From Linear Programming to Graph Theory: Standardization of the Algebraic Model of Timed Event Graphs

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8 juillet 2009
Plan

1. Motivations and Objectives
2. Timed Event Graphs
3. Problem of standardization
4. Technique 1 using linear programming
5. Dual technique 2
6. Properties
7. Conclusion
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Motivations and Objectives

- Description of Timed Event Graphs with the form $Ax \leq b$.
  → analogous to the state equation of automatic control in continuous systems.

- Development of a path theory but completely defined in the standard algebra.

  → Possible application of polyvalent algorithms of linear programming like the simplex

- Objective: **Standardization of the Algebraic Model of Timed Event Graphs**

  → Avoid the useless calculations in the calculation of the state trajectory
Motivations and Objectives

Timed Event Graphs

Problem of standardization

Technique 1 using linear programming

Dual technique 2

Properties

Conclusion
A Timed Event Graph is a Petri Net such that each place $p \in P$ has an upstream transition and a downstream transition.
Timed Event Graphs

Model

1. A Timed Event Graph is a Petri Net such that each place $p \in P$ has an upstream transition and a downstream transition.

2. Each place $p_l \in P$: a temporisation $T_l \in R^+$ and an initial marking $m_l$. 
Timed Event Graphs

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Inequations

1. Dater: each variable $x_i(k)$ is the $k^{\text{ème}}$ firing date of transition $x_i$. 

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Inequations

1. Dater: each variable $x_i(k)$ is the $k^{ème}$ firing date of transition $x_i$.

2. FIFO behavior of places.
**Timed Event Graphs**

### Model

1. A Timed Event Graph is a Petri Net such that each place \( p \in P \) has an upstream transition and a downstream transition.

2. Each place \( p_l \in P \): a temporisation \( T_l \in R^+ \) and an initial marking \( m_l \).

### Inequations

1. **Dater**: each variable \( x_i(k) \) is the \( k^{\text{eme}} \) firing date of transition \( x_i \).

2. **FIFO behavior of places.**

3. \( \forall p_l : T_l + x_j(k - m_l) \leq x_i(k) \iff x_j(k - m_l) - x_i(k) \leq -T_l \).


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Example of Timed Event Graph

**Figure:** Timed Event Graph

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Initial model

Internal inequalities.

\[
\begin{pmatrix}
A_{1} & A_{0}
\end{pmatrix}
\begin{pmatrix}
x(k-1) \\
x(k)
\end{pmatrix}
\leq -T^A
\] (1)

Input inequalities.

\[
\begin{pmatrix}
B_1 & B_0
\end{pmatrix}
\begin{pmatrix}
u(k) \\
x(k)
\end{pmatrix}
\leq -T^B
\] (2)

Output inequalities.

\[
\begin{pmatrix}
C_1 & C_0
\end{pmatrix}
\begin{pmatrix}
x(k) \\
y(k)
\end{pmatrix}
\leq -T^C
\] (3)

Each row of matrices \( A = \begin{pmatrix} A_{1} & A_{0} \end{pmatrix} \), \( B = \begin{pmatrix} B_1 & B_0 \end{pmatrix} \) and \( C = \begin{pmatrix} C_1 & C_0 \end{pmatrix} \), is null except two coefficients 1 and \(-1\).

We assume that the set of input and output places presents a null initial marking.
Example of Timed Event Graph.

The relevant matrices are as follows: \( A_{.,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \),

\[
A_{.,0} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\text{ and } T^A = \begin{pmatrix} 10 \\ 8 \\ 10 \\ 8 \\ 3 \end{pmatrix}.
\]

\( B_1 = 1, \quad B_0 = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix} \text{ and } T^B = 0. \)

\( C_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \text{ and } C_0 = -1 \text{ and } T^C = 3. \)
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Internal inequalities.

\[
\begin{pmatrix}
A_{:,1} & A_{:,0}
\end{pmatrix}
\begin{pmatrix}
x(k-1) \\
x(k)
\end{pmatrix} \leq -T^A
\] (4)

Input inequalities.

\[
\begin{pmatrix}
B_1 & B_0
\end{pmatrix}
\begin{pmatrix}
u(k) \\
x(k)
\end{pmatrix} \leq -T^B
\] (5)

Output inequalities.

\[
\begin{pmatrix}
C_1 & C_0
\end{pmatrix}
\begin{pmatrix}
x(k) \\
y(k)
\end{pmatrix} \leq -T^C
\] (6)

Each row of matrices \(A = \begin{pmatrix} A_{:,1} & A_{:,0} \end{pmatrix}\), \(B = \begin{pmatrix} B_1 & B_0 \end{pmatrix}\) and \(C = \begin{pmatrix} C_1 & C_0 \end{pmatrix}\), is null except two coefficients 1 and \(-1\).

We assume that the set of input and output places presents a null initial marking.
Objective : Final model using incidence matrices

\[
\begin{pmatrix}
W_{x\rightarrow x}^+ & -W_{x\rightarrow x}^-
\end{pmatrix}
\begin{pmatrix}
x(k-1) \\
x(k)
\end{pmatrix}
\leq -T_{x\rightarrow x} \hspace{1cm} (7)
\]

\[
\begin{pmatrix}
W_{u\rightarrow x}^+ & -W_{u\rightarrow x}^-
\end{pmatrix}
\begin{pmatrix}
u(k) \\
x(k)
\end{pmatrix}
\leq -T_{u\rightarrow x} \hspace{1cm} (8)
\]

\[
\begin{pmatrix}
W_{x\rightarrow y}^+ & -W_{x\rightarrow y}^-
\end{pmatrix}
\begin{pmatrix}
x(k) \\
y(k)
\end{pmatrix}
\leq -T_{x\rightarrow y} \hspace{1cm} (9)
\]

The matrices of this new model are well-known ingoing/outgoing incidence matrices used in the fundamental relation of marking. A Petri net can directly be deduced from this model.
Initial model

**Figure**: Timed Event Graph (initial)
Final model
Advantages

Connections with the incidence matrices of fundamental equation of marking

\[
W^+ = \begin{pmatrix}
W_{u \rightarrow x}^+ & 0 & 0 \\
0 & W_{x \rightarrow x}^+ & 0 \\
0 & W_{x \rightarrow y}^+ & 0
\end{pmatrix}
\quad \text{and} \quad
W^- = \begin{pmatrix}
0 & W_{u \rightarrow x}^- & 0 \\
0 & W_{x \rightarrow x}^- & 0 \\
0 & 0 & W_{x \rightarrow y}^-
\end{pmatrix}
\]

, for a vector of transitions \((u^t \ x^t \ y^t)^t\).

The temporisations are \(T_{u \rightarrow x}\), \(T_{x \rightarrow x}\) and \(T_{x \rightarrow y}\).

Each internal place → initial marking equal to one

Each input/output place → initial marking equal to zero.
Connections with the incidence matrices of fundamental equation of marking

\[
W^+ = \begin{pmatrix}
W_{u \rightarrow x}^+ & 0 & 0 \\
0 & W_{x \rightarrow x}^+ & 0 \\
0 & W_{x \rightarrow y}^+ & 0
\end{pmatrix}
\quad \text{and} \quad
W^- = \begin{pmatrix}
0 & W_{u \rightarrow x}^- & 0 \\
0 & W_{x \rightarrow x}^- & 0 \\
0 & 0 & W_{x \rightarrow y}^-
\end{pmatrix}
\]

, for a vector of transitions \((u^t \ x^t \ y^t)^t\).

The temporisations are \(T_{u \rightarrow x}\), \(T_{x \rightarrow x}\) and \(T_{x \rightarrow y}\).

Each internal place → initial marking equal to one

Each input/output place → initial marking equal to zero.

Calculations

This model allows an efficient calculation of the state (knowing the past state and the control)→ it avoids the repetition of the useless calculations in the iterative calculation of the state.
Example: the initial Timed Event Graph.

The relevant matrices are as follows: $A_{.,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$A_{.,0} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $T^A = \begin{pmatrix} 10 \\ 8 \\ 10 \\ 8 \\ 3 \end{pmatrix}$.

$B_1 = 1$, $B_0 = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}$ and $T^B = 0$.

$C_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ and $C_0 = -1$ and $T^C = 3$. 
Example: the final Timed Event Graph.

\[ W_{u \rightarrow x}^+ = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad -W_{u \rightarrow x}^- = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

\[ T_{u \rightarrow x} = \begin{pmatrix} 10 \\ 18 \end{pmatrix} \]

\[ W_{x \rightarrow x}^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad -W_{x \rightarrow x}^- = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

\[ (T_{x \rightarrow x})^t = \begin{pmatrix} 10 & 20 & 28 & 8 & 16 & 3 \end{pmatrix} \]
### Earliest state trajectory.

<table>
<thead>
<tr>
<th>k</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>0</td>
<td>10</td>
<td>20</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>20</td>
<td>30</td>
<td>45</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>28</td>
<td>38</td>
<td>53</td>
</tr>
<tr>
<td>u</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>35</td>
</tr>
<tr>
<td>y</td>
<td>3</td>
<td>31</td>
<td>41</td>
<td>56</td>
</tr>
</tbody>
</table>
Technical point

\[
\begin{pmatrix}
A_{1,1} & A_{1,0} \\
A_{0,1} & A_{0,0}
\end{pmatrix}
\begin{pmatrix}
x(k - 1) \\
x(k)
\end{pmatrix}
\leq -T^A
\tag{10}
\]

\[
\begin{pmatrix}
0 & A_{0,0} \\
A_{1,1} & A_{1,0}
\end{pmatrix}
\begin{pmatrix}
x(k - 1) \\
x(k)
\end{pmatrix}
\leq \begin{pmatrix}
-T^A_0 \\
-T^A_1
\end{pmatrix}
\tag{11}
\]

Elimination of the relations connected the entries of state vector \(x(k)\) for given \(k\)

\[A_{0,0}x(k) \leq -T^A_0 \tag{12}\]
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The possible effects on date $x_i(k)$ are produced by:
- the firing dates of a control transition $u_j(k)$ (case a) and also, produced by
- the firing dates of the upstream transitions of places whose initial marking is one $x_j(k - 1)$ (case b).
Case a) The minimal effect is the minimal difference
\[ x_i(k) - u_j(k) \]

or
\[ \min(c'x) \]

for the following constraints
\[
\begin{pmatrix}
0 & A_{0,0} \\
B_1 & B_0
\end{pmatrix}
\begin{pmatrix}
u(k) \\
x(k)
\end{pmatrix}
\leq
\begin{pmatrix}
-T_0^A \\
-T_B
\end{pmatrix}
\]

where \( c' \) is a null row-vector except \( c'_i = 1 \) et \( c'_j = -1 \)

For each pair \((x_i, u_j)\), the resolution of this problem gives the minimal difference \( \Delta T \)

\[ x_i(k) - u_j(k) \geq \Delta T \] or \[ x_i(k) \geq u_j(k) + \Delta T. \]
• Case b) The minimal effect is the minimal difference $x_i(k) - x_j(k - 1)$ or $\min(c'x)$ for the following constraints

$$\begin{pmatrix} 0 & A_{0,0} \\ A_{1,1} & A_{1,0} \end{pmatrix} \cdot \begin{pmatrix} x(k - 1) \\ x(k) \end{pmatrix} \leq \begin{pmatrix} -T_0^A \\ -T_1^A \end{pmatrix}$$

(13)

where $c'$ is a null row-vector except $c'_i = 1$ and $c'_j = -1$

For each pair $(x_i, x_j)$, the resolution of this problem gives the minimal difference $\Delta T x_i(k) - x_j(k - 1) \geq \Delta T$ or $x_i(k) \geq x_j(k - 1) + \Delta T$
Example

a) The minimal difference $x_i(k) - u_j(k)$ is

$$
\begin{pmatrix}
0 \\
10 \\
18
\end{pmatrix}
$$

So,

$$W^+_{u \to x} = \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix},
-W^-_{u \to x} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
$$

and

$$-T_{u \to x} = \begin{pmatrix}
0 \\
-10 \\
-18
\end{pmatrix}$$
b) The minimal difference \( x_i(k) - x_j(k - 1) \) is

\[
\begin{pmatrix}
10 & -\infty & -\infty \\
20 & 8 & -\infty \\
28 & 16 & 3
\end{pmatrix}
\]

So, \( W_{x\rightarrow x}^+ = \)

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\(-W_{x\rightarrow x}^- = \)

\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
0 & 0 & -1 \\
0 & 0 & -1
\end{pmatrix}
\]

and

\(-T_{x\rightarrow x} = \)

\[
\begin{pmatrix}
-10 \\
-20 \\
-28 \\
-8 \\
-16 \\
-3
\end{pmatrix}
\]
Dual technique 2

Theorem of duality.
Primal problem (P) : $\min \ y.b$ with $y \in \mathbb{R}^n_+$, $y.A = c$ and $y$ real positive.
and
Dual problem (D) : $\max \ c.x$ with $x \in \mathbb{R}^m$, $A.x \leq b$ and $x$ real. ■

Case b)

$$\max y. \begin{pmatrix} T^A_0 \\ T^A_1 \end{pmatrix} \text{ with } y \in \mathbb{R}^m_+(y \geq 0) \text{ under constraints}$$

$$y. \begin{pmatrix} 0 & A_{0,0} \\ A_{1,1} & A_{1,0} \end{pmatrix} = -c'$$

(14)

and $y \geq 0$ where $c'$ is a null row-vector except $c'_i = 1$ and $c'_j = -1$.
$i$ and $j$ are respectively the indexes of outgoing transition $x_i$ and ingoing transition $x_j$. 
Case a)

$$\text{max } y \cdot \begin{pmatrix} T_A^0 \\ T_B \end{pmatrix} \text{ with } y \in \mathbb{R}^m_+ (y \geq 0) \text{ under constraints}$$

$$y \cdot \begin{pmatrix} 0 & A_{0,0} \\ B_1 & B_0 \end{pmatrix} = -c' \quad (15)$$

where $c'$ is a null row-vector except $c'_i = 1$ et $c'_j = -1$.

$i$ and $j$ are respectively the index of internal transition $x_i$ and input transition $u_j$. 


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Properties

- Approach 1. determination of the minimal time difference between the same vertices.
- Approach 2. determination of the greatest paths in graph theory. Integer vector $y$ can only choose a unique path from transition $x_s$ to transition $x_e$ and its coefficients are zero or one.

$$y_{opt} \geq 0$$

- Solution $y_{opt}$ of the linear programming (P) is integer.
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We generalize the technique of Roy (see book of Gondran and Minoux) to the modeling of Event Graphs

Connections of our model using daters $Ax \leq b$, with incidence matrices of equation of marking

Two dual approaches which solve the problem of standardisation in linear programming.

In (max, +) algebra, an equivalent technique exists (Kleene star–problem P)

A perspective is the application to the model checking (Calculation of the polyhedrons in state classes)