From monotone inequalities to Model Predictive Control

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Abstract

The dater equalities constitute a well-known tool which allows the description of Timed Event Graphs in the field of (max, +) algebra. This paper gives an equivalent model in the standard algebra which can describe Timed Event Graphs with inputs and outputs. Concepts of componentwise order and monotone inequalities prove the existence of unique minimum, or maximum, solution in special system $Ax \leq b$. The approach is applied to just in time control and Model Predictive Control. Connections with linear programming are made.

1. Introduction

The dater equalities [2] constitutes a well-known tool which allows the description of Timed Event Graphs and P-time Event Graphs in the field of (max, +) algebra. These graphs are convenient tools to model systems with durations. Operation times for P-time Event Graphs, are included between a minimum and a maximum duration. Paper [5] proposes a new model which describes P-time Event Graphs. In that study, special incidence matrices are introduced. Considering 1-periodic behavior, the application of a variant of Farkas' Lemma allows the determination of upper and lower bounds of the production rate and conditions of consistency. In [6], the production rate is calculated by two efficient approaches using the duality theorem of linear programming.

In this paper, we complete the model used in these studies by introducing inputs and outputs and we make the connection with the well-known incidence matrix used in the fundamental relation on marking. We moreover consider two problems:

- The earliest operation of Timed Event Graphs which allows the simulation of the system knowing the inputs;

- The control synthesis which computes an optimal controller under a just in time criterion, in the sense that the controller delays as much as possible such that system output occurs before a desired output on a given horizon.

In [4], we propose solutions of these two problems using (max,+) algebra. In this paper, we show that the resolution can also been made in standard algebra. In fact, we need to use a special type of linear inequalities which has been studied by different authors out the field of discrete event systems. The basic concepts are lattices and componentwise order. A pioneer is G.B. Dantzig who analysis dynamic Leontief System in 1955. R.W. Cottle [3] has shown a correspondence between linear inequalities and lattices. A recent research is M. Queyranne 2006 [9]. Using these mathematical results, a perspective is the analysis of the connections between (max, +) algebra and standard algebra in the field of Discrete Event Systems. The outline of this paper is as follows. We first complete [5] and [6] by introducing inputs and outputs. Then, we present the concept of monotone inequalities which allows the resolution of linear inequalities. Finally, the framework is applied to the synthesis of a just in time control of Timed Event Graphs and model predictive control. We discuss the connections between componentwise order and objective function of linear programming.

2 Model

A Petri net is a pair (G, M_0), where $G = (R, V)$ is a bipartite graph with a finite number of nodes (the set V) which are partitioned into the disjoint sets of places P and transitions TR (transitions are denoted t while temporization are denoted T,); R consists of pairs of the form (p_i,q_i) and (q_i,p_i) with $p_i \in P$ and $q_i \in TR$. The initial marking M_0 is a vector of dimension $\mid P \mid$ whose elements denote the number of initial tokens in the respective places. Each place $p_l \in P$ is associated with an initial marking (initial number of tokens) denoted m_l. The set $^\bullet p$ is the set of input transitions of p and $p^\bullet$ is the set of output transitions of place $p \in P$. The set $^\bullet t_i$ (respectively, $t_i^\bullet$) is the set of the input (respectively, output) places of the transition $t_i \in TR$.

For a Petri net with $\mid P \mid$ places and $\mid TR \mid$ transitions, the incidence matrix $W = [W_{ij}]$ is an $\mid P \mid \times \mid TR \mid$ matrix of integers and its entry is given by $W_{ij} = W_{ij}^+ - W_{ij}^-$ where W_{ij}^+ is the weight of the arc from transition j

to an output place i and W_{ij}^- is the weight of the arc to transition j from an input place i [8].

In a Petri net, a firing sequence from marking M, implies a string of successive markings. The characteristic vector s of a firing sequence S is such that each component is an integer corresponding to the number of firings of the corresponding transition. Then marking M reached from initial marking M_0 by firing of sequence S can be calculated by the fundamental relation: $M = M_0 + W \times s$.

A Petri net is called an Event Graph if each place has exactly one upstream and one downstream transition. Timed Petri nets allow the modeling of discrete event systems with sojourn time constraints of the tokens inside the places. Consistent with the dioid $\overline{\mathbb{R}}_{max}$ (see [2]), we associate a temporization defined in $\mathbb{R}^+$ with each place. Each place $p_l \in P$ is associated with an temporization T_l, and, an initial marking denoted m_l.

2.1 Preliminary inequalities of a Timed Event Graph

We consider the "dater" type well-known in the (max, +) algebra: each variable $x_i(k)$ represents the date of the k^{th} firing of transition x_i. If we assume a FIFO functioning of the places which guarantees that the tokens do not overtake one another, a correct numbering of the events can be carried out. The evolution can be described by the following inequalities expressing relations between the firing dates of transitions. Let us recall that an Event Graph can be considered as a set of subgraphs made up of a place p_l linked with one upstream transition $\{t_j\} = ^\bullet p$ and one downstream transition $\{t_i\} = p^\bullet$.

Using temporization T_l we can write the following inequality for each place p_l where $(j, i) = (^\bullet p, p^\bullet)$:

$T_l + x_j(k - m_l) \leq x_i(k)$ or equivalently, $x_j(k - m_l) - x_i(k) \leq -T_l$.

Weight 1 of $x_j(k - m_l)$ (respectively, -1 of $x_i(k)$) is the weight of the arc going from t_j to place p_l (respectively, the arc going from place p_l to transition t_i) which is equal to W_{lj}^+ (respectively, $-W_{li}^-$).

2.2 Matrix expression of a Timed Event Graph

Let m be the maximum number of initial tokens. The set of the previous inequalities which describes a Timed Event Graph, can be expressed with the following form: Column-vector $-T$ is a vector of temporization where T_l is the temporization of place p_l.

$$\begin{pmatrix} G \end{pmatrix} \times \begin{pmatrix} x(k-m) \\ x(k-m+1) \\ .... \\ x(k-1) \\ x(k) \end{pmatrix} \leq \begin{pmatrix} -T \end{pmatrix} \qquad (1)$$

where $G = [G_m G_{m-1} G_{m-2} .... G_1 G_0]$ with a dimension equal to $\mid P \mid . \, (m+1) . \mid TR \mid$.

Each place corresponds to a row of G which contains the weights of its entering and outgoing arcs. Particularly, matrix G_m for $i \in [1, m]$ contains the weights of the arcs entering the places with tokens ($m \geq 1$). Matrix G_0 contains:

1. the weight of arc entering the places with no token ;
2. also, the weight of the arc outgoing from each place (usually expressed by W^-).

From the above description on the weight of the arcs, we can deduce the following relation with the incidence matrix W:

$$W = \sum_{i=0}^{i=m} G_i$$

Now let us express system inequalities (1) on a reduced horizon. Such a form will simplify the calculation. The objective is to build an equivalent model such that each place of the graph contains only zero or one token.

As a place contains a maximum number of m tokens, the general idea is to split each place containing m tokens into m places, where each place contains only one token. A systematic procedure is as follows.

Let us introduce new variable X , that is:

$X(k) = \begin{pmatrix} X_0(k) & .... & X_i(k) & .... & X_{m-1}(k) \end{pmatrix}$ with $X_i(k) = x(k - m + i + 1)$. By construction, we have: $X_{m-1}(k) = x(k)$ and $X_i(k) = X_{i+1}(k-1)$ for i going from 0 to $m - 2$.

So, system (1) becomes:

$$\begin{pmatrix} G_1' & G_0' \end{pmatrix} \times \begin{pmatrix} X(k-1) \\ X(k) \end{pmatrix} \leq (-T)$$

where: $G_1' = \begin{pmatrix} G_m & 0 & \ldots & \ldots & 0 \end{pmatrix}$
and $G_0' = \begin{pmatrix} G_{m-1} & G_{m-2} & \ldots & G_1 & G_0 \end{pmatrix}$.

By completing the system with following $(m-1) \times |TR|$ relations,

$$X_i(k) - X_{i+1}(k-1) \leq 0$$

for $i = 0$ to $m - 2$. The system can be written as follows:

$$\begin{pmatrix} V_{01} & V_{00} \end{pmatrix} \times \begin{pmatrix} X(k-1) \\ X(k) \end{pmatrix} \leq (0)$$

where matrix V_{01} of dimension $((m-1) \times |TR| \times m)$ is an subdiagonal of identity matrices immediately above the main diagonal, while the matrix V_{00} is a diagonal of negative identity matrices.

Finally, the concatenation of the two systems gives the algebraic form:

$$\begin{pmatrix} H \end{pmatrix} \times \begin{pmatrix} X(k-1) \\ X(k) \end{pmatrix} \leq \begin{pmatrix} -T \\ 0 \end{pmatrix}$$

where: $H = \begin{pmatrix} G_1' & G_0' \\ V_{01} & V_{00} \end{pmatrix}$.

2.3 Input/output model

Internal transitions, input transitions and output transitions are denoted x, u and y respectively. The corresponding set of transitions are denoted TR_x, TR_u and TR_y. The set of places P is the union of three sets , that is $P = P_{u\to x} \cup P_{x\to x} \cup P_{x\to y}$ where $P_{u\to x}$ is the set of places between inputs and internal transitions, $P_{x\to x}$ is the set of places between internal transitions and $P_{x\to y}$ is the set of places between internal and output transitions. This graphical notation is kept for relevant matrices and roughly speaking, we note $W^+_{u\to x}$. This abuse of notation facilitates the writing of the matrices.

Without reduction of generality, we assume that the initial marking of places of $P_{u\to x}$ and $P_{x\to y}$ is null. Moreover, to simplify the presentation, we assume that the initial marking of places of $P_{x\to x}$ is equal to one and the set of places linking inputs to outputs is empty. Therefore, we can write the relevant inequalities which express relations between the firing dates of internal, input and output transitions.

One obtains for $P_{x\to x}$, $P_{u\to x}$ and $P_{x\to y}$, respectively:

$$\begin{pmatrix} W^+_{x\to x} & -W^-_{x\to x} \end{pmatrix} \begin{pmatrix} x(k-1) \\ x(k) \end{pmatrix} \le (T_{x\to x}) \qquad (2)$$

$$\begin{pmatrix} W^+_{u\to x} & -W^-_{u\to x} \end{pmatrix} \begin{pmatrix} u(k) \\ x(k) \end{pmatrix} \le (-T_{u\to x}) \qquad (3)$$

$$\begin{pmatrix} W^+_{x\to y} & -W^-_{x\to y} \end{pmatrix} \begin{pmatrix} x(k) \\ y(k) \end{pmatrix} \le (-T_{x\to y}) \qquad (4)$$

where dimensions of incidences matrices are as follows. dim($W^+_{x\to x}$) $= dim(W^-_{x\to x}) = |P_{x\to x}|.|TR_x|$, dim($W^+_{u\to x}$) $= |P_{u\to x}|.|TR_u|$, dim($W^-_{u\to x}$) $= |P_{u\to x}|.|TR_x|$, dim($W^+_{x\to y}$) $= |P_{x\to y}|.|TR_x|$ and dim($W^-_{x\to y}) = |P_{x\to y}|.|TR_y|$.

Remark.

1. From the above description on the weight of the arcs, we can deduced that $W^+_{u\to x}$, $W^+_{x\to y}$, $W^-_{u\to x}$, $W^-_{x\to y}$ are submatrices of the incidence matrices W^+ and W^-.

2. No row of incidence matrices is null as we express relations of each place of the Event Graph. Similarly, no column of W^+_{ux} and W^-_{xy} is null knowing that each input/output transition is supposed connected to the graph.

Example 1 :

Graph $G = (P, TR)$ in figure (1) illustrates the Input/output model.

$P = P_{u\to x} \cup P_{x\to x} \cup P_{x\to y}$ where : $P_{u\to x} = \{P_1\}$, $P_{x\to x} = \{P_2, P_3\}$ and $P_{x\to y} = \{P_4\}$.

$TR = TR_u \cup TR_x \cup TR_y$ where : $TR_u = \{u\}$, $TR_x = \{x_1, x_2\}$ and $TR_y = \{y\}$.

Let us suppose that $M_0 = (0, 1, 1, 0)^t$.

The input/output model is as follows :

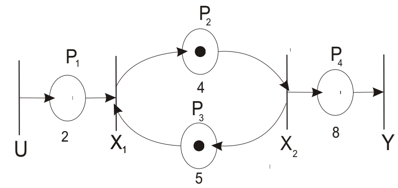

Figure 1. Timed event graph

$$\begin{pmatrix} W^+_{x\to x} & -W^-_{x\to x} \end{pmatrix} \begin{pmatrix} x_1(k-1) \\ x_2(k-1) \\ x_1(k) \\ x_2(k) \end{pmatrix} \le \begin{pmatrix} -4 \\ -5 \end{pmatrix}$$

where :

$W^+_{x\to x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $W^-_{x\to x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} W^+_{u\to x} & -W^-_{u\to x} \end{pmatrix} \begin{pmatrix} u(k) \\ x_1(k) \end{pmatrix} \le \begin{pmatrix} -2 \end{pmatrix}$

where :

$W^+_{u\to x} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $W^-_{u\to x} = \begin{pmatrix} 0 & 1 \end{pmatrix}$

$\begin{pmatrix} W^+_{x\to y} & -W^-_{x\to y} \end{pmatrix} \begin{pmatrix} x_2(k) \\ y(k) \end{pmatrix} \le \begin{pmatrix} -8 \end{pmatrix}$

where :

$W^+_{x\to y} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $W^-_{x\to y} = \begin{pmatrix} 0 & 1 \end{pmatrix}$ ■

3 Monotone inequalities

In this paper, we focus on partial order $\le$ defined on a set S which is defined componentwise: $x \le y \quad iff$ $x_i \le y_i$ for all $i \in \{1, 2, ..., card(x)\}$. This part presents vocabulary (see part 4.3.1 of [2]) and theoretical results used in parts 5 and 6.

A maximum (minimum) of a subset is an element of the subset which is greater (lower) than any other element of the subset. If it exists, it is unique.

A majorant (minorant) of a subset is an element not necessarily belonging to the subset which is greater (lower) than any other element of the subset. If a majorant belongs to the subset, it is the maximum (minimum) element. Majorant (minorant) is also called upper (lower) bound.

Upper bound (lower bound) is the least majorant (greatest minorant). When 'majorant' is called 'upper bound', this notion is called ' least upper bound'. Respectively, when 'minorant' is called 'lower bound', this notion is called 'greatest lower bound'.

Sup-semilattice (respectively, inf-semilattice) $(S, \le)$ if an ordered set such that there exists an upper (respectively lower) bound in S for each pair of elements. It is called lattice if it is both a inf-semilattice and a sup-semilattice.

In this part, we present theoretical results used in parts 5 and 6.

The intersection of a finite number of closed half spaces in $\mathbb{R}^n$ is called a convex polyhedron or simply a polyhedron. Equivalently, a polyhedron is a subset Γ of $\mathbb{R}^n$ which can be represented as the solution set to a system of linear inequalities : $\Gamma = \{x \in \mathbb{R}^n : Ax \le b\}$

where A is an $m \times n$ real matrix and b is a real m dimensional vector.

In this paper, we focus on a particular class of linear systems defined as follows.

Definition 1 *The system of linear inequalities $Ax \leq b$ is inf-monotone (respectively, sup-monotone) if each row of matrix A has one strictly negative (respectively positive) coefficient at the most.*

Example 2 :
System $Ax \leq b$ is as follows :
where : $A = \begin{pmatrix} -3 & -2 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$
We remark that each row of matrix A has one strictly positive coefficient at the most. So the system is sup-monotone ■

Definition 2 *If an inequality is inf-monotone and sup-monotone at the same time, this inequality is said bi-monotone. A system is called bi-monotone if each inequality is bi-monotone.*

Therefore, each row of matrix A has one strictly positive coefficient at the most and one strictly negative coefficient at the most. Another definition is as follows.

Definition 3 *A bimonotone linear inequality is of the form $a_i x_i + a_j x_j \leq c$, for some i,j with the product $a_i a_j$ nonpositive.*

Example 3:
Let system $Ax \leq b$ with:
A = $\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ and b= $\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$. The product of the coefficients of each inequality is nonpositive, what means, inequalities are bimonotone. Each row of matrix A has one strictly positive and negative coefficient at the most ■

If the initial marking is null, $m = 0$ and $W = G_0$. We can write $W \times x(k) \leq -T^-$for a Timed Event Graph. This system is bi-monotone as each row of incidence matrix W contains zero elements except two non-null coefficients 1 and -1. Let us note that matrix A of a bi-monotone system can also contain rows with one non-null coefficient.

4 Monotone system having a least element

Theorem 1 *[13] [3]. Let Γ be the set of solutions of a inf-monotone (respectively sup-monotone)system $Ax \leq b$. The following are equivalent :*

1. *Set Γ is a inf-semilattice (respectively sup-semilattice).*
2. *If x and y are two elements of Γ then their minimum $x \wedge y$ (respectively, maximum $x \vee y$) belongs to Γ.*

Property 1 *A set of solutions of a bi-monotone system $Ax \leq b$ is a lattice.*

The proof is immediate as a inf-semilattice which is sup-semilattice, is a lattice. ■

The following Theorem is important as it guarantees the existence of minimal or maximal solution of sets Γ . It is applied in the following parts as it proves the existence of earliest trajectory or greatest control in the following problems.

Theorem 2 *Set Γ has a largest (respectively, least) element if the set is non-empty and has a majorant (respectively, minorant).*

Example 4 :
$Ax \leq b$ with A = $\begin{pmatrix} 2 & -1 \\ -1 & 2 \\ -1 & 0 \end{pmatrix}$ and b= $\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$
We remark that the system is bimonotone. Set
Γ is non-empty as $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \Gamma$ and $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ is a majorant of Γ. The largest element is: $\begin{pmatrix} 8/3 \\ 10/3 \end{pmatrix}$. Equally $\begin{pmatrix} -5 \\ -6 \end{pmatrix}$ is a minorant of Γ. The least element is $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ ■

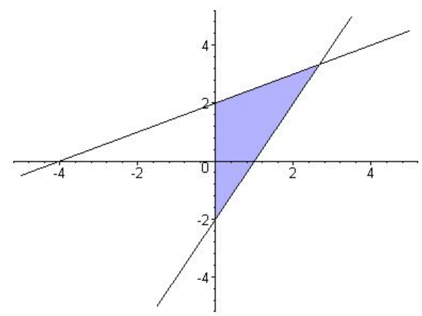

Figure 2. curves of system example 4

5 Earliest trajectory

In (max, +) algebra, it is well-known that Timed Event Graphs has an earliest trajectory expressed by an equality. In this part, we interpret this by using the proposed model defined in standard algebra. Knowing initial state x_0 and control u in $\mathbb{R}$, the aim is to prove the existence of the minimum state trajectory. Input and state inequalities are as follows.

For $k \geq 1$,

$$W^-_{u \to x}\, x(k) \geq (T_{u \to x}) \;+ W^+_{u \to x}.u(k)$$

and

$$\begin{pmatrix} I \\ W^-_{x \to x} \end{pmatrix} x(k) \geq \begin{pmatrix} 0 \\ T_{x \to x} \end{pmatrix} + \begin{pmatrix} I \\ W^+_{x \to x} \end{pmatrix} x(k-1)$$

with $x(0) = \mathrm{x}_0$ in $\mathbb{R}$.

The first row expresses non-decreasing characteristic of the state. The two last inequalities are a system of linear inequalities which are expressed in form $Ax \geq b$.

$$\begin{pmatrix} W^-_{u\to x} \\ I \\ W^-_{x\to x} \end{pmatrix} x(k) \geq \begin{pmatrix} T_{u\to x} \\ 0 \\ T_{x\to x} \end{pmatrix} + \begin{pmatrix} 0 & W^+_{u\to x} \\ I & 0 \\ W^+_{x\to x} & 0 \end{pmatrix} \begin{pmatrix} x(k-1) \\ u(k) \end{pmatrix} \quad (5)$$

Property

Proposition 1 *Knowing the control, there is a minimum state and output for initial state $x(0) = x_0$.*

Proof

Let A= $\begin{pmatrix} W^-_{u\to x} \\ I \\ W^-_{x\to x} \end{pmatrix}$

and

$b = \begin{pmatrix} T_{u\to x} \\ 0 \\ T_{x\to x} \end{pmatrix} + \begin{pmatrix} 0 & W^+_{u\to x} \\ I & 0 \\ W^+_{x\to x} & 0 \end{pmatrix} \begin{pmatrix} x(k-1) \\ u(k) \end{pmatrix}$.

- This system $Ax \geq b$ is inf-monotone as each row of matrix $A = \begin{pmatrix} W^-_{u\to x} \\ I \\ W^-_{x\to x} \end{pmatrix}$ has one strictly positif coefficient.

- This set is non-empty as we can easily calculate an arbitrary value $x(k)$ in $\mathbb{R}$ for any initial state and control as each row of matrix A contains a only non-null coefficient.

- Moreover, we know that $\mathtt{x}_0$ is a minorant of $\mathtt{x(1)}$ as the state trajectory is non-decreasing: this property is expressed in the state inequality. We can generalize to an arbitrary state trajectory which has a minorant.

We can deduce from Theorem 2 that the space solution set has a minimum state $x(k)$. In other words, a Timed Event Graph has a unique earliest trajectory.

Output inequality is $W^-_{x\to y}.y(k) \geq T_{x\to y} + W^+_{x\to y} x(k)$ and the same result holds as $x(k)$ is deduced.

Finally, we can deduce that the minimum state and output exist knowing initial state x_0 and control u in $\mathbb{R}$.■

Remark. This result also holds for autonomous case (without input transitions) as the input inequality can be removed from the previous proof. An application is for instance the determination of 1-periodic trajectory.

Optimal solution

Knowing initial state $x(0) = x_0$ and control $u(k)$ on an arbitrary known horizon, different algorithms can give the state and output trajectories. For instance, the general Fourier-Motzkin algorithm can give an arbitrary solution but also the optimal solution after an adaptation [10]. A more specific algorithm is given in [1]: it solves systems of linear inequalities with two variables per inequality in polynomial time. See also [11] for the same type of systems.

In fact, the optimal solution is also solution of special linear programming problem as indicated by the following result [3] [7]. Set $\Gamma = \{x \in \mathbb{R}^n : Ax \leq b\}$ has a maximum element x_0 if and only if x_0 is optimal for the problem max $\{cx$, such that $Ax \leq b$ for any $c > 0\}$. Therefore, approaches using componentwise order relation give the same results as the relevant problems using linear programming for $c > 0$ if the inf-monotone (respectively, sup-monotone) system define a non-void set with a minorant(respectively, majorant).

6 Control synthesis

We propose here a method to compute an optimal controller under just in time criterion, in the sense that the controller delays as much as possible such that system output occurs before a reference trajectory z on horizon $[k_s, k_f]$. As we have no information after event k_f, there is no demand on the production and we take $z(k) = +\infty$ (or an arbitrary large value) for $k > k_f$. So the problem is the determination of the greatest control so that $y(k) \leq z(k)$ on horizon $[k_s, k_f]$. We assume that the event graph is structurally observable and controllable (definitions are given below). A well-known solution of this problem exists in (max, +) algebra and is described in part 5.6 of [2].

More formally, the system is described below by the inequalities of model and the constraint on the output:

$$W^-_{u\to x}\, x(k) \geq (T_{u\to x}) + W^+_{u\to x}.u(k) \quad (6)$$

$$\begin{pmatrix} I \\ W^-_{x\to x} \end{pmatrix} x(k) \geq \begin{pmatrix} 0 \\ T_{x\to x} \end{pmatrix} + \begin{pmatrix} I \\ W^+_{x\to x} \end{pmatrix} x(k-1) \quad (7)$$

$$W^-_{x\to y}.y(k) \geq T_{x\to y} + W^+_{x\to y}.x(k) \quad (8)$$

and

$$y(k) \leq z(k) \quad (9)$$

Let us write an equivalent system under form $Ax \leq b$.

From (8) and (9), we can write :

$$W^+_{x\to y}.x(k) \leq -T_{x\to y} + W^-_{x\to y}.y(k) \leq -T_{x\to y} + W^-_{x\to y}.z(k)$$

as $W^-_{x\to y}$ is non-negative. From (7), we obtain:

$$\begin{pmatrix} I \\ W^+_{x\to x} \end{pmatrix} x(k) \leq \begin{pmatrix} 0 \\ -T_{x\to x} \end{pmatrix} + \begin{pmatrix} I \\ W^-_{x\to x} \end{pmatrix} x(k+1)$$

Finally, we must solve the following system inequalities expressed in form $Ax \leq b$.

$\forall k \in [k_s, k_f]$

$$\begin{pmatrix} W^+_{x\to y} \\ I \\ W^+_{x\to x} \end{pmatrix} x(k) \leq \begin{pmatrix} -T_{x\to y} \\ 0 \\ -T_{x\to x} \end{pmatrix} + \begin{pmatrix} W^-_{x\to y} & 0 \\ 0 & I \\ 0 & W^-_{x\to x} \end{pmatrix} \begin{pmatrix} z(k) \\ x(k+1) \end{pmatrix} \quad (10)$$

$W^+_{u\to x}.u(k) \leq -T_{u\to x} + W^-_{u\to x}.x(k)$

In (max, +) algebra, the structural observability [2] gives a condition to observe an effect at the output due to one internal transition at least. In this definition, the dates of firing of the outputs are not especially known.

Definition 4 *[2] An event graph is structurally observable if from every internal transition, there exists a path to one output transition at least. The minimal initial number of tokens of these paths to internal transition t_i is denoted $\pi_{i,out}$.*

In Petri nets, another definition is used. A transition is said to be observable if the relevant dates of firing is known. Let us assume that 1) every output is observable 2) the event graph is structurally observable 3) the date of firing of an internal transition can be modified. Therefore, an arbitrary large date of firing of this internal transition yields an arbitrary large date of firing of an output transition. In other words, in a structurally observable event graph, every internal transition have a structural influence on one output transition at least. This structural influence can be see if every output is observable.

The assumption of structural observability is used in the proposition below.

Property

There is a maximum date $x_i^+(k) \in \mathbb{R}$ on horizon $[k_s, k_f - \pi_{i,out}]$ satisfying (10) to each internal transition.

Proof :

This system of linear inequalities is sup-monotone as each row of matrix $A = \begin{pmatrix} W^+_{x\to y} \\ I \\ W^+_{x\to x} \end{pmatrix}$ has one strictly positif coefficient per row.

The resolution of the following system gives a majorant:

$W^+_{x\to y}.x(k_f) \leq -T_{x\to y} + W^-_{x\to y}.z(k_f)$ and $x(k) \leq x(k+1)$ for $k < k_f$.

However, as each internal transition is not directly connected to output transition, some columns of $W^+_{x\to y}$ can be null and only some components of $x(k_f)$ has a finite majorant if $z(k_f)$ is finite. If we assume that the system is structurally observable, then there exists from every internal transition x_i at least a path to one output transition y_i in the event graph. The resolution follows the opposite direction and there is a relation linking values of an output transition and an internal transition but with a shift in the numbering of events. For every internal transition, its minimal value $\pi_{i,out}$ exists as the event graph is structurally observable. Therefore, a finite value $z(k)$ generates a majorant of $x_i(k - \pi_{out})$. Therefore, we must consider the observable values of internal transitions as they present a majorant.

Finally, the set reduced to variables with majorant is non-empty as we can easily calculate a solution (if condition x(0)= x_0 is removed). From Theorem 2, we can deduce that the solution set has a maximum element.

Moreover, inequality (6) gives $W^+_{u\to x}.u(k) \leq -T_{u\to x} + W^-_{u\to x}.x(k)$ which is also sup-monotone. The same reasoning can be made for the inputs: as every input transition is connected to the system and, we can deduce a majorant from majorant of state (or maximum state). The greatest control trajectory exists and can be deduced from the above inequality.■

The determination of the optimal control can use the algorithms cited in part 5. Starting from k_f, the algorithm can successively solve system (10) following a backward iteration from k_f to k_s . In the initial step, $x(k_f + 1)$ is equal to infinite (or an arbitrary large value).

Let us note that the concept of structural controllability is not used in the calculation of the greatest state and control. A possible effect on dates of internal transitions can be made only if there a path between an input transition and each internal transition. This condition is named structural controllability.

Definition 5 *[2] An event graph is structurally controllable if, every internal transition can be reached by a path from one input transition at least.*

If the event graph is structurally controllable, the possible effect of an input transition on an internal transition t_i is made with a shift in the numbering of events. Its minimal value is denoted $\pi_{i,in}$. To sum up, the assumption of structurally observable and controllable event graph yields a pair (π_{in}, π_{out}) for each internal transition t_i.

Example 5 :

Continuation of Example 1, on horizon $[1, 10]$ reference trajectory z is given in the following table:

k	1	2	3	4	5	6	7	8	9	10
$z(k)$	42	42	42	90	90	90	138	138	138	186

From $z(k)$, the solution set of inequality (10) has a maximum element, which can be calculate by the Fourier-Motzkin algorithm. This greatest state corresponds to vector denoted x^+given in the following table:

k	1	2	3	4	5	6	7	8	9	10
	30	30	69	78	78	117	126	126	174	$+\infty$
$x^+(k)$	25	34	34	73	82	82	121	130	130	178

We can calculate control u from x^+:

k	1	2	3	4	5	6	7	8	9	10
$u(k)$	28	28	67	76	76	115	124	124	172	$+\infty$

Finally, given initial state

$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and control u, we solve inequality (5) by using Fourier-Motzkin algorithm. The calculated minimum represents the earliest trajectory, given in the following table:

k	1	2	3	4	5	6	7	8	9	10
	30	30	69	78	78	117	126	126	174	$+\infty$
$x(k)$	4	34	34	73	82	82	121	130	130	178

The relevant earliest trajectory y is:

k	1	2	3	4	5	6	7	8	9	10
$y(k)$	12	42	42	81	90	90	129	138	138	186

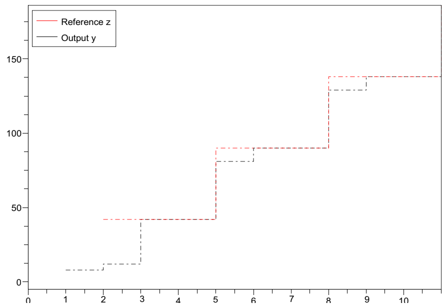

Figure 3. Trajectory simulation

7 Model Predictive control

Now, the optimal control can be extended to on-line approach and particularly Model Predictive Control. After an optimal control on horizon $[k_s, k_s + h]$, the horizon is shifted and becomes $[k_s + 1, k_s + h + 1]$: the problem is updated with new information of the measurements and a new optimization at step k_s+1 must be performed. Below, we detail the general technique of the sliding horizon in Model Predictive control.

7.1 Data and sliding horizon

We assume that each event date of transition firing is available for current number of event k: at step $k = k_s$,u_{k_s} and x_{k_s} are known. A future control sequence $u(k)$ for $k \in [k_s+1, k_s+h]$ is determined such that this control is the optimal solution of the problem. The first element of the optimal sequence (here $u(k_s + 1)$) is applied to the process. At the next number of event $k_s + 1$, the horizon is shifted: the problem is updated with new information of the measurements and a new optimization at step k_s+1 is performed. Two approaches using componentwise order relation or objective function can solve this problem.

7.1.1 Approach using objective function

- Input cost criterion

The objective can be the maximization of the sum of the components of the control on the horizon. A consequence is that the internal buffer levels are kept as low as possible.

$$J_{in} = \sum_{k=k_s+1}^{k_f} \sum_{i=1}^{\dim(u)} u_i(k)$$

- Tardiness

If initial state $x(k_s) = x_{k_s}$ and control $u(k)$ are known on an arbitrary horizon, the resolution of part 5 gives the earliest state denoted χ and earliest output trajectories. The control synthesis generates a greatest state which is denoted $\xi(k)$. The difference $\xi(k) - \chi(k)$ expresses the margin but the two trajectories must be coherent and satisfies constraint $\chi{\leq}\xi$ otherwise, the Timed Event Graph cannot (provisionally) obeys the Just-in-time criteria $y{\leq}z$ as state ξ is the greatest possible trajectory. If we have to pay a penalty for every delay by respect to the desired output z, an interesting cost criterion is the tardiness which is equal to $y_i - z_i$ if $y_i \geq z_i$ or 0 if $y_i < z_i$. If the tardiness is denoted v, we can write

$$\begin{pmatrix} I \\ 0 \end{pmatrix} y(k) - \begin{pmatrix} I \\ I \end{pmatrix} v(k) \leq \begin{pmatrix} z(k) \\ 0 \end{pmatrix}$$

and we can minimize the sum of the components of the tardiness on the horizon

$$J_{out} = \sum_{k=k_s+1}^{k_f} \sum_{i=1}^{\dim(y)} v_i(k)$$

- Approach

Considering input cost and tardiness, the problem of minimization of $J = J_{out} - \lambda.J_{in}$ with λ a nonnegative scalar can be solved by linear programming .

Remark. Contrary to the state equation of (max, +) algebra, the model of this paper is defined by inequalities. Let us express the earliest functioning with A and b defined in part 5. Recall that the earliest functioning is expressed by a simple equality in (max, +) algebra. As each row contains only one non-null coefficient, the calculation of the minimum of $x_i(k)$ uses the rows j such that $A_{j,i} \neq 0$ and takes the maximal value calculated on this relevant inequality.

$$\prod_{A_{j,i}\neq 0} (A_{j,i} x_i(k) - b_j) = 0$$

Expressing the earliest functioning of a Timed Event Graph, this equality contains a product of sums. In [12] it is proved that this condition can be neglected for a slightly more general problem .

7.1.2 Approach using componentwise order relation

- Input criterion

This criterion is considered in part 6. Therefore, approaches using componentwise order relation give the same results as the relevant problems using linear programming (see part 5) if the sup-monotone system define a non-void set with a majorant.

- Tardiness

We also desire to solve the tardiness case. A priori, an approach is to apply the technique already applied in part 6 based on a maximization. A convenient form is a sup-monotone inequality $Ax \leq b$. As we want to minimize $v(k)$ or maximize $-v(k)$, the convenient form is:

$$\begin{pmatrix} I \\ 0 \end{pmatrix} y(k) + \begin{pmatrix} I \\ I \end{pmatrix} (-v(k)) \leq \begin{pmatrix} z(k) \\ 0 \end{pmatrix}$$

Unfortunately, the system which define the tardiness is not sup-monotone as there is rows with two positive coefficients: we cannot directly apply a technique based only on componentwise order. The following multi-step approach solves this problem.

- Approach A possibility is to delay the desired output with a realistic desired output which can be followed by the Timed Event Graph. Therefore, the technique is as follows:

a) Step a is the calculation of the earliest possible trajectory y_p^- such that the Timed Event Graph can start from initial condition x_0. The earliest trajectory can simply be calculated by application of approach described in part 5.

Known data are initial condition x_{k_s} and minimal control $u(k) = u_{k_s}$ for $k \geq k_s$.

b) Step b is the replacement of desired output z by modified desired output $max(\ (y_p^-(k))_i, (z(k))_i\ )$ for each component i. The tardiness is minimized as trajectory y_p^- is minimal. c) Step c is to apply the control synthesis described above.

Example 6 Let us consider the following problem which can only be solved by predictive control approaches: the just in time criteria ($y(k) \leq z(k)$) cannot be satisfied for the following current state and desired output. Horizon h is equal to 10 and we consider 10 steps of the algorithm. The desired output is $z(k) = 42 + 24.E(\frac{k-1}{3})$ for $k \geq 1$. We assume that $x(0) = \begin{pmatrix} 50 \\ 50 \end{pmatrix}$. From the

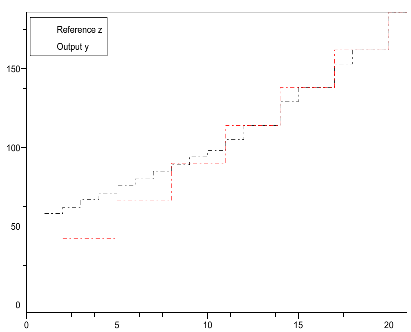

Figure 4. Trajectory simulation of Model Predictive Control

model, we deduce that $y(1) \geq 62 > z(1) = 42$ for any control and $y(k) \not\leq z(k)$ for $k \geq 1$. In this situation, Figure 4 shows the performance of the Model Predictive Control: output y converges to desired output z at $k = 11$.

8 Conclusion

We propose in this paper, an input/ouput model based on new particular incidence matrices relevant to input, internal and output places. The connection between these matrices and the classical incidence matrix (usually denoted W) of the fundamental relation on marking is shown. We give a new state inequality system which describes Timed Event Graphs in standard algebra. Let us recall that the model of Timed Event Graph is a state equation in (max,+) algebra. We prove that the minimum state trajectory exists.

We also consider the problem of just in time control of Timed Event Graphs. Classically, this problem is solved in (max,+) algebra by using a componentwise order and residuation theory. Using the new model, we show that this problem can also be solved in standard algebra with componentwise order. The resolution can be made with Fourier-Motzkin algorithm. This fact suggests a connection between approaches using standard algebra and (max, +) algebra in context of control.

The results are based on the structure of inequalities which are inf-monotone or sup-monotone. In part 5, the determination of the earliest trajectory leads to a inf-monotone system while the determination of the greatest control yields a sup-monotone system in part 6. Application of Theorem 2 shows that minimal or maximal solution exists. The study is generalized to Model Predictive Control. A simple example illustrates the approach.

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