ESTIMATION, PREDICTION AND CONTROL IN (MAX,+) SYSTEMS

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Abstract: Discrete event systems undergo perturbations such as failures that disrupt the control system and reduce the anticipation capacities of the future evolution of the process. Using the (max,+) algebra, processes modelled by a timed event graph may be represented by a linear model. The knowledge of the model and of the initial conditions enable to characterise the state vector with a state equation iteration but perturbations may generate a misappreciation of the state vector. The aim of this paper is to estimate the unknown state, to predict the output trajectory and to calculate the control when the state vector is unknown.

Keywords: discrete-event dynamic systems, supervisory control, estimation, prediction, perturbation.

1 INTRODUCTION

Discrete Event Dynamic Systems (DEDS) can represent a great number of systems characterised as being concurrent, asynchronous, distributed or parallel such as flexible manufacturing systems, multiprocessor systems, and transportation networks. Among formalisms used to represent DEDS, Timed Event Graphs with constant firing form a subclass playing an important role because of its deterministic temporal behaviour. Its evolution is described by linear systems defined on a dioïd. The interpretation of each variable is, for example, of "dater" type for (max,+) algebra: each function $x_i(k)$ represents the date of the k^{th} firing of transition x_i ; $\oplus$ stands for the max. operation while the usual addition plays the role of the multiplication, denoted $\otimes$. In the dioïd (max,+), the state variable representation of the system is of the form:

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k) \qquad (1)$$
$$y(k) = C \otimes x(k)$$

where the state x , the control u, the output y and are defined on $\mathfrak{R} \cup \{-\infty\}$. In the (max,+) literature, $\otimes$ is usually replaced by . or is omitted.

This state variable representation has some conceptual advantages. The future behaviour is governed by a simple equation and the properties of the system are determined by the constant matrices A, B and C that are deduced from the holding times of the Petri net. The study of the system can be carried out in the field of matrix theory which is well developed. The transfer representation is another point of view that expresses clearly the trajectories and theirs properties like the important property of periodicity and makes it possible the association of Petri Nets (Gaubert, 1992).

When faced with the problem of controlling a system, the control theory designs are often based on the assumption that the state vector of the system to be controlled is available. For example, let us recall the classical problem. Let us suppose that some events are designated as controllable, meaning that their input transitions may be delayed from firing until some arbitrary time. The delayed enabling times $u_i(k)$ for the controllable events are to be provided by a supervisor. Let us suppose that we wish to slow the system down as much as possible without causing any event to occur later than some sequence of execution times Z. It can be proved that, for the system which dater equations give the lowest solution (the earliest times), the solution (the latest times) is explicitly given by the backward recursive equations where the co-state vector plays the role of the state vector. The consideration of a vector of temporal objectives (Cohen, et al.,1993) (Baccelli, et al., 1992) generalises the classical solutions (PERT, potentials/tasks graph) (Gondran and Minoux, 1995) enabling the definition of an execution calendar of a given project. An improvement is to consider the past evolution of the process (Declerck, 1997). The results can be given by two algorithms providing the earliest and the latest times of the tasks respectively. The differences between the co-state and the state represent the "spare time" or the "margin" which is available for the firing of the transitions. A negative difference prevents the future deadlines from being achieved. In such a case, it is possible to compensate the effects of unfavourable past evolution of the system: each output $y(k)$ occurs at the latest before $z(k)$, otherwise at the earliest after this time (Declerck, 1997). In other words, the problem is, for the control synthesis, to consider " external " perturbations as the variations of the desired output or the variations of the system supplying for products and spare parts.

Thus, this approach requires the knowledge of the vector state values. However, the state vector is not always available from the information system if the process has for example an human aspect. The knowledge of the initial conditions and of the model enables to characterise the state vector with a state equation iteration. Unfortunately, this solution must start from a known state and disregards unavoidable model errors produced by the following " internal " perturbations.

- the physical and human aspects of the process entails the variation of the holding times.
- this situation also occurs when the process undergoes a failure and must be recovered.(Declerck and Staroswiecki, 1992)
- the aim of the maintenance is to prevent any breakdowns. This pre-emptive maintenance produces a necessary perturbation of the production.

In these cases, applications of theories which assume that the state vector is known is limited. In this paper it is shown how the available system inputs and outputs may be used to construct an estimate of the system state vector. The system which reconstructs the state vector is called an " observer " by analogy with the automatic control in the statistical case (Kalman in (Borne, et al., 1990)) or in the non-statistical case (Luenberger, 1964; 1966). A past description rupture of the classical model which generates a misappreciation of the state vector is considered and, in this case, the problem is:

- to estimate the past values of the state from the known values of the input and output
- to compute the latest firing dates of the input transitions in such a way that the output events occur at the latest before the desired dates.
- to predict the future evolution of the output and to verify the existence of the optimal solution of the control synthesis.

This paper is organised as follows. The state estimation and the prediction are presented. Then, the estimation is integrated in the general frame of the control. These parts are preceded by notations and by a recall of previous results.

2 PRELIMINARY

A semi-ring S is a triplet $(S, \oplus, \otimes)$ where $(S, \oplus)$ and $(S, \otimes)$ are monoïds, $\oplus$ is commutative, $\otimes$ is distributive with respect to $\oplus$ and the zero element of $\oplus$ is the absorbing element of $\otimes$. A dioïd is an idempotent semi-ring. The set $\mathbb{R} \cup \{-\infty\}$ provided with max. denoted $\oplus$ and with addition denoted $\otimes$ is usually called (max,+) algebra and is an example of dioïd which is denoted D.

$\mathbb{R}_{max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ with

$a \oplus b = max(a,b)$; $\quad \varepsilon = -\infty$ is the zero element of $\oplus$

$a \otimes b = a+b$; $\quad e = 0$ is the identity element of $\otimes$

$a \oplus a = a \quad$ (idempotency of $\oplus$)

$a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon \quad$ (absorbing element ε)

$a \backslash b = max \{x \ / \ ax \leqslant b\}$ is called the left residuated of b by a (also called the subsolution of equality $ax=b$). The set of n.n matrices endowed with the operation $\oplus$ and $\otimes$ is also a dioïd which is denoted $D^{n.n}$. $D^{n.n}$ is not commutative.

$$(A \wedge B) \oplus C = (A \oplus C) \wedge (B \oplus C)$$

(right-distributivity of $\oplus$ with respect to $\wedge$)

$$A \otimes (B \wedge C) \leqslant (A \otimes B) \wedge (A \otimes C)$$

(left-subdistributivity of $\otimes$ with respect to $\wedge$)

A^t is the transpose of A. If $A \in \Re_{\max}^{m.n}$, $B \in \Re_{\max}^{m}$, $x \in \Re_{\max}^{n}$ then $A \backslash B = A^t \odot B$, where $\odot$ is a matrix product where operation $\oplus$ and $\otimes$ respectively of $\Re_{\max}$ are replaced by $\wedge$ (minimum) and \ (left residuated) of $\Re_{\max}$.

The following equation can be deduced from the state equation.

$$Y_{k+i,i} = \mathbb{F}_i \, . \, x(k) \oplus \mathbb{G}_i \, . \, U_{k,i-1} \qquad (2)$$

$$\text{With } Y_{k,i} = \begin{pmatrix} y_{k-i} \\ \\ \\ y_k \end{pmatrix} \quad U_{k,i-1} = \begin{pmatrix} u_{k-i+1} \\ \\ \\ u_k \end{pmatrix}$$

$$\mathbb{F}_i = \begin{pmatrix} C \\ CA \\ \\ \\ CA^i \end{pmatrix}$$

$$
\mathbb{G}_i = \begin{pmatrix}
\varepsilon & \varepsilon & \dots & \varepsilon & \varepsilon \\
CB & \varepsilon & \dots & \varepsilon & \varepsilon \\
CAB & CB & \dots & \varepsilon & \varepsilon \\
. & . & & . & . \\
. & . & & . & . \\
. & . & & . & . \\
CA^{i-1}B & CA^{i-2}B & \dots & CB & \varepsilon \\
CA^{i-2}B & CA^{i-3}B & \dots & CAB & CB
\end{pmatrix}
$$

For the state, the formal serie $X(\gamma) = \bigoplus_{i=k_s}^{+\infty} x(i)\, \gamma^i$ is taken with $k_s \geqslant 0$ where γ is the backward shift operator in numbering and $[\, X(\gamma)\,]_i = x(i)$.

From the state equation (1), the use of the γ-transform leads to the following representation

$$X = (\, A\gamma)\, X \oplus B\, U \oplus x(k_s)\, \gamma^{k_s} \qquad (3)$$
$$Y = C\, X$$

where X, U and Y are the γ-transform of x, u and y respectively in the interval $[k_s, +\infty[$.

The least solution of the above equation is given by

$$Y = C(\, A\gamma)^*.\, (B\, U \oplus x(k_s)\, \gamma^{k_s}\ )$$

with $(\, A\gamma)^* = \bigoplus_{i=0}^{+\infty} (\, A\gamma)^i$

3 ESTIMATION, PREDICTION AND CONTROL

This part contains an estimation of the past values of the state from the known values of the input and output, an prediction of the future evolution of the output and the control synthesis of the process in the case of an unknown state. The model is supposed to be known and invariant on a defined horizon: for example, the holding times of the associated Petri Net are constant. The control synthesis is based on the estimation of the past values of the state from the known values of the input and output and, the prediction of the future output evolution.

3.1 State estimation

The following properties propose an estimation of the greatest state dates but also the lowest state dates in order to complete the study.

The hypothesis that the known data are in the interval $[k_s,\ k_0]$ is taken and the objective is to estimate $x(k)$ from $k = k_s$ to $k = k_0$. In other words, this situation corresponds to a problem of smoothing: given all observations available up to number of event k_0, one tries to estimate $x(k)$ with $k \leqslant k_0$.

Let W be the trajectory of the known output values from k_s to k_0:

$w(k) = y(k)$ for $k \leqslant k_0$

$w(k) = T$ for $k > k_0$ with $T = +\infty$

Proposition 1

With the observation horizon from k_s to k_0 with $k_s \leqslant k_0$, a frame of $x\,(k_s)$ is

$$\hat{x}^-\ (k_s) \leqslant x\,(k_s) \leqslant \hat{x}^+\ (k_s))$$

with $\hat{x}^-\ (k_s) = B\, u(k_s)$

and $\hat{x}^+\ (k_s) = \gamma^{-k_s}\ [C\,(\, A\gamma)^*]\setminus W$

Proof

<u>Calculus of $\hat{x}^-\ (k_s)$</u>

$X = (\, A\gamma)^*.\, (B\, U \oplus x(k_s)\, \gamma^{k_s}\ )$ is the least solution of (3).

So $X \geqslant (\, A\gamma)^*.\, B\, U$ and $\hat{X}^- = (\, A\gamma)^*.\, B\, U$

Particularly, $x(k_s) \geqslant B\, u(k_s)$ and $\hat{x}^-\ (k_s) = B\, u(k_s)$

<u>Calculus of $\hat{x}^+\ (k_s)$</u>

$$Y = C\,(\, A\gamma)^*.\, (B\, U \oplus x(k_s)\, \gamma^{k_s}\ )$$

$$Y \geqslant C\,(\, A\gamma)^*.x(k_s)\, \gamma^{k_s}$$

$$x(k_s)\, \gamma^{k_s} \leqslant [C\,(\, A\gamma)^*]\setminus Y$$

with $\quad Y(\gamma) = \bigoplus_{i=k_s}^{+\infty} y(i)\, \gamma^i$

and $C\,(\, A\gamma)^* = \bigoplus_{i=0}^{+\infty} C\, A^i\, \gamma^i$

From the known output, the upper bound can be deduced:

$$\hat{x}^+\ (k_s) = \gamma^{-k_s}\ [C\,(\, A\gamma)^*]\setminus W$$

with $\quad W(\gamma) = \bigoplus_{i=k_s}^{+\infty} w(i)\, \gamma^i$

Now, the extended calculus of the lower and upper bounds for $k \geqslant k_s$ is considered.

Proposition 2

For j from 0 to i with $i = k_0 - k_s$

$$\hat{x}^{+}(k_s+j) = A^j . [\mathbb{F}_i \backslash Y_{k_s+i,i}] \oplus \bigoplus_{k=k_s+1}^{k_s+j} A^{k_s+j-k} . B\, u(k) \text{ and}$$

$$\hat{x}^{-}(k_s+j) = \bigoplus_{k=k_s+1}^{k_s+j} A^{k_s+j-k} . B\, u(k)$$

The horizon of the calculations is finite. The introduction of each piece of data will entail a new calculation of the state that produces a minimisation of the values because of the influence of the estimate $\hat{x}^{+}(k_s)$. Naturally, if no piece of information is available, the estimation of the state vector is infinite.

3.2 Prediction of the output trajectory

The problem is now to anticipate the effects of the control and the past evolution on the output. In that aim, the exploitation of the estimation of $x(k_s)$ gives a frame of the output: the prediction of the earliest and latest firing date of the output. As the complete output trajectory is considered, the following result includes the smoothing for $k \leqslant k_0$ and the prediction for $k > k_0$. In this part, the hypothesis that the control is already calculated by a control system is taken.

Proposition 3

A framing of Y is

$$Yp^- \leqslant Y \leqslant Yp^+$$

with $Yp^- = C\,(A\gamma)^*.\ B\ U \oplus C\,(A\gamma)^*.\ B\ u(k_s)\ \gamma^{k_s}$ and $Yp^+ = C\,(A\gamma)^*.\ (B\ U \oplus [C\,(A\gamma)^*] \backslash W)$

3.3 Application to the Control synthesis

Problem. Knowing the dates' values of the control and the output, the number of events belonging to $[k_s, k_0]$ and a sequence of the desired output $z(k)$, k ranging from k_0+1 to k_f, the problem is to determine the greatest control sequence $u(k)$ such as each output $y(k)$ occurs at the latest before $z(k)$.

Approach

Step 1: Control

The calculation of the greatest control does not need to know the value of the state and can be realised by the following equations [1]. ξ is called the co-state. This step does not guarantee the existence of the solution.

For $k_0 \leqslant k \leqslant k_f$

$$\xi(k) = A \backslash \xi(k+1) \wedge C \backslash z(k)$$

with $z(k) = +\infty$ and $\xi(k) = +\infty$ for $k > k_f$

$$u(k) = B \backslash \xi(k)$$

Step 2: estimation/prediction of the upper and lower bounds:

$$\hat{x}^{-}(k_s) = B\, u(k_s)\ ;\ \hat{x}^{+}(k_s) = \mathbb{F}_i \backslash Y_{k_s+i,i}$$

For $k \geqslant k_s$

$$x_p^-(k+1) = A\, x_p^-(k) \oplus B\, u(k+1)$$
$$x_p^+(k+1) = A\, x_p^+(k) \oplus B\, u(k+1)$$

with $x_p^-(k_s) = \hat{x}^{-}(k_s)$; $x_p^+(k_s) = \hat{x}^{+}(k_s)$

step 3: guaranteed desired output

The desired output $Z \oplus Y_p^+$ is guaranteed in any case (Declerck, 1997).

If $y_p^+(k) \leqslant z(k)$ for $k_0+1 \leqslant k \leqslant k_f$ then the previous calculated control is optimal and the desired output Z is guaranteed.

If it exists k such that $y_p^-(k) > z(k)$ for $k_0+1 \leqslant k \leqslant k_f$ then the desired output Z is not reachable with the previous control.

Remark 1. As the control is applied after the calculation of the control synthesis, the calculated dates must be later than the initial data. Consequently, the causality condition $u(k_0+1) > u(k_0) \oplus y(k_0)$ must be verified. Also $\forall\, k > k_0\ y^-(k) \geqslant y(k_0)$

Remark 2. If there exists a k such that $z(k) < y^+(k)$ et $y^-(k) \neq y^+(k)$, then there is no conclusion on the existence of the optimal control calculated at step 1. However, a deterministic behaviour can be considered if a bounded gradient is chosen (Declerck, 1999) and in this case, the control is minimised.

4 RELATED WORK IN THE CLASSICAL AUTOMATIC CONTROL

In this part, the connections between the (max,+) " observer " and the classical observers is discussed. A first well-known approach is the discrete-time state observer that uses the deterministic least squares framework (Ling and Lim, 1999). From the equation set $Y_{k,N} = M_N\, x_{k,N}$, the classical solution is directly obtained.

$$\hat{x}_{k,N} = (M_N^T M_N)^{-1} M_N^T\, Y_{k,N}$$

and the fact that the resolution of the equation set $Y_{k_s+i,i} = \mathbb{F}_i . x(k_s)$ is also direct can be noticed.

Another approach is the Luenberger's observer (Luenberger, 1964). Instead of requiring that the observer reconstructs the state vector itself, the Luenberger's observer reconstruct some constant linear transformation of the state vector ($\hat{x} = Tx$) ; it would then be possible to reconstruct the state vector itself, provided that the transformation T is invertible. For the continuous systems, the general form of the observer for the continuous system is

$$\frac{d\,\hat{x}(t)}{dt} = A_2\,\hat{x}(t) + L_2\,y(t) + B_2\,u(t)$$

and a result is

$$\hat{x}(t) - T\,x(t) = \exp(L_2\,C\,t)\,[\hat{x}(0) - T\,x(0)]$$

The equation of the sample form of the Luenberger's observer is

$$\hat{x}(k+1) = A\,\hat{x}(k) + B\,u(k) + K\,[y(k) - C\,\hat{x}(k)] \quad (4)$$

and the equation of the estimation error is

$$x(k+1) - \hat{x}(k+1) = (A - K\,C)\,[x(k) - \hat{x}(k)]$$

In the classical observers, the last estimation gives a picture of the current state. Similarly, a similar form is proposed below. The horizon is increasing and for example y_{k_s+i+1} with $k_s + i = k_0$ is added to the previous data $y_{k_s}, \ldots, y_{k_0}$. The new horizon is $[k_s, k_0+1]$ and one tries to estimate $x(k_0+1)$.
The following relations are directly deduced.

$$\hat{x}^+(k_s) = \mathbb{F}_{i+1} \backslash Y_{k_s+i+1,i+1}$$

$$\hat{x}^+(k_s+i+1) = A^{i+1}.\,[\mathbb{F}_{i+1} \backslash Y_{k_s+i+1,i+1}] \oplus \bigoplus_{j=k_s+1}^{k_s+i+1} A^{k_s+i+1-j}.\,B\,u(j)$$

Let $\hat{x}^+(k|[k_s,k])$ be the estimate of $x(k)$ with $[k_s,k]$ the horizon of the known values.

The estimate $\hat{x}^+(k+1\,|[k_s,k+1])$ and $\hat{x}^+(k|[k_s,k])$ are connected in the following inequality that integrates the new information y_{k+1}.

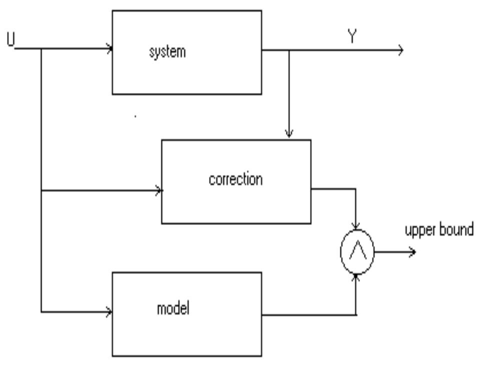

Fig. 1 : Structure of the (max,+) " observer "

Proposition 4
For $k \geqslant k_s$

$$\hat{x}^+(k+1|[k_s,k+1]) \leqslant [A.\,\hat{x}^+(k|[k_s,k]) \oplus B\,u(k+1)] \wedge [A^{k+1-k_s}(C\,A^{k+1-k_s})\backslash Y_{k+1,0} \oplus \hat{x}^-(k+1|[k_s,k+1])] \quad (5)$$

$$\hat{x}^-(k+1|[k_s,k+1]) = A.\,\hat{x}^-(k) \oplus B\,u(k+1)$$

with $\hat{x}^+(k_s|[k_s,k_s]) = C\backslash y(k_s)$ and $\hat{x}^-(k_s|[k_s,k_s]) = \varepsilon$

Proof: see annex

The figure 1 shows that the structure of the proposed "observer" is similar to the Luenberger 's filter in the case of a continuous, time-invariant linear system in the classical algebra. Two expressions are calculated:
- a direct iteration of the state equation from the preceding estimation
- a correction term which is calculated from the new piece of information

The minimisation of the proposed "observer" is replaced by the usual addition in the equation (4) and that the form of the second term is a difference whose concept does not exist in the (max,+) algebra.

5 CONCLUSION

Perturbations can generate the loss of the state vector that is necessary to make the control synthesis of systems described by Timed Event Graphs. In the case of an unknown state, we propose an estimation of the past values of the state from the known values and an prediction of the future evolution of the output. The observation/prediction system is integrated into the control synthesis. The model is supposed to be known and invariant on a defined horizon. Estimation and prediction are together based on a framing of respectively the state and the output trajectory that depends on past evolution of the system.

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7 ANNEX

$$\hat{x}^{+}(k+1/[k_s,k+1]) = A^{i+1}.\ \hat{x}^{+}(k_s) \oplus \bigoplus_{j=k_s+1}^{k+1} A^{k+1-j}.\ B\,u(j) =$$

$$A^{i+1}.\ [\mathbb{F}_{i+1} \backslash Y_{k+1,i+1}] \oplus \bigoplus_{j=k_s+1}^{k+1} A^{k+1-j}.\ B\,u(j) =$$

$$A^{i+1}.[\begin{pmatrix} F_i \\ CA^{i+1} \end{pmatrix} \backslash \begin{pmatrix} Y_{k,i} \\ Y_{k+1,0} \end{pmatrix}] \oplus \bigoplus_{j=k_s+1}^{k+1} A^{k+1-j}.B\,u(j)$$

$$= A^{i+1}.\ [\ \mathbb{F}_i \backslash Y_{k,i} \wedge (CA^{i+1}) \backslash Y_{k+1,0}\ ] \oplus \bigoplus_{j=k_s+1}^{k+1} A^{k+1-j}.\ B\,u(j)$$

$$\preccurlyeq [A^{i+1}.\ \mathbb{F}_i \backslash Y_{k,i} \wedge A^{i+1}.(CA^{i+1}) \backslash Y_{k+1,0}\ ] \oplus \bigoplus_{j=k_s+1}^{k+1} A^{k+1-j}.\ B\,u(j)$$

(subdistributivity of $\otimes$ with respect to $\wedge$)

$$= [A^{i+1}.\ \mathbb{F}_i \backslash Y_{k,i} \oplus \bigoplus_{j=k_s+1}^{k+1} A^{k+1-j}.\ B\,u(j)\ ] \wedge [A^{i+1}.(CA^{i+1}) \backslash Y_{k+1,0} \oplus \bigoplus_{j=k_s+1}^{k+1} A^{k+1-j}.\ B\,u(j)]$$

(distributivity of $\oplus$ with respect to $\wedge$)

As $\bigoplus_{j=k_s+1}^{k+1} A^{k+1-j}.\ B\,u(j) = \bigoplus_{j=k_s+1}^{k} A^{k+1-j}.\ B\,u(j) \oplus \bigoplus_{j=k+1}^{k+1} A^{k+1-j}.\ B\,u(j)$

$$= A \bigoplus_{j=k_s+1}^{k} A^{k-j}.\ B\,u(j) \oplus B\,u(k+1)$$

$$\hat{x}^{+}(k+1/[k_s,k+1]) \preccurlyeq [A.\ A^{i}.\ \mathbb{F}_i \backslash Y_{k,i} \oplus A \bigoplus_{j=k_s+1}^{k} A^{k-j}.\ B\,u(j) \oplus B\,u(k+1)] \wedge [A^{i+1}.(CA^{i+1}) \backslash Y_{k+1,0} \oplus \bigoplus_{j=k_s+1}^{k+1} A^{k+1-j}.\ B\,u(j)]$$

As the last estimation is

$$\hat{x}^{+}(k/[k_s,k]) = A^{i}.\ [\mathbb{F}_i \backslash Y_{k,i}] \oplus \bigoplus_{j=k_s+1}^{k} A^{k-j}.\ B\,u(j)$$

we can deduce the following relation

$$\hat{x}^{+}(k+1/[k_s,k+1]) \preccurlyeq [A.\ \hat{x}^{+}(k/[k_s,k]) \oplus B\,u(k+1)] \wedge [A^{i+1}.(CA^{i+1}) \backslash Y_{k+1,0} \oplus \bigoplus_{j=k_s+1}^{k+1} A^{k+1-j}.\ B\,u(j)]$$

As $\hat{x}^{-}(k+1/[k_s,k+1]) = \bigoplus_{j=k_s+1}^{k+1} A^{k+1-j}.\ B\,u(j)$,

$$\hat{x}^{+}(k+1/[k_s,k+1]) \preccurlyeq [A.\ \hat{x}^{+}(k/[k_s,k]) \oplus B\,u(k+1)] \wedge [A^{k-k_s+1}(CA^{k-k_s+1}) \backslash Y_{k+1,0} \oplus \hat{x}^{-}(k+1/[k_s,k+1])\ ]$$