Offline analysis of the relaxed upper boundedness for online estimation of optimal event sequences

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Abstract

The aim of this paper is the analysis of the property of the relaxed structurally boundedness of the unobservable subnet of the Petri net, which brings a condition guaranteeing the finitude of all possible sequence lengths in the context of an on-line estimation in Partially Observable Petri Nets relevant to a sliding horizon or a receding horizon starting from the initial marking. Based on specific invariants defined over the real numbers, the approach focuses on an offline structural analysis, that is, the determination of the parts of the unobservable subnet, where an online estimation for any criterion can be made. The decomposition-composition technique is based on a block triangular form obtained with any technique. The composition of the substructures leads to a propagation of the relaxed structurally boundedness property through the structure. The study of a large-scale manufacturing system shows that the direct treatment of the large system system can be avoided and that the triangular form brings a sequential treatment allowing a computation based on smaller systems independently of the resolution of the complete system.

Keywords: Petri nets, Partially Observable, Estimation, Event sequences, Sliding Horizon, Receding horizon, Invariants, Large-scale systems, Triangular form.

I. Introduction

In this paper, we focus on the analysis of the modelling conditions such that any possible unobservable event sequence leading to an observation presents a finite length in the estimation problem. Finite lengths are necessary in fault diagnosis [27] and diagnosability problem [5],

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where all the sequences which end in an observation or a fault event must be finite. Indeed, if the estimation considers an observation relevant to a given finite time, the length of all the possible sequences starting from the initial marking and finishing in this event must also be finite, otherwise, the modelling of the estimation problem is not sufficient to characterize the relevant unobservable sequences. This situation which can affect the estimation and the fault detection, can be produced by the lack of sensors or inadequate positions of the sensors in the Petri net. The presence of a finite length can be interpreted not as a condition of the uniqueness of the trajectory [1] [3], but as a necessary condition contributing to the estimation. Let us develop the motivations of this paper. The context is the sequence estimation problem (Problem 1) whose main points are presented below. The problem traited in this paper is described in the sequel (Problem 2).

1) The event sequences (the events considered in the sequences are only the *transition firings* throughout the article) are represented with the counter form which is an efficient tool developed in the field of dioid algebra (max-plus algebra, min-plus algebra,. . .) for originally Timed Event Graphs. Based on [29], the study [14] shows that for estimation in Timed Petri nets, all the unobservable sequences (for any length and rank of the events) can be described with a *unique* polyhedron which can be easily deduced from the structure of the Petri net. This possibility is extended to Untimed Petri nets under a weak condition in [15] which allows to avoid the combinatorial building of a reachability graph or a similar graph.
2) As a unique polyhedron which describes all the sequences can be established, a classical technique of optimization which generates optimal solutions can be applied. A natural objective is to optimize an arbitrary criterion which can present the form of a linear weighting of the transition firing numbers. The aim is not to make the time-consuming computation of all the possible event sequences but only the pertinent sequences useful for the considered problem. The well-developed theory of optimization can be exploited and standard algorithms of linear programming and integer linear programming can be used. Possibly leading to the state space explosion, the burdensome enumeration of all the numerical solutions as all possible markings reachable from the initial marking is avoided.
3) A practical advantage of an estimation based on transition firings is the possibility to

define different criteria. The criterion can be a minimum or maximum number of firings in the fault detection presented in [11]. But criteria outside diagnosis can be defined as a criterion can be a balance-sheet, that is a global price depending on the costs and gains provided by the tasks [15]. Another criteria is the least cost, where each transition has a nonnegative cost in labeled Petri nets [37] [28]. Potentially, each criteria can lead to a practical application of the estimation as the sequences can model faulty behaviors (simple faulty event, multiple faults, intermittent faults, faulty behaviors,....) but also the repair of a system after the occurrence of a fault and any normal behaviors [25].

Therefore, an interesting aim is the on-line estimation of optimal current subsequences in Partially Observable Petri Net, that is, the determination of sequences of unobservable transition firings which are coherent with the observed label sequence produced by the observable transitions and are (preferably) optimal with respect to a criterion which can present the form of a linear weighting of the transition firing numbers (Problem 1).

An important point is that the estimation of optimal sequences is based on the existence of a time horizon (corresponding to a sequence length) which is sufficient to describe all the sequences and to write a finite set of relations describing the model with the counter form (System (8) in [15]). In the *timed* case, the successive dates of observations are known and the time horizon is simply the difference between the dates relevant to the beginning and the end of the desired horizon [14]. This technique is used in [16] which presents an application in P-time Petri nets which facilitates the schedulability of estimated sequences, that is, the determination of possible firing dates of a given sequence without token deaths. But, this way cannot be used in the *untimed* case, where the dates of observations are unknown by assumption. A possible solution is to take an arbitrary large horizon but this technique does not guarantee that this value is sufficient for each considered estimation problem and increases the size of the treated system and consequently the execution time of the corresponding computations. So, the aim is to determine appropriate horizons which allow to express every subsequence. The paper [15] presents Integer Linear Programming Problems and problems relaxed over the real numbers which provide temporary guaranteed horizons adapted to the current evolution.

However, to exploit the previous results, the computation of these problems must converge to a finite horizon. In that case, *any possible sequence* presents a finite length in the estimation problem: a finite set of relations describing the model with the counter form can be written and any criterion can be considered. As [15] shows that the horizon is always finite when the unobservable induced subnet is Relaxed Structurally Bounded (RSB), the objective of this paper is the analysis of the Relaxed Structurally Boundedness property (RSB property: the same acronym is taken) based on an offline analysis of the specific invariants defined over the real numbers (Problem 2). Particularly, different questions emerge in the following cases.

- When the incidence matrix describing the unobservable induced subnet is non-RSB, the resolution treating the complete system is not guaranteed. However, the analysis of some examples shows that an estimation for any criterion can be made on some subsystems and an objective is the determination of the RSB parts. A corollary question is the improvement of the non-RSB parts.

- Petri nets can describe large scale systems as production systems [6] and transportation systems (railway systems [19]). This case cannot be treated by the estimation approaches which suffer of the state explosion particularly when the estimation tries to determine all the numerical solutions of the state space. An efficient way to cope with the complexity in modelling large-scale systems consists in decomposing it into subsystems and refining the results [35] [2] [4] [36]. Clearly, a resolution considering smaller systems improves the numerical efficiency of the chosen technique.

In this context, this paper proposes a structural approach presenting two phases.

1) The decomposition of the structure into substructures leading to a triangular form, where the blocks can be non-square (Tables II and IV present this triangular form). Obtained with any technique, this practical structure can be obtained if the Petri net is an association of Petri subnets. This triangular form can also be deduced with the Dulmage-Mendelsohn decomposition [22] [21] (DM decomposition) presented in the appendix which completes the paper.
2) The composition of the algebraic substructures and the relevant analysis of the RSB property. The goal is to integrate the substructures together to determine the property of the whole system.

Related problems.

Let us describe some other problems related to Problem 2. As the definition of relaxed structurally boundedness concerns the sequences, this definition is fundamentally different from the classical definitions characterizing the marking as the (marking) boundedness, the structurally boundedness (the number of tokens in each place does not exceed a finite natural number for any marking reachable from any finite initial marking) and deadlock structurally boundedness (it leads to a deadlock state from any finite initial marking) even if the relevant algebraic expressions are close [31]. Moreover, the definition of RSB which is over the real numbers and not the integers considers the unobservable induced subnet and not the complete Petri net. More details can be found in [15] and Example 1 below. A problem presenting a similarity with Problems 1 and 2 is the study [27] which proposes the maximal size of all the elementary firing sequences by determining offline the maximal length of the paths in the induced unobservable graph which is obtained according to the reachability graph or the coverability graph. Howevever, the construction of these graphs suffers from the well-known state explosion problem related to the exponential space and time complexity which limits the application of this technique to modest-size Petri nets [23]. The approach proposed in this paper is complementary to this approach as the online computation of the guaranteed horizons provides different temporary bounds which are adapted to the current evolution. In addition, the assumption of the existence of a finite horizon is not taken as the objective of problem 2 is precisely the analysis of this existence in the context of the paper.

Another similar problem is diagnosability which is the property of a partially observed dynamical system that asserts whether it is always possible to diagnose with certainty the occurrence of an anticipated failure. This problem focuses on the detection of a given event [5] or a given event pattern [25]. Problem 1 presents another aim as: the objective is the determination of a current majorant which depends on the observations and must be computed online. Another problem is observability which presents different definitions in the literature as sequence estimation or marking estimation can be considered. In max-plus algebra, the structural observability expresses only a dependence between the unobservable events and the observations [13] while different papers consider a constraining definition based on the uniqueness of the marking for any known initial marking [34] or the uniqueness of the unobservable sequence and marking [1] [3]. Contrary to these previous studies, the general aim of this paper is the determination of a possible optimal sequence in the solution space and not the detection of a unique sequence of transition firings. Observable places are not considered in this paper and the initial marking is assumed to be known.

The paper is organized as follows. The preliminary notations are given in Section II and Table I which summarizes the main notations. The beginning of Section III presents the principle of the computation of a guaranteed horizon, that is, a horizon which is sufficiently large such that any possible sequence is modelled in the estimation problem [15]. This section is completed by different results. A comparison of the definition of RSB, structural boundedness and deadlock structurally bounded Petri nets is made in Section III-C. Then, the composition of substructures and the propagation of the RSB property through the triangular form is analyzed in Section IV. The elementary Example 1 illustrates the main concepts of the problem, while the other examples focus on the structural aspect. Well-known in the literature, the case study of Example 3 is a classical manufacturing system which is a large scale system. The reading of the paper is largely independent of the appendix which presents the DM decomposition yielding a triangular form as in Tables II and IV. As the paper focuses on the sequence length, the state estimation studied in many references is out the scope of this paper [20] [26] [8] [12] [7]. Examples of criteria can be found in [11] [15] [28] [37].

II. Preliminary notations

A. General notations

The notation $|Z|$ is the cardinality of set Z and A^Tcorresponds to the transpose of matrix A. The entry in the $i-th$ row and $j-th$ column of a matrix A is denoted $A(i,j)$. The notation $A(i,.)$ represents the row i of A while $A(.,j)$ expresses the column j. Symbol $\setminus$ is the set difference, that is, $U \setminus V$ is the set formed by the elements of set U that are not in set V. The 1-norm of vector u is equal to the sum of the absolute values of the vector elements and is denoted as $\parallel u \parallel_1$. The notation $\lfloor x \rfloor$ represents the greatest integer less than or equal to x. A Place/Transition (P/TR) net is the structure $N = (P, TR, W^+, W^-)$, where P is a set of $|P|$ places and TR is a set of $|TR|$ transitions. The matrices W^+ and W^- are respectively the $|P| \times |TR|$ post and pre-incidence matrices over $\mathbb{N}$, where each row $l \in \{1, ..., |P|\}$ specifies the weight of the incoming and outgoing arcs of the place $p_l \in P$, respectively. The incidence matrix is $W = W^+ - W^-$. The preset and postset of the node $v \in P \bigcup TR$ are denoted by ${}^\bullet v$ and $v^\bullet$, respectively. A source transition $tr_i \in TR$ satisfies ${}^\bullet tr_i = \emptyset$. The notation Ω^* represents the set of firing sequences, denoted as σ, consisting of transitions of the set $\Omega \subset TR$. The vector $\overline{\sigma}$ of dimension $|TR|$ expresses the firing vector or count vector of the sequence $\sigma \in TR^*$, where the

i-th component $\overline{\sigma_i}$ is the firing number of the transition $tr_i \in TR$ which is fired $\overline{\sigma_i}$ times in the sequence σ. A source transition $tr_i \in TR$ satisfies $^\bullet tr_i = \emptyset$ and its firing count can be infinite.

The marking of the set of places P is a vector $M \in \mathbb{N}^{|P|}$ that assigns to each place $p_i \in P$ a non-negative integer number of tokens M_i, represented by black dots. The i-th component M_i is also written as $M(p_i)$. The marking M reached from the initial marking M^0 by firing the sequence σ can be calculated by the fundamental relation: $M = M^0 \ + W.\overline{\sigma}$. The transition tr_i is enabled at M if $M \geq W^-(., tr_i)$ and may be fired yielding the marking $M' = M + W(., tr_i)$ for a unique firing. We write $M[\sigma \succ$ to denote that the sequence of transitions σ is enabled at M, and we write $M[\sigma \succ M'$ to denote that the firing of σ yields M'. To easily describe the Petri net with the incidence matrices W^+ and W^-, we assume that there is at most a unique arc between a place p_l and each input (resp. output) transition x_i of this place: each weight $(W)^+_{l,i} \neq 0$ (respectively, $(W)^-_{l,i} \neq 0$) corresponds to a unique arc in the Petri net. Otherwise a simple modification of the Petri net yields the desired form.

B. Notations for estimation

A labeling function $L : TR \rightarrow AL \cup \{\varepsilon\}$ assigns to each transition $tr_i \in TR$ either a symbol from a given alphabet AL or the empty string ε. In a partially observed Petri net, we consider that the set of transitions TR can be partitioned as $TR = TR_{obs} \bigcup TR_{un}$, where the set TR_{obs} (respectively, TR_{un}) is the set of observable transitions associated with a label of AL (respectively, the empty string ε). In this paper, we assume that the same label of AL cannot be associated with more than one transition of TR_{obs} (named Assumption $AS-4$ below).

The $TR_{un}-$induced subnet of the Petri net N is defined as the new net $N_{un} = (P, TR_{un}, W^+_{un}, W^-_{un})$, where W^+_{un} and W^-_{un} (respectively, W^+_{obs} and W^-_{obs}) are the restrictions of W^+ and W^- to $P \times TR_{un}$ (respectively, $P \times TR_{obs}$). This $TR_{un}-$induced subnet is also named unobservable induced subnet. Therefore, $W_{un} = W^+_{un} - W^-_{un}$ (respectively, $W_{obs} = W^+_{obs} - W^-_{obs}$). A reorganization of the columns with regards to TR_{obs} and TR_{un} yields $W = \begin{pmatrix} W_{obs} & W_{un} \end{pmatrix}$.

Notation x_i expresses a unobservable transition belonging to TR_{un}, while an observable transition belonging to TR_{obs} is denoted as y_i. The notation of the count vectors is taken for $\overline{x}$ of dimension $|TR_{un}|$ and $\overline{y}$ of dimension $|TR_{obs}|$. The reorganization of the components of $\overline{\sigma}$ yields $\overline{\sigma} = \begin{pmatrix} \overline{x}^T & \overline{y}^T \end{pmatrix}^T$.

The estimation of the current unobservable sequence is based on the treatment of the data produced by the observed transitions successively in an on-line procedure. For step $< k >$, the subsequence $x^{<k>}$ relevant to the unobservable transitions of TR_{un} separates two successive observations and precisely leads to the subsequence $y^{<k>}$ corresponding to the observable transition of TR_{obs}. From $M^{<k>}$, the subsequence $\sigma^{<k>} = x^{<k>}y^{<k>}$ allows the establishment of marking $M^{<k+1>}$: formally, $M^{<k>}[\sigma^{<k>} \succ M^{<k+1>}$ for $k \geq 1$, where $M^{<1>}$ represents the initial marking M^0. Moreover, $\overline{\sigma}^{<k>} = \begin{pmatrix} (\overline{x}^{<k>})^T & (\overline{y}^{<k>})^T \end{pmatrix}^T$, where notations $\overline{x}^{<k>}$ and $\overline{y}^{<k>}$ represent the count vector of $x^{<k>}$ and $y^{<k>}$ respectively. We assume that $\overline{x}^{<k>} = 0$ and $\overline{y}^{<k>} = 0$ for $k \leq 0$. So, the estimation limited to one step must consider $M^{<1>}[x^{<1>}y^{<1>} \succ M^{<2>}$ for the first step $< 1 >$, then $M^{<2>}[x^{<2>}y^{<2>} \succ M^{<3>}$ for step $< 2 >$ and so on. The generalization to a horizon composed of several steps is immediate. Note that *these notations are not cumulative* as we can have $\overline{x}^{<3>} = 0$ but $\overline{x}^{<1>} \neq 0$ and $\overline{x}^{<2>} \neq 0$: the condition $\overline{x}^{<1>} \leq \overline{x}^{<2>} \leq \overline{x}^{<3>}$ does not hold. The notations $\overline{x}^{<0>\to<k>} = \sum_{k'=0,\dots,k} \overline{x}^{<k'>}$ and $\overline{y}^{<0>\to<k>} = \sum_{k'=0,\dots,k} \overline{y}^{<k'>}$ allow to write shorter expressions.

III. Guaranteed horizons

A. Assumptions

In this paper, we assume the following assumptions for the different Petri nets under investigation:

- Assumption $\mathcal{AS}{-}1$: the incidence matrices and the initial marking M^0 are known.

- Assumption $\mathcal{AS}{-}2$: the Petri net is live.

- Assumption $\mathcal{AS}{-}3$: the firing number of each observable transition is finite.

- Assumption $AS - 4$: the observations are distinguishable, that is, the same label cannot be associated with more than one observable transition (the case of indistinguishable events is treated in [11]).

- Assumption $AS - 5$. A unique firing of a transition on all transitions occurs at each time (Assumption $AS - 5$ is a facility to express the sequences under a form without concurrency).

The assumptions about cyclicity of the structure and boundedness of the marking are not considered in this article contrary to many papers in this topic.

TABLE I

MAIN NOTATIONS

Notation	Description
P	Set of places $p_i \in P$
TR	Set of transitions $tr_j \in TR$
TR_{obs}	Set of observable transitions $y_i \in TR_{obs}$
TR_{un}	Set of unobservable transitions $x_i \in TR_{un}$
W	Incidence matrix
W^+ (resp. W^-)	Post-incidence matrix (resp. pre-incidence matrix)
W_{un}	Incidence matrix of the unobservable induced subnet
$W(i,.)$	Row i of W
$W(.,j)$ (resp. $W(.,tr_j)$)	Column j of W (resp. column of transition tr_j)
M_i	Marking of place p_i with $i \in \{1,\ldots,\|P\|\}$ $(M_i = M(p_i))$
M^0	Initial marking (M_3^0 : initial marking of p_3)
$< k >$	Step k of the estimation
$M^{<k>}$	Marking at step $< k >$ $(M^{<1>} = M^0)$
$x^{<k>}$ (resp. $y^{<k>}$)	Subsequence relevant to the unobservable transitions at step $< k >$ (resp. observable transitions)
$\overline{x}^{<k>}$ (resp. $\overline{y}^{<k>}$)	Count vectors of $x^{<k>}$ (resp. $y^{<k>}$)
$\overline{x}^{<0>\rightarrow<k>}$ (resp. $\overline{y}^{<0>\rightarrow<k>}$)	Sum of $\overline{x}^{<k>}$ (resp. $\overline{y}^{<k>}$) on horizon $\{0,1,\ldots,k\}$
$\parallel \overline{x}^{<k>} \parallel_1$	Length of sequence $x^{<k>}$

B. Computation of a guaranteed horizon

Given a sequence of the observed firing events of the transitions of TR_{obs} generated by the activity of the Petri net, we desire to find a guaranteed horizon such that any possible sequence $x^{<k>}y^{<k>}$ of the Petri net can be expressed for step $< k >$. The principle is to take a pessimistic point of view of the behavior of the Petri net. Precisely, we consider the worst case in terms of number of firings for the unobservable transitions, which corresponds to the greatest possible length of any unobservable sequence $x^{<k>}$ when Assumption $AS-5$ is taken. Therefore, for step $< k >$ and its relevant sequence $x^{<k>}y^{<k>}$, a guaranteed horizon denoted as $h_g^{<k>}$ is given by $h_g^{<k>} = \max \parallel \overline{x}^{<k>} \parallel_1 +1 = max(c.\overline{x}^{<k>}) + 1$ with $\overline{x}^{<k>}$ over the integers and $c_{1\text{x}|TR_{un}|}$ unitary, where 1 corresponds to the unique observation $y^{<k>}$. A consequence is the possibility to treat an estimation problem for any criterion [15]. This worst case treats all the transitions even if a given criterion exploits a subset of transitions.

For step $< k = 1 >$, a majorant relevant to the unobservable transitions is given by $\max \| \overline{x}^{<k>} \|_1 = max(c.\overline{x}^{<k>})$ with $c_{1\text{x}|TR_{un}|}$ unitary for the system [15]

$$-W_{un}.\overline{x}^{<1>} \leq b^{<1>} \tag{1}$$

with $\overline{x}^{<1>} \geq 0$ and $b^{<1>} = M^{<1>} - W^-_{obs}.\overline{y}^{<1>}$.

For the following steps $k \geq 2$, we can consider the maximization $max(c.\overline{x}^{<k>})$ for a sliding horizon reduced to $< k >$ or $max(c.\overline{x}^{<0>\rightarrow<k>})$ for a receding horizon going from $< 0 >$ to $< k >$. A possible system [15] is

$$\begin{pmatrix} -W_{un} & 0 \\ -W_{un} & -W_{un} \end{pmatrix} . \begin{pmatrix} \overline{x}^{<0>\rightarrow<k-1>} \\ \overline{x}^{<k>} \end{pmatrix} \leq \begin{pmatrix} M^{<1>} + W_{obs} \cdot \overline{y}^{<0>\rightarrow<k-1>} \\ b^{<k>} \end{pmatrix} \tag{2}$$

with $\overline{x}^{<0>\rightarrow<k>} \geq 0$ and $b^{<k>} = M^{<1>} + W_{obs} \cdot \overline{y}^{<0>\rightarrow<k-1>} - W^-_{obs}.\overline{y}^{<k>}$. Naturally, more complex systems including the firing conditions can be considered but the advantage of these forms is the fixed dimensions of the matrices, which only depend on the numbers of places and transitions of the Petri net (and not the number of observations) even if greater values are obtained. With that in mind, a simplified form of (2), which facilitates the analysis is studied in the sequel.

C. *Upper boundedness of the problem*

If the computed value $max(c.\overline{x}^{<k>})$ is finite, the problem is upper-bounded, and the horizon length presents a majorant. So, we now analyze the conditions such that the above problems of optimization using (1) or (2) converge to a finite solution. With that aim, we consider the second line of (2), which includes inequality (1) as for $k = 1$, $\overline{x}^{<0>\rightarrow<0>} = 0$ and $b^{<1>} = M^{<1>} + W_{obs} \cdot \overline{y}^{<0>\rightarrow<0>} - W^-_{obs}.\overline{y}^{<1>} = M^{<1>} - W^-_{obs}.\overline{y}^{<1>}$ since $\overline{y}^{<0>\rightarrow<0>} = \overline{y}^{<0>} = 0$. We consider the Integer Linear Programming problem (ILP problem)

$$max(c.\overline{x}^{<0>\rightarrow<k>}) \text{ such that} - W_{un}.\overline{x}^{<0>\rightarrow<k>} \leq b^{<k>} \tag{3}$$

with $\overline{x}^{<0>\rightarrow<k>} \geq 0$ over $\mathbb{Z}$ and $b^{<k>} = M^{<1>} + W_{obs} \cdot \overline{y}^{<0>\rightarrow<k-1>} - W^-_{obs}.\overline{y}^{<k>}$ for the succession of steps going from $< 0 >$ to $< k >$.

Moreover, the ILP problem (3) is relaxed over $\mathbb{R}$ and become a Linear Programming Problem (LP problem). The relevant maximization can converge to a greater value than the initial ILP problem as the space is not limited to the natural numbers. Formally, $\max_{\mathbb{N}}(c.\overline{x}^{<0>\rightarrow<k>}) \leq \max_{\mathbb{R}}(c.\overline{x}^{<0>\rightarrow<k>})$ for $-W_{un}.\overline{x}^{<0>\rightarrow<k>} \leq b^{<k>}$. In addition, if the relaxed problem provides a finite solution, the same conclusion holds for ILP problem, which is more restrictive.

A standard theorem of linear programming is that the dual problem of the primal problem $max(c.x)$ subject to $A.x \leq b$ and $x \geq 0$ is $min(y.b)$ subject to $y.A \geq c$ and $y \geq 0$. Consequently, we can deduce the LP problem dual to the ILP problem (3) relaxed over $\mathbb{R}$, which is

$$\min(z.b^{<k>}) \text{ such that } z.W_{un} \leq -c \text{ with } c \text{ unitary and } z \geq 0 \text{ over } \mathbb{R} \tag{4}$$

Let $\overline{x}_{opt}^{<0>\rightarrow<k>}$ and z_{opt}, be the optimal solutions relevant to the ILP problem (3) relaxed over $\mathbb{R}$ and LP problem (4).

Theorem 1: [15] The maximization of problem (3) relaxed over $\mathbb{R}$ is upper-bounded if there is $z \geq 0$ over $\mathbb{R}$ satisfying (4). Moreover, $c.\overline{x}^{<0>\rightarrow<k>} \leq z.b^{<k>}$ and $c.\overline{x}_{opt}^{<0>\rightarrow<k>} = z_{opt}.b^{<k>}$. ■

Therefore, the ILP problem (3) relaxed over $\mathbb{R}$ can be replaced by the problem (4).

D. New results on the computation of a guaranteed horizon

Theorem 1 is completed by the following theorems 2 and 3, which focus on the computation of a guaranteed horizon $h_g^{<0>\rightarrow<k>}$, which represents a guaranteed receding horizon relevant to the succession of steps going from $< 0 >$ to $< k >$.

Theorem 2: Assume that the space defined by $z.W_{un} \leq -c$ with c unitary and $z \geq 0$ over $\mathbb{R}$ is non-empty. For the succession of steps going from $< 0 >$ to $< k >$, a guaranteed horizon is $h_g^{<0>\rightarrow<k>} = \lfloor c.\overline{x}_{opt}^{<0>\rightarrow<k>} \rfloor + k = \lfloor z_{opt}.b^{<k>} \rfloor + k$, where $\overline{x}_{opt}^{<0>\rightarrow<k>}$ and z_{opt} are the optimal solutions relevant to the ILP problem (3) relaxed over $\mathbb{R}$ and LP problem (4).

Proof. Theorem 1, which considers the real numbers says that, if the space defined by $z.W_{un} \leq -c$ is non-empty (formally, $\exists z \geq 0$ over $\mathbb{R}$ satisfying $z.W_{un} \leq -c$), then $c.\overline{x}^{<0>\rightarrow<k>}$ for $-W_{un}.\overline{x}^{<0>\rightarrow<k>} \leq b^{<k>}$ is upper-bounded by $z.b^{<k>}$. The maximization of the ILP problem converges to a lower value than the relevant relaxed problem as the space of $-W_{un}.\overline{x}^{<0>\rightarrow<k>} \leq b^{<k>}$ is restricted to the natural numbers. Formally, $\max_{\mathbb{N}}(c.\overline{x}^{<0>\rightarrow<k>}) \leq \max_{\mathbb{R}}(c.\overline{x}^{<0>\rightarrow<k>})$ for $-W_{un}.\overline{x}^{<0>\rightarrow<k>} \leq b^{<k>}$. In addition, as c is an integer row-vector and the maximization over $\mathbb{N}$ provides an integer solution, $\max_{\mathbb{N}}(c.\overline{x}^{<0>\rightarrow<k>})$ is integer and $\lfloor\max_{\mathbb{N}}(c.\overline{x}^{<0>\rightarrow<k>})\rfloor = \max_{\mathbb{N}}(c.\overline{x}^{<0>\rightarrow<k>}) \leq \lfloor\max_{\mathbb{R}}(c.\overline{x}^{<0>\rightarrow<k>})\rfloor$. Moreover, $\max_{\mathbb{R}}(c.\overline{x}^{<0>\rightarrow<k>}) = c.\overline{x}_{opt}^{<0>\rightarrow<k>}$ by definition and Theorem 1 gives $c.\overline{x}_{opt}^{<0>\rightarrow<k>} = z_{opt}.b^{<k>}$. Finally, variable k, that is, the k observations $y^{<k>}$ for the succession of steps going from $< 0 >$ to $< k >$ under Assumption $AS-5$, must be added to obtain $h_g^{<0>\rightarrow<k>}$. ■

Degraded horizons $h_g^{<0>\rightarrow<k>}$ can be proposed if a non-optimal z is taken or if the vector $b^{<k>}$ is modified.

***Theorem** 3:* Assume that the space defined by $z.W_{un} \leq -c$ with c unitary and $z \geq 0$ over $\mathbb{R}$ is non-empty. Some possible guaranteed horizons $h_g^{<0>\rightarrow<k>} = \lfloor z.b^{<k>}\rfloor + k$ with z over $\mathbb{R}$ can be deduced if z is taken as follows:

1) z is an arbitrary solution of $z.W_{un} \leq -c$
2) z is deduced from the optimization $\min(z.d)$ for $z.W_{un} \leq -c$, where d is an arbitrary positive row-vector.
3) A possible guaranteed horizon is $\lfloor z.b'^{<k>}\rfloor + k$ with $b'^{<k>} = M^{<1>} + W_{obs} \cdot \overline{y}^{<0>\rightarrow<k-1>}$, where z is provided by Point 1 or 2.

Proof. Point 1) The second point of Theorem 1 says that any solution z to (4) satisfies $c.\overline{x}^{<0>\rightarrow<k>} \leq z.b^{<k>}$. So, the relevent $z.b^{<k>}$ is a possible majorant of $c.\overline{x}_{opt}^{<0>\rightarrow<k>}$ and Theorem 2 can be applied.

Point 2) As point 1) says that any solution z satisfying $z.W_{un} \leq -c$ can be used, we can choose a minimization of $z.d$ with an arbitray $d > 0$ as it always converges to an arbitrary finite solution as $z \geq 0$ and $d > 0$.

Point 3) $b^{<k>} = M^{<1>} + W_{obs} \cdot \overline{y}^{<0>\rightarrow<k-1>} - W_{obs}^{-}.\overline{y}^{<k>} \leq M^{<1>} + W_{obs} \cdot \overline{y}^{<0>\rightarrow<k-1>}$ as $W_{obs}^{-}.\overline{y}^{<k>} \geq 0$ ($W_{obs}^{-} \geq 0$ and $\overline{y}^{<k>} \geq 0$). So, $b^{<k>} \leq b'^{<k>}$ and $z.b^{<k>} \leq z.b'^{<k>}$ as $z \geq 0$. So, $\lfloor z.b^{<k>}\rfloor \leq \lfloor z.b'^{<k>}\rfloor$. Each vector z given by Point 1) or 2) provides a possible guaranteed horizon $\lfloor z.b'^{<k>}\rfloor + k$. ■

Let us consider an execution of the computation made with (4). In Point 1), a horizon can be computed without waiting the end of this computation by taking an intermediate solution z while an arbitrary z for any step $< k >$ can be computed with a simple criterion in Point 2). As $b'^{<k>}$ does not contain $\overline{y}^{<k>}$ in Point 3), a degraded horizon can be computed at the end of step $< k-1 >$ without waiting the occurrence of observation $y^{<k>}$ and the knowledge of $\overline{y}^{<k>}$. Knowing $\overline{y}^{<k>}$, a better result is obtained with $z.b^{<k>}$.

E. RSB property

The following definition introduced in [15] modifies the condition in Theorem 1 by replacing $z.W_{un} \leq -c$ by $z.W_{un} < 0$.

***Definition** 1:* The unobservable induced subnet is Relaxed Structurally Bounded (RSB) when there exists a non-negative vector z over $\mathbb{R}$ such that $z.W_{un} < 0$. Formally,

$$\exists z \geq 0 \text{ such that } z.W_{un} < 0 \text{ with } z \text{ over } \mathbb{R}$$

Example 1

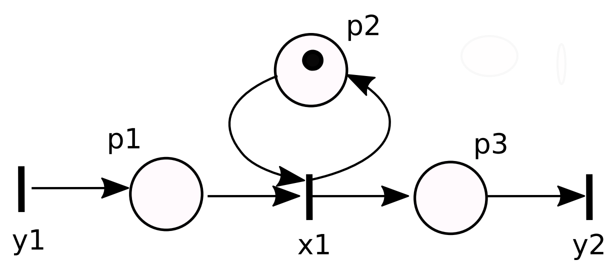

Fig. 1. Elementary Petri net of example 1

Let us consider the Petri net of Fig. 1, which is an elementary event graph containing a self-loop, a source transition y_1 and a sinks transition y_2. We have: $P = \{p_1, p_2, p_3\}$; $TR = TR_{obs} \bigcup TR_{un}$ with $TR_{obs} = \{y_1, y_2\}$ and $TR_{un} = \{x_1\}$. So, $W_{obs} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$ and $W_{un} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. The unobservable induced subnet is RSB as $\exists z = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \geq 0$ such that $z.W_{un} = -1 < 0$ with z over $\mathbb{R}$. Note that the Petri net is neither structurally bounded ($\exists z \in \mathbb{N}^3$ with $z > 0$ and $z.W \leq 0$), nor deadlock structurally bounded ($\exists z \in \mathbb{N}^3$ with $z > 0$ and $z.W < 0$) as the first column of W is non-negative. Remember that these definitions concern the marking and not the sequences. Now if the first element of the first column becomes -1, the unobservable induced subnet remains RSB but the Petri net becomes structurally bounded and deadlock structurally bounded (for $z = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}$, $z.W = \begin{pmatrix} -2 & -1 & -1 \end{pmatrix} < 0$). Also note that, if we assume $W_{obs} = 0$ in a general Petri net and add the conditions z over $\mathbb{N}$ and

$z > 0$ to a RSB unobservable induced subnet, the Petri net becomes structurally bounded and deadlock structurally bounded. ■

The following theorem makes the connection between the primal problem and the RSB property.

Theorem 4: [15] The maximization $max(c.\overline{x}^{<0>\rightarrow<k>})$ with $c > 0$ unitary over $\mathbb{R}$ for system (3) is upper-bounded if the unobservable induced subnet is RSB.

Example 1 continued.

a) Let us analyze the upper-boundedness of the count number $\overline{x}_1$ of the unobservable transition x_1. Note that the analysis of the Petri net in Fig. 1 shows that, for p_1, if $\overline{y}_1$ is upper-bounded, then $\overline{x}_1$ is also upper-bounded. The resolution of (1) for the observation sequence y_1y_2 gives $\overline{x}_1^{<1>} = 0$ for step $< k = 1 >$ (no firing of x_1 before observation y_1) and the resolution of (2) yields $\overline{x}_1^{<1>} + \overline{x}_1^{<2>} = 1$ for step $< k = 2 >$ (firing of x_1 at step $< 1 >$ or $< 2 >$ before observation y_2).

b) Let us consider the dual problem (4). Theorem 4 implies that $\overline{x}_1$ is upper-bounded as the unobservable induced subnet is RSB. These results, which are consistent are obtained despite that the self-loop is not represented in the incidence matrices W_{obs} and W_{un}. In fact, the same computation can be made without place p_2 and its initial token. ■

IV. Composition of substructures and propagation of the RSB property

A. Introduction

Considering *the primal problem* (3) relaxed over $\mathbb{R}$, where the objective is $max(c.\overline{x}^{<k>})$ with c unitary, a necessary condition to define a guaranteed horizon is that each component $\overline{x}_i$ must be upper-bounded. Indeed, if $max(c_i.\overline{x}_i) = +\infty$ with $c_i \neq 0$, the variable is not upper-bounded and a relevant guaranteed horizon cannot be computed. However, a possibility is to consider not the complete set of unobservable transitions but a subset. Therefore, the objective of Section IV is to determine the set of variables, which are upper-bounded. An interesting phenomenon, which may be studied is a possible propagation of the upper-boundedness: if the event numbers of all input transitions of a place are upper-bounded, the event numbers of all the output transitions of this place are also upper-bounded.

In the rest of the paper, the objective is not to analyze the primal problem (3) relaxed over $\mathbb{R}$ but *the dual problem* (4) and particularly the propagation of the RSB property through the composition of substructures in W_{un}.

B. Preliminary results

Let us analyze the columns of W_{un}.

***Theorem** 5:* A sufficient condition such that W_{un} is non-RSB is the existence of a non-RSB column at least.

Proof. Consider a matrix $A = W_{un}$ containing a non-RSB column denoted A_2. If A_2 is non-RSB then $\nexists z_2 \geq 0$ such that $z_2.A_2 < 0$. The consideration of the other columns of A, which increases the constraints cannot improve this impossibility. ■

This result does not depend on the number of the non-RSB columns and their positions, which can be separate. A practical way to reduce this difficulty is to add sensors such that the relevant unobservable transitions become observable.

***Theorem** 6:* Let $A = W_{un} = \begin{pmatrix} A_1 \\ B \end{pmatrix}$. If the substructure A_1 is RSB, then A is also RSB.

Proof. If the substructure A_1 is RSB, there is $z_1 \geq 0$ such that $z_1.A_1 < 0$. Let us build a new vector $\begin{pmatrix} z_1 & z_2 \end{pmatrix} \geq 0$ such that $\begin{pmatrix} z_1 & z_2 \end{pmatrix} . \begin{pmatrix} A_1 \\ B \end{pmatrix}$ is negative. If $z_2 = 0$, the product is equal to $z_1.A_1$ and the vector $\begin{pmatrix} z_1 & z_2 \end{pmatrix} \geq 0$ is obtained. ■

In other words, a structure remains RSB if some rows are added to it.

C. Triangular structure

Let us consider more complex structures. We now assume that a reorganization of the rows and the columns of W_{un} has established a specific triangular structure, where the blocks can be non-square and the top right corner contains null elements only. For each block, a rank can be numbered from the upper left corner to the lower right corner of the table.

We below consider the RSB characteristic of each block of the structure, that is, the blocks of the main diagonal and also the substructures making the connection between the blocks.

Before considering the general case, the block-triangular form is composed of two blocks below. Let

$$A = W_{un} = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}, \tag{5}$$

where the substructures A_1, B and A_2 are rectangular in general. Matrix B makes the connection with the two substructures A_1 and A_2.

Considering the primal problem, the resolution in the following example illustrates the propagation of the upper-boundedness of $\overline{x}^{<0>\to<k>}$ inside the structure. The propagation of the upper-boundedness on the count variables is represented by the propagation of the RSB characteristic in the blocks. This propagation follows the increasing order of the blocks.

Example 2.

Let $A_1 = \begin{pmatrix} -1 & 1 \\ -1 & 3 \\ 1 & -3 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & 1 \\ 1 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$ in (5). Let us consider $-W_{un}.\overline{x}^{<0>\to<k>} \leq b^{<k>}$ in the primal problem (3). To facilitate the writing of the expressions, we take $\overline{x} = \overline{x}^{<0>\to<k>}$ and $b = b^{<k>} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix}^T$. For $-A_1.\begin{pmatrix} \overline{x}_1 & \overline{x}_2 \end{pmatrix}^T \leq \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}^T$, $\overline{x}_1$ and $\overline{x}_2$ are upper-bounded as some obtained relations are $\overline{x}_2 \leq (b_1 + b_3)/2$ and $\overline{x}_1 \leq b_2 + 3.\overline{x}_2$ (also, $\overline{x}_1 \leq b_1 + \overline{x}_2$).

For $-A_2.\begin{pmatrix} \overline{x}_3 & \overline{x}_4 \end{pmatrix}^T \leq \begin{pmatrix} b_4 & b_5 \end{pmatrix}^T$, $\overline{x}_3$ and $\overline{x}_4$ are upper-bounded as a resolution yields $\overline{x}_4 \leq (b_4 + b_5)/3$ and $\overline{x}_3 \leq b_4 + \overline{x}_4$.

Now, for the complete rows $-\begin{pmatrix} B & A_2 \end{pmatrix}.\overline{x} \leq b$, a similar resolution gives $\overline{x}_4 \leq (b_4 + b_5 + 4.\overline{x}_2)/3$ and $\overline{x}_3 \leq b_4 + \overline{x}_1 + 2.\overline{x}_2 + \overline{x}_4$. So, $\overline{x}_3$ and $\overline{x}_4$ are upper-bounded as $\overline{x}_1$ and $\overline{x}_2$ are upper-bounded. ■

Considering the structure of A, a first analysis shows that A_1 can be treated without considering the rows of B and A_2. The dependence between A_1 and A_2 in (5) is expressed by each non-null component of B. When $B = 0$, there is no dependence with A_1 and a resolution can treat the subsystem A_2 independently of the treatment of A_1, which can be RSB or not. So, A_2 is independent when $B = 0$. Note that a transition depending of an unbounded part can be upper-bounded as a dependence with another part, which is upper-bounded is possible.

***Theorem** 7:*

If the substructures A_1 and A_2 are RSB in (5), then A is RSB.

Proof. Let us consider the left columns of A. If the substructure A_1 is RSB, there is $z_1 \geq 0$ such that $z_1.A_1 < 0$. Let us build a new vector $\begin{pmatrix} z_1 & z_2 \end{pmatrix} \geq 0$ such that $\begin{pmatrix} z_1 & z_2 \end{pmatrix}.\begin{pmatrix} A_1 \\ B \end{pmatrix} = z_1.A_1 + z_2.B$ is negative.

We now show that for a given z_1 such that $z_1.A_1 < 0$, we can make a product by $\lambda > 0$ such that $\lambda.z_1.A_1 \leq -u$, where u is a unitary row-vector with the adapted dimension. Indeed, if $-1 < z_1.(A_1)_{.,i} < 0$ for the column $(A_1)_{.,i}$, then we can always multiply z_1 by $\lambda_i = 1/|z_1.(A_1)_{.,i}|$ with $z_1.(A_1)_{.,i} \neq 0$, which implies $(\lambda_i.z_1).(A_1)_{.,i} = -1$. The result is kept if we take a greater $\lambda_i : (\lambda_i'.z_1).(A_1)_{.,i} < -1$ for $\lambda_i' > \lambda_i$. The reasoning can be repeated for all columns satisfying $-1 < z_1.(A_1)_{.,i} < 0$. So, we can make a multiplication of z_1 by taking the maximum $\lambda_{\max}$ on all the obtained values λ_i and the desired result is obtained. With $z_1' = \lambda_{\max}.z_1$, we have $z_1'.A_1 + z_2.B \leq -u + z_2.B$ and we can take the new objective $-u + z_2.B < 0$, which implies $z_1'.A_1 + z_2.B < 0$. So, for the left columns of A, we must have $z_2.B < u$ with $z_2 \geq 0$ and also, $z_2.A_2 < 0$ for the right columns of A. So, to prove that A is RSB, the problem is equivalent to search $z_2 \geq 0$ such that

$$\begin{cases} z_2.A_2 < 0 \\ z_2.B < u \end{cases}.$$

As A_2 is RSB, there is an arbitrary vector $z_2 \geq 0$ such that $z_2.A_2 < 0$, which is modified below. In general, the product $z_2.B_{.,i}$ for each column $B_{.,i}$ can be in the intervals $[1, +\infty[$ (case 1), $[0, 1[$ (case 2) or $]-\infty, [0$ (case 3).

Case 1. If the product satisfies $z_2.B_{.,i} \geq 1$, then the scalar $z_2.B_{.,i}$ is positive, and we can always divide z_2 by a positive scalar $\mu_i > z_2.B_{.,i}$ such that $(z_2/\mu_i).B_{.,i} < 1$. Note that $\mu_i > 1$. The same reasoning holds for all the columns $B_{.,i}$ satisfying the case 1 and we can make a division of z_2 by taking the maximum $\mu_{\max}$ on all the obtained values $\mu_i > 1$. This implies that the relations $(z_2/\mu_{\max}).B_{.,i} < 1$ are kept for the columns $B_{.,i}$ in the case 1. Let us show that the columns presenting the cases 2 and 3 satisfy $(z_2/\mu_{\max}).B_{.,i} < 1$ with this new vector $z_2/\mu_{\max}$.

Case 2. If a column $B_{.,i}$ satisfies $0 \leq z_2.B_{.,i} < 1$, then $0 \leq (z_2/\mu_{\max}).B_{.,i} < z_2.B_{.,i} < 1$ as $\mu_{\max} \geq 1$.

Case 3. If a column $B_{.,i}$ satisfies $z_2.B_{.,i} < 0$, then the sign of $(z_2/\mu_{\max}).B_{.,i}$ remains negative as $\mu_{\max} > 0$.

Finally, we have build a desired vector $z_2' = z_2/\mu_{\max} \geq 0$ such $z_2'.B < u$ and consequently, $\begin{pmatrix} z_1' & z_2' \end{pmatrix} \geq 0$ with $z_1' = \lambda_{\max}.z_1$ such that $\begin{pmatrix} z_1' & z_2' \end{pmatrix} . \begin{pmatrix} A_1 \\ B \end{pmatrix}$ is negative and $\begin{pmatrix} A_1 \\ B \end{pmatrix}$ is RSB. In addition, $z_2.A_2 < 0$ implies $z_2'.A_2 < 0$ as $\mu_{\max} > 0$ and the complete structure A is RSB. ■

Example 2 continued.

The two substructures A_1 and A_2 are RSB: $z_1.A_1 = \begin{pmatrix} -1 & -1 \end{pmatrix}$ with $z_1 = \begin{pmatrix} 2 & 2 & 3 \end{pmatrix}$ and $z_2.A_2 = \begin{pmatrix} -1/2 & -1/4 \end{pmatrix}$ with $z_2 = \begin{pmatrix} 3/4 & 1/4 \end{pmatrix}$. We have $\begin{pmatrix} z_1 & z_2 \end{pmatrix}.W_{un} = \begin{pmatrix} -1/2 & +1 & -1/2 & -1/4 \end{pmatrix} \not< 0$ but the new vector $\begin{pmatrix} z_1 & z_2/\mu \end{pmatrix}$ with $\mu = 4$ gives $\begin{pmatrix} z_1 & z_2/\mu \end{pmatrix}.W_{un} = \begin{pmatrix} -7/8 & -1/2 & -1/8 & -1/16 \end{pmatrix} < 0$. So, W_{un} is RSB and Theorem 4 implies that all the components of the vector $\overline{x}^{<0> \to <k>}$ are upper-bounded. This result is coherent with the direct analysis made above. ■

Theorem 7 is now generalized by considering a triangular form, where the main diagonal contains more than two blocks.

Let $RSB(A)$ be the matrix expressing the RSB property of each submatrix of A, which is denoted by Y if RSB, N if non-RSB, $-$ if indifferent and 0 if the submatrix is null.

***Theorem** 8:* If the substructures of the main diagonal are RSB, then A is RSB.

Proof. Let us consider a structure A, where the main diagonal contains three RSB blocks: $RSB(A) = \begin{pmatrix} Y & 0 & 0 \\ - & Y & 0 \\ - & - & Y \end{pmatrix}$. Matrix A can be rewritten as $A = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ with $RSB(C_{11}) = \begin{pmatrix} Y & 0 \\ - & Y \end{pmatrix}$, $RSB(C_{12}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $RSB(C_{21}) = \begin{pmatrix} - & - \end{pmatrix}$ and $RSB(C_{22}) = (Y)$. The application of Theorem 7 implies that C_{11} is RSB. Moreover, the same theorem shows that A is RSB. The reiteration of this reasoning for greater dimensions gives the desired result. ■

V. Case study: a manufacturing system (Example 3)

This example is a classical automated manufacturing system, which has been presented in Section VII.A of [30] and is recognized to be significant in the literature since slight variations of it have already been considered by different authors. The plant produces two different types of products from two types of raw materials. It consists of five machines (*MA1* to *MA5*), four robots (*R01* to *R04*), a finite capacity buffer (*BU*), two inputs of raw parts and two automated guided vehicle systems (*AGV1* and *AGV2*). Presented in [30], the description of the plant and its layout is out the scope of the paper.

The Petri net Fig. 2 has 38 places and 26 transitions and corresponds to the Petri net Fig. 3 page 979 in Section VII.A of [30] with the same initial marking. Remember that observable

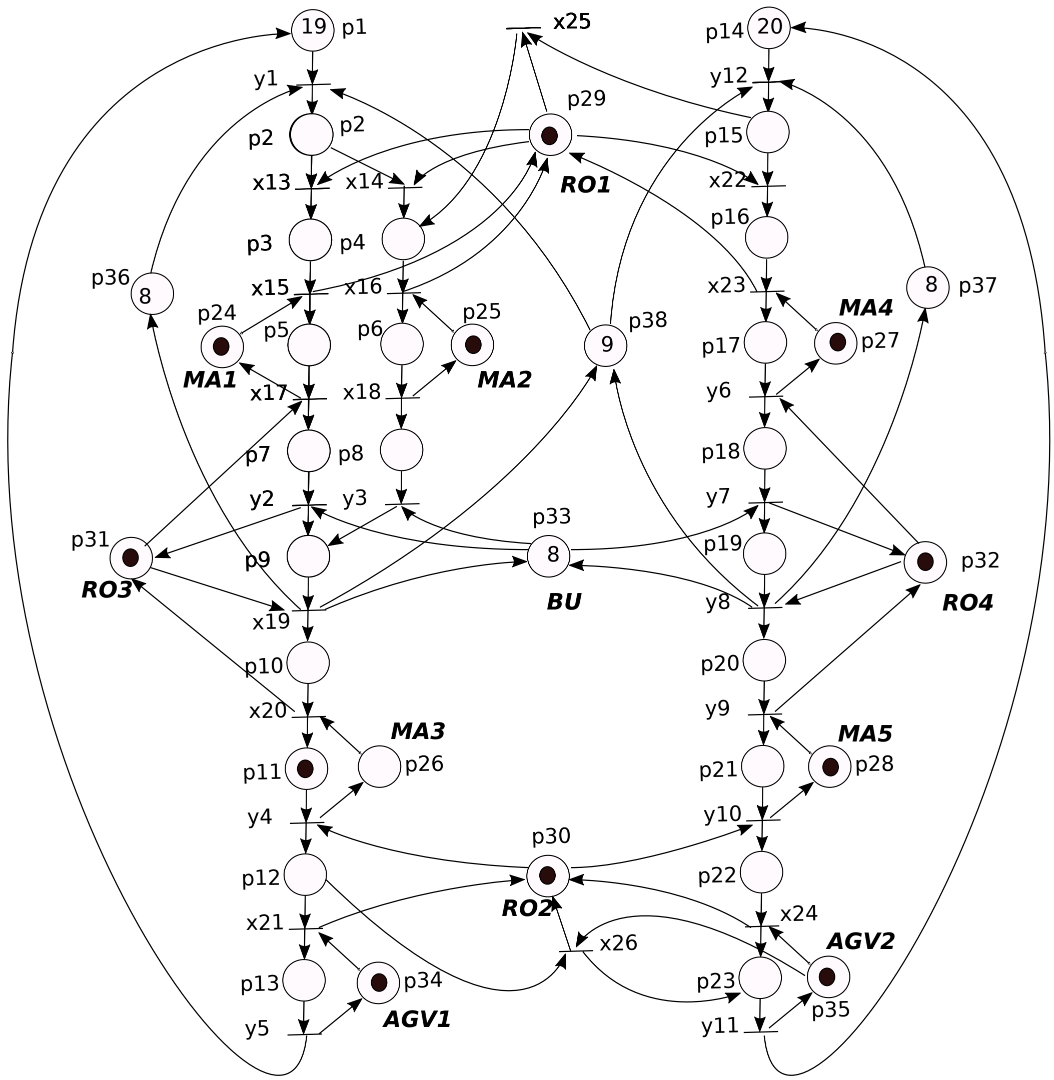

Fig. 2. Petri net of a manufacturing system (Example 3) [30]. The number in a place represents its initial marking.

transitions are $y_1, \ldots, y_{12}$ while $x_{13}, \ldots, x_{26}$ correspond to unobservable transitions. Despite a complex aspect, the structural approach proposed in this paper shows that this Petri net can be represented under a more approachable form in a context of estimation: in fact, the incidence matrix of the unobservable induced subnet is large (38×14) but sparse (by lake of space, this complex matrix is not given), and a judicious reorganization of the rows and columns can provide a clearer view under a block-triangular form presented below.

Corresponding to places p_1, p_{14}, p_{18}, p_{19}, p_{20}, p_{21}, p_{28}, p_{32} and p_{37} the null rows of the unobservable incidence matrix cannot affect the sequence estimation and can be disregarded. As shown by Table II, the reorganization of the rows and columns shows that the unobservable incidence matrix presents two disconnected parts denoted as S_A and S_B (named simply connected structures in graph theory, these disconnected parts can be found by a standard search of paths between the vertices). System S_A decribes the connections between 19 places and 11 unobservable transitions while system S_B in the bottom right corner is about 7 places and 3 unobservable transitions: $S_A = (\{p_{33}, p_{36}, p_{38}, p_{26}, p_{11}, p_{33}, p_9, p_{10}, p_5, p_{24}, p_3, p_2, p_7, p_{31}, p_8, p_6, p_{27}, p_{16}, p_{15}, p_{17}, p_4, p_{29}, p_{25}\} \times$

$\{x_{19}, x_{20}, x_{15}, x_{17}, x_{13}, x_{14}, x_{18}, x_{16}, x_{23}, x_{22}, x_{25}\})$ and $S_B = (\{p_{34}, p_{30}, p_{22}, p_{12}, p_{23}, p_{13}, p_{35}\} \times \{x_{21}, x_{26}, x_{24}\})$. The unobservable induced subnet presents some circuits $(x_{16}p_6x_{18}p_{25}x_{16}$, $x_{15}p_5x_{17}p_{24}x_{15}, x_{13}p_3x_{15}p_{29}x_{13})$ but system S_B has no circuit. This first decomposition suggests that the consideration of these systems can reduce the size of the matrices and vectors used in the estimation.

Presented in the appendix, the DM decomposition is a possible approach leading to a triangular form. Let consider an application of this technique presented in Table II and analyze the obtained subsystems.

System $\mathbf{S}_A$

System $\mathbf{S}_A$ is composed of S_1, S_2, S_3 and S_4 defined as follows:

$S_1 = \{p_{33}, p_{36}, p_{38}, p_{26}, p_{11}, p_{33}, p_9, p_{10}\} \times \{x_{19}, x_{20}\}, S_2 = \{p_5, p_{24}, p_3, p_2, p_7\} \times \{x_{15}, x_{17}, x_{13}, x_{14}\}$, $S_3 = \{p_{31}, p_8\} \times \{x_{18}\}$ and $S_4 = \{p_6, p_{27}, p_{16}, p_{15}, p_{17}, p_4, p_{29}, p_{25}\} \times \{x_{16}, x_{23}, x_{22}, x_{25}\}$

Highlighted in grey in the table, the four incidence matrices $W_{un,1}$ for S_1, $W_{un,2}$ for S_2, $W_{un,3}$ for S_3 and $W_{un,4}$ for S_4 can easily be expressed and are RSB. Indeed, the relevant possible row-vector z_i satisfying $z_i.W_{un,i} < 0$ are $z_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix}$, $z_2 = \begin{pmatrix} 3 & 1 & 3 & 4 & 1 \end{pmatrix}$, $z_3 = \begin{pmatrix} 2 & 1 \end{pmatrix}$ and $z_4 = \begin{pmatrix} 1 & 2 & 2 & 3 & 1 & 2 & 0 \end{pmatrix}$. So, we can deduce that $\mathbf{S}_A$ is RSB (Theorem 8) and that all count variables of structure $\mathbf{S}_A$ are upper bounded (Theorem 4).

System S_B

The unique substructure of S_B is $S_5 = \{p_{34}, p_{30}, p_{22}, p_{12}, p_{23}, p_{13}, p_{35}\} \times \{x_{21}, x_{26}, x_{24}\}$.

The incidence matrix $W_{un,5}$ for S_5 is RSB as a possible z_5 satisfying $z_5.W_{un} < 0$ is $z_5 = \begin{pmatrix} 1 & 0 & 2 & 3 & 0 & 0 & 1 \end{pmatrix}$.

Finally, the analysis of $\mathbf{S}_A$ and S_B shows the that the Petri net has good properties as the unobservable induced incidence matrix W_{un} is completely RSB and all count variables of $\mathbf{S}_A$ and S_B are upper bounded.

TABLE II

INCIDENCE MATRIX OF THE UNOBSERVABLE INDUCED PETRI NET IN FIG. 2 (EXAMPLE 3) AFTER REORGANIZATION OF THE ROWS AND COLUMNS

W_{un}	x_{19}	x_{20}	x_{15}	x_{17}	x_{13}	x_{14}	x_{18}	x_{23}	x_{22}	x_{25}	x_{16}	x_{21}	x_{26}	x_{24}
p_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{18}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{19}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{20}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{21}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{28}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{32}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{37}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{33}	1	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{36}	1	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{38}	1	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{26}	0	−1	0	0	0	0	0	0	0	0	0	0	0	0
p_{11}	0	1	0	0	0	0	0	0	0	0	0	0	0	0
p_{33}	1	0	0	0	0	0	0	0	0	0	0	0	0	0
p_9	**−1**	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{10}	1	**−1**	0	0	0	0	0	0	0	0	0	0	0	0
p_5	0	0	1	**−1**	0	0	0	0	0	0	0	0	0	0
p_{24}	0	0	**−1**	1	0	0	0	0	0	0	0	0	0	0
p_3	0	0	−1	0	**1**	0	0	0	0	0	0	0	0	0
p_2	0	0	0	0	−1	**−1**	0	0	0	0	0	0	0	0
p_7	0	0	0	1	0	0	0	0	0	0	0	0	0	0
p_{31}	−1	1	0	−1	0	0	**−1**	0	0	0	0	0	0	0
p_8	0	0	0	0	0	0	1	0	0	0	0	0	0	0
p_{27}	0	0	0	0	0	0	0	**−1**	0	0	0	0	0	0
p_{16}	0	0	0	0	0	0	0	−1	**1**	0	0	0	0	0
p_{15}	0	0	0	0	0	0	0	0	−1	**−1**	**0**	0	0	0
p_4	0	0	0	0	0	1	0	0	0	1	**−1**	0	0	0
p_{17}	0	0	0	0	0	0	0	1	0	0	0	0	0	0
p_6	0	0	0	0	0	0	−1	0	0	0	1	0	0	0
p_{29}	0	0	1	0	−1	−1	0	1	−1	−1	1	0	0	0
p_{25}	0	0	0	0	0	0	1	0	0	0	−1	0	0	0
p_{34}	0	0	0	0	0	0	0	0	0	0	0	**−1**	0	0
p_{30}	0	0	0	0	0	0	0	0	0	0	0	1	**1**	0
p_{22}	0	0	0	0	0	0	0	0	0	0	0	0	0	**−1**
p_{12}	0	0	0	0	0	0	0	0	0	0	0	−1	−1	0
p_{23}	0	0	0	0	0	0	0	0	0	0	0	0	1	1
p_{13}	0	0	0	0	0	0	0	0	0	0	0	1	0	0
p_{35}	0	0	0	0	0	0	0	0	0	0	0	0	1	−1

We now show that the computation of a guaranteed horizon can be applied on a subsystem.

System S_1

Let us consider the subsystem S_1 at step $< 1 >$ and the relevant estimation of the sequence length for transitions x_{19} and x_{20}. In inequality (1), we can consider the rows relevant to the places of the set $\{p_{33}, p_{36}, p_{38}, p_{26}, p_{11}, p_{33}, p_9, p_{10}\}$, the relevant marking $M_R = \begin{pmatrix} M_{33} & M_{36} & M_{38} & M_{26} & M_{11} & M_{33} & M_9 & M_{10} \end{pmatrix}^T$ (R as Reduced, remember that $M_i = M(p_i)$) and the relevant rows of the incidence matrices: $W_{un,R} = \begin{pmatrix} W_{un}(33,.) \\ W_{un}(36,.) \\ W_{un}(38,.) \\ W_{un}(26,.) \\ W_{un}(11,.) \\ W_{un}(33,.) \\ W_{un}(9,.) \\ W_{un}(10,.) \end{pmatrix}$ and $W^-_{obs,R} = \begin{pmatrix} W^-_{obs}(33,.) \\ W^-_{obs}(36,.) \\ W^-_{obs}(38,.) \\ W^-_{obs}(26,.) \\ W^-_{obs}(11,.) \\ W^-_{obs}(33,.) \\ W^-_{obs}(9,.) \\ W^-_{obs}(10,.) \end{pmatrix}$. Moreover, we can consider the columns of $\{x_{19}, x_{20}\}$ and remove the null columns of $W_{un,R}$, which becomes $W_{un,1}$ already introduced. We obtain

$$-W_{un,1} \cdot \overline{x}_R^{<1>} \leq b_R^{<1>}$$

with $\overline{x}_R = \begin{pmatrix} \overline{x}_{19} & \overline{x}_{20} \end{pmatrix}^T$ and $b_R^{<1>} = M_R^{<1>} - W^-_{obs,R}.\overline{y}^{<1>}$. For the following steps $< k >$, the same reasoning implies that relation (3) becomes

$$-W_{un,1}.\overline{x}_R^{<0>\to<k>} \leq b_R^{<k>} \tag{6}$$

with $b_R^{<k>} = M_R^{<1>} + W_{obs,R} \cdot \overline{y}^{<0>\to<k-1>} - W^-_{obs,R}.\overline{y}^{<k>}$

The size of vectors $\overline{x}_R^{<0>\to<k>}$, $M_R^{<1>}$, $\overline{y}^{<0>\to<k-1>}$and $\overline{y}^{<k>}$ are (2×1), (8×1), (12×1) and (12×1) respectively. The size of matrices $W_{un,1}, W_{obs,R}$ and $W_{obs,R}^{-}$ are (8×2), (8×12), and (8×12) respectively. Except $W_{obs,R}$, which is given below, all the matrices can easily be deduced.

$$W_{obs,R} = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In the case of a receding horizon going from $<0>$ to $<k>$, the application of Theorem 2 allows to compute the guaranteed horizon with the following linear programming problem: $h_g^{<0>\to<k>} = \lfloor \max(c.\overline{x}_R^{<0>\to<k>}) \rfloor + k$ for (6) relaxed over $\mathbb{R}$, where $\overline{x}_R^{<0>\to<k>}$ and (6) replace $\overline{x}^{<0>\to<k>}$ and $-W_{un}.\overline{x}^{<0>\to<k>} \leq b^{<k>}$ in (3), respectively. Taking the initial marking and the first observed words of Table I in [30], the results are presented in Table III. The execution time of each computation based on function glpk() of GNU Octave is negligible.

TABLE III

GUARANTEED HORIZON FOR SUBSYSTEM S_1 (EXAMPLE 3)

$<k>$	Observed word	$\lfloor \max(c.\overline{x}_R^{<0>\to<k>}) \rfloor$	$h_g^{<0>\to<k>}$
1	y_1	0	1
2	$y_1 y_1$	0	2
3	$y_1 y_1 y_2$	0	3
4	$y_1 y_1 y_2 y_{12}$	1	5
5	$y_1 y_1 y_2 y_{12} y_3$	1	6
6	$y_1 y_1 y_2 y_{12} y_3 y_{12}$	2	8
7	$y_1 y_1 y_2 y_{12} y_3 y_{12} y_3$	2	9
8	$y_1 y_1 y_2 y_{12} y_3 y_{12} y_3 y_6$	3	11

Table III already gives a rough estimate of the sequences as $\overline{x}_{19} + \overline{x}_{20} \leq \lfloor \max(c.\overline{x}_R^{<0>\to<k>}) \rfloor$. When $\lfloor \max(c.\overline{x}_R^{<0>\to<k>}) \rfloor = 0$, the firings of x_{19} and x_{20} are not possible. Knowing the guaranteed horizon for the subsystem S_1 and each step, the sequence estimation presented in Section 3.2 [15] can be applied: using the counter form, the finite set of relations can be established and the timed sequences relevant to transitions x_{19} and x_{20} can be estimated with any criterion.

The same remarks hold for subsystems S_2 and S_5, which are independent. When the majorants of the count variables relevant to S_1 and S_2 are obtained, the structure suggests that the majorant of the count variable $\overline{x}_{18}$ of S_3, which depends on S_1 and S_2 can be estimated in a sequential way. The same remark holds for the count variable$\{\overline{x}_{16}, \overline{x}_{23}, \overline{x}_{22}, \overline{x}_{25}\}$ of S_4. Finally, the sequential treatment of each reduced system with smaller dimensions facilitates the computation and brings local results independently of the resolution of the complete system.

VI. CONCLUSION

In Section III, we have introduced new results as Theorems 2 and 3, which focus on the computation of a guaranteed horizon $h_g^{<0>\to<k>}$ that is, a guaranteed receding horizon relevant to the succession of steps going from $<0>$ to $<k>$. We have analyzed the structure of the incidence matrix of the unobservable induced subnet in Section IV and have shown that a structure presenting at least a non-RSB column is non-RSB. The composition of the RSB blocks shows a propagation of the RSB property through the structure proving that a part of the transitions relevant to the RSB blocks is upper bounded even if the complete system is non-RSB. The algebraic analysis can be applied to any triangular form, which can be obtained with any technique as the Dulmage-Mendelsohn decomposition or a variant: a double application of this decomposition has been made in Section V. The case study shows that the direct treatment of the large system system can be avoided and that the triangular form brings a sequential treatment allowing a computation based on smaller systems independently of the resolution of the complete system. The adaptation to any known estimation approach is clearly a perspective. As this proposed paper shows that the consideration of the triangular form of the incidence matrix is an useful way to analyze a Petri net, another perspective is the adaptation of this approach to analyze the properties of structurally boundedness, and deadlock structurally boundedness.

VII. APPENDIX

This appendix presents the main lines of the Dulmage-Mendelsohn decomposition [21] [22], which brings a partition of the transitions and places in notable sub-structures. Generalizing the classical triangular form, where the substructures are square, this structural analysis is based on a canonical decomposition of any bipartite graph (and its relevant table, which is rectangular).

The DM decomposition presents a large scope of applications as it has been exploited in many fields such that: the resolution of large scale systems [33] [32] [10]; the simulation of continuous systems, where the debugging of software based on modeling languages is necessary [9]; the fault detection in continuous systems [17] [18]. Similar to bond graphs, the study [24] improves this structural approach by taking into account integral and differential causal interpretations for differential constraints.

The classical algorithms of maximum matching as the classical "Hungarian method" (developed by H. Kuhn) and the algorithm of permutation of the rows and columns of a matrix [32] [33] [17] [18] leading to its DM decomposition are out the scope of the appendix.

Let us define the initial table and adapt the DM decomposition to the Petri nets.

As the resolution focuses on the unknown variables, we separate the transitions of TR and the relevant columns in two sets:

- The set of observable transitions TR_{obs}, which correspond to the known variables $\overline{y}$.

- The set of unobservable transitions TR_{un}, which are relevant to unknown variables $\overline{x}$.

Therefore, a permutation of the columns allows to establish the incidence matrix $W = \begin{pmatrix} W_{obs} & W_{un} \end{pmatrix}$. Moreover, a structural point of view of the unobservable induced subnet is taken. Each (oriented) arc of the Petri net is replaced by a (non-oriented) edge, which is non-valuated, that is, the valuations and the orientations of the arcs of the Petri net are neglected: only the existence of a connection (which is often represented by the symbol $\times$ or 1 in the literature) is considered in this appendix. We now show that the analysis of this non-oriented bipartite graph of the unobservable induced subnet allows to determine notable sub-systems, which are the support of a possible resolution and an algebraic interpretation.

The matching between the relations P and the unobservable transitions TR_{un} is defined as follows:

***Definition** 2:* A matching C is a set of pairs (p_i, x_j), where:

- Each place p_i is associated with a transition x_j at the most.

- Each transition $x_j \in TR_{un}$ is associated with a place p_i at the most.

So, in a matching C, a unique transition is associated to each place and a unique place is associated to each transition at the most. In the bipartite graph, the matching is represented by a set of edges without common vertices.

As the non-oriented bipartite graph of the unobservable induced subnet is considered, a possible pair can be composed of a place and one of its input or output transitions. Moreover, we focus on maximum matchings, where the number of its pairs is maximum. Different maximum matchings can be obtained but all of them have the same cardinality. In the tables II and IV, each pair of the matching is expressed by a symbol in bold.

In this context, the maximum matching is the support of the canonical decomposition developed by A. L. Dulmage and N. S. Mendelsohn [21] [22] in graph theory, where a structural decomposition of the table leads to a diagonal of specific block substructures. Three distinct canonical structures named Just-, Over- and Under-structures are highlighted and the block substructures of the Just-structure are square. If the matching C is maximum, there is a unique partition of rows of P and columns of TR_{un} denoted as X such that: $P = P^{>} \cup P^{=} \cup P^{<}$ and $X = X^{>} \cup X^{=} \cup X^{<}$ with empty intersections. This partition highlights three important sub-structures: the Over-structure $S^{>} = (P^{>}, X^{>})$, the Just-structure $S^{=} = (P^{=}, X^{=})$ and the Under-structure $S^{<} = (P^{<}, X^{<})$. Moreover, we have $|C| = |C^{>}| + |C^{=}| + |C^{<}|$ (expression $|.|$ denotes the number of pairs in the matching), where $C^{>}, C^{=}$ and $C^{<}$ satisfy the following points.

- For the Over-structure, the maximum matching $C^{>}$ satisfies $|C^{>}| = |X'| < |P'|$. All elements of X' are matched but there is at least a non-matched element in P'.
- For the Just-structure, the maximum matching $C^{=}$ satisfies $|C^{=}| = |P'| = |X'|$. All elements of P' and X' are matched in the case of a Just-structure, which can be decomposed in square blocks.
- For the Under-structure, the maximum matching $C^{<}$ satisfies $|C^{<}| = |P'| < |X'|$. All elements of P' are matched but there is at least a non-matched element in X'.

Given a matching, an alternating path is a path whose edges belong alternatively to the matching and not to the matching. Using these notions, the following theorem transposed from [21] [22] [17] [18] allows to determine different notable substructures. To facilitate the presentation of the results, a direction is added to the edges of the non-oriented bipartite graph. Each pair (p_i, x_j) of the maximum matching C is oriented from p_i to x_j (graphically, $p_i \xrightarrow{C} x_j$) and in the opposite direction when $(p_i, x_j) \notin C$ (graphically, $p_i \leftarrow x_j$).

***Theorem** 9:* Let us assume that the matching is maximum.

- The places and transitions of an alternating path belong to the Over-structure $S^{>} = (P^{>}, X^{>})$ when this path starts from a matched place and finishes in a non-matched place.
- The places and transitions of an alternating path belong to the Under-structure $S^{<} = (P^{<}, X^{<})$ when this path starts from a non-matched transition and finishes in a matched transition.
- The Just-structure is defined by $S^{=} = (P^{=}, X^{=})$ with $P^{=} = P \backslash (P^{>} \cup P^{<})$ and $X^{=} = TR_{un} \backslash (X^{>} \cup X^{<})$. ■

Example 4.

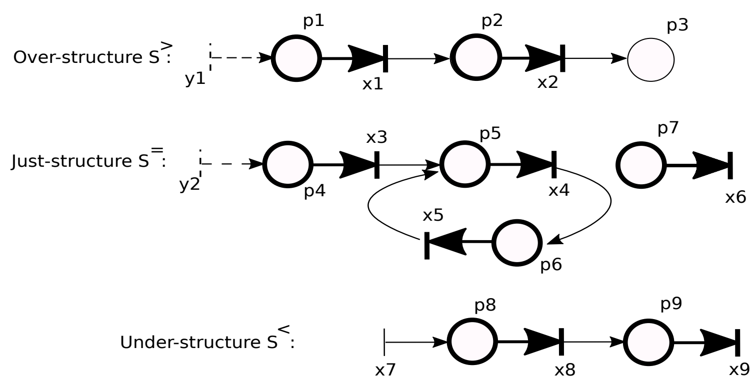

Fig. 3. Petri net of example 4: the matched pairs (place and its output transition) and the relevant outgoing arcs are in bold; Over-, Just- and Under-substructures are represented with bold and thin lines.

In the Petri net of example 4, the sets of observable and unobservable transitions are $\{y_1, y_2\}$ and$\{x_1, x_2, \ldots, x_8\}$ respectively (Fig. 3). Remember that the firing number of y_1, y_2 is finite (Assumption $\mathcal{AS}-3$). Table IV presents the non-oriented form of the incidence matrix $W = \begin{pmatrix} W_{obs} & W_{un} \end{pmatrix}$ for the Petri net in Fig. 3: The values 1 and -1 are replaced by 1, which represents the presence of a connection between the relevant place and the unobservable transition. For clarity, the labels of the places and transitions in Petri nets Fig. 3 have been chosen such that they illustrates the DM decomposition without making a reorganization of the columns and rows of the table. A possible maximum matching C whose cardinality is 8 is

TABLE IV

CANONICAL DECOMPOSITION OF THE STRUCTURE OF THE INCIDENCE MATRIX OF THE UNOBSERVABLE INDUCED PETRI NET IN FIG. 3 (EXAMPLE 4). EACH VALUE 1 REPRESENTS A CONNECTION FOR A PAIR (p_i, x_j) WHILE 0 EXPRESSES AN ABSENCE OF LINK.

	y_1	y_2	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
p_1	1	0	**1**	0	0	0	0	0	0	0	0
p_2	0	0	1	**1**	0	0	0	0	0	0	0
p_3	0	0	0	1	0	0	0	0	0	0	0
p_4	0	1	0	0	**1**	0	0	0	0	0	0
p_5	0	0	0	0	1	**1**	1	0	0	0	0
p_6	0	0	0	0	0	1	**1**	0	0	0	0
p_7	0	0	0	0	0	0	0	**1**	0	0	0
p_8	0	0	0	0	0	0	0	0	1	**1**	0
p_9	0	0	0	0	0	0	0	0	0	1	**1**

represented in bold (this matching is suggested by the orientation of the Petri net): $C = \{(p_1, x_1), (p_2, x_2), (p_4, x_3), (p_5, x_4), (p_6, x_5), (p_7, x_6), (p_8, x_8), (p_9, x_9)\}$. Other maximum matchings are available as $\{(p_2, x_1), (p_3, x_2), (p_4, x_3), (p_5, x_5), (p_6, x_4), (p_7, x_6), (p_8, x_7), (p_9, x_8)\}$.

As C is maximum, we can apply the DM decomposition (Theorem 9). The edges of the non-oriented bipartite graph are now oriented with the direction presented above and the oriented path can be interpreted. In this simple example, the orientation deduced from the chosen maximum matching corresponds to the orientation of the Petri net. For $S^>$, an alternating path starts from p_1 (matched) and finishes in p_3 (non-matched): $p_1 \xrightarrow{C} x_1 \longrightarrow p_2 \xrightarrow{C} x_2 \longrightarrow p_3$. For $S^<$, an alternating path starts from x_7 (non-matched) and finishes in x_9 (matched): $x_7 \longrightarrow p_8 \xrightarrow{C} x_8 \longrightarrow p_9 \xrightarrow{C} x_9$. For $S^=$, an alternating path starts from p_4 (matched) and finishes in x_5 (matched): $p_4 \xrightarrow{C} x_3 \longrightarrow p_5 \xrightarrow{C} x_4 \longrightarrow p_6 \xrightarrow{C} x_5$. Another one starts from p_7 (matched) and finishes in x_6 (matched): $p_7 \xrightarrow{C} x_6$. ■

Let us consider the Just-structure $S^=$. Let us assume that a fictitious self-loop of null length connecting x_i to x_i is added to each vertex $x_i \in X^=$. For any pair of matched vertexes (x_i, x_j) with $x_j, x_i \in X^=$, we can focus on the case, where there is a path from x_i to x_j and a path from x_j to x_i (mutual dependence of x_i and x_j). This case defines a pair of dependent transitions corresponding to a circuit and we can define a substructure composed of transitions, where each transition is connected to any transition of the substructure with a circuit composed of places.

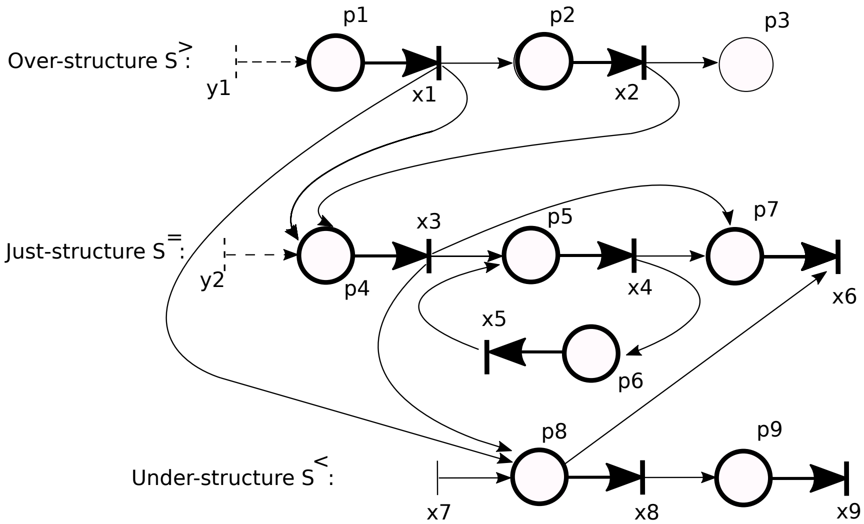

Fig. 4. Variant of Petri net Fig. 3 with the same DM decomposition.

Formally, the substructure contains a directed path from to x_i to x_j and a directed path from x_j to x_i for every pair of vertices x_i, x_j. These substructures are usually named *strongly connected* substructure and, *irreducible* substructure for the corresponding representation in the table. As this type of substructure is remarkable, the approach is based on the determination of all these substructures, which leads to a partition of the Just-structure $S^=$.

Example 4 continued. Represented in grey in the Just-structure $S^=$ of Table IV, substructures $(\{p_4\},\{x_3\})$, $(\{p_5,p_6\},\{x_4,x_5\})$ and $(\{p_7\},\{x_6\})$ are irreducible. ■

Note that each DM decomposition is common to a set of Petri nets. In fact, each filling of the lower-left corner of W_{un} defines a new Petri net with the same DM decomposition.

Example 4 continued. The Petri net in Fig. 4 is a variant of the Petri net in Fig. 3, which presents the same DM decomposition. The difference is the addition of components in the lower-left corner of W_{un}: $(W_{un})_{4,1} = 1$, $(W_{un})_{4,2} = 1$, $(W_{un})_{7,3} = 1$, $(W_{un})_{7,4} = 1$, $(W_{un})_{8,1} = 1$, $(W_{un})_{8,3} = 1$, $(W_{un})_{8,6} = -1$. ■

Example 3 continued (case study). The determination of a maximum matching gives a matching, where the non-matched places are $p_{33}, p_{36}, p_{38}, p_{26}, p_{11}, p_{33}, p_7, p_8, p_{17}, p_4, p_{29}, p_{25}$, p_{34}, p_{30}, p_{13}, p_{35} (in Table II, the matched components are in bold). The application of the DM decomposition of W_{un} is a priori unsuccessful as it leads to a unique under-determined structure, which does not generate smaller subsystems. Keeping only the rows of the matched places as

p_9, p_{10}, p_5,..., a second DM decomposition leads to a just-structure, which is a triangular form (shaded in darker grey) suggesting an order of resolution. The size of all the blocks in the main diagonal except the substructure $\{p_5, p_{24}\} \times \{x_{15}, x_{17}\}$ is 1×1. The addition of the rows of the non-matched places proposes the subsystems in darker grey and light grey. Note that, as the DM decomposition has not been applied in the classical form, this decomposition of subsystems is not unique a priori. ■

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VIII. Compliance with Ethical Standards

The authors have no conflict of interest to declare that are relevant to this article.