model based on the firing event numbers of transitions which allows to describe the evolution of a place of the P-timed PN. For each place $p_l \in P$ which is associated with a time duration T_l, we can write that the output flow of tokens at time $\theta \in \{t-h+1, t-h+2, \ldots, t\}$ is lower than or equal to the addition of the initial marking of p_l, that is, $M_l^{<1>}$, and the input flow with a delay $T_l \geq 0$.

$$W^-_{l,.}\overline{x(\theta)} \leq W^+_{l,.}\overline{x(\theta - T_l)} + M_l^{<1>} \ . \tag{3}$$

As explained in [9] [10], the set of the previous inequalities can be rewritten without reduction of generality such that each equality is associated with a time duration of a place which is equal to zero or one. Now, we can deduce the polyhedron (4) which must be satisfied by the trajectories of the Petri net for a sliding horizon $h^{<k-1>\rightarrow<k>}$of length $h^{<k-1>} + h^{<k>}$ relevant to the step horizon $\{< k-1 >, < k >\}$. If $h^{<k-1>\rightarrow<k>}$ is simply denoted h', the dimension of vector $\overline{\varkappa}^{<k-1>\rightarrow<k>}$ is denoted by $n = (h'+1).|TR_{un}|$ while the dimension of vector $\overline{\gamma}^{<k-1>\rightarrow<k>}$ is $(h'+1).|TR_{obs}|$. Below the form generalizes the polyhedron exploited in [12]. Moreover, the time durations can be unitary but also null.

$$\mathbf{A}_1 \cdot \overline{\varkappa}^{<k-1>\rightarrow<k>} \leq \mathbf{b}_1 \tag{4}$$

where $\mathbf{b}_1 = \mathbf{C}_1 - \mathbf{B}_1.\overline{\gamma}^{<k-1>\rightarrow<k>}$ with $\mathbf{C}_1 = \begin{pmatrix} (M^{<1>})^T & (M^{<1>})^T & \ldots & (M^{<1>})^T \end{pmatrix}^T$. The components of matrices $\mathbf{A}_1$ and $\mathbf{B}_1$ depend on the components of W^+_{un}, W^-_{un} and W^+_{obs}, W^-_{obs}, respectively [10] [12]. Note that no marking must be estimated as vector $\mathbf{C}_1$ only depends on the initial marking. Depending of the current step horizon $\{< k-1 >, < k >\}$, all the matrices and vectors can easily be established under the condition that the horizons $h^{<k-1>}, h^{<k>}$ are known. The dimensions of matrices $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$ and column vector $\mathbf{b}_1$ are respectively $(h'.|P|$ x $n)$, $(h'.|P|$ x $(h'+1).|TR_{obs}|)$, $(h'.|P|$ x $1)$ and $(h'.|P|$ x $1)$.

In the optimization with system (4), different criteria can be exploited. Some criteria can be used in diagnosis [5]. Generalizing a criterion presented in [12], we can focus on the minimum or maximum number of events of a sequence inside the step $< k >$, that is, more formally, the sum of the event numbers with $c_{1\text{x}|TR_{un}|}.\overline{x}^{<k>}$ where $c_{1\text{x}|TR_{un}|}$ is unitary. As $\overline{x}^{<k>} = \overline{x(t)} - \overline{x(t-h)}$, we can also compute the difference between $(\overline{x(t)})$ and $(\overline{x(t-h)})$ by taking $c^{<k>}.\overline{\varkappa}$ with $c^{<k>} = \begin{pmatrix} -c_{1\text{x}|TR_{un}|} & 0 & \ldots & 0 & c_{1\text{x}|TR_{un}|} \end{pmatrix}$ where $c_{1\text{x}|TR_{un}|}$ is unitary. The relevant criterion is $c_{diff}.\overline{\varkappa}^{<k-1>\rightarrow<k>}$ with c_{diff}=$\begin{pmatrix} c^{<k-1>} & c^{<k>} \end{pmatrix}$ and $c^{<k-1>} = 0$ (diff: difference).

E. Example 2

Let us consider the Labelled Petri Net of Fig. 1.

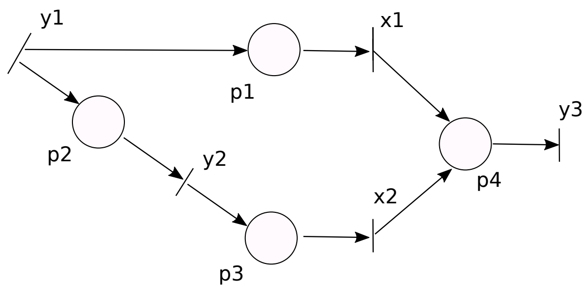

Fig. 1. Example 3: a simple Petri net

Let $TR_{un} = \{x_1, x_2\}$ and $TR_{obs} = \{y_1, y_2, y_3\}$. The alphabet AL is $\{a, b, c\}$ and $L(y_1) = a$, $L(y_2) = b$ and $L(y_2) = c$. Hence, $|TR_{un}| = 2$, $|TR_{obs}| = 3$, $|AL| = 3$, $|P| = 4$.

The incidence matrices W_{un}, W_{obs}, W^-_{obs} and the initial marking are as follows: $W_{un} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}$, $W_{obs} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $W^-_{obs} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $M^{<1>} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T$. All the places present unitary time durations.

Consider the sequence of observable transitions $y_1 y_2 y_3$ where the observations are not simultaneous. The sequence which starts from the initial marking $M^{<1>}$ corresponds to three steps.

a) In the first instance, we consider a simplified form of the above problem by considering only a unique step $< 3 >$ and an optimization with c unitary. Let $t^{<2>} = 3$, $t^{<3>} = 6$. So, $h^{<k>} = t^{<k>} - t^{<k-1>} = 3$, $\overline{\gamma}^{<3>} =$ $\begin{pmatrix} (\overline{y(t-3)})^T & (\overline{y(t-2)})^T & (\overline{y(t-1)})^T & (\overline{y(t)})^T \end{pmatrix}^T$ and $\overline{\varkappa}^{<3>} =$ $\begin{pmatrix} (\overline{x(t-3)})^T & (\overline{x(t-2)})^T & (\overline{x(t-1)})^T & (\overline{x(t)})^T \end{pmatrix}^T$. From the sequence of observable transitions $y_1 y_2 y_3$, we obtain $\overline{\gamma}^{<3>} = \left(\begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \right)^T$ which expresses the firings of y_1 and y_2 at the latest time $t-3$ ($(\overline{y(t-3)})^T = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T$), and the firing of y_3 at time t. Taking $t = 5$ arbitrarily, we must solve the system

$$\begin{cases} p_1 : \overline{x}_1(\theta) \leq \overline{y}_1(\theta - 1) \\ p_2 : \overline{y}_2(\theta) \leq \overline{y}_1(\theta - 1) \\ p_3 : \overline{x}_2(\theta) \leq \overline{y}_2(\theta - 1) \\ p_4 : \overline{y}_3(\theta) \leq \overline{x}_1(\theta - 1) + \overline{x}_2(\theta - 1) \end{cases} \tag{5}$$

for $\theta = \{3,4,5\}$ with $\overline{x}_1(\theta) \geq \overline{x}_1(\theta-1) \geq 0$ and $\overline{x}_2(\theta) \geq \overline{x}_2(\theta-1) \geq 0$. If we search $\min(c.\overline{\varkappa}^{<3>})$ with c unitary, two optimal solutions are $\overline{\varkappa}^{<3>} = \left(\begin{pmatrix} 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\right)^T$ and $\overline{\varkappa}^{<3>} = \left(\begin{pmatrix} 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}\right)^T$ which correspond to the firing of x_1 and x_2 at $\theta = 4$, respectively. Completed with $\overline{\gamma}^{<3>}$ observation y_3, the possible subsequences are x_1y_3 and x_2y_3 which can be deduced from the analysis of the Petri net. Note that a solution is also $\overline{\varkappa}^{<3>} = \left(\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\right)^T$ but this solution is not optimal.

b) As in the proposed approach, we now consider the step horizon composed of successive steps $<2>, <3>$. Let $t^{<1>} = 1$, $t^{<2>} = 3$ and $t^{<3>} = 6$. So, $h^{<2>} = t^{<2>} - t^{<1>} = 2$, $h^{<3>} = t^{<3>} - t^{<2>} = 3$, $\overline{\gamma}^{<2>\to<3>} = \begin{pmatrix} \overline{y(1)}^T & \overline{y(2)}^T & \overline{y(3)}^T & \overline{y(4)}^T & \overline{y(5)}^T & \overline{y(6)}^T \end{pmatrix}^T =$ $(\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix})^T$ and $\overline{\varkappa}^{<2>\to<3>} =$ $\begin{pmatrix} \overline{x(1)}^T & \overline{x(2)}^T & \overline{x(3)}^T & \overline{x(4)}^T & \overline{x(5)}^T & \overline{x(6)}^T \end{pmatrix}^T$.

System (4) is defined by System (5) for $\theta = \{2,3,4,5,6\}$. If we search $\min(c.\overline{\varkappa}^{<3>})$ with c unitary, two optimal solutions are $\overline{\varkappa}^{<2>\to<3>} =$ $\left(\begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}\right)^T$. and $\overline{\varkappa}^{<2>\to<3>} =$ $\left(\begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}\right)^T$ which are relevant to the firings of x_1 and x_2 at $\theta = 5$ in step $<3>$, respectively.

Finally, $M^{<3>} = M^{<1>} + W.\overline{\sigma(t^{<2>})} = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}^T$ as $\overline{\sigma(t^{<2>})} =$ $\begin{pmatrix} (\overline{x(t^{<2>})})^T & (\overline{y(t^{<2>})})^T \end{pmatrix}^T = \begin{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \end{pmatrix}^T$ ■

F. Intermediate conclusion

Up to now, time sequences for a given step horizon $\{<k-1>, <k>\}$ and P-timed PNs have been considered. If the current horizons $h^{<k-1>}, h^{<k>}$ are known (which is the case when the dates of observations are known), the above vectors and matrices can be exploited. A consequence is that the estimation of a time trajectory $\overline{\varkappa}^{<k-1>\to<k>}$ can be made with an optimization of $c.\overline{\varkappa}^{<k-1>\to<k>}$ with c a row-vector over $\mathbb{R}$ where the constraint system is (4). From the estimate $\overline{\varkappa}^{<k-1>\to<k>}$ and $\overline{\gamma}^{<k-1>\to<k>}$, a direct extraction (Section IV-C) allows to provide the untimed sequences $\sigma^{<k-1>}, \sigma^{<k>}$. We also obtain the count vectors $\overline{\sigma}^{<k-1>}$, $\overline{\sigma}^{<k>}$(remember: $\overline{\sigma}^{<k>} = \begin{pmatrix} (\overline{x}^{<k>})^T & (\overline{y}^{<k>})^T \end{pmatrix}^T$) and the starting marking $M^{<k>}$ with
$M^{<k>} = M^{<1>} + W.\overline{\sigma}^{<k-1>}$. To summarize, we have built a candidate pair $(M^{<k>}, \sigma^{<k>})$ which is checked below.

V. Phase 2: Schedulability analysis for P-time Petri nets

A. Initial Problem of Phase 2.

The aim in Phase 2 is now to determine if the considered P-time PN can follow a pair $(M^{<k>}, \sigma^{<k>})$ proposed by Phase 1 for a relevant P-timed PN under a logical point of view. Precisely, a way to solve this problem is to estimate possible firing dates of the known (untimed) sequence $\sigma^{<k>}$ starting from the starting marking $M^{<k>}$ without token deaths. The existence of these dates proves the consistency of the pair $(M^{<k>}, \sigma^{<k>})$, that is, the possible schedulability of $\sigma^{<k>}$ for $M^{<k>}$ with respect to the data of the sliding horizon of the problem. With that aim, we adapt the transition firings with the condition that no token coming of the starting marking or providing by the creation of the firings dies.

For the sake of simplicity, we now take Assumption $\mathcal{AS}-7$: only ordinary PNs are now considered, that is, the arc weights are element of $\{0, 1\}$ and we assume that there is a unique place between two transitions at most.

B. Notations

Let $EV^{<k>}$ be a set of events relevant to step $< k >$ and $\sigma = e_1e_2e_3, ..., e_s$ be a sequence of s ordered events $e_i \in EV^{<k>}$. The index represents the event rank in the sequence and the length of σ is s. For simplicity, the events are not indexed by $< k >$. An event represents a transition firing in the context of the paper.

We assume that we can associate with event e_i and its rank i a unique transition j. More formally, an event function $\mathcal{F}$: $EV^{<k>} \to TR$ assigns to each event e_i from set $EV^{<k>}$ a unique transition $tr_j \in TR$. As a transition j can present more than one firing, each transition can correspond to more than one rank i in the sequence. So, $\mathcal{F}(e_i)$ represents a unique transition of TR but a transition can have no antecedent, one antecedent or more than one antecedent. Note that the events are possibly associated with the labels of AL if $L(\mathcal{F}(e_i)) \in AL$, that is, the label of an observable transition or an unobservable transition if $L(\mathcal{F}(e_i)) = \varepsilon$ (silent event).

The date relevant to the emission of the event e_i is denoted $\underline{e}_i$. In the initial sequence, the order of the events $i < j$ implies that the relevant dates must satisfy $\underline{e}_i \leq \underline{e}_j$ for each pair of events (e_i, e_j). So, $\underline{e}^T = \begin{pmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 & \dots & \underline{e}_s \end{pmatrix}$ where $\underline{e}_s = t^{<k>}$.

C. Principle

The estimation of the possible firing dates needs to have the exact relations in a dater form and to make the connection between the transition firings.

Knowing the transition firing and the solicited places, we can build a table where each column is relevant to an event with a specific date and contains the input places where the consumption of the tokens is necessary to the firing and the output places where some tokens are created. Therefore, the firing is only possible if previous firings have produced the necessary tokens. Each input place of a fired transition must be the output place of a transition (at least) whose event has occurred before (except if the starting marking gives the necessary tokens). Therefore, following the increasing order of the sequence, the analysis of the table facilitates the establishment of the connections between the transition firings. However, the marking does not contain the data permitting to know the token origin and the problem is to improve the accuracy of the information. Knowing all the input and output places solicited by the unobservable transition, we now show that we can describe the token evolution on the horizon and deduce the relations allowing a schedulability analysis.

Let us note that the creation of tokens can be directly deduced from the sequence σ. As the definition of the Petri net leads to a deterministic behavior, each event e_i in the sequence implies the creation of tokens when $(\mathcal{F}(e_i))^{\bullet} \neq \emptyset$, which are directly deduced. Indeed, all the tokens created by the ith firing of transition $\mathcal{F}(e_i)$ appear in places $(\mathcal{F}(e_i))^{\bullet}$.

But even if the sequence is known, the exact firing providing the tokens and leading to the consumption of the relevant created tokens is not known at this step and the relevant relations cannot be written. So, the problem is to make the connections between the creation and consumption of tokens.

With this aim in mind, we now analyze the firing conditions of each transition $\mathcal{F}(e_i)$ for $i \in \{1, \dots, s\}$ which must be satisfied. (Note that the firing condition is always satisfied for the source transitions (${}^{\bullet}(\mathcal{F}(e_i)) = \emptyset$)).

D. Starting marking

Let us integrate the starting marking in the approach with respect to step $< k >$. The tokens of the starting marking has been created by the transition firings at the past steps and the last creation corresponds to the firing of an observable transition whose firing occurs at $t^{<k-1>}$. Formally, $M^{<k-1>}[x^{<k-1>}y^{<k-1>} \succ M^{<k>}$ for step $< k-1 >$. Without considering the origin of the tokens of $M^{<k>}$, we introduce $M_l^{<k>}$ fictive input transitions of ${}^{\bullet}(p_l)$ which create these $M_l^{<k>}$ initial tokens in each place p_l presenting the initial marking $M_l^{<k>} \neq 0$. These fictive input transitions denoted $x_i^{F,l}$ for $i = 1, \ldots, M_l^{<k>}$ (F : Fictive) do not present an incoming place (${}^{\bullet}(x_i^{F,l}) = \emptyset$) and are source transitions with respect to step $< k >$. The relevant fictive events are included in the sequence starting at e_1 until e_m with $m = \sum_{l=1,\ldots,|P|} M_l^{<k>}$ before the events corresponding to the transition firings. Therefore, we have built a null starting marking ($M^{<k>} = 0$). Keeping the same notations for simplicity, the augmented sequence σ with $s = |\sigma|$ is defined as follows: $\underline{e}_m = t^{<k-1>}$, $\underline{e}_s = t^{<k>}$ and $h^{<k>} = h = \underline{e}_s - \underline{e}_m$.

E. Internal relations

The objective is now to highlight the internal relations connecting the firing transitions and to deduce possible event dates $\underline{e}_i$. As the sequence is relevant to a subset of transitions, only these transitions must be considered in the resolution. The firing of each transition needs to consider their input and output places, and to treat a relevant subnet of the Petri net in general.

For each firing of transition $\mathcal{F}(e_i)$ in the sequence with ${}^{\bullet}(\mathcal{F}(e_i)) \neq \emptyset$, and for each input place p_l of ${}^{\bullet}(\mathcal{F}(e_i))$, the firing of a unique transition $\mathcal{F}(e_j)$ in the set ${}^{\bullet}(p_l)$ is necessary to create a token. In that case, we must have $|{}^{\bullet}(p_l)| \neq 1$ (as $M^{<k>} = 0$) and $\mathcal{F}(e_j) \in {}^{\bullet}(p_l)$ in the order of the sequence $j < i$ (j before i), and we can write

$$\underline{e}_j + T_l^- \leq \underline{e}_i \leq \underline{e}_j + T_l^+ \tag{6}$$

If ${}^{\bullet}(\mathcal{F}(e_i)) = \emptyset$ for a transition $\mathcal{F}(e_i)$ in the sequence, this transition is a source transition and its firing does not depend of the constraints of the Petri net except the order of the sequence.

The pairs of indices (j, i) consistent with the sequence which must be used in the relations are unknown and the problem is the determination of these index numbers. We now propose a solution to this problem.

F. Structure.

Let us consider a transition which is not a source transition. Formally, ${}^{\bullet}\mathcal{F}(e_i) \neq \emptyset$. For each input place $p_l \in^{\bullet} \mathcal{F}(e_i)$, only one firing of a transition $\mathcal{F}(e_j) \in {}^{\bullet}(p_l)$ is necessary and we desire to determine it, that is the exact event e_j for the considered event e_i and place

$p_l \in^\bullet \mathcal{F}(e_i)$. We must search all e_j such that $j < i$ and $\mathcal{F}(e_j) \in^\bullet p_l$ and a choice must be made when this set is not a singleton. Note that a transition can satisfy $\mathcal{F}(e_j) = \mathcal{F}(e_{j'})$ with $j \neq j\prime$ and provides two possible events or more.

The final choice is expressed by the component $TOK(j,i)$ which contains the place p_l which is the support of the chosen evolution, that is, the creation of a token by the jth firing and the consumption by the ith firing in the sequence with $j < i$. As we consider an ordered sequence σ, $TOK(j,i) = \emptyset$ for $j > i$ where e_j, e_i are two events of the sequence. Note that a more explicit notation is $TOK(e_j, e_i)$.

Improving the concept of marking, a technique based on the token tracking can be developed. With Assumption $\mathcal{AS}-7$, a transition firing creates a unique token for each output place in the Petri net. We can associate with each token a label which is the event which has produced the token. Formally, we define the set $J_l^{<k>}$ with $|J_l^{<k>}| = M(p_l)$ which is the set of current events where each event $e_j \in J_l$ has produced a token which can be consumed by the future firings of transitions. For simplicity, we take $J_l^{<k>} = J_l$. When a token is consumed by the firing of transition $\mathcal{F}(e_i)$, the relevant event must be removed from J_l. Each choice of a creation/consumption of a token allows the modification of the sets J_l and the establishment of an element of TOK. The principle of the algorithm for step $< k >$ is to consider an event e_i in the increasing order of the sequence, to treat each incoming place p_l in $^\bullet\mathcal{F}(e_i)$ when $^\bullet\mathcal{F}(e_i) \neq \emptyset$ (respectively, each outgoing place p_l in $\mathcal{F}(e_i)^\bullet$ when $\mathcal{F}(e_i)^\bullet \neq \emptyset$) and to manage the events of the relevant J_l. Algorithm 1 providing the set TOK for step $< k >$ is as follows.

Algorithm 1

Input: $\sigma = e_1e_2e_3, ...e_s$; Output: TOK.

Initialization: $TOK = \emptyset$ (no choice) ; $J_l = \emptyset$ (no event) for any $p_l \in P$

$i = 0$

Repeat

- $i \leftarrow i + 1$
- For any $p_l \in^\bullet \mathcal{F}(e_i)$ with $^\bullet\mathcal{F}(e_i) \neq \emptyset$,

a) choose $e_j \in J_l$,

b) remove e_j from J_l ($J_l \leftarrow J_l \backslash \{e_j\}$)

c) and build $TOK(j,i) = \{p_l\}$ //consumption

- For any $p_l \in \mathcal{F}(e_i)^\bullet$ with $\mathcal{F}(e_i)^\bullet \neq \emptyset$, add e_i ($J_l \leftarrow J_l \cup \{e_i\}$) //creation

Until $i = s$ ■

As $|J_l| \neq 1$ in general, different tokens can be consumed and different evolutions are possible. The classical FIFO assumption below suggests a unique choice.

FIFO assumption

For each event i, the FIFO assumption implies that we take the smallest rank j with $j < i$ in J_l for $p_l \in^\bullet \mathcal{F}(e_i)$ as it avoids the token overtaking. The smallest rank is unique by construction of J_l which is based on the creation of a unique event for each firing. The first choice of consumption is based on the first creation, that is, the oldest token in the place (as $i < i'$ implies $\underline{e}_i \leq \underline{e}_{i'}$ for the creation dates). Providing a specific choice, this behavior rule is generated when the step a) "Choose $e_j \in J_l$" in Algorithm 1 is replaced by:

- Choose $e_j \in J_l$ such that $j = min(i)$ such that $e_i \in J_l$

Example 2 continued.

Considering that Fig. 1 represents a P-time PN, we desire to make the schedulability analysis of the untimed sequence x_1y_3 for step $< 3 >$ with $M^{<3>} = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}$.

The two tokens of $M^{<3>}$ are created by the firings of two fictive transitions y_4 and y_5. So, $\sigma = e_1e_2e_3e_4$ with $\mathcal{F}(e_1) = y_4$, $\mathcal{F}(e_2) = y_5, \mathcal{F}(e_3) = x_1$, and $\mathcal{F}(e_4) = y_3$.

We follow below the increasing order of the sequence.

e_1 : the fictive firing of y_4 creates a token in p_1. So, J_1 becomes $\{e_1\}$.

e_2 : the fictive firing of y_5 creates a token in p_3. So, $J_3 \leftarrow \{e_2\}$.

e_3 : the firing of transition x_1 consumes the token relevant to e_1 in $J_1 = \{e_1\}$ for p_1. So, $J_1 \leftarrow \{\varepsilon\}$), and creates a token in p_4 (so, $J_4 \leftarrow \{e_3\}$).

$\rightarrow TOK(e_1, e_3) = \{p_1\}$ from the creation by e_1/consumption by e_3 of token in p_1

e_4 : the firing of transition y_3 consumes the token relevant to e_3 in $J_4 = \{e_3\}$ for p_1 ($J_4 \leftarrow \{\varepsilon\}$).

$\rightarrow TOK(e_3, e_4) = \{p_4\}$ from the creation by e_3/consumption by e_4 of token in p_4 ■

G. Computation graph

As the previous step has established a global choice expressed by TOK for step $< k >$, a more accurate form of (6) can now be written. The system describing the creation/consumption of the tokens is

$$\begin{gathered} \underline{e}_j + T_l^- \leq \underline{e}_i \leq \underline{e}_j + T_l^+ \\ \text{for } TOK(j,i) = \{p_l\}. \end{gathered} \tag{7}$$

Let us show that a solution can easily be computed as System (7) presents strong properties. Indeed, we can deduce from (7) a computation graph denoted $\mathcal{CG}$ defined as follows. We associate with each event a new transition $\mathcal{F}(e_i)$ denoted with an abuse of notation e_i which appears once in the new graph by construction. The relevant set of transition is denoted $\mathcal{TR}$ with $|\mathcal{TR}| = s$. So, a transition $tr_i \in TR$ in the Petri net presenting more than one firing event in the sequence σ can be duplicated in two transitions or more $\mathcal{F}(e_i) \in \mathcal{TR}$ in $\mathcal{CG}$. For each pair (j, i) satisfying $TOK(j, i) = \{p_l\}$ we introduce a new place corresponding to p_l with time interval $[T_l^-, T_l^+]$ but appearing once in the computation graph $\mathcal{CG}$. To avoid a confusion with the places $p_l \in P$, they are indexed with a letter in upper case of the Latin alphabet (or a word if necessary) and are denoted p_{letter}. The relevant set is denoted $\mathcal{P} = \{p_A, p_B, p_C, ...\}$. So, a place $p_l \in P$ in the Petri net satisfying $TOK(j, i) = \{p_l\}$ can be duplicated in two places or more places $p_{letter} \in \mathcal{P}$ in $\mathcal{CG}$ but each place $p_{letter} \in \mathcal{P}$ appears once. The relevant time durations are also duplicated.

Property *3:* The computation graph $\mathcal{CG}$ is conflict-free.

Proof. Algorithm 1 has chosen for each event e_i and incoming place $p_l \in^\bullet \mathcal{F}(e_i)$, a unique event e_j and, has built $TOK(j, i) = \{p_l\}$ where p_l is replaced by a place $p_{letter} \in \mathcal{P}$ in $\mathcal{CG}$ relevant to the pair (i, j). By construction, each considered place $p_{letter} \in \mathcal{P}$ has a unique incoming transition and a unique outgoing transition, that is, ${}^\bullet p_{letter} = \mathcal{F}(e_j)$ and $p_{letter}^\bullet = \mathcal{F}(e_i)$, which defines the conflict-free structure of an event graph. ■

The following property shows that the complete approach always builds an acyclic system.

Property *4:* The computation graph $\mathcal{CG}$ is acyclic.

Proof. As the indices satisfy $j < i$ in (7) by construction, the computation graph is acyclic. ■

Moreover, the rewriting of (7) leads to a polyhedron $A'.\underline{e} \leq b'$ which is inf-monotone and sup-monotone (the system of linear inequalities $A'.\underline{e} \leq b'$ is inf-monotone (respectively, sup-monotone) if each row of matrix A' has one strictly negative (respectively positive) element at most [10]). From the relevant theorems in [10], we can say that the least solution exists since the problem is lower bounded as $\underline{e} \geq 0$ and is unique. The greatest solution exists and is unique as $\underline{e}_s = t^{<k>}$ is an upper bound. The analysis of the consistency can be made by a minimization or maximization with $c'.\underline{e}$ with c' unitary. The computation is efficient as standard algorithms of linear programming as the Simplex

(Section IV.D. in [10]).

Example 2 continued.

The relations can directly be deduced from TOK relevant to sequence x_1y_3 for step $< 3 >$ with $M^{<3>} = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}$. $\begin{cases} TOK(e_1, e_3) = \{p_1\} : \underline{e}_1 + T_1^- \leq \underline{e}_3 \leq \underline{e}_1 + T_1^+ & (A) \\ TOK(e_3, e_4) = \{p_4\} : \underline{e}_3 + T_4^- \leq \underline{e}_4 \leq \underline{e}_3 + T_4^+ & (B) \end{cases}$
where $\underline{e}_1 = t^{<2>}$ and $\underline{e}_4 = t^{<3>}$.

We can deduce system $A'.\underline{e} \leq b'$ with $\underline{e}^T = \begin{pmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 & \underline{e}_4 \end{pmatrix} \geq 0$, $\underline{e}_3 \leq \underline{e}_4$, $\underline{e}_2 \leq \underline{e}_3, \underline{e}_1 \leq \underline{e}_3$

$(b')^T = \begin{pmatrix} -T_1^- & -T_4^- & T_1^+ & T_4^+ \end{pmatrix}$ and $A' = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$. A direct resolution gives $max(t^{<2>} + T_1^-, t^{<3>} - T_4^+) \leq \underline{e}_3 \leq \min(t^{<2>} + T_1^+, t^{<3>} - T_4^-)$. For $t^{<2>} = 3$, $t^{<3>} = 12$, $\left(T_1^-, T_1^+\right) = (3, 5)$, $\left(T_2^-, T_2^+\right) = (1, 12)$, $\left(T_3^-, T_3^+\right) = (1, 3)$ and $\left(T_4^-, T_4^+\right) = (4, 13)$, we obtain $6 \leq 8$ which is consistent and shows the schedulability of the sequence x_1y_3 for step $< 3 >$. Now if we take $t^{<3>} = 8$, the system becomes inconsistent $(6 \nleq 4)$ and the sequence x_1y_3 cannot occur. The same approach applied to the sequence x_2y_3 shows the existence of a schedule for $t^{<3>} = 8$. ■

VI. Conclusion and perspectives

As the complexity of the problem implies the use of different descriptions of the trajectories developed on a given horizon, the Phase 1 exploits a relaxation of the time model which must be checked in the Phase 2 where the constraints of the P-Time PN are considered. In Phase 2, a technique based on a token tracking generates a matrix description of the creation/consumption of the tokens that permits to build an acyclic conflict-free computation graph describing the candidate sequence which can easily be checked.

Providing an optimal solution for each subproblem, the proposed approach uses efficient standard tools of linear programming contrary to the classical techniques which are based on the generation of marking sets which are generally time and space costly. Note that this generation of marking sets is an *estimation* of sets and not an exact determination as some markings can be inconsistent with a future evolution: the relevant markings which can be basis markings must be withdrawn with a procedure [6]. The resolution of Phase 2 is effective as the obtained system presents strong properties and standard algorithms as the Simplex can be applied directly. Similarly, for Phase 1, the paper [10] has presented some classes of models where solutions over $\mathbb{Z}$ or $\mathbb{R}$ are equal [13].

This paper tackles different issues of estimation but many perspectives extending the proposed two-staged approach can be considered at middle term. 1) As the principle of the approaches based on a sliding horizon is to consider a limited amount of information which guarantees the efficiency of the algorithms but can limit the accuracy of the results, a numerical analysis allowing the study of the compromise between accuracy of the estimate and execution time in function of the horizon can be made. 2) As each phase of the proposed approach exploits a respective optimization, the proposed technique which is suboptimal is a bi-criteria optimization: a deeper insight into the global optimality and suboptimality for different criteria can be the subject of a theoretical study. Starting from an obtained solution, a search of a better local solution can complete the technique. 3) Finally, a natural perspective is the extension of this technique to models as Time Stream PNs which generalizes P-time PNs.

References

[1] T. Becha, R. Kara, S. Collart-Dutilleul, J. J. Loiseau, ”Modelling, Analysis and Control of Electroplating Line Modelled by P-Time Event Graphs,” 6th International Conference on Management and Control of Production and Logistics, Fortaleza, Ceará, Brazil, pp. 311-316, September 11-13, 2013.

[2] P. Bonhomme, ”Marking estimation of P-Time Petri Nets with unobservable transitions,” IEEE Transactions on Systems, Man, and Cybernetics: Systems, Volume 45, Issue 3, pp. 508 - 518, 2015.

[3] P. Bonhomme, ”Decentralized state estimation and diagnosis of P-time labeled PNs systems,” Journal of Discrete Event Dynamic Systems, 31(1), pp 137-162, 2021.

[4] A. Boussif, B. Liu and M. Ghazel, ”Diagnosability Analysis of Intermittent Faults in Discrete Event Systems,” International Journal of Control, Automation and Systems 17(X), pp. 1-14, 2019.

[5] A. Chouchane, P. Declerck, A. khedher and A. kamoun, Diagnostic Based on Estimation Using Linear Programming for Partially Observable Petri Nets with Indistinguishable Events, International Journal of Systems Science: Operations & Logistics, December 2018

[6] A. chouchane and P. Declerck, ”Diagnosis on a sliding horizon for partially observable Petri nets,” Kybernetika, International journal of Institute of Information Theory and Automation of The Czech Academy of Science, to appear.

[7] S. Collart-Dutilleul, A. Mhalla, E. Craye and M. Benrejeb, ”Active Robustness of a Milk Manufacturing Workshop with Time Constraints,” International Journal of Production Research, 51 (1), pp. 9-25, 2013.

[8] P. Declerck, ”From extremal trajectories to consistency in P-time Event Graphs,” IEEE Transactions on Automatic Control, Volume 56, pp. 463-467, 2011.

[9] P. Declerck. Discrete Event Systems in Dioid Algebra and Conventional Algebra, Focus Series in Automation & Control, ISTE Ltd and John Wiley, 2013.

[10] P. Declerck and P. Bonhomme, ”State Estimation of Timed Labeled Petri Nets with Unobservable Transitions,” IEEE Transactions on Automation Science and Engineering, Special Issue on Discrete Event Systems for Automation, Vol. 11, No. 1, ITASC9, pp. 103-110, January 2014.

[11] P. Declerck, A. Chouchane, P. Bonhomme, ”A strategy for Estimation in Timed Petri nets,” CoDIT'17, Barcelona, Spain, April 5-7, 2017.

[12] P. Declerck, ”Counter approach for the estimation of optimal sequences in Partially Observable Untimed Petri Nets,” 31, pp.489–512 , Journal of Discrete Event Dynamic Systems, 2021.

[13] M. Dotoli, M. P. Fanti and A. M. Mangini, ”Fault detection of discrete event systems by Petri nets and integer linear programming,” Automatica, 45(11), pp. 2665-2672, 2009.

[14] J. Komenda, S. Lahaye and J-L. Boimond, ”Determinization of Timed Petri nets behaviors,” Journal of Discrete Event Dynamic Systems, 26(3), pp. 413–437, 2016.

[15] J. Komenda, S. Lahaye, J-L. Boimond and T. van den Boom, ”Max-plus algebra in the history of discrete event systems,” Annual Reviews in Control. 45. 10.1016/j.arcontrol.2018.04.004, 2018.

[16] Liang Li, Yaqiong Li, Bin Liu, and Weimin Wu, ”Least-Cost Transition Sequence Estimation in Labeled Time Petri Net Systems with Unobservable Transitions,” International Journal of Control, 1. 10.1080/00207179.2022.2121764, 2022.

[17] H. Yue, S. Xu, G. Zhou, H. Hu, Y. Guo, and J. Zhang, “Estimation of least-cost transition firing sequences in labeled Petri nets by using basis reachability graph,” IEEE Access, pp. 1-1. 10.1109/ACCESS.2019.295205, 2019.