Control synthesis in interval systems

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ABSTRACT

Control system is an important problem in the field of discrete event systems. The subclass of Timed event graphs can be modelled in state equations using the (max, +) algebra. A realistic hypothesis is that temporization values belong to known intervals. In this type of model, this document proposes an approach of constraint propagation type using Kleene's star in order to solve the control synthesis problem.

I. Introduction

Discrete Event Dynamic Systems (DEDS) can represent a great number of processes characterized as being concurrent, asynchronous, distributed or parallel, such as flexible manufacturing systems, multiprocessor systems or transportation networks. Among the different formalisms used to represent DEDS, Timed Event Graphs with constant temporizations form a subclass of Petri nets, which plays an important role because of their deterministic temporal behavior.

The evolution of the system is described by linear equations defined on a dioid. The interpretation of each variable is as follows, e.g., for "dater" type in (max,+) algebra: each variable $x_i(k)$ represents the date of the kth firing of transition x_i ; $\oplus$ stands for the maximum operation while the usual addition plays the role of the multiplication, denoted $\otimes$. In the dioid (max,+), the state variable representation of the system is of the form:

$$\begin{cases} x(k) = & A \bigotimes x(k-1) \bigoplus B \bigotimes u(k) \\ y(k) = & C \bigotimes x(k) \end{cases} \tag{1}$$

where the control u, the state x and the output y are defined on $\Re \cup \{-\infty\}$. Equality arises from the assumption that there is no extra delay for firing transitions whenever tokens are all available. In the (max,+) literature, $\otimes$ is usually replaced by . or is omitted.

For instance, the firing of transition u_i in production system means that an object is given to the process to be manufactured. The firing of transition y_i means that an object has just ended.

The element $[A]_{ij}$ represents the temporization of a place which links the transition x_j to the transition x_i and contains one token . The element $[B]_{ij}$ (respectively $[C]_{ij}$) is the temporization of a place which links the input u_j to the transition x_i (respectively the transition x_j to the output y_i) and contains no token.

Let us assume that some events are stated as controllable, meaning that the corresponding input transitions may be delayed from firing until some arbitrary time $u_i(k)$ provided by a supervisor. Let us assume that we wish to slow down the system as much as possible without causing any event to occur later than some sequence of execution times Z. We can express this objective with

$\forall\ k \in [k_s\ k_f]\ y(k) \leq z(k)$. We assume that the desired output $z(k)$ is known in the horizon $[k_s$ $k_f]$. As we have no piece of information for $k > k_f$, there is no demand on the production and we take $z(k) = T$ with $T = +\infty\ \forall k > k_f$. So, the problem is the determination of the greatest output $u_i(k)$ so that $y(k) \leq z(k)$ in the horizon $[k_s\ k_f]$.

When the matrices A,B and C are known , the classical "backward" approach gives the solution if the initial condition x(0)=ε with $\varepsilon = -\infty$ is hold. It can be proved that for the system which dater type equations give the least solution (the earliest times) of the process evolution, the greatest solution (the latest times) of the control problem is explicitly given by the "backward" recursive equations where the co-vector plays the role of the state vector. The state equations and the "backward" recursive equations provide the earliest and the latest times of the tasks respectively. The differences between the co-state and the state represent the "spare time" or the "margin" which is available for the firing of the transitions. The existence of a negative difference prevents the future deadlines from being achieved.

However, the "backward" approach arises from the assumption that the holding times are perfectly known. A more realistic hypothesis is that temporization values belong to intervals. This system can be named an interval system. A basic assumption that allows us to model the system, is that places are First In First Out (FIFO) channels. A place p_i is FIFO if the k^{th} token to enter this place is also the k^{th} token that becomes available in this place. The interpretation is that tokens cannot overtake one another which is a necessary numbering condition of the events. So, the problem is to make control synthesis in interval systems with this assumption. With this aim in view, we propose to apply the resolution approach of E.Walkup-G.Borriello [WAL 98] which is based on two concepts, the propagation constraint and the Kleene's star.

The paper is structured as follows. We give initially, the notations and some previous results (Baccelli, Cohen, Olsder & Quadrat, 1992). Then, we propose a framing of the optimal solution. We describe the algorithm which calculates the greatest solution of a special type of inequations and give a new initialization which improves the approach. These concepts are finally applied in the control field in the last section.

II. Préliminaires

A monoid is a couple $(S, \oplus)$ where the operation $\oplus$ is associative and presents a neutral element. A semi-ring S is a triplet $(S, \oplus, \otimes)$ where $(S, \oplus)$ and $(S, \otimes)$ are monoids, $\oplus$ is commutative, $\otimes$ is distributive relatively to $\oplus$ and the zero element of $\oplus$ is the absorbing element of $\otimes$. A dioid D is a idempotent semi-ring. Let us notice that contrary to the structures of group and ring, monoid and semi-ring do not have a property of symmetry on S.

The unit $\Re \cup \{-\infty\}$ provided with the maximum operation denoted $\oplus$ and the addition denoted $\otimes$ is usually called (max, +) algebra and is an example of dioid. We have : $\Re_{max} = (\Re \cup \{-\infty\}, \oplus, \otimes)$ with

$a \oplus b = \max(a, b)$; $\varepsilon = -\infty$ is the neutral element of $\oplus$

$a \otimes b = a + b$; $e = 0$ is the identity element of $\otimes$

$a \oplus a = a$ (idempotency of $\oplus$)

$a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$ (absorbing element ε)

Definition 1 *A dioid D is complete if it is closed for infinite sums and the distributivity of the multiplication with respect to addition extends to infinite sums :* $(\forall\ c \in D\ )\ (\forall\ A \subseteq D)\ c \otimes (\bigoplus_{x \in A} x) = \bigoplus_{x \in A} c \otimes x$

Thereafter, a $\otimes$ b could be noted $a.b$ or simply ab . Lest us remind that in the $(max, +)$ notation , the following equality stands for with ordinary minus sign: $[C_{i,j}]^{\otimes(-1)} = -[C_{i,j}]$. The sum and the product of the matrices operate as in the usual algebra:

$$(\mathrm{A} \oplus \mathrm{B})_{ij} = \mathrm{A}_{ij} \oplus \mathrm{B}_{ij}$$

$$(\mathrm{A} \otimes \mathrm{B})_{ij} = \bigoplus_{k=1}^{n} \mathrm{A}_{ik} \otimes \mathrm{B}_{kj}$$

Kleene's star is defined by: $A^* = \bigoplus_{i=0}^{+\infty} A^i$

Noted as G(A), an induced graph of a square matrix A is deduced from this matrix by associating

- a node i to the column i and line i

- an arc from the node j towards the node i if $A_{ij} \neq \varepsilon$.

Like the theorem 4.75 in [BAC 92], the following theorem considers the resolution of $x = A \otimes x \oplus B$ but differs because every circuit has a strictly negative weight.

Theorem 2 *(theorem 3.17 in [BAC 92])For matrix A with induced graph $G(A)$, if the cycle weights in $G(A)$ are all strictly negative, then there is a unique solution to the equation $x = A \otimes x \oplus B$ which is given by $A^* \otimes B$.*

The left residuation of b by a is defined by:

$a \backslash b = \max\{x \in D \text{ such that } ax \leq b\}$

Proposition 3 *In the complete dioid $\bar{\Re}_{\max}= (\Re \cup \{\pm\infty\}, \oplus, \otimes)$, $a \backslash b$ equals $b - a$ when $a \neq \varepsilon$, and equals $T = +\infty$ if $a = \varepsilon$.*

The operator $\wedge$ defines the minimum between the elements a and b by:

$a \wedge b = \min(a, b)$

Given A and B two matrices in the complete dioid, the residuation of B (dimensions $n.q$) by A (dimensions $n.p$) is given by:

$$(A \backslash B)_{ij} = \bigwedge_{k=1}^{n} A_{ki} \backslash B_{kj}$$

Theorem 4 *(theorem 4.73 in [BAC 92]) Given a and b in a complete dioid and consider the equation $x = a \backslash x \wedge b$ and the inequality $x \leq a \backslash x \wedge b$. Then, for these expressions : $a^* \backslash b$ is the greatest solution ; every solution x verifies $x = a^* \backslash x$; ε is the least solution of the inequality and it is also the least solution of the equality provided that $a \neq \varepsilon$ or $b = \varepsilon$.*

Definition 5 *A* min-max *function of type $(n, 1)$ is any function $f : \Re^n \longrightarrow \Re^1$, which can be written as a term in the following grammar:*

$f = x_1, x_2, \ldots, x_n \mid f + a \mid f \wedge f \mid f \oplus f$

The vertical bars separate the different ways in which terms can be recursively constructed.

Definition 6 *A* min − max *function of type* (n, m) *is any function* $F : \Re^n \longrightarrow \Re^m$, *such that each component* F_i *is a* min-max *function of type* $(n, 1)$.

Min-max functions include max-plus linear maps which can be written as:

$g(x)_i = \underset{1\leq j\leq n}{\oplus} (A_{ij} + x_j)$

Min-plus linear maps, defined dually, are also special min-max functions :

$h(x)_i = \underset{1\leq j\leq n}{\wedge} (B_{ij} + x_j)$

where A and B are a $n \times n$ matrix with entries in $\Re \cup \{-\infty\}$

Using the mutual distributivity of $\otimes$ and $\wedge$, we can write any min-max function as:

$$F(x) = \underset{g\in G}{\wedge} g(x) = \underset{h\in H}{\oplus} h(x)$$

where G and H are finite sets of max-plus and min-plus linear maps, respectively.

III. Resolution

A. Objective

In this part, we show how to find the maximum solution of an arbitrary system of s variables et s' inéquations, the form of which is X$\leq F(X)$ with F a min-max function of type (n,m). Below, we need that every inequation has the form of an upper bound constraint which we define now.

Définition 2 An upper bound constraint or UBC has form $x_{\tau(i)} \leq \bigoplus_{j=1}^{s} A_{i,j}\ x_j$ for i $\in [1, s']$ where $\tau(i)$ is used to index the variable on the left-hand term of this i^{th} such UBC. A single $x_{\tau(i)}$ may be the target, or left-hand side of several UBCs. So, we can have $x_{\tau(i)} = x_{\tau(j)}$ with $i \neq j$, $i, j \in [1, s']$.

If an inequality has the form $x_{\tau(i)} \leq \bigoplus_{j=1}^{s} A_{i,j}\ x_j \oplus B_i$, we introduce the variable x_0 with $x_0 = 0$ in the following manner : the UBC is $x_{\tau(i)} \leq \bigoplus_{j=1}^{s} A_{i,j}\ x_j \oplus B_i x_0$.

We can simplify the constraint UBC i :

- if $0 \leq A_{\tau(i),i}$, remove the UBC which is always verified
- if $A_{\tau(i),i} < 0$ and $A_{i,j} \neq \varepsilon$ with $j \neq_{\tau(i)}$, replace $A_{\tau(i),i}$ by ε
- if $A_{\tau(i),i} < 0$ and $A_{i,j} = \varepsilon$ for $j \neq_{\tau(i)}$, the system is not feasible.

The objective of this part is the determination of the greatest solution which checks the complete set of s' inequations UBC and the equality $x_0 = 0$. Before, we make a framing of the optimal solution.

B. Framing

A framing of the optimal solution can be obtained with the following processings of the UBC set.

1. Lower system

Each constraint UBC is replaced by $x_{\tau(i)} \leq \bigwedge_{j=1}^{s} A_{i,j}\ x_j$. As $\bigwedge_{j=1}^{s} A_{i,j}\ x_j \leq \bigoplus_{j=1}^{s}\ A_{i,j}\ x_j$, this new set of constraints is more strict and minimize the solution which is lower than or equal the optimal one. We can slit the right-hand term and write

$(\forall i, j \in [1, s])\ x_{\tau(i)} \leq A_{i,j}\ x_j$

2. Upper system

The new set is a subset of the original UBC set: we keep each UBC which contains only one element $A_{i,j}$ different from ε. We define $E = \{i \in N | \exists j \in N$ such that $A_{i,j} \neq \varepsilon$ and it does not exist $j \neq j' \in N$, $A_{i,j'} \neq \varepsilon\}$. We have $\mathrm{card}(E) \leq s'$.

$x_{\tau(i)} \leq A_{i,j}\ x_j$ for $i \in ES$

Consequently, the new set contains fewer constraints and the corresponding solution is greater than or equal the optimal one.

3. Calculation of the bounds

The two simplifications gives the same type of inequations : $X \leq f(X)$ with $f(X)$, a (min,+) equation which can be written $x_{\tau(i)} \leq A_{i,j}\ x_j$

We can make the following usual soustraction and obtain $x_{\tau(i)} - A_{i,j} \leq\ x_j$ or $x_{\tau(i)} \otimes (-A_{i,j}) \leq x_j$ for each $A_{i,j} \neq \varepsilon$ for the strict simplification and $i \in E$ for the second one. As the variable $x_{\tau(i)}$ may be the left-hand term of some UBC , we can have $x_{\tau(i)} = x_{\tau(j)}$ with $i \neq j$. We note $B_{\tau(i),,j} = \bigoplus_i (-A_{i,,j})$ such that $\tau(i) = \tau(i')$ and otherwise, $B_{\tau(i),,j} = \varepsilon$ if it does not exist $A_{i,j} \neq \varepsilon$. So, $x_{\tau(i)} B_{\tau(i),,j} \leq x_j$ .The simplification of the notation gives $\bigoplus_{j=1}^{s} B_{i,j}\ x_j \leq x_i$

The simplification gives for the lower system (respectively for the upper system) the following structure of inequations:

$MX \leq X$ (respectively $NX \leq X$)

As $x_0 = 0$, we add $X \leq P$ with $P = (e, T, T, ..., T)^t$ and $T = +\infty$

The resolution gives respectively the greatest solutions $M^*\backslash P$ and $N^*\backslash P$ and the framing of the optimal solution.

$$M^*\backslash P \leq X_{opt.} \leq N^*\backslash P$$

Elementary example:

$x_1 \leq 1$, $x_1 \leq 2 + x_2$ and $x_2 \leq \max(1 + x_1, 0)$. The direct resolution shows that the greatest solution is $x_1 = 1$ and $x_2 = 2$.

$$M = \begin{pmatrix} \varepsilon & -1 & e \\ \varepsilon & \varepsilon & -1 \\ \varepsilon & -2 & \varepsilon \end{pmatrix} \quad N = \begin{pmatrix} \varepsilon & -1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & -2 & \varepsilon \end{pmatrix}$$

$$M^* = \begin{pmatrix} e & -1 & e \\ \varepsilon & e & -1 \\ \varepsilon & -2 & \varepsilon \end{pmatrix} \quad N^* = \begin{pmatrix} e & -1 & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & -2 & e \end{pmatrix}$$

$$M^*\backslash P = \begin{pmatrix} e \\ 1 \\ e \end{pmatrix} \quad N^*\backslash P = \begin{pmatrix} e \\ 1 \\ T \end{pmatrix}$$

We conclude that $x_1 = 1$ and $0 \leq x_2$.

IV. Resolution by Kleene 's star and constraint propagation

Using the Kleene's star, the technique of resolution of Walkup and Borriello (1998) makes it possible to calculate the greatest solution of general systems of equations and inequations. It can solve linear (max, +) equations of the form $A \otimes x \oplus B = C \otimes x \oplus D$, inequations of the type

$Ax \leqslant B$ and more generally any system which can be reduced to a UBC form. The solution appears in Walkup-Borriello's approach at the end of the decrease of an upper limit with a finite number of steps. Let us notice that this approach does not deal with the (max, +) polynomial forms as the algorithm solving the ELCP problem (Extended Linear Complementarity Problem) of De Schutter (1996) who generalizes the linear complementary problem by using the usual algebra. However, this last approach does not treat equations of the type $A \otimes x \oplus B = C \otimes x \oplus D$ and is NP-complete. The two approaches share the fact of having an original idea to treat the equalities of the (max, +) equations while proposing different solutions. The broad outlines of Walkup-Borriello's approach a detailed description of which is in (Walkup, 1995) (Walkup & Borriello, 1998), are given below.

A. Method

The technique includes three stages:

- choose a particular subset of UBC constraints;

- apply Kleene's star to calculate the greatest solution;

- use the preceding solution which guides the choice of the new constraint and thus minimize the greatest solution. The convergence is achieved when a solution checks all the constraints on the assumption of existence of a solution.

B. Resolution of a subset of UBC constraints

The variables are reorganized such as x_0 is the first variable. The greatest solution will be found when $x_0 = e$.

Definition 7 *A targeting subset of a system of UBC constraints on $s+1$ variables is a set of s UBC constraints such as each variable x_i different from x_0 is the target of exactly one UBC.*

A system of UBCs may be expressed in matrix form as follows. We first make note of the fact that $a \leqslant b$ and $a \oplus b = b$ are equivalent. Consequently, the r inequalities $x_{\tau(i)} \leqslant \bigoplus_{j=0}^{s} M_{i,j}\ x_j$ may also be written as $x_{\tau(i)} \oplus \bigoplus_{j=0}^{s} M_{i,j}\ x_j = \bigoplus_{j=0}^{s} M_{i,j}\ x_j$. We obtain the matrix form

$(J \oplus M) \bigotimes x = M \otimes x$ with J a Boolean matrix $r.(s+1)$ the inputs of which are equal to e or ε . To choose s UBCs among r not targeting x_0, it is then enough to multiply by a Boolean matrix P of dimensions $s.r$ such as each row contains one e exactly and ε otherwise so that PJ (dimension: $s.s+1$) is a concatenation of a column of ε relative to x_0 and the identity matrix of dimension $s.s$. Inequalities which are not selected correspond to columns of P containing only elements ε. We obtain $(P.J \oplus P.M) \bigotimes x = P.M \otimes x$ (dimension of $P.M$: $s.s+1$). To be able to apply Kleene's star, the matrix $P.M$ needs however to be square. The addition of a top row containing only elements ε makes it possible to obtain a matrix P' of dimension $(s+1).r$ such that the matrix $P' \otimes J = E'$ is an identity matrix of dimension $(s+1).(s+1)$ except the element $(E')_{1.1} = \varepsilon$. The result is $(P'.(J \oplus M)) \bigotimes x = (P'.M) \otimes x$. The top row of the products $P'.J$ and $P'.M$ are all ε-elements. Thus, the variable x_0 is not coupled with any inequality.

Definition 8 *A targeting subset is called a safe targeting if all circuits in the graph induced by the targeting subset $P'.M$ have a strictly negative weight.*

The previous form is solved by considering the form $(P.J) \bigotimes x = (P.M) \otimes x$. We add the equality $x_0 = 0$ in order to obtain a square system. This system is equivalent to $x = (P'.M) \otimes x \oplus d$ with d a vector column in which only the first element $(d)_1$ is different from ε and equal to e. Theorem 1 states that its single solution is $(P'M)^*_{i,0}$. The following lemma declares that this solution is also the greatest solution of the targeting subset, i.e. of $(P'.(J \oplus M)) \bigotimes x = (P'.M) \otimes x$.

Lemma 9 *(Walkup and Borriello, 1998) Given a safely targeting subset of constraints UBC. The vector $(l_0, l_1, ..., l_s)^t$ where $l_i = (P'M)^*_{i,0}$ is the maximum solution to the constraint subset when $x_0 = 0$.*

C. Resolution of the complete system by iteration

As the solution of the preceding section results from the resolution of a targeting subsystem, it is not guaranteed to be a solution of the complete system. We can however maintain that this solution is an upper limit for any solution of the system. If the solution l to the current safe targeting subsystem is not a solution to the complete system, then there must be at least one inequation UBC u_i which is not yet satisfied and is not in the safe targeting subset. If one replaces the constraint previously coupled with $x_\tau(i)$ by the new constraint u_i , one shows that this new system is also safe and that its resolution carries out a minimization. A new solution is obtained l' such as for any $j \neq \tau(i)$, $l_j \geqslant l'_j$ and $l_{\tau(i)} > l'_{\tau(i)}$.

The found solution will have to also check the UBC type inequalities coupled with the variable x_0 in order to check all the inequalities. The system has no solution if a UBC inequality coupled with x_0 imposes a negative value on x_0. Futhermore, it is possible to show the convergence in a finite number of steps.

D. Initialization

The initialisation is an important step which make it possible the start of the algorithm. As the algorithm is non-decreasing, we must take $+\infty$ a priori, for every variable. But, we can find examples which show that the algorithm does not act. For example, if we consider $x_1 \leq -2 + x_2$ and take $x_1 = x_2 = +\infty$, we obtain $+\infty \leq +\infty$ which does not change x_1. Let ν be larger than the maximum of the values of the x_i's maximum solution. But, if we augment the initial UBC system with constraints of the form, $x_i \leq x_0 + \nu$ then the addition of this set does not change the maximum solution. As we do not know a priori suitable value for ν, we consider ν as a finite but very large element of $\Re$, for which $b < \nu + a$ for all $a, b \in \Re$ wih $a \neq \varepsilon$. In other words , we make a formal calculation with ν and we check the above UBC by $\nu \leq -2 + \nu$. Clearly, x_1 must be minimize to $-2 + \nu$.

As, the framing of the optimal solution has given an upper bound $N^* \backslash P$, we can reduce this formal calculation by considering the finite values. So, we augment the initial set with the following UBCs.

$x_i \leq x_0 + w_i$ with $w_i = (N^* \backslash P)i \wedge \nu$

The initial target matrix V is without circuit and the kleene's star of V can always be expressed even though if $w_i = \nu$.

$$V = \begin{pmatrix} \varepsilon & \varepsilon & \dots & \varepsilon \\ w_1 & \varepsilon & \dots & \varepsilon \\ \vdots & \vdots & & \vdots \\ w_s & \varepsilon & \dots & \varepsilon \end{pmatrix} \text{ and } V^* = \begin{pmatrix} e & \varepsilon & \dots & \varepsilon \\ w_1 & e & \dots & \varepsilon \\ \vdots & \vdots & & \vdots \\ w_s & \varepsilon & \dots & e \end{pmatrix}$$

Consequently, an initial safe targeting subset can always be found which allows the application of the following steps.

V. Control synthesis

The determination of the greatest control $u(k)$ entails the calculation of the greatest state and output by reason of the application of the isotonicity of laws $\oplus$ and $\otimes$ on the state and output equations. The solution can be obtained if we calculate the greatest u and x cheking the following constraints. We note these values u^+ and x^+respectively:

$$\begin{cases} x(k) & = & Ax(k-1) \bigoplus Bu(k) \\ y(k) & = & Cx(k) \\ y(k) & \leq & z(k) \end{cases} \tag{2}$$

with $k \in [k_s \ k_f]$, $\dim(x) = n, \dim(u) = q, \dim(y) = \dim(z) = m,$

$\dim(A) = n \times n, \dim(B) = n \times q, \dim(C) = m \times n$

Let us assume that the desired trajectory $z(k)$ is known. Let us assume that the temporization values are partially known and belongs to intervals, i.e : $A^- \leq A \leq A^+$; $B^- \leq B \leq B^+$; $C^- \leq C \leq C^+$. With these data, the problem is the determination of the control u^+ and the state x^+ .The next theorem make it possible the introduction of a new formulation of this main problem.

Theorem 2. The control problem is the calculation of the greatest control u and x checking the following constraints:

$$\begin{cases} x(k) \leq [A^+ x(k-1) \oplus B^+ u(k)] \wedge [A^- \backslash x(k+1)] \wedge [C^+ \backslash z(k)] \\ u(k) \leq B^- \backslash x(k) \end{cases}$$

Proof:

If we replace $y(k)$ by its expression, the process 2 become :

$$\begin{cases} x(k) & = & Ax(k-1) \bigoplus Bu(k) \\ Cx(k) & \leq & z(k) \end{cases}$$

First, let us remind that :

If $a \leq b$ and $c \leq d$ then $\begin{matrix} a \oplus b \leq c \oplus d \\ k \otimes a \leq k \otimes b \end{matrix}$

We introduce the lower and upper bounds of the matrices A and B in the first inequation of the system (2) :

$$A^- x(k-1) \oplus B^- u(k) \leq Ax(k-1) \oplus Bu(k) \leq A^+ x(k-1) \oplus B^+ u(k)$$

$$A^- x(k-1) \oplus B^- u(k) \leq x(k) \leq A^+ x(k-1) \oplus B^+ u(k)$$

$$\Longrightarrow \begin{cases} x(k) \leq A^+ x(k-1) \oplus B^+ u(k) \\ A^- x(k-1) \leq x(k) \\ B^- u(k) \leq x(k) \end{cases}$$

In the second inequation of the system (2), we apply the same processing:

$$C^- x(k) \leq Cx(k) \leq C^+ x(k)$$

The constraint $\mathbf{y(k)} \leq \mathbf{z(k)}$ gives:

$$\begin{cases} C^- x(k) \leq z(k) \\ C^+ x(k) \leq z(k) \end{cases} \Longrightarrow C^+ x(k) \leq z(k) \text{ because } (C^- x(k) \leq C^+ x(k))$$

Finally, we obtain the following system:

$$\begin{cases} x(k) \leq A^+ x(k-1) \oplus B^+ u(k) \\ x(k-1) \leq A^- \backslash x(k) \\ u(k) \leq B^- \backslash x(k) \\ x(k) \leq C^+ \backslash z(k) \end{cases} \Longleftrightarrow \begin{cases} x(k) \leq C^+ \backslash z(k) \\ x(k) \leq A^+ x(k-1) \oplus B^+ u(k) \\ x(k) \leq A^- \backslash x(k+1) \\ u(k) \leq B^- \backslash x(k) \end{cases} \quad (3)$$

$$\Longleftrightarrow \begin{cases} x(k) \leq [A^+ x(k-1) \oplus B^+ u(k)] \wedge [A^- \backslash x(k+1)] \wedge [C^+ \backslash z(k)] \\ u(k) \leq B^- \backslash x(k) \end{cases} \quad \blacksquare$$

Remark 1 • *The final result uses the operators* min, max, *and the residuation. The variable $x(k)$ is the minimization of a "backward " term $[A^- \backslash x(k+1)] \wedge [C^+ \backslash z(k)]$ and a "forward " term $[A^+ x(k-1) \oplus B^+ u(k)]$ wich corresponds to the state equation.*

- *The expressions contains lower and upper bounds like A^-, A^+, B^-, B^+ and C^+ which show the complexity of the problem. Only, C^- is not present in the inequations.*

From the system (3), we finally obtain the set of the UBC constraints in the following manner. We develop the previous system and we replace the residuation by the minus sign in accordance with the proposition (3).

$$\begin{cases} (4-1) \quad x_j(k) \leq -[C^+]_{ij} + z_i(k) \text{ with } [C+]_{ij} \neq \varepsilon \\ (4-2) \quad x_i(k) \leq \overset{n}{\underset{j=1}{\oplus}} [A^+]_{ij} x_j(k-1) \oplus \overset{q}{\underset{s=1}{\oplus}} [B^+]_{is} u_s(k) \\ (4-3) \quad x_j(k-1) \leq -[A^-]_{ij} + x_i(k) \text{ with } [A-]_{ij} \neq \varepsilon \\ (4-4) \quad u_s(k) \leq -[B^-]_{is} + x_i(k) \text{ with } [B]_{ij} \neq \varepsilon \end{cases}$$

This set contains at the most, $n \times m$ relations (state/desired output), n relations (state/control), $n \times n$ relations (state/state) and $n \times q$ relations (state/input). If $[C]_{ij}$, $[A]_{ij}$ or$[B]_{ij}$ equals ε, the UBC constaint has the form $x \leq T = +\infty$ and can be remove from the system.

After the transformation of the initial problem into a new set of UBC constraints modelling the process and the control objective, we can apply the framing of the optimal solution and the algorithm of the previous section in order to calculate the greatest values of the state $x(k)$ and the control $u(k)$.

A. Example

From the corresponding timed event graph, we deduce the following state equation.

$$\begin{cases} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = \begin{pmatrix} a & e \\ \varepsilon & b \end{pmatrix} \begin{pmatrix} x_1(k-1) \\ x_2(k-1) \end{pmatrix} \oplus \begin{pmatrix} e \\ 1 \end{pmatrix} u(k) \\ \begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} = \begin{pmatrix} 2 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \end{cases} \quad \text{with } a \in [1\ 3] \text{ and } b \in [e\ 3].$$

We want to calculate the greatest state and input such that the desired outputs are:

k	2	1
z_1	18	15
z_2	15	12

Let $X = \begin{pmatrix} x_0 & x_1(0) & x_1(1) & x_1(2) & x_2(0) & x_2(1) & x_2(2) & u(1) & u(2) \end{pmatrix}^t$

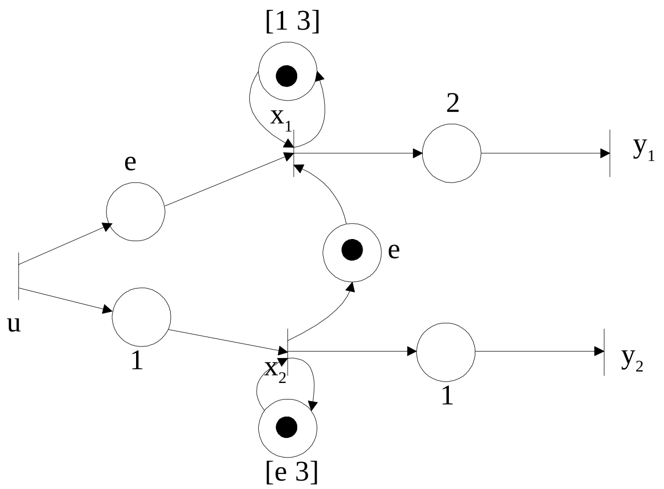

Figure 1: Timed event graph

The system (3) of the chapter V allows us to write the following inequation set:

$$\begin{cases} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \leq \begin{pmatrix} 2 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} \diagdown \begin{pmatrix} z_1(k) \\ z_2(k) \end{pmatrix} \\ \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \leq \begin{pmatrix} 3 & e \\ \varepsilon & 3 \end{pmatrix} \begin{pmatrix} x_1(k-1) \\ x_2(k-1) \end{pmatrix} \oplus \begin{pmatrix} e \\ 1 \end{pmatrix} u(k) \\ \begin{pmatrix} x_1(k-1) \\ x_2(k-1) \end{pmatrix} \leq \begin{pmatrix} 1 & e \\ \varepsilon & e \end{pmatrix} \diagdown \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \\ u(k) \leq \begin{pmatrix} e \\ 1 \end{pmatrix} \diagdown \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \end{cases}$$

We deduce the matrices M and N .

$$M = \begin{pmatrix} \varepsilon & \varepsilon & -13 & -16 & \varepsilon & -11 & -14 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & -3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon & -3 & e & \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & -3 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e & e & \varepsilon & -3 & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & 1 \\ \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & -1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & -1 & \varepsilon & \varepsilon \end{pmatrix}$$

$$N = \begin{pmatrix} \varepsilon & \varepsilon & -13 & -16 & \varepsilon & -11 & -14 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & 1 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

The framing of the optimal solution is:

$$X^- = \begin{pmatrix} e \\ 9 \\ 10 \\ 11 \\ 10 \\ 11 \\ 12 \\ 10 \\ 11 \end{pmatrix} \leq X_{opt.} = \begin{pmatrix} e \\ 12 \\ 13 \\ 16 \\ 11 \\ 11 \\ 14 \\ 10 \\ 11 \end{pmatrix} \leq X^+ = \begin{pmatrix} e \\ 12 \\ 13 \\ 16 \\ 11 \\ 11 \\ 14 \\ 10 \\ 13 \end{pmatrix}$$

VI. Conclusion

In this paper, we have proposed a new approach of control system in the field of discrete event systems. We have first considered that temporization values belong to known intervals contrary to the "backward" approach. A new formulation of the problem have given an expression which uses the operators min, max, and the residuation. The state variable is the minimization of a "backward " term and a "forward " term which is similar to the state equation. Showing the complexity of the problem, the expressions contain lower and upper bounds of matrices .

Moreover, this document have presented the application of an approach of constraint propagation type using Kleene's star in order to calculate the greatest values of the state and the control. We have also proposed a framing of the optimal solution the greatest bound of which can be applied to the initialization step. Some recent papers have shown that development of new algorithms is possible and the study of non-decreasing approaches seems particularly, a promising perspective.

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