Cycle Time of a P-time Event Graph with Affine-Interdependent Residence Durations

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Abstract

In this paper, we widen the class of P-time Event graphs by introducing affine-interdependent residence durations. This new class is studied through a general algebraic model. Considering a periodic behavior, we provide conditions of existence of a trajectory and propose a technique allowing the determination of extremal solutions. We show that the cycle time is intrinsic to this new model: it depends on the circuits of an associated graph but also on more complex structures.

Index Terms

P-time Petri nets, cycle time, linear programming

I. Introduction

In discrete event systems, Petri nets allow the modeling of transportation networks, multi-processor systems, and manufacturing systems. An interesting model is the P-time Event Graph [12], whose evolution can undergo token deaths which express the loss of resources or parts and

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failures to meet time specifications. Contrary to Time Petri nets where a temporal interval of firing is associated with each transition enabled by the marking, P-time Event Graphs depend on the time evolution of the tokens which leads to the firing of the transitions: a temporal interval of availability is associated with each token which enters a place in P-Time Petri nets. Applications of P-time Event Graphs can be found in production systems [12], the food industry [6] and transportation systems [9].

In this paper, we propose a generalization of P-time Event Graphs by introducing links among residence durations. Indeed, in some practical examples, some tasks must compensate for the undesirable effects of other operations such as the warming of a part or an incomplete achievement. This new Petri net needs an algebraic model which describes the trajectories of the dates of transition firings that will allow its treatment. In this paper, we will present the model and show that the relevant matrices generalize the incidence matrices obtained in previous works [5] and [6]. To the best of our knowledge, the considered Petri net is original: a first description can be found in [8].

As we cannot say a priori that this new model can follow a periodic behavior, the second objective is the analysis of the cycle time (production rate) of the general algebraic model (1). Classically, the cycle time is based on the consideration of the circuits and is determined by the calculation of the maximum of the ratios defined by the sum of temporizations to the sum of the number of the initial tokens, for each elementary circuit [16]. However, this technique, which can be applied to Timed Event Graphs, cannot be applied to our new model since the considered matrices are not incidence matrices and so cannot be associated with a simple graph. The well-known theorem [16] based on the elementary circuits of the associated graph and Karp's algorithm [11] in the digraphs, cannot be applied. In fact, the cycle time depends on more complex structures. As graph theory in its classical form cannot be applied, we propose to use linear programming with two objectives: 1) A first goal is the analysis of the consistency of the problem 2) A second goal is the determination of the extremal cycle times.

Considering Timed Event Graphs and using the concept of circuits, different papers [13] [2] [4] [10] [14] also apply linear programming in performance evaluation. To the best of our knowledge, these studies do not use the theorems and lemmas of Farkas which allow an analysis of solution existence of linear systems.

The paper is organized as follows: Section III generalizes the P-time Event Graph described in [5] [6] and gives its algebraic model. A first version of this model can be found in Chapter 3 in [8]. Using a variant of Farkas' Lemma, Section IV gives conditions of the existence of an arbitrary 1-periodic behavior. Assuming that these conditions are fulfilled, we finally propose a technique of linear programming allowing the determination of the bounds on the cycle time (Section V). Two pedagogical examples illustrate the different concepts. The reader can find in the Appendix the technique allowing the reduction of the horizon associated with the model.

II. Preliminary

The notation $|E|$ stands for the cardinality of the set E and the notation $A_{i,.}$ corresponds to the row i of matrix A.

A Petri net is a pair (GR, M_0), where $GR = (V, AR)$ is a bipartite graph defined as follows: the set V is a finite number of nodes which are partitioned into the disjoint sets of places P and transitions TR; the set AR consists of pairs of the form (p_i,x_j) or (x_j,p_i) with $p_i \in P$ and $x_j \in TR$. The initial marking M_0 is a vector of dimension $|P|$ whose elements denote the number of initial tokens in their respective places. The set ${}^\bullet p_l$ (respectively, $p_l^\bullet$) is the set of input (respectively, output) transitions of the place $p_l \in P$. The set ${}^\bullet x_i$ (respectively, $x_i^\bullet$) is the set of the input (respectively, output) places of the transition $x_i \in TR$.

For a Petri net with $|P|$ places and $|TR|$ transitions, the incidence matrix $W = [W_{ij}]$ is a $|P| \times |TR|$ matrix of integers and its entry is given by $W_{ij} = W_{ij}^+ - W_{ij}^-$ where W_{ij}^+ is the weight of the arc from the transition j to the place i, and W_{ij}^- is the weight of the arc from the place i to the transition j [15]. In this paper, we consider that the weight of each arc is is equal to 1 which implies that $W_{ij} \in \{-1, 0, 1\}$.

In a Petri net, a firing sequence from a marking M, implies a string of successive markings. The characteristic vector s of a firing sequence S is such that each component of s is a natural number corresponding to the number of firings of the corresponding transition. A marking M reached from initial marking M_0 by the firing of a sequence S can be calculated by the fundamental relation: $M = M_0 + W \times s$.

***Definition** 1:* A Petri net is called an Event Graph if each place has exactly one input transition and exactly one output transition.

P-time Petri nets allow the modeling of discrete event systems with time constraints for tokens to remain in place. We associate a temporal interval defined in $\mathbb{R}^+ \times (\mathbb{R}^+ \cup \{+\infty\})$ with each place: each place $p_l \in P$ is associated with an interval $[T_l^-, T_l^+]$, where T_l^- is the lower bound and T_l^+ is the upper bound. Its initial marking is denoted $(M_0)_l$.

***Definition** 2:* A P-time Event Graph is a triple (GR, M_0, f) where GR is an Event Graph, M_0 is the initial marking and the mapping f is defined by $p_l \mapsto [T_l^-, T_l^+]$ with $0 \leq T_l^- \leq T_l^+$ from P to $\mathbb{R}^+ \times (\mathbb{R}^+ \cup \{+\infty\})$.

The interval $[T_l^-, T_l^+]$ is the time interval of a token in place p_l. The token must stay in this place during the minimum residence duration T_l^-. Before this duration, the token is in a state of unavailability for firing the output transition. The value $T_l^+$is a maximum residence duration after which the token must leave place p_l (and can contribute to the enabling of the output transition). If not, the system finds itself in a token-dead state. The token is therefore available to fire the output transition in the time interval $[T_l^-, T_l^+]$.

III. Algebraic model

We consider the "dater" representation well-known in the (max, +) algebra [1]: each variable $x_i(k)$ over $\mathbb{R}$ represents the date of the k^{th} firing of transition $x_i \in TR$. Let $m \in \mathbb{N}$ be the maximum number of initial tokens: $m = max\{(M_0)_l \mid l \in [1, |P|]\}$. In this paper, we consider the following algebraic model defined over $\mathbb{R}$

$$\begin{pmatrix} G^- \\ G^+ \end{pmatrix} \times \begin{pmatrix} x(k-m) \\ x(k-m+1) \\ \vdots \\ x(k-1) \\ x(k) \end{pmatrix} \leq \begin{pmatrix} -T^- \\ T^+ \end{pmatrix}, \tag{1}$$

where: the dimension of $x(k)$ is equal to $|TR|$; the dimension of vectors T^-, T^+ is equal to $|P|$; the dimension of G^- and G^+ is equal to $|P| \ \times (m+1).|TR|$. In general, vectors T^-, T^+ describe durations as temporizations of place and matrices $G^-$and G^+ express time connections inside the time Petri net. We can easily add the constraints $x(k-i) \geq x(k-i-1)$ for $i \in [0, m-1]$ in the model so that the trajectory is non-decreasing. We assume the consistency of the model, that is, the existence of a state trajectory over $\mathbb{R}$ on an infinite horizon.

We will show below that a P-time Event Graph, but also a P-time Event Graph with affine-interdependent residence durations, can be expressed with the previous form (1). In Sections IV and V, the algebraic model (1) is used directly.

A. Matrix expression of a P-time Event Graph

Let us express the firing interval for each transition of P-time Event Graphs, guaranteeing the absence of token deaths. If we assume a FIFO functioning of the places which guarantees that the tokens do not overtake one another, a correct numbering of the events can be carried out. Since an Event Graph is composed of places where each place p_l links one input transition $\{x_j\} = {}^{\bullet}p_l$ and one output transition $\{x_i\} = p_l^{\bullet}$, we can write the following system for each place p_l using the lower bound T_l^- and the upper bound T_l^+:

$$\begin{cases} x_j(k-(M_0)_l) - x_i(k) \leq -T_l^- \\ -x_j(k-(M_0)_l) + x_i(k) \leq T_l^+ \end{cases} . \tag{2}$$

- The unitary weight of the arc going from x_j to place p_l leads to the coefficient 1 of $x_j(k-(M_0)_l)$ in the first inequality and the coefficient -1 of $x_j(k-(M_0)_l)$ in the second inequality.
- The unitary weight of the arc going from place p_l to transition x_i yields the coefficient -1 of $x_i(k)$ in the first inequality and the coefficient 1 of $x_i(k)$ in the second inequality.

Example 1.

Let us consider the P-time Event Graph of Fig. 1. The inequalities relevant to place p_1 are $x_1(k-2)+3 \leq x_2(k) \leq x_1(k-2)+5$ which are rewritten as $x_1(k-2) - x_2(k) \leq -3$ and $-x_1(k-2) + x_2(k) \leq +5$. We have $l = 1$ (place p_1), $j = 1$ (transition x_1), $i = 2$ (transition x_2), $(M_0)_{l=1} = 2$, $T_{l=1}^- = 3$ and $T_{l=1}^+ = 5$. ■

The set of the previous inequalities which describes a P-time Event Graph, can be expressed with the previous symmetrical form (1): column-vectors $-T^-$ and T^+ are vectors of temporiza-

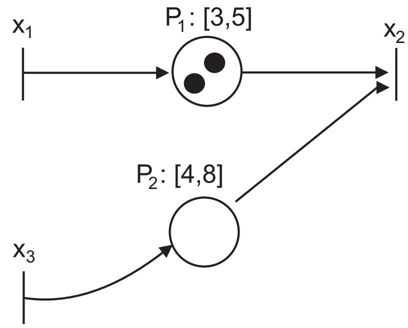

Fig. 1. Example 1: An elementary P-time Event graph.

tion where $[T_l^-, T_l^+]$ is the time interval of place p_l. Naturally, we have $0 \leq T^- \leq T^+$.

Each place corresponds to a row of G^- which contains the weights of its incoming and outgoing arcs which are usually expressed by W^+ and W^-. Remember that the weight of each arc is equal to 1. Let $G^- = [\ G^-_m \quad G^-_{m-1} \quad G^-_{m-2} \quad \cdots \quad G^-_1 \quad G^-_0\ ]$ and $G^+ = [\ G^+_m \quad G^+_{m-1} \quad G^+_{m-2} \quad \cdots \quad G^+_1 \quad G^+_0\ ]$ where the dimension of G^-_i and G^+_i is equal to $|P| \ \times |TR|$ for $i \in \{0, \ldots, m\}$. These matrices are built as follows.

- The entries of the matrices are initially null.
- The weight of an arc going from transition x_j to place p_l with an initial marking $(M_0)_l = r \in [0, m]$ is added to the entry (l, j) of matrix G^-_r.
- Finally, the weight of the arc going from place p_l to transition x_i for any initial marking $(M_0)_l \in [0, m]$ is subtracted from the entry (l, j) of the matrix G^-_0 .

So, $G^-_r \geq 0$ for $r \in [1, m]$ and a coefficient of the matrix G^-_0 can be null, negative or positive: $G^-_0 \geq -W^-$. The interpretation of matrix G^+ is similar to G^- but with a change of sign of coefficients $G^+ = -G^-$.

Example 1 continued. We have $m = 2$, $|P| = 2$, $|TR| = 3$ and system (1) yields $\begin{pmatrix} +1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 \end{pmatrix} \times \begin{pmatrix} x(k-2) \\ x(k-1) \\ x(k) \end{pmatrix} \leq \begin{pmatrix} -3 \\ -4 \\ +5 \\ +8 \end{pmatrix}$. We have $G_2^- = \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $G_1^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $G_0^- = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$, $T^- = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $T^+ = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$. We can also verify that $G_2^- = -G_2^+$, $G_1^- = -G_1^+$ and $G_0^- = -G_0^+$. ■

From the above description on the weight of the arcs, we can deduce the following relation expressing the incidence matrix W:

$$W_{l,.} = \sum_{r=0}^{m} (G_r^-)_{l,.} = - \sum_{r=0}^{m} (G_r^+)_{l,.} \tag{3}$$

for each place l of the P-time event graph.

Remark 1: Let us consider that the initial marking of each place p_l is unitary, i.e., $(M_0)_l = 1$. We have $m = 1$, $G_1^- = W^+$ and $G_0^- = -W^-$.

Remark 2: Let us consider that the initial marking is null, i.e., $M_0 = 0$. We have $m = 0$, $G_0^- = W$ and there is no G_i^- with $i \geq 1$.

B. Matrix expression of a P-time Event Graphs with affine-interdependent residence durations.

We can also consider the case of residence duration of a token in place p_l which determines the temporization of another place. For instance, a product which has been cooked cannot be put immediately in a package and needs to cool down. In the food industry, specific cooling systems are used. We can suppose the temporization of the cooling of a part is an affine relation depending on its cooking time. These residence durations are called *affine-interdependent* and are now described.

Using lower bound T_l^- we can write the following inequality for a place p_l where $(x_j, x_i) = ({}^\bullet p_l, p_l^\bullet)$: $T_l^- + x_j(k - (M_0)_l) \leq x_i(k)$. We assume that place p_l has no finite upper bound: $T_l^+ = +\infty$.

Let us consider another place p_h where $(x_{j'}, x_{i'}) = ({}^\bullet p_h, p_h^\bullet)$: the residence duration of a token is $x_{i'}(k) - x_{j'}(k - (M_0)_h) \geq 0$.

Now, we say that place p_l is *affinely-dependant* on place p_h with an affine function if we can write

$$T_l^- = \alpha.(x_{i'}(k) - x_{j'}(k - (M_0)_h)) + \beta \ , \tag{4}$$

where α and $\beta \in \mathbb{R}$. Note that β is coherent with a time and we have $\alpha > 0$ in a cooling system.

Therefore, the obtained inequality for place p_l is

$$\alpha.(x_{i'}(k) - x_{j'}(k - (M_0)_h)) + x_j(k - (M_0)_l) - x_i(k) \leq -\beta \ , \tag{5}$$

which is a relation among four variables.

If we only consider the term $+x_j(k - (M_0)_l) - x_i(k)$ in (5), the entries of the row $(G^-)_{l,.}$ of a place p_l, which is affinely-dependant on a place p_h, are defined in the same way as in the case of a place of a P-time Event Graph with respect to place p_l (see previous section). Moreover, the affine-dependence between places p_l and p_h leads to the following additions and subtractions to the row $(G^-)_{l,.}$:

- The arc going from transition $x_{j'}$ to place p_h with an initial marking $(M_0)_h = r \in [0, m]$ leads to the substraction of the α value from the entry (l, j') of the matrix G_r^-.

- The arc going from place p_h to transition $x_{i'}$ for any initial marking $(M_0)_h \in [0, m]$ leads to the addition of the α value to the entry (l, i') of the matrix G_0^-.

The relevant entry in $-T_l^-$ is $-\beta$ and we have $(G^+)_{l,.} = 0$ and $T_l^+ = +\infty$.

A slight simplification of relation (5) is possible in the following case: when the output transition $x_{i'}$ of the place p_h is the input transition x_j of the place p_l, we have $x_{i'} = x_j$. With the condition $(M_0)_h = (M_0)_l = 0$, relation (5) becomes $(1+\alpha).x_{i'}(k) - \alpha.x_{j'}(k) - x_i(k) \leq -\beta$ which is a relation among three variables.

Therefore, the inequalities (5) can contain three or four variables contrary to the bi-variable inequalities (2) of the P-time Event Graphs. Moreover, these new relations are more complex as the coefficients depend not only on the entries of the incidence matrix but also on the coefficients of the affine function.

Example 2

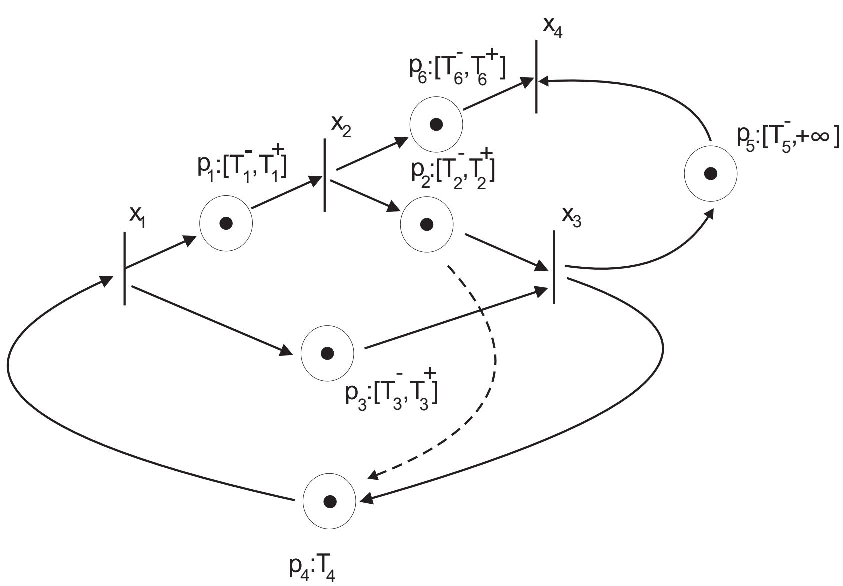

Fig. 2. Example 2: P-time Event graph with interdependent residence durations (dotted line).

The state is $x(k)=\begin{pmatrix} x_1(k) & x_2(k) & x_3(k) & x_4(k) \end{pmatrix}^t$. The temporal intervals are: $[T_1^-,T_1^+]=[3,10]$, $[T_2^-,T_2^+]=[3,20]$, $[T_3^-,T_3^+]=[1,2]$, $[T_5^-,+\infty]=[11.5,+\infty]$ and $[T_6^-,T_6^+]=[1,5]$. Place p_4 describes the cooling down of the product which has been cooked in an oven (place p_2): $T_4=\alpha.(x_3(k)-x_2(k-1))+\beta$ with $\alpha=5$ and $\beta=3$. The matrices of the algebraic model are as follows:

$$T^-=\begin{pmatrix} 3 & 3 & 1 & \beta & 11.5 & 1 \end{pmatrix}^t,\ T^+=\begin{pmatrix} 10 & 20 & 2 & +\infty & +\infty & 5 \end{pmatrix}^t,$$

$$G_1^-=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\alpha & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ G_0^-=\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$G_1^+=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \text{ and } G_0^+=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.\ \blacksquare$$

IV. Analysis of a 1-periodic behavior

The previous part shows that the algebraic model (1) can describe the class of the P-time Event Graphs with affine-interdependent residence durations. The objective is now the analysis of the algebraic model (1) following a 1-periodic trajectory defined by $x(k+1)=\lambda\times u+x(k)$ for $k\geq 1$ where λ is the cycle time and $u=(1,1,...,1)^t$. Vector $x(1)$ expresses the first firing date of the transitions.

In Section IV-A, we rewrite the problem under a simple form and we analyze the existence of the cycle time in Section IV-B.

A. A simple form

Without loss of generality, we suppose in the rest of the paper that $m = 1$ (see Appendix). System (1) for a 1-periodic trajectory can be rewritten as follows:

$$\begin{pmatrix} G_1^- & G_0^- \\ G_1^+ & G_0^+ \end{pmatrix} \times \begin{pmatrix} x(k) \\ \lambda \times u + x(k) \end{pmatrix} \leq \begin{pmatrix} -T^- \\ T^+ \end{pmatrix} ,$$

or

$$\begin{pmatrix} (G_1^- + G_0^-) \\ (G_1^+ + G_0^+) \end{pmatrix} \times x(k) + \begin{pmatrix} G_0^- \\ G_0^+ \end{pmatrix} \times \lambda \times u \leq \begin{pmatrix} -T^- \\ T^+ \end{pmatrix} . \tag{6}$$

If we simplify the writing with $x(k) = x$, we obtain the following system which presents the general form $A \times x \leq b$,

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \times \begin{pmatrix} x \\ \lambda \end{pmatrix} \leq \begin{pmatrix} -T^- \\ T^+ \end{pmatrix} , \tag{7}$$

where $A_{11} = G_1^- + G_0^-$, $A_{12} = G_0^- \times u$, $A_{21} = G_1^+ + G_0^+$ and $A_{22} = G_0^+ \times u$.

Example 2 continued

$$A_{11} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & -\alpha & 1+\alpha & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}, A_{12} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \alpha - 1 \\ -1 \\ -1 \end{pmatrix}, A_{21} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \text{ and}$$

$$A_{22} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \blacksquare$$

Let us consider the important particular case of a P-time event graph. The following property makes the connection with the incidence matrices.

***Property** 1:* For a P-time Event Graph, we have $A_{11} = W = -A_{21}$, $A_{12} = -M_0 = -A_{22}$ and system (7) becomes

$$\begin{pmatrix} W & -M_0 \\ -W & M_0 \end{pmatrix} \times \begin{pmatrix} x \\ \lambda \end{pmatrix} \leq \begin{pmatrix} -T^- \\ T^+ \end{pmatrix}. \tag{8}$$

Proof.

System (7) can be rewritten because we deduce from (3) that $G_1^- + G_0^- = W$ and $G_1^+ + G_0^+ = -W$. So, we can deduce that $A_{11} = W$, $A_{21} = -W$.

Let us note that $G_0^- = -W^-$ and $G_1^- = W^+$ when the P-time Event Graph initially has one token per place. Thus, $G_0^- \times u = -W^- \times u = -M_0$ which represents the initial marking. This result is also true when $m = 1$, that is, each place initially has one token at most. Indeed, each place without a token is represented by a line of G_0^- which contains the weights 1 and -1 corresponding to its incoming and outgoing arcs. The similar reasoning holds for the lower part of the system and hence we obtain $G_0^+ \times u = W^- \times u = M_0$. ■

B. Existence

After obtaining a linear inequalities system (7) having the form $A \times x \leq b$, we study the existence of a 1-periodic behavior by applying a known result of linear programming. If this trajectory exists, we can conclude that the model is consistent on an infinite horizon, in other words, the model has a trajectory on an infinite horizon.

As vector b in $A \times x \leq b$ is finite, we assume that each infinite bound $T_l^+ = +\infty$ is replaced by a finite but arbitrarily large number which neutralizes the constraint $x_i(k) - x_j(k - m_l) \leq T_l^+ = +\infty$. The following variables and matrices are over $\mathbb{R}$ and y is a row-vector of non-negative real numbers.

***Lemma** 1:* Farkas' Lemma (variant), Corollary 7.1.e in [18]. Let A be a matrix and let b be a vector. Then the system $A \times x \leq b$ of linear inequalities has a solution x, if and only if, $y \times b \geq 0$ for each row vector $y \geq 0$ with $y \times A = 0$. ■

We consider below the set Y of row vectors $y \geq 0$ such that

$$y \times \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = 0. \tag{9}$$

The following partition of the set $Y = Y^- \cup Y^= \cup Y^+$ is used in Theorem 1.

$$Y^- = \{y \in Y \mid y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} < 0\}$$
$$Y^= = \{y \in Y \mid y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = 0\} \tag{10}$$
$$Y^+ = \{y \in Y \mid y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} > 0\}$$

***Theorem** 1:* The system (1) with $m = 1$ can follow a 1-periodic behavior for some given cycle time λ, if and only if the two following conditions are satisfied:

1)
$$(\forall y \in Y^=)\ y \times \begin{pmatrix} -T^- \\ T^+ \end{pmatrix} \geq 0 \tag{11}$$

2)
$$(\forall y_i \in Y^-)\ (\forall y_j \in Y^+) \frac{y_i \times \begin{pmatrix} -T^- \\ T^+ \end{pmatrix}}{y_i \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}} \leq \frac{y_j \times \begin{pmatrix} -T^- \\ T^+ \end{pmatrix}}{y_j \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}}. \tag{12}$$

Proof.

System (7) can be rewritten as

$$\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} \times x \leq \begin{pmatrix} -T^- \\ T^+ \end{pmatrix} - \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} \times \lambda. \tag{13}$$

From Farkas' Lemma, we can deduce that this system has a solution x, if and only if, $y \times (\begin{pmatrix} -T^- \\ T^+ \end{pmatrix} - \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} \times \lambda) \geq 0$ for each row vector $y \in Y$. So, we deduce that $y \times$

$\begin{pmatrix} -T^- \\ T^+ \end{pmatrix} \geq y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} \times \lambda = \lambda \times y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}$ and different cases arise from the analysis of $y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}$ which can be null, negative or positive. Using the partition $Y = Y^- \cup Y^= \cup Y^+$ defined above, we have the three following cases:

1) $y \times \begin{pmatrix} -T^- \\ T^+ \end{pmatrix} \geq 0$ if $y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = 0$, that is $y \in Y^=$;

2) $\dfrac{y \times \begin{pmatrix} -T^- \\ T^+ \end{pmatrix}}{y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}} \leq \lambda$ if $y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} < 0$, that is $y \in Y^-$;

3) $\dfrac{y \times \begin{pmatrix} -T^- \\ T^+ \end{pmatrix}}{y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}} \geq \lambda$ if $y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} > 0$, that is $y \in Y^+$.

Case 1) is directly the first condition of existence. Cases 2) and 3) define a set of lower and upper bounds of λ which must satisfy the second condition since λ is defined by these bounds.

■

Therefore, the theorem provides a way of checking the consistency of (7) and the existence of a 1-periodic trajectory for the extended P-time Event Graphs. Moreover, the second condition of existence can give an interval of the cycle time given by the maximum of the lower bounds and the minimum of the upper bounds. Note that the first condition of existence (11) does not define a bound of λ.

Remark 3: Row vector y is not a P-invariant (or P-semiflow) [15] even if the concepts are close. Indeed, the coherence of equality (9) where the number of rows of $\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}$ is $2 \times |P|$ implies that the dimension of the row vector y is also $2 \times |P|$. Moreover, y is not a vector of integers but is a vector of real numbers. Indeed, it depends on the incidence matrix expressed in A_{11} and A_{21} but also on the affine functions of the interdependent residence durations. The determination of the set of vectors $y \geq 0$ with $y \in \mathbb{R}^{2\times|P|}$ can be deduced from the techniques used for calculation of the P-invariants ([3], [19], techniques applying Fourier-Motzkin, etc.) if the restriction on the integers is relaxed: integer linear programming is not necessary in this paper.

We now introduce a generalized associated graph which provides a graphical interpretation.

Associated graph of generalized P-time Event Graph.

In the case of a P-time Event Graph, equation (9) becomes $y \times \begin{pmatrix} W \\ -W \end{pmatrix} = 0$ and an associated graph can be associated with matrix $\begin{pmatrix} W \\ -W \end{pmatrix}$. As W is the incidence matrix of an event graph where each row contains the two entries $W_{lj} = 1$ and $W_{li} = -1$, each row l of this matrix can be associated with an arc coming from vertex j to vertex i. The lower matrix $-W$ leads to the same arcs but in the opposite direction. It is well known that each vector y defines a circuit (or a set of circuits) which determines the minimal and maximal bounds of the cycle time [12] [5] [6].

However, this graphical interpretation does not hold for a P-time Event Graph with affine-interdependent residence durations: we have $A_{11} \neq W$ and $A_{21} \neq -W$, and a row cannot be associated with an arc as it can contain more than two entries. Moreover, the entries of vector y cannot be normalized to 1 in general, which implies that the selection of the rows by the non-null entries of vector y, and the relevant selected subgraph, presents a weighting in $\mathbb{R}$. In fact, each vector y expresses a general dependance among the rows of the relevant submatrix.

We can define a more general associated graph expressing the connections between the rows and the columns of matrix $A_{.,1} = \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}$ as follows: each row $l \in 2 \times |P|$ is associated with a specific element (a black dot in Fig. 3) linked to the different vertices corresponding to the $|TR|$ columns (a vertical line like a transition). Each positive entry $(A_{.,1})_{l,i}$ is associated with an incoming arc from the vertical line x_i to the black dot l and each negative entry is associated with an outgoing arc from the black dot l to the vertical line x_i. Following the relation (7), we can associate the pair $(-T_l^-, (A_{12})_l) \in (\mathbb{R} \times \mathbb{R})$ with the black dot relevant to the row l of the upper matrix A_{11} and the pair $(T_l^+, (A_{22})_l) \in (\mathbb{R} \times \mathbb{R})$ with the black dot relevant to the row l of the lower matrix A_{21}. Showing that only a subset of vectors $y \in Y$ corresponds to a subset of circuits in the associated graph of generalized P-time Event Graph (see Fig. 3), the following example highlights original structures.

Example 2 continued

More general than the matrices obtained in the studies [5] [6], the matrices G_1^- and G_0^- are almost (but are not) the incidence matrices W^+ and W^- of an event graph, since the fourth row contains two non-null entries (see III-B). We also have $A_{11} \neq W$ and $A_{21} \neq -W$ (see IV-A). The analysis of the independent row-vectors y

$$\begin{pmatrix} 1 & 6 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2.5 & 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0.33 & 0 & 0 & 1.66 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(remember that $\alpha = 5$) highlights the following relevant structures (see Fig. 4)

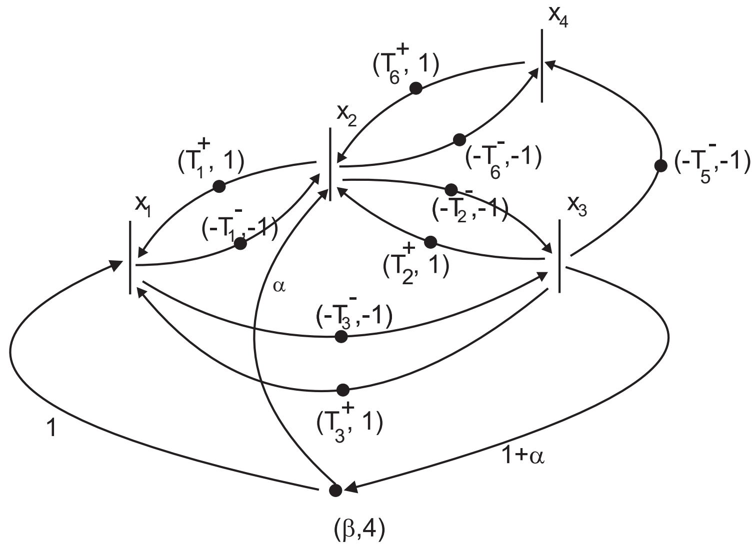

Fig. 3. Graph associated with Example 2 given in Fig. 2: Each row corresponds to a black dot and each column is expressed by a vertical line. The structure of the Event Graph can easily be recognized, but the dot $(\beta, 4)$ is connected to three vertices x_1, x_2 and x_3.

y	structure	bound
$y_1 \in Y^-$	$-T_1^-, -T_2^-$ and β	lower bound: 8
$y_2 \in Y^-$	$-T_2^-, -T_5^-$ and T_6^+	lower bound: 9.5
$y_3 \in Y^-$	$-T_2^-, -T_3^-$ and β	lower bound: 9.5
$y_4 \in Y^+$	$-T_3^-, \beta$ and T_1^+	upper bound: 13.66

if we neglect this weighting of row-vectors y. The substructure selected by y_2 corresponds to a circuit ($-T_2^-$,$-T_5^-$ and T_6^+) in the associated graph and the proposed technique calculates the ratio $\frac{-T_2^- - T_5^- + T_6^+}{-1-1+1}$ which can also be given by the classical theorem [16] generalized to P-time Event Graphs [12]. Contrary to this substructure relevant to y_2, the substructures selected by y_1, y_3 and y_4 present non-disjointed circuits (see Fig. 4). As a direct analysis of the submatrices of A_{11} and A_{21} selected by y_1, y_3 and y_4 can show that they cannot be decomposed in elementary circuits corresponding to vectors $y \in Y$, the procedure has select, original basic substructures.

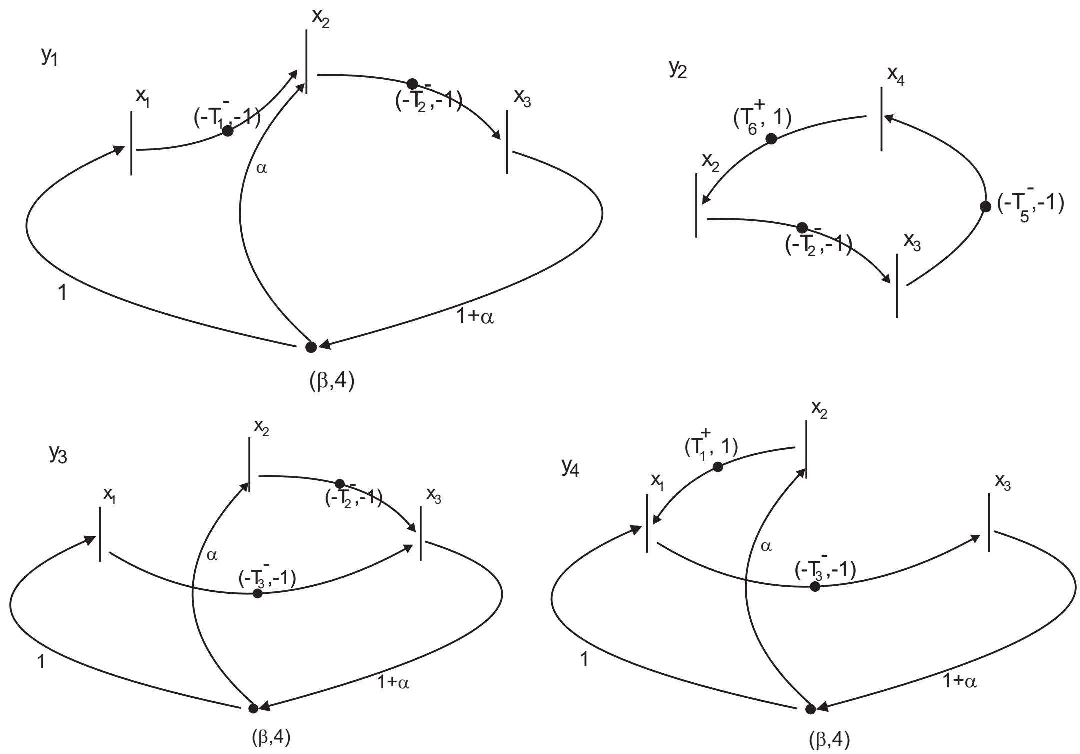

Fig. 4. Example 2: Substructures selected in the associated graph given in Fig. 3

We obtain $\max(8,\, 9.5,\, 9.5) \leq \lambda \leq \min(13.66)$ and the interval of the possible values is $[9.5,$ $13.66]$. Note also that vectors y_2 and y_4, which define a lower and an upper bound respectively of the cycle time, select a combination of lower and upper bounds of temporizations and not only a set of lower (respectively, upper) bounds of temporizations. ■

Now we consider a slight modification of Example 2 where the P-time Event Graphs with affine-interdependent residence durations cannot follow a 1-periodic trajectory.

Example 2 modified

Now let us assume that the initial marking of place p_5 is null. So, $(M_0)_5 = 0$ and the matrices G_i are identical except $(G_1^-)_{5,.} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$ $(G_0^-)_{5,.} = \begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix}$. The matrix $A_{11} = G_1^- + G_0^-$ is not modified but $(A_{12})_5 = (G_0^-)_5.u = 0$.

Therefore, the determination of the row-vectors y satisfying (9) gives the same results. As above, we have y_1, $y_3 \in Y^-$ and $y_4 \in Y^+$ since these vectors do not select the row $(A_{12})_5$. The condition (12) is satisfied: $\max(8,\, 9.5) \leq \lambda \leq \min(13.66)$

Let us consider $y_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. Now, we have $y_2 \in Y^=$ since $y_2 \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = 0$ but the condition (11) is not satisfied as $y_2 \times \begin{pmatrix} -T^- \\ T^+ \end{pmatrix} = -3 - 11.5 + 5 = -9.5 \ngeq 0$. Therefore, the system (1) cannot follow a 1-periodic trajectory. In fact, the P-time Event Graphs with affine-interdependent residence durations is not temporally live [7]: the sub-structure highlighted by y_2 leads to the following incoherent system $\begin{cases} x_2(k-1) + T_2^- \leq x_3(k) \\ x_3(k) + T_5^- \leq x_4(k) \\ x_4(k) \leq T_6^+ + x_2(k-1) \end{cases}$ since the condition $T_2^- + T_5^- \leq T_6^+$ is not satisfied. ■

V. Computation of extremal cycle times

The objective of the previous Part IV-B is the analysis of the existence of a 1-periodic trajectory. It also allows for the calculation of the bounds of the cycle time but it requires the computation of all y vectors. For this second objective, we propose a more efficient technique allowing the determination of the minimal/maximal cycle time. Deduced from the previous Theorem 1, the optimization problem is as follows: the lower bound $\lambda_{\min}$ is the maximum of the expressions $\frac{y \times \begin{pmatrix} -T^- \\ T^+ \end{pmatrix}}{y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}}$ $(\forall y_i \in Y^-)$ while the upper bound $\lambda_{\max}$ is based on the minimum of the same expressions $(\forall y_i \in Y^+)$. So,

$$\lambda_{\min} = \max_{y \in \mathbb{R}^{2\times|P|}} \frac{y \times \begin{pmatrix} -T^- \\ T^+ \end{pmatrix}}{y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}} \tag{14}$$

such that

$$\begin{cases} y \geq 0 \\ y \times \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = 0 \\ y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} < 0 \end{cases} . \tag{15}$$

Vector y is defined for a positive scalar factor as we can replace y by $\mu \times y$ with $\mu > 0$ without modifying the ratio in (14) and the fulfilment of the three relations in (15). Therefore, we can set $y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = -1$ and the problem can be rewritten under the following simple forms.

***Property** 2:*

$$\lambda_{\min} = \max_{y \in \mathbb{R}^{2\times|P|}} -y \times \begin{pmatrix} -T^- \\ T^+ \end{pmatrix} \text{ such that} \tag{16}$$

$$\begin{cases} y \geq 0 \\ y \times \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = 0 \\ y \times \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = \gamma \end{cases} \tag{17}$$

$$\text{with } \gamma = -1. \tag{18}$$

***Property** 3:* Symmetrically, $\lambda_{\max} = \min_{y \in \mathbb{R}^{2\times|P|}} y \times \begin{pmatrix} -T^- \\ T^+ \end{pmatrix}$ such that $\gamma = +1$ in the above constraints.

A standard linear programming form can easily be written:

***Property** 4:*

$$\lambda_{\min} = \max_{y \in \mathbb{R}^{2\times|P|}} -y \times \begin{pmatrix} -T^- \\ T^+ \end{pmatrix} \text{such that} \tag{19}$$

$$y. \begin{pmatrix} -I_{|P|\times|P|} & 0 & A_{11} & -A_{11} & A_{12} & -A_{12} \\ 0 & -I_{|P|\times|P|} & A_{21} & -A_{21} & A_{22} & -A_{22} \end{pmatrix} \leq \begin{pmatrix} 0_{|P|} & 0_{|P|} & 0_{|TR|} & 0_{|TR|} & \gamma & \gamma \end{pmatrix}$$

with $\gamma = -1$.

***Property** 5:* Symmetrically,

$$\lambda_{\max} = \min_{y \in \mathbb{R}^{2\times|P|}} y \times \begin{pmatrix} -T^- \\ T^+ \end{pmatrix} \text{ such that } \gamma = +1 \text{ in the above constraints.} \tag{20}$$

Example 2 continued

The results are for $\lambda_{\min}$ and $\lambda_{\max}$ respectively:

$y = \begin{pmatrix} 0 & 2.5 & 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $\lambda_{\min} = 9.5$;
$y = \begin{pmatrix} 0 & 0 & 2 & 0.33 & 0 & 0 & 1.66 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $\lambda_{\max} = 13.66$. These vectors respectively correspond to y_3 and y_4, calculated in Section IV-B. Note that the relevant structures are not simple circuits. The interval of the possible values for the cycle time is identical to the interval obtained in Section IV-B ($\lambda \in [9.5, 13.66]$) where all y vectors and bounds are considered. ■

The application of classical algorithms of linear programming can efficiently give the optimal solutions. Let us recall that, although some artificial examples show exponential running times, the simplex is efficient in practice as it has polynomial-time *average-case* complexity in some general cases [18]. The modern algorithms of linear programming are polynomial (the complexities of the ellipsoid algorithm of Khashiyan and the interior point algorithm of Karmarkar are respectively $O(n^4 \times L)$ and $O(n^{3.5} \times L)$ where n is the number of variables and L is the number of bits necessary in the storage of the data [18] [17]).

VI. Conclusion

In this paper, we consider a general algebraic model (1) which can describe the class of the P-time Event Graphs with affine-interdependent residence durations. Such a class generalizes the class of the P-time Event graphs [12]. Note that the results of this paper consider the general algebraic model (1) and not the more specific algebraic model of the generalized P-time Event Graphs. A perspective is the determination of the different classes of Time Petri nets which can be described by the algebraic model (1).

Unlike the matrices of a P-time Graph where there is some correspondence between matrices G_0^-, G_1^-, G_0^+ and G_1^+ and incidence matrices of the Petri net, the matrices G_0^- and G_1^- in example 2 have some entries which depend on time parameters and a row can contain more than one entry. As a consequence, the fourth row of matrix $\begin{pmatrix} A_{11} & A_{12} \end{pmatrix}$ in Example 2 of Section IV presents four entries. Therefore, the system cannot be completely analyzed with a classical graph theory as in the case of a Timed Event Graph or a P-time Event Graph: a natural approach is the use of linear programming.

The application of Farkas' Lemma gives conditions of consistency of the algebraic model (1) for a 1-periodic behavior. It also gives an interval limiting the cycle time depending only on the matrices of the system. Showing that the concept of cycle time is intrinsic to the new model, this result widens the class of models where the cycle time can be defined. We show in Example 2 that the cycle time can be associated with specific structures (non-disjointed circuits) which are more general than the classical circuits in the associated graph. We finally provide a simple technique which allows the bounds on the cycle time to be determined in polynomial time.

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VII. Appendix

Let us express system inequalities (1) on a reduced horizon. Such a form will simplify the calculations. The objective is to establish an equivalent model such that each place of the graph initially contains only zero or one token.

Roughly speaking, the general idea is to split each place containing i tokens into i places, where each place contains only one token (a place can initially contain a maximum number of m tokens). A systematic procedure is as follows.

Let us introduce new variable X, that is,

$X(k) = \begin{pmatrix} X_0^t(k) & X_1^t(k) & \dots & X_i^t(k) & \dots & X_{m-1}^t(k) \end{pmatrix}^t$ with $X_i(k) = x(k-m+i+1)$. By construction, we have $X_{m-1}(k) = x(k)$ and $X_i(k) = X_{i+1}(k-1)$ for i going from 0 to $m-2$.

So, system (1) becomes

$$\begin{pmatrix} G_1'^- & G_0'^- \\ G_1'^+ & G_0'^+ \end{pmatrix} \times \begin{pmatrix} X(k-1) \\ X(k) \end{pmatrix} \leq \begin{pmatrix} -T^- \\ T^+ \end{pmatrix},$$

where $G_1'^- = \begin{pmatrix} G_m^- & 0 & \dots & \dots & 0 \end{pmatrix}$, $G_0'^- = \begin{pmatrix} G_{m-1}^- & G_{m-2}^- & \dots & G_1^- & G_0^- \end{pmatrix}$, $G_1'^+ = \begin{pmatrix} G_m^+ & 0 & \dots & \dots & 0 \end{pmatrix}$ and $G_0'^+ = \begin{pmatrix} G_{m-1}^+ & G_{m-2}^+ & \dots & G_1^+ & G_0^+ \end{pmatrix}$.

This system is completed by $X_i(k) = X_{i+1}(k-1)$ for i going from 0 to $m-2$ which is equivalent to the following inequalities

$$\begin{cases} X_{i+1}(k-1) - X_i(k) \leq 0 \\ -X_{i+1}(k-1) + X_i(k) \leq 0 \end{cases}$$

for $i = 0$ to $m-2$. The relevant matrix form is as follows:

$$\begin{pmatrix} H_{01}^- & H_{00}^- \\ H_{01}^+ & H_{00}^+ \end{pmatrix} \times \begin{pmatrix} X(k-1) \\ X(k) \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the dimension of the matrices $H_{01}^- = -H_{01}^+$ and $H_{00}^- = -H_{00}^+$ is $((m-1).|TR| \times m.|TR|)$. The matrix H_{01}^- is a subdiagonal of identity matrices immediately above the main diagonal and the matrix H_{00}^- is a main diagonal of negative identity matrices.

Finally, the concatenation of the two systems gives the algebraic form

$$\begin{pmatrix} G^- \\ \\ G^+ \end{pmatrix} \times \begin{pmatrix} X(k-1) \\ X(k) \end{pmatrix} \leq \begin{pmatrix} -T^- \\ \\ 0 \\ \\ T^+ \\ \\ 0 \end{pmatrix} ,$$

where $G^- = \begin{pmatrix} G_1'^- & G_0'^- \\ \\ H_{01}^- & H_{00}^- \end{pmatrix}$ and $G^+ = \begin{pmatrix} G_1'^+ & G_0'^+ \\ \\ H_{01}^+ & H_{00}^+ \end{pmatrix}$.