OPTIMAL CONTROL SYNTHESIS IN INTERVAL DESCRIPTOR SYSTEMS APPLICATION TO TIME STREAM EVENT GRAPHS

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Abstract: This paper presents a new class of model in the field of topical algebra whose time evolution belongs to intervals. The lower and upper bounds depend on the maximization, minimization and addition operations, simultaneously. A motivation for the current study is the modelling of Time Petri nets like Time Stream Petri Nets and P-time Petri nets which extend Timed Petri Nets and generalize the semantic of its synchronization. The final objective is to make optimal control synthesis on this new model and to generalize the classical "backward" equations applied to Timed Event Graphs.

Keywords: (min, max, +) functions, cycle-time vector, fixed point, P-time Petri Nets, Time Stream Petri Nets

1. INTRODUCTION

Discrete event dynamics systems involving synchronization can be modeled by several types of Petri nets (PNs). Among these PNs, we can quote P-time Petri nets (P-time PNs), Time stream Petri nets (Time Stream PNs),.... which extend the application field of Timed Event Graphs. Time Stream Event Graphs for example, allow to specify the synchronization requirements of multimedia applications (Diaz and Owezarski, 1997) and can describe complex synchronizations. The Time Stream Petri Nets present different types of semantic as "And", "Weak-And", "Strong-Or", "Or", "Master" and theirs variations (Courtiat *et al.*, 1996) which correspond to different temporal evolutions. In (Declerck and Alaoui, 2004), we show that P-time Event Graphs can be modeled by a new class of systems called interval descriptor system for which the time evolution is not strictly deterministic but belongs to intervals. The lower and upper bounds depends on the maximization, minimization and addition operations, simultaneously. Moreover, the liveness can be studied in the topical algebra using the spectral theory.

Let us assume that some events are stated as controllable, meaning that the corresponding transitions (input) may be delayed from firing until some arbitrary time provided by a supervisor. Given a desired behavior of some transitions (output) of the interval descriptor system such as a sequence of execution times, we wish to slow down the system as much as possible without causing any event to occur later than this sequence. So, the problem is to determine whether there exist control actions which will restrict the system to that behavior. If this is possible, we wish to determine the greatest input in order to obtain the output before the desired date. The objective of this paper is to make optimal control for Time Stream Event Graphs and to generalize the classical "backward" equations applied to Timed Event graphs. Let us recall that for Timed

Event Graphs, the greatest solution (the latest times) of the control problem is explicitly given by the "backward" recursive equations where the co-vector plays the role of the state vector, whereas dater equations give the least solution (the earliest times) of the process evolution (Baccelli *et al.*, 1992)(Declerck, 1999).

The paper is structured as follows. We give initially, the notations and some previous results. We then introduce the modelling of Time Stream Event Graphs in the (min, max, +) algebra and study the optimal control problem of interval (min, max, +) systems. The objective of control is the determination of the existence of a solution and the effective calculation of the greatest control. Lastly, the approach is applied to a simple example.

2. PRELIMINARIES

A monoid is a couple $(S, \oplus)$ where the operation $\oplus$ is associative and presents a neutral element. A semi-ring S is a triplet $(S, \oplus, \otimes)$ where $(S, \oplus)$ and $(S, \otimes)$ are monoids, $\oplus$ is commutative, $\otimes$ is distributive relatively to $\oplus$ and the zero element ε of $\oplus$ is the absorbing element of $\otimes$ ($\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$) . A dioid D is an idempotent semi-ring (the operation $\oplus$ is idempotent, that is $a \oplus a = a$). Let us notice that contrary to the structures of group and ring, monoid and semi-ring do not have a property of symmetry on S. The unit $\Re \cup \{-\infty\}$ provided with the maximum operation denoted $\oplus$ and the addition denoted $\otimes$ is an example of dioid. We have : $\Re_{max} = (\Re \cup \{-\infty\}, \oplus, \otimes)$. The neutral elements of $\oplus$ and $\otimes$ are represented by $\varepsilon = -\infty$ and $e = 0$ respectively. The absorbing element of $\otimes$ is ε. Isomorphic to the previous one by the bijection: $x \longmapsto -x$, another dioid is $\Re \cup \{+\infty\}$ provided with the minimum operation denoted $\wedge$ and the addition denoted $\odot$. The neutral elements of $\wedge$ and $\odot$ are represented by $t = +\infty$ and $e = 0$ respectively. The absorbing element of $\odot$ is ε. The following convention is taken: $t \otimes \varepsilon = \varepsilon$ and $t \odot \varepsilon = t$. The expression $a \otimes b$ and $a \odot b$ are identical if at least either a or b is a finite scalar. The partial order denoted $\leq$ is defined as follows: $x \leq y \iff x \oplus y = y \iff x \wedge y = x \iff x_i \leq y_i$, for i from 1 to n in $\Re^n$. Notation $x < y$ means that $x \leq y$ and $x \neq y$. A dioid D is complete if it is closed for infinite sums and the distributivity of the multiplication with respect to addition extends to infinite sums : $(\forall\ c \in D\ )\ (\forall\ A \subseteq D)\ c \otimes (\bigoplus_{x \in A} x) = \bigoplus_{x \in A} c \otimes x$. For example, $\bar{\Re}_{max} = (\Re \cup \{-\infty\} \cup \{+\infty\}, \oplus, \otimes)$ is complete. The set of $n.n$ matrices with entries in a complete dioid D endowed with the two operations $\oplus$ and $\otimes$ is also a complete dioid which is denoted $D^{n.n}$. The elements of the matrices in the (max,+) expressions (respectively (min,+) expressions) are either finite or ε ((respectively t). We can deal with nonsquare matrices if we complete by rows or columns with entries equals to ε (respectively t). The different operations operate as in the usual algebra: The notation $\odot$ refers to the multiplication of two matrices in which the $\wedge$-operation is used instead of $\oplus$. The mapping f is said residuated if for all $y \in D$, the least upper bound of the subset $\{x \in D \mid f(x) \leq y\}$ exists and lies in this subset. The mapping x$\in (\bar{\Re}_{max})^n \mapsto A \otimes x$ defined over $\bar{\Re}_{max}$is residuated (Baccelli *et al.*, 1992) and the left $\otimes-$residuation of B by A is denoted by: $A \backslash B = \max\{x \in (\bar{\Re}_{max})^n$ such that $A \otimes x \leq B\}$.

In $(\oplus,\ \otimes)$ algebra, Kleene's star is defined by: $A^* = \bigoplus_{i=0}^{+\infty} A^i$. Denoted $G(A)$, an induced graph of a square matrix A is deduced from this matrix by associating: a node i to the column i and line i ; an arc from the node j towards the node i if $A_{ij} \neq \varepsilon$. The weight of a path p, $\mid p \mid_w$ is the sum of the labels on the edges in the path. The length of a path p, $\mid p \mid_l$ is the number of edges in the path. A circuit is a path which starts and ends at the same node.

Theorem 2.1 (Baccelli *et al.*, 1992) For matrix A with induced graph $G(A)$, if the cycle weights in $G(A)$ are all strictly negative, then there is a unique solution to the equation $x = A \otimes x \oplus B$ which is given by $A^* \otimes B$.

Definition (Cochet-Terrasson *et al.*, 1999)
A (min, max, +) function of type $(n, 1)$ is any function $f : \Re^n \longrightarrow \Re^1$, which can be written as a term in the following grammar: $f = x_1, x_2, \ldots, x_n \mid f \otimes a \mid f \wedge f \mid f \oplus f$ where a is an arbitrary real number ($a \in \Re$). The vertical bars separate the different ways in which terms can be recursively constructed. A (min,max,+) function of type (n, m) is any function $f : \Re^n \longrightarrow \Re^m$, such that each component f_i is a (min, max, +) function of type $(n, 1)$. The set of (min, max, +) function of type (n, m) is noticed $F(n, m)$ and is a special class of topical functions which are homogeneous, monotonic and nonexpansive. Only homogeneity ($\forall \lambda \in \Re, \forall x \in \Re^n\ f(\lambda \otimes x) = \lambda \otimes f(x)$ in the usual vector-scalar convention: $(\lambda \otimes x)_i = \lambda \otimes x_i$) will be used. They include (max, +) linear maps and (min, +) linear maps which can be written respectively as: $g(x)_i = \bigoplus_{1 \leq j \leq n} (A_{ij} \otimes x_j)$ where A is a $n \times n$ matrix with entries in $\Re \cup \{-\infty\}$ $h(x)_i = \bigwedge_{1 \leq j \leq n} (B_{ij} \otimes x_j)$ where B is a $n \times n$ matrix with entries in $\Re \cup \{+\infty\}$

Let $f \in F(n, n)$. A subset $S \subset F(n, n)$ is said to be a max-representation of f if S is a finite set of (max,+) functions such that $f = \bigwedge_{h \in S} h$. A subset $T \subset F(n, n)$ is said to be a min-representation of f if T is a finite set of (min,+) functions such that $f = \bigwedge_{h \in T} h$. The mutual distributivity of $\otimes$ and $\wedge$ ($(x \oplus y) \wedge z = (x \wedge z) \oplus (y \wedge z)$ and $(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$) entails that every (min, max, +) function have a max-representation and a min-representation.

The set of (min, max, +) functions $F(n, n)$ has a natural representation as an n-fold cartesian product: $F(n, n)$ =$F(n, 1)$x...$F(n, 1)$. Let R_i the set $\{h_i$ such that $h \in S\}$. The rectangularisation of S, denoted $rec(S)$, is defined by $rec(S)$ =R_1xR_2x...xR_n. In other words, a set S of min-max functions is rectangular if for all h, $h' \in S$, and for all $i = 1, ..., n$ the function obtained by replacing the i-th component of h by the i-th component of h' belongs to S. So, $rec(S)$ is finite when S is finite and $S \subset rec(S)$.

Dynamics of the form are considered: $x(k) = f(x(k-1))$, $\forall k \geq 1$ and $x(0) = \xi \in \Re^n$ where f is a (min, max, +) function of type (n, n) $\Re^n \rightarrow \Re^n$. The cycle time vector is defined by $\chi(f) = \lim_{k \to \infty} x(k)/k$ if it exists. It does not depend on ξ. In the following theorem, the notion of cycle time which always exists in $F(n, n)$ makes it possible to check the existence of a solution of different inequalities and equalities.

Theorem 2.2 (Cochet-Terrasson *et al.*, 1999) (Cochet-Terrasson, 2001) Let $f \in F(n, n)$. The two following conditions are equivalent: (i) It exists a finite x such that $x \leq f(x)$ (respectively, $x \geq f(x)$) (ii) $\chi(f) \geq 0$ (respectively, $\chi(f) \leq 0$)

The calculation of the spectral vector can be realized as follows. If c is a circuit, its cycle mean, denoted $m(c)$ is defined by $m(c) =\mid c \mid_w / \mid c \mid_l$(the notation / represents the classical division). A node j is upstream from i, denoted $i \Longleftarrow j$, if either $i = j$ or there is a path in $G(A)$ from j to i. Vector $\mu(A) \in \Re^n$ is defined by $\mu_i(A) = \max\{m(c) \mid i \Longleftarrow c\}$. If $f \in F(n, n)$ is max-only and A is the associated matrix over $\Re_{max}$, then $\chi(f)$ exits and $\chi(f) = \mu(A)$. The result is identical for min-only function. If S and T are rectangular max and min-representations, respectively, of $f \in F(n, n)$, then $\chi(f) = \bigwedge_{h \in S} \chi(h) = \bigoplus_{g \in T} \chi(g)$.

3. MODELLING OF TIME STREAM EVENT GRAPHS

Time Stream PNs are an extension of Petri nets which allows to represent complex synchronizations and temporal compositions of the tasks or processes which are carried out (Courtiat *et al.*, 1996)(Diaz and Owezarski, 1997). Time Stream PNs directly extend P-time PNs. We consider Event Graphs which constitute a subclass of Petri nets of which each place has exactly one upstream and one downstream transition.

Definition (Time Stream Event Graph) Let I_j a set of upstream arcs of a transition j and P_j the corresponding set of upstream places. A Time Stream Event Graph is an Event-Graph such as: an interval $[\alpha_i, \beta_i]$ $(Q^+ \cup 0) \times (Q^+ \cup +\infty)$ is associated to each $a_i \in I_j$; defined below, a special semantic of firing is associated to each transition.

Considering one outgoing arc from a given place, when a token is received by that place at time x , the token should remain in the place during an amount of time defined by a value inside the range $[x+\alpha, x+\beta]$ associated with the arc. As the firing time of a transition depends on the nature of the processes which will be synchronized, different semantics of firing may be associated to a transition. In this paper, we consider two types of semantics, And and Weak-And, which we will use later. They are defined by a couple $[x + \alpha_i, x + \beta_i]$ associated to each ingoing arc.

Definition For a transition i , let I_i denote a set of upstream arcs and P_i the corresponding set of upstream places. A transition i of the type "And" and "Weak-And" is firing at absolute time x_i if and only if the two following conditions are satisfied:

1) transition i is enabled for the current marking: every upstream place j of P_i contains at least one token. Let x_j the entrance date of the token which is also the date of firing of the upstream transition of this place.

2) For the **semantic And,** the value of x_i is such as: $(x_j + \alpha_j) \leq x_i \leq (x_j + \beta_j)$ for every upstream place $p_j \in P_i$ and arc $a_j \in I_i$ (every time condition has to be fulfilled).

For the **semantic Weak-And,** the value of x_i is such as: $(x_i + \alpha_i) \leq x_i$ for every upstream place $p_j \in P_i$ and arc $a_j \in I_i$ and $\exists j \in P_{i,}\ x_i \leq (x_j + \beta_i)$ (the firing may wait until the last time interval).

Now, let us consider the variable $x_i(k)$ as the date of the kth firing of transition i and P_i the set of the upstream places of this transition. If we take the assumption of functioning FIFO of the transition i which guarantees the condition of non overtaking of the tokens between them, a numbering of the events can be carried out and the model can be written as follows: Given n_j the number of the present tokens in each place p_j at the instant $t = 0$ (initial marking), for each transition , $\bigoplus_{j \in P_i} (x_j(k - n_j) + \alpha_j) \leq x_i(k) \leq \bigwedge_{j \in P_i} (x_j(k - n_j) + \beta_j)$ if the semantic is And; $\bigoplus_{j \in P_i} (x_j(k - n_j) + \alpha_j) \leq x_i(k) \leq \bigoplus_{j \in P_i} (x_j(k - n_j) + \beta_j)$ if the semantic is Weak-And.

Let us notice that the inequalities of P-time Event Graph correspond to semantic And.

4. CONTROL SYNTHESIS

One can represent the date $x(k)$ by the following formal power series in γ: $x(\gamma) = \bigoplus_{k \in Z} x(k)\gamma^k$. Variable may also be regarded as the backward shift operator in event domain (formally, $\gamma x(k) = x(k-1)$) and γ-transforms of functions can express this effect. Reciprocally, the dater algebraic function $\Phi_{f,l}(X)$ associated to a formal $f(x(\gamma))$ on a horizon l is a function obtained by developing $f(x(\gamma))$ algebraically with dater variables over the appropriate dimensions. It describes every connection which links the different variables which composed $X = (x(k), x(k+1), ..., x(k+l))^t$ with (min, max, +) functions. The evolution of the system is described by the following equations where f^- and f^+ are formal (min, max, +) functions on the set of sequences over $\Re \cup \{\pm\infty\}$ $f^-(x(\gamma), u(\gamma)) \leq x(\gamma) \leq f^+(x(\gamma), u(\gamma))$. The vectors x and u are respectively the state and the input. We can also introduce the output y by $y(\gamma) = C(\gamma) \otimes x(\gamma)$ without reduction of generality. As the type of the system is defined by the types of the functions f^- and f^+, we can characterize the model by the following couple (type of f^-, type of f^+). The type ((min,max,+), (min,max,+)) represents naturally the more general mathematical case. Under the assumption of the existence of a solution, they define corresponding classes of interval descriptor systems. If the lower bound defined by f^- is a (max,+) function and the upper bound is infinite, the classical (max,+) systems can be obtained after some classical manipulations. In this case, equality arises from the assumption that there is no extra delay for firing transitions whenever tokens are all available.

For stream flow Petri nets for semantics And and Weak-And, $f^-(x(\gamma), u(\gamma))$ can be a (max, +) function and $f^+(x(\gamma), u(\gamma))$ a (min, max, +) function. We can write: $f^-(x(\gamma), u(\gamma)) = A^- \otimes x(\gamma) \oplus B^- u(\gamma)$ and $f^+(x(\gamma), u(\gamma)) = \bigwedge_{i=1}^{j_1} A_i^+ \otimes x(\gamma) \oplus B_i^+ u(\gamma)$

The just-in-time objective is to calculate the greatest control u such that $y \leq z$ with $y(\gamma) = C \otimes x(\gamma)$

As function f^- is residuated , the determination of the greatest solution $(x(\gamma), u(\gamma))^t$ of the following inequality set, will give the greatest control.

We can write $\begin{pmatrix} x(\gamma) \\ u(\gamma) \end{pmatrix} \leq h \begin{pmatrix} x(\gamma) \\ u(\gamma) \end{pmatrix}$ with

$$h \begin{pmatrix} x(\gamma) \\ u(\gamma) \end{pmatrix} = \begin{pmatrix} (A^- \backslash x(\gamma)) \wedge (C \backslash z(\gamma)) \wedge \\ \bigwedge_{i=1}^{j_1} (A_i^+ \otimes x(\gamma) \oplus B_i^+ u(\gamma)) \\ B^- \backslash x(\gamma) \end{pmatrix} \quad (1)$$

Clearly, this set contains (min,max,+) functions. Notice, that the first expression presents an usual backward part $(A^- \backslash x(\gamma)) \wedge (C \backslash z(\gamma))$ but also, in the case where A_i^+ and B_i^+ have positive exponents a forward part $(\bigwedge_{i=1}^{j_1} (A_i^+ \otimes x(\gamma) \oplus B_i^+ u(\gamma)))$ which increases the complexity of the problem and forbids the writing of simple equations on a short horizon as the classical backward equations. In other words, we must solve a (min, max, +) fixed-point problem of type $x \leq f(x)$ over the horizon of the desired output z. Naturally, if the matrices A_i^+ and B_i^+ have only negative exponents, the control can be calculated by an iterative backward approach. However, the resolution must still consider a (min, max, +) system. If $f^+(x(\gamma), u(\gamma))$ is a (min,+) function, h has only a (min,+) type: P-time Event graphs lead to this formulation (Declerck and Alaoui, 2004). When the matrices A_i^+ and B_i^+ does not exist, the resolution is reduced to the classical approach and for the daters, h is a (min,+) function. As the function h contains the desired output z, h is not a (min, max, +) function of type (n,m) which is homogeneous. The form of our practical problem is to find a greatest x (if x exists) such that

$$x \leq f(x) \quad (2)$$

with f a non-homogeneous (min, max, +) function which can be defined by the following grammar: $f = b, x_1, x_2, \ldots, x_n \mid f \otimes a \mid f \wedge f \mid f \oplus f$ where a, b are arbitrary real numbers $(a, b \in \Re)$. In the aim of applying the spectral theory about these functions, we will realize a relaxation by associating in the above definition a variable x_0 to b such that b is replaced by $b \otimes x_0$. So, the problem is to find a greatest $y = (x_0, x_1, ..., x_n)^t$ (if y exists) such that $x_0 = 0$ and $x_i \leq f_i(x_0, x_1, ..., x_n)^t)$ for $i \neq 0$. If we introduce the obvious inequality $x_0 \leq x_0$, the general problem becomes: find a greatest $y = (x_0, x_1, ..., x_n)^t$ (if y exists) such that $x_0 \leq x_0$ and $x_i \leq f_i(x_0, x_1, ..., x_n)^t)$ for $i \neq 0$ with $x_0 = 0$. In other terms, we have to solve the new system

$$y \leq g(y) \quad (3)$$

with g an homogeneous function of type $(n+1, n+1)$.

Suppose that a non-homogeneous (min, max, +) function is given in the form: $f_i(x) = \bigwedge_{v \in V(i)} A_{iv} x$ (i from 1 to n) and $f_0(x) = 0$ with $x = (x_0, x_1, ..., x_n)^t$ where $V(1), ..., V(n)$ are finite sets of indexes and A_{iv} are row vectors with entries in $\Re_{\max}$. We say that a policy is a map $\pi : \{0, ..., n\} \rightarrow \cup_{0 \leq i \leq n} V(i)$, such that $\pi(i) \in V(i)$, for all $0 \leq i \leq n$. The corresponding policy matrix $A[\pi]$ is defined by $A[\pi]_i = A_{i\pi(i)}$. Each h belonging to $rec(S)$ define a policy π_h. So, $h : x \in R^n \rightarrow h(x)=A[\pi_h] \otimes x$. A function $h \in rec(S)$ and the corresponding policy π_h are said safe if all cycles in the graph induced by $A[\pi_h]$ have strictly negative weight. The resolution of the system has an unique solution given by the first column of $A^*[\pi_h]$. To consider the corresponding homogeneous function, we introduce the expressions $V(0)$ and A_{00} by $V(0) = \{0\}$ and $A_{00} = (e, ..., \varepsilon)$. In this case, a function h and the corresponding policy are safe if it is true in the non-homogeneous case.

The following proposition allows us to transpose the results from non-homogeneous case to homogeneous case.

Proposition 4.1 For f and g respectively defined by (2) and (3) $\{x$ such that $x \leq f(x)$ $\} = \{x$ such that $x_i = (-y_0) \otimes y_i$ for y satisfying $y \leq g(y)$ $\}$

Proof Given x such that $x \leq f(x)$. The variable $y = (0, x_1, ..., x_n)^t$ is clearly a solution of $x_0 \leq x_0$, $x_i \leq f_i(x_0, x_1, ..., x_n)^t)$ for $i \neq 0$ with $x_0 = 0$ or $y \leq g(y)$ with $y_0 = 0$. Reciprocally, given y a solution of $y \leq g(y)$, the homogeneity entails that $\lambda \otimes y$ is also a solution. Particularly, we can take $\lambda = (-y_0)$ and the solution $x_i = (-y_0) \otimes y_i$ satisfies $x_0 = 0$ and $x_i \leq f_i(x_0, x_1, ..., x_n)^t)$ for $i \neq 0$ □

Consequently, the resolution and analysis of existence of a solution can be applied to the homogeneous system and transpose to the initial system by multiplication $\otimes$ of $(-y_0)$ to each component. Now, we consider the case of structures which satisfies $\chi(\Phi_{f,l}(.)) = 0$. It leads to two structural propositions about the resolution.

Proposition 4.2 Given $f(x) = \bigwedge_{g \in G} g(x)$ with $g(x)$ a max-only function. Each solution of a structure (max,+) with $\chi(g) = 0$ satisfies $f(x) \leq x$.
Proof $f(x) = \bigwedge_{g \in G} g(x)$ with $g(x)$ a max-only function. If $\chi(g) = 0$, then there exists a solution denoted α which satisfies $\alpha = g(\alpha)$. So, $f(\alpha) \leq g(\alpha) = \alpha$. □

Now, we consider the important particular case of safe structure.
Proposition 4.3 The solution of a safe (max,+) structure $A(\pi)$ for $x_0 = 0$ is an upper bound of the solution space of $x \leq f(x)$ and $x = f(x)$.
Proof Given $f(x) = \bigwedge_{g \in G} g(x)$ with $g(x)$ a max-only function. A safe (max,+) structure $A(\pi)$ for $x_0 = 0$ has a unique solution denoted α. It is also the greatest solution satisfying $x \leq g_\pi(x)$ and $x = g_\pi(x)$. As each solution must satisfy $x \leq g_i(x)$, it must satisfy particularly $x \leq g_\pi(x)$. Consequently, each solution is lower or equal α. □

Now, we apply the spectral theory to the control synthesis problem.
Theorem 4.4 Given a system of ((max, +), (min, max, +)) type defined formally by

$$\begin{pmatrix} x_0 \\ x(\gamma) \\ u(\gamma) \end{pmatrix} \leq h \begin{pmatrix} x_0 \\ x(\gamma) \\ u(\gamma) \end{pmatrix} \quad (4)$$

with

$$h \begin{pmatrix} x_0 \\ x(\gamma) \\ u(\gamma) \end{pmatrix} = \begin{pmatrix} x_0 \\ (A^- \backslash x(\gamma)) \wedge (C \backslash (z(\gamma) \otimes x_0)) \wedge \\ \bigwedge_{i=1}^{j_1} (A_i^+ \otimes x(\gamma) \oplus B_i^+ u(\gamma)) \\ B^- \backslash x(\gamma) \end{pmatrix}$$

It exists a solution satisfying the system (4) on horizon l if and only if $\chi(\Phi_{h,l}(.)) \geq 0$
Proof The final inequality set presents the general form $x(\gamma) \leq \varphi(x(\gamma))$ and is associated to the algebraic inequality $X \leq \Phi_{h,l}(X)$. The spectral vector is here $\chi(\Phi_{h,l}(X))$. The system of ((max, +), (min, max, +)) type is reduced to a ($-\infty$, (min, max, +)) type and can be analyzed by the relevant theorem (2.2). If the cycle time satisfies the corresponding condition of existence, it describes a compatible interval descriptor system. □

Now, we consider the expression of the spectral vector and its underlined structure.

In the function h, we can consider the classical "backward" part which corresponds to the Just-In-Time problem in Timed Event Graphs.

$$\begin{pmatrix} x_0 \\ x(\gamma) \\ u(\gamma) \end{pmatrix} \leq h_1 \begin{pmatrix} x_0 \\ x(\gamma) \\ u(\gamma) \end{pmatrix} \text{ with } h_1 \begin{pmatrix} x_0 \\ x(\gamma) \\ u(\gamma) \end{pmatrix} = \begin{pmatrix} x_0 \\ (A^- \backslash x(\gamma)) \wedge (C \backslash (z(\gamma) \otimes x_0)) \\ B^- \backslash x(\gamma) \end{pmatrix}$$

The structural observability (respectively controllability) gives a condition to observe an effect in the output (resp. transition) whose origin comes from at least one internal transition (resp. input) and allows us to introduce the following propositions.
Definition(Baccelli *et al.*, 1992) An event graph is structurally controllable if, every internal transition can be reached by a path from at least one input transition. An event graph is structurally observable if, from every internal transition, there exists a path to at least one output transition.
Proposition 4.5 A structurally controllable and observable event graph satisfies $\chi(\Phi_{h_1,l}(.)) = 0$ and the solution of classical "backward" approach satisfies $h(x) \leq x$ and gives an upper bound of the solutions of $x \leq h(x)$ and $x = h(x)$

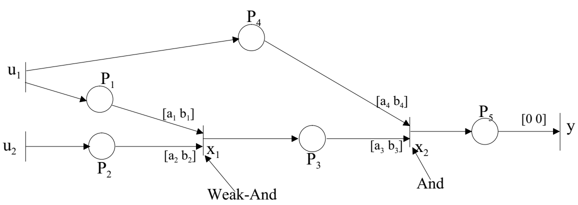

Fig. 1. A Time Stream Event Graph

Theorem 4.6 In a structurally controllable and observable event graph, there exists a solution satisfying the equality $\begin{pmatrix} x_0 \\ x(\gamma) \\ u(\gamma) \end{pmatrix} = h \begin{pmatrix} x_0 \\ x(\gamma) \\ u(\gamma) \end{pmatrix}$ or there exists no solution in inequality (4)

Let us notice that the effective calculation of the greatest control can be made by a classical iterative algorithm (Zad *et al.*, 1999) which is known to be pseudo-polynomial. The resolution of x$\leq f(x)$ is given by the iterations of $x_{k+1} \leftarrow f(x_k) \wedge x_k$ if the starting point is greater than the final solution. In the control problem, application of the classical "backward" approach in a first step allows to give this starting point before an optimal minimization.

Application

A simple example of Time Stream Petri nets in the static case and without specifications, is given in figure 1.

$x(\gamma) = \begin{pmatrix} x_1(\gamma) \\ x_2(\gamma) \end{pmatrix}$, $u(\gamma) = \begin{pmatrix} u_1(\gamma) \\ u_2(\gamma) \end{pmatrix}$, $A^- = \begin{pmatrix} \epsilon & \epsilon \\ a_3 & \epsilon \end{pmatrix}$, $B^- = \begin{pmatrix} a_1 & a_2 \\ a_4 & \epsilon \end{pmatrix}$, $j_1 = 2$, $A_1^+ = \begin{pmatrix} \epsilon & \epsilon \\ b_3 & \epsilon \end{pmatrix}$, $B_1^+ = \begin{pmatrix} b_1 & b_2 \\ \epsilon & \epsilon \end{pmatrix}$, $A_2^+ = \epsilon$, $B_2^+ = \begin{pmatrix} \epsilon & \epsilon \\ b_4 & \epsilon \end{pmatrix}$ and $C = (\epsilon \ \ e)$

$X = (x_0, x_1(k), x_2(k), u_1(k), u_2(k))^t$. We take $z = 14$.

Case 1: $[a_1 \ b_1] = [1,5]$, $[a_2 \ b_2] = [2,6]$, $[a_3 \ b_3] = [2,8]$ and $[a_4 \ b_4] = [3,12]$.

$$\begin{cases} x_0 \leq x_0 \\ x_1(k) \leq (2 \setminus x_2(k)) \wedge [(u_1(k)+5) \oplus (u_2(k)+6)] \\ x_2(k) \leq (z + x_0) \wedge (x_1(k)+8) \wedge (u_1(k)+12) \\ u_1(k) \leq (1 \setminus x_1(k)) \wedge (3 \setminus x_2(k)) \\ u_2(k) \leq 2 \setminus x_1(k) \end{cases}$$

As $\chi(\Phi(X)) = (0,0,0,0,0)^t \geq 0$, there is a solution X which is equals to
$(x_0, x_1(1), x_2(1), u_1(1), u_2(1))^t = (0,12,14,11,10)^t$

Case 2: $[a_1 \ b_1] = [0,3]$, $[a_2 \ b_2] = [1,2]$, $[a_3 \ b_3] = [2,4]$ and $[a_4 \ b_4] = [7,12]$. The problem has no solution because $\chi(\Phi(X)) = (0, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})^t$

5. CONCLUSION

In this paper, we have shown that Time Stream Petri Nets can be modeled under the form of a new model, the interval descriptor system based on (min,max,+) functions. The interval descriptor system can describe the time behaviour of a lot of models as Timed Event Graphs but also P-time Petri Nets. These results lead us to think that the interval descriptor system will bring an unified model and a new subject in the field of Time and Timed Petri Net.

The second part considers the problem of optimal control synthesis with specifications for these models. The resolution leads to an inequality set whose form includes the optimal approach used in Timed Event Graphs. The application of fixed point theory makes it possible to calculate the greatest solution. Moreover, the spectral theory gives conditions of existence of a solution which satisfies the corresponding equality when the event graph is structurally controllable and observable.

REFERENCES

Baccelli, F., G. Cohen, G.J. Olsder and J.P. Quadrat (1992). *Synchronization and linearity. An Algebra for Discrete Event Systems.* Wiley. New York.

Cochet-Terrasson, J. (2001). *Algorithmes d'itération sur les politiques pour les applications monotones contractantes.* thèse, Ecole Nationale Supérieure des Mines de Paris.

Cochet-Terrasson, J., S. Gaubert and J. Gunawardena (1999). A constructive fixed point theorem for min-max functions, dynamics and stability of systems. *Birches. J.* **14**, 407–433.

Courtiat, J.P., M. Diaz, R.C. De Oliviera and P. Sénac (1996). Formal models for the description of timed behaviors of multimedia and hyper media distributed systems. *Computer Communications* **19**, 1134–1150.

Declerck, P. (1999). Predictability and control synthesis in timed event graphs. *Kybernetika, Academy of Sciences of the Czech Republic* **35**, 25–38.

Declerck, P. and K. Didi Alaoui (2004). Modelling and liveness analysis of p-time event graphs in the (min, max, +) algebra. *IEEE International Conference on Systems, Man and Cybernetics, The Hague, The Netherlands.*

Diaz, M. and P. Owezarski (1997). From multimedia models to multimedia transport protocols. *Computer Networks and ISDN Systems* **29**, 745–758.

Zad, S.H., R.H. Kwong and W.M. Wonham (1999). Supremum operators and computation of supremal elements in system theory. *Siam J. Control Optm.* **37**, 695–709.