Liveness and acceptable trajectories in P-time Event Graphs

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***Abstract*— This paper presents a modelling and an analysis of P-time Event Graphs in the field of (max, +) algebra. Under the hypothesis of the logical liveness of the event graph, temporal liveness is defined by the existence of a trajectory. Based on a particular serie of matrices, the extremal trajectories starting from an initial interval are deduced. The liveness of the static part and dynamic part are analysed.**

Keywords: P-time Petri Nets, Timed Event Graph, (max,+) algebra, token death, Kleene'star, control synthesis, fixed point.

I. Introduction

In an algebraic point of view, P-time Event Graphs can be modelled by a new class of systems called interval descriptor systems [9] for which the time evolution is not strictly deterministic but belongs to intervals. For interval descriptor systems, lower and upper bounds of the intervals depends on the maximization, minimization and addition operations, simultaneously in the general case. The algebraic model of P-time Event Graphs corresponds to the semantic "And" of Time Stream Event Graph [6] where the lower and upper bound constraints of P-time Event Graphs are respectively (max,+) and (min,+) functions. Also, it includes P-Time**d** Event Graphs (which is different of P-tim**e** Event Graphs).

An important characteristic of P-time Event Graphs is the possible deaths of tokens if a synchronization is not fulfilled. In this case, the initial algebraic model in the topical algebra, cannot be used. Some authors apply performance evaluation to determine the set of constraints guaranteeing the liveness of tokens in the strongly connected case [1]. Analysis of token liveness can be realized through the spectral vector [9] in the general case.

Let us assume that the initial state belongs to an interval. The aim of this paper is the determination of acceptable trajectories satisfying this initial condition. In other words, the problem is the determination if there is an acceptable trajectory starting from a given interval and the calculation of the corresponding extremal trajectories. This problem has been already considered but in the particular case of Timed Event Graphs [12]. It has been shown that the initial condition must verify a condition such as the trajectory is nondecreasing in the counting representation.

The paper is structured as follows: notations and some previous results are first given. We then introduce the modelling of P-time Event Graphs in the (max,+) algebra in the "dater" form. We study its behavior with the help of a special serie of matrices [7] and the extremal trajectories obeying to an initial condition defined on an interval, are deduced. Lastly, a simple example illustrates the approach.

In this paper, no hypothesis is taken on the structure of the Event Graph which can be non-strongly connected. The initial marking must only satisfy the classical liveness condition and the usual hypothesis that places must be First In First Out (FIFO) is taken.

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II. Preliminaries

A monoid is a couple $(S, \oplus)$ where the operation $\oplus$ is associative and presents a neutral element. A semi-ring S is a triplet $(S, \oplus, \otimes)$ where $(S, \oplus)$ and $(S, \otimes)$ are monoids, $\oplus$ is commutative, $\otimes$ is distributive relatively to $\oplus$ and the zero element ε of $\oplus$ is the absorbing element of $\otimes$ ($\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$). A dioid D is an idempotent semi-ring (the operation $\oplus$ is idempotent, that is $a \oplus a = a$). Let us notice that contrary to the structures of group and ring, monoid and semi-ring do not have a property of symmetry on S. The unit $\mathbb{R} \cup \{-\infty\}$ provided with the maximum operation denoted $\oplus$ and the addition denoted $\otimes$ is an example of dioid. We have: $\mathbb{R}_{max}$ = $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$. The neutral elements of $\oplus$ and $\otimes$ are represented by $\varepsilon = -\infty$ and $e = 0$ respectively. The absorbing element of $\otimes$ is ε. Isomorphic to the previous one by the bijection: $x \longmapsto -x$, another dioid is $\mathbb{R} \cup \{+\infty\}$ provided with the minimum operation denoted $\wedge$ and the addition denoted $\odot$. The neutral elements of $\wedge$ and $\odot$ are represented by $T = +\infty$ and $e = 0$ respectively. The absorbing element of $\odot$ is ε. The following convention is taken: $T \otimes \varepsilon = \varepsilon$ and $T \odot \varepsilon = T$. The expression $a \otimes b$ and $a \odot b$ are identical if at least either a or b is a finite scalar. The partial order denoted $\leqslant$ is defined as follows: $x \leqslant y \iff x \oplus y = y \iff x \wedge y = x \iff x_i \leqslant y_i$, for i from 1 to n in $\mathbb{R}^n$. Notation $x < y$ means that $x \leqslant y$ and $x \neq y$. A dioid D is complete if it is closed for infinite sums and the distributivity of the multiplication with respect to addition applies to infinite sums: ($\forall\ c \in D$) ($\forall\ A \subseteq D$) $c \otimes (\bigoplus_{x \in A} x) = \bigoplus_{x \in A} c \otimes x$. For example, $\overline{\mathbb{R}}_{max}$ = $(\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, \oplus, \otimes)$ is complete. The set of $n.n$ matrices with entries in a complete dioid D provided with the two operations $\oplus$ and $\otimes$ is also a complete dioid which is denoted $D^{n.n}$. The elements of the matrices in the (max,+) expressions (respectively (min,+) expressions) are either finite or ε ((respectively T). We can deal with nonsquare matrices if we complete by rows or columns with entries equal to ε (respectively T). The different operations operate as in the usual algebra: The notation $\odot$ refers to the multiplication of two matrices in which the $\wedge-$operation is used instead of $\oplus$. The mapping

f is said residuated if for all $y \in D$, the least upper bound of the subset $\{x \in D \mid f(x) \leq y\}$ exists and lies in this subset. The mapping $x \in (\overline{\mathbb{R}}_{max})^n \mapsto A \otimes x$ defined over $\bar{\mathbb{R}}_{max}$is residuated (see [2]) and the left $\otimes-$residuation of B by A is denoted by: $A \backslash B = \max\{x \in (\overline{\mathbb{R}}_{max})^n$ such that $A \otimes x \leqslant B\}$.

Kleene's star is defined by: $A^* = \bigoplus_{i=0}^{+\infty} A^i$. Denoted as $G(A)$, an induced graph of a square matrix A is deduced from this matrix by associating: a node i with the column i and the line i ; an arc from the node j towards the node i with $A_{ij} \neq \varepsilon$. The weight of a path p, $\mid p \mid_w$ is the sum of the labels on the edges in the path. The length of a path p, $\mid p \mid_l$ is the number of edges in the path. A circuit is a path which starts and ends at the same node.

Theorem 2.1 (Theorem 4.75 part 1 in [2]) Given A and B in a complete dioid D, A^*B is the least solution of the equation $x = A \otimes x \oplus B$, and the inequality $x \geq A \otimes x \oplus B$

Theorem 2.2 (Theorem 4.73 part 1 in [2]) Given A and B in a complete dioid D, $A^* \backslash B$ is the greatest solution of the equation $x = A \backslash x \wedge B$, and the inequality $x \leq A \backslash x \wedge B$

III. Definition and modelling of P-time Event Graphs

The P-time Petri nets makes it possible to model the discrete event dynamic systems with time constraints of stay of the tokens inside the places. Consistent with the dioid $\overline{\mathbb{R}}_{max}$, we associate for each place a temporal interval defined in $\mathbb{R}^+ \times (\mathbb{R}^+ \cup \{+\infty\})$.

Definition 3.1 (p-time Petri nets) A P-time Petri net is a pair $< R, IS >$ where R is a marked Petri nets

$IS : P \longrightarrow \mathbb{R}^+ \times (\mathbb{R}^+ \cup \{+\infty\})$

$p_i \longrightarrow IS_i = [a_i, b_i]$ with $0 \leq a_i \leq b_i$

IS_i is the static interval of residence time or duration of a token in place p_i belonging to the set of places P. The token must stay in the place p_i during the minimum residence duration a_i. Before this duration, the token is in state of unavailability to firing the transition t_j. The value b_i is a maximum residence duration after which the token must thus leave the place p_i. If not, the system is found in a token-dead state. So, the token is available to firing the transition t_j in the interval time $[a_i, b_i]$.

For Event Graphs, we will express the interval of shooting of each transition from the system which will guarantee an functioning without token-dead state. The set ${}^{\cdot}p$ is the set of input transitions of P, $p^{\cdot}$ is the set of output transitions of P. The set ${}^{\cdot}t_i$ (respectively $t_i^{\cdot}$) is the set of the input (respectively output) places of the transition t_i. Let us consider the variable $x_i(k)$ as the date of the kth firing of transition t_i. For each place p_k, we associate an interval $[a_{ij}, b_{ij}]$ with a_{ij} the lower bound and b_{ij} the upper bound with $t_i \in {}^{\cdot} p$ and $t_j \in p^{\cdot}$. As $|{}^{\cdot}p| = |p^{\cdot}| = 1$, the set of upstream (respectively downstream) transitions of t_i is noted ${}^{\leftarrow}t_i = {}^{\cdot}({}^{\cdot}t_i)$ (respectively $t_i^{\rightarrow} = (t_i^{\cdot})^{\cdot}$).

We consider the "dater" type in (max,+) algebra: each variable $x_i(k)$ represents the date of the kth firing of transition x_i. The usual assumption of functioning FIFO of the places is taken: it guarantees the condition of nonovertaking of the tokens between them and the correct numbering of the events. So, the evolution is described by the following inequalities which expresses relations between the dates of firing of transitions:

$(\forall\ t_j \in^{\leftarrow} t_i\ )\ x_i(k) \geq x_j(k - m_{ij}) + a_{ij}$

with a_{ij} the lower bound of an upstream place of t_i and m_{ij} the corresponding number of tokens present initially.

Respectively, $(\forall\ t_j \in^{\leftarrow} t_i)\ x_i(k) \leq x_j(k - m_{ij}) + b_{ij}$

with b_{ij} the upper bound of an upstream place of x_i, which is equivalent to $(\forall\ t_j \in t_i^{\rightarrow})\ x_j(k + m_{ji}) - b_{ji} \leq x_i(k)$

Consequently, the model can be described by the following expression in the (max,+) dioid.

$x_i(k) \geq \bigoplus_{t_j \in^{\leftarrow} t_i} x_j(k - m_{ij}) \otimes a_{ij}$

$x_i(k) \geq \bigoplus_{t_j \in t_i^{\rightarrow}} x_j(k + m_{ji}) \otimes (-b_{ji})$ or

$$x_i(k) \geq \bigoplus_{t_j \in^{\leftarrow} t_i} a_{ij}^- \otimes x_j(k - m_{ij}) \oplus \bigoplus_{t_j \in t_i^{\rightarrow}} a_{ij}^+ \otimes x_j(k + m_{ji}) \tag{1}$$

with $a_{ij}^- = a_{ij}$ and $a_{ij}^+ = -b_{ji}$, $a_{ij}^- \in \mathbb{R}^+$, $a_{ij}^+ \in \mathbb{R}^-$

Let us notice the above set is also, equivalent to the following "interval descriptor system" : $\bigoplus_{t_j \in^{\leftarrow} t_i} a_{ij} \otimes x_j(k - m_j) \leq x_i(k) \leq \bigwedge_{t_j \in^{\leftarrow} t_i} b_{ij} \odot x_j(k - m_j)$ with m_j the number of the present tokens in each place p_j at the instant $t = 0$ (initial marking). The lower bound (respectively upper bound) is a (max, +) function (respectively (min, +) function) and this model is an example of $((max, +), (min, +))$ type of interval descriptor system. This form can be used but need the use of two dioids which complicates its treatment.

Some transitions can be considered as inputs. They are usually associated to transitions i such that ${}^{\leftarrow}t_i = \emptyset$ and describe for instance the input of a part. Similarly, some transitions can be considered as outputs. They are usually associated to transitions i such that $t_i^{\rightarrow} = \emptyset$ and describe for instance the departure of a finished product.

The output places of each input transition denoted u, are without token otherwise a place witout token is added.

For $t_j \in t_i^{\rightarrow}$, $x_j(k) \geq u_i(k) + a_{ij}$

and $x_j(k) \leq u_i(k) + b_{ij}$

or $u_i(k) \geq x_j(k) - b_{ji}$

Similarly, the input places of each output transition denoted y, are without token otherwise a place witout token is added.

For $t_j \in^{\leftarrow} t_i$, $y_i(k) \geq x_j(k) + a_{ji}$

and $y_i(k) \leq x_j(k) + b_{ji}$

or $x_j(k) \geq y_i(k) - b_{ji}$

Naturally, for each input transition t_i, $|{}^{\leftarrow} t_i| = 0$ and $| t_i^{\rightarrow} | = 1$ and for each output transition t_i, $|{}^{\leftarrow} t_i| = 1$ and $| t_i^{\rightarrow} | = 0$. In the (max,+) algebra, an equivalent inequality set is:

$(t_j \in t_i^{\rightarrow})\ x_i(k) \geq b_{ij}^- \otimes u_j(k)$, $u_i(k) \geq b_{ij}^+ \otimes x_j(k)$

$(t_j \in^{\leftarrow} t_i)\ y_i(k) \geq c_{ij}^- \otimes x_j(k)$, $x_i(k) \geq c_{ij}^+ \otimes y_j(k)$

with $b_{ij}^- = a_{ij}$, $b_{ij}^+ = -b_{ji}$, $c_{ij}^- = a_{ij}$, $c_{ij}^+ = -b_{ji}$.

Additional input and output places

In relation to the input and output transitions, the following additions to the initial Event Graph do not modify its behavior and make it possible to alleviate the notations and expressions without reduction of generality. Now, the previous transitions denoted u and y are considered as simple transitions and are denoted x.

To each input transition, an input place and its input transition denoted u are added such that the place is without token and has an interval $[0,0]$.

For $t_j \in t_i^{\rightarrow}$, $x_j(k) \geq u_i(k-m_{ij})+a_{ij}$ with $a_{ij}=0$
and $x_j(k) \leq u_i(k-m_{ij})+b_{ij}$ with $b_{ij}=0$
or $u_i(k) \geq x_j(k+m_{ij})-b_{ji}$

Similarly, to each output transition, an output place and its corresponding output transition denoted y are added such that the place is without token and has an interval $[0,0]$.

For $t_j \in {}^{\leftarrow}t_i$, $y_i(k) \geq x_j(k-m_{ji})+a_{ji}$ with $a_{ji}=0$
and $y_i(k) \leq x_j(k-m_{ji})+b_{ji}$ with $b_{ji}=0$
or $x_j(k) \geq y_i(k+m_{ji})-b_{ji}$

Naturally, for each input transition t_i, $|{}^{\leftarrow}t_i| = 0$ and $|t_i^{\rightarrow}| = 1$ and for each output transition t_i, $|{}^{\leftarrow}t_i| = 1$ and $|t_i^{\rightarrow}| = 0$. In the (max,+) algebra, an equivalent inequality set is:

$(t_j \in t_i^{\rightarrow})$ $x_i(k) \geq b_{ij}^{-} \otimes u_j(k)$, $u_i(k) \geq b_{ij}^{+} \otimes x_j(k)$
$(t_j \in {}^{\leftarrow}t_i)$ $y_i(k) \geq c_{ij}^{-} \otimes x_j(k)$, $x_i(k) \geq c_{ij}^{+} \otimes y_j(k)$
with $b_{ij}^{-}=a_{ij}=0$, $b_{ij}^{+}=-b_{ji}=0$, $c_{ij}^{-}=a_{ij}=0$, $c_{ij}^{+}=-b_{ji}=0$

IV. Models in (max,+) algebra

One can represent the date sequence $x(k) \in \overline{\mathbb{R}}_{max}$ with $k \in \mathbb{Z}$ by the following formal power series in one variable γ and coefficients in $\overline{\mathbb{R}}_{max}$: $x(\gamma) = \underset{k \in \mathbb{Z}}{\oplus} x(k)\gamma^k$. Variable γ may be regarded as the backward shift operator in event domain (formally, $\gamma x(k) = x(k-1)$) and γ-transform of functions is analogous to the Z-transform used in discrete-time classical control theory. Denoted $\overline{\mathbb{R}}_{max}[[\gamma]]$, the set of formal series in γ constitutes a dioid which brings a synthetic representation of trajectory $x(k) \in \overline{\mathbb{R}}_{max}$ with $k \in \mathbb{Z}$.

The state inequalities are deduced from 1. As the trajectory x is non-decreasing, condition $x \geq \gamma^1 x$ is introduced into A_1^{-}.

$$\begin{cases} x \geq \bigoplus\limits_{0 \leq i \leq M} A_i^{-} \otimes \gamma^i x \\ x \geq \bigoplus\limits_{0 \leq i \leq M} A_i^{+} \otimes \gamma^{-i} x \end{cases} \tag{2}$$

with: $M = \bigoplus\limits_{i \in P} m_i$; for $k = m({}^{\cdot}(t_i))$, if $t_j \in {}^{\leftarrow}t_i$,$(A_k^{-})_{ij} = e \oplus a_{ij}^{-}$ if $k=1$, $i=j$ and $(A_k^{-})_{ij} = a_{ij}^{-}$ otherwise; for $k = m((t_i)^{\cdot})$, $(A_k^{+})_{ij} = a_{ij}^{+}$ if $t_j \in t_i^{\rightarrow}$.

$$\begin{cases} x \geq B^{-} \otimes u \text{ and } u \geq B^{+} \otimes x \\ y \geq C^{-} \otimes x \text{ and } x \geq C^{+} \otimes y \end{cases} \tag{3}$$

In this part, we will successively, consider the static parts of the system based on the transitions denoted x and on the transitions denoted u and y.

We consider the forward part of 2.

$x \geq \bigoplus\limits_{0 \leq i \leq M} A_i^{-} \otimes \gamma^i x \oplus B^{-} \otimes u$ (respectively, $x \geq \bigoplus\limits_{0 \leq i \leq M} A_i^{+} \otimes \gamma^{-i} x$)

If the Event Graph is live, there is no circuit without token and consequently the matrix $(A_0^{-})^*$ (respectively, $(A_0^{+})^*$) converges because they are no circuits.

If we directly transpose the approach used for Timed Event Graph, an inequation set is

$$\begin{cases} x \geq (A_0^{-})^* [\bigoplus\limits_{1 \leq i \leq M} A_i^{-} \otimes \gamma^i x \oplus B^{-} \otimes u] \\ x \geq A_0^{-} \otimes x \end{cases} \tag{4}$$

Respectively,

$$\begin{cases} x \geq (A_0^{+})^* [\bigoplus\limits_{1 \leq i \leq M} A_i^{+} \otimes \gamma^{-i} x \oplus C^{+} \otimes y] \\ x \geq A_0^{+} \otimes x \end{cases} \tag{5}$$

Therefore, we can a priori deduce the following expression for a P-time Event Graph.

$$\begin{aligned} x \geq (A_0^{-})^* [\textstyle\bigoplus_{1 \leq i \leq M} A_i^{-} \otimes \gamma^i x \oplus B^{-} \otimes u] \\ \oplus (A_0^{+})^* [\textstyle\bigoplus_{1 \leq i \leq M} A_i^{+} \otimes \gamma^{-i} x] \end{aligned} \tag{6}$$

However, this right hand term does not represent the least solution of the two initial inequalities which must be taken together. It can be formulated with the following equivalent inequality.

$$\begin{aligned} x \geq & \bigoplus_{0 \leq i \leq M} A_i^{-} \otimes \gamma^i x \oplus B^{-} \otimes u \oplus \\ & \bigoplus_{0 \leq i \leq M} A_i^{+} \otimes \gamma^{-i} x \oplus C^{+} \otimes y \\ = & \\ & (A_0^{-} \oplus A_0^{+}) \otimes x \oplus \bigoplus_{1 \leq i \leq M} A_i^{-} \otimes \gamma^i x \oplus \\ & \bigoplus_{1 \leq i \leq M} A_i^{+} \otimes \gamma^{-i} x \oplus B^{-} \otimes u \oplus C^{+} \otimes y \end{aligned}$$

So,

$$\begin{cases} x \geq (A_0^{-} \oplus A_0^{+})^* [\bigoplus\limits_{1 \leq i \leq M} A_i^{-} \otimes \gamma^i x \oplus \\ \qquad \bigoplus\limits_{1 \leq i \leq M} A_i^{+} \otimes \gamma^{-i} x \oplus B^{-} \otimes u \oplus C^{+} \otimes y] \\ x \geq (A_0^{-} \oplus A_0^{+}) \otimes x \end{cases} \tag{7}$$

As $(A_0^{-} \oplus A_0^{+})^* \geq (A_0^{-})^*$ and $(A_0^{-} \oplus A_0^{+})^* \geq (A_0^{+})^*$, we can deduce that the right hand term of the first inequality of 7 which represents the least solution of the previous inequality, is greater than the corresponding right hand term of 6.

$$\begin{aligned} & x \geq (A_0^{-} \oplus A_0^{+})^* [\bigoplus_{1 \leq i \leq M} A_i^{-} \otimes \gamma^i x \oplus \bigoplus_{1 \leq i \leq M} A_i^{+} \otimes \gamma^{-i} x] = \\ & (A_0^{-} \oplus A_0^{+})^* [\bigoplus_{1 \leq i \leq M} A_i^{-} \otimes \gamma^i x] \oplus (A_0^{-} \oplus A_0^{+})^* [\bigoplus_{1 \leq i \leq M} A_i^{+} \otimes \gamma^{-i} x] \\ & \geq (A_0^{-})^* [\bigoplus_{1 \leq i \leq M} A_i^{-} \otimes \gamma^i x \oplus B^{-} \otimes u] \oplus (A_0^{+})^* [\bigoplus_{1 \leq i \leq M} A_i^{+} \otimes \gamma^{-i} x] \end{aligned}$$

A consequence is that 6 includes trajectories which are not consistent with 7 and the p-time event graph.

Example

The following system (figure 1) is not live.

$A_0^{-} = \begin{pmatrix} \varepsilon & \varepsilon \\ 4 & \varepsilon \end{pmatrix}$ $A_1^{-} = \begin{pmatrix} \varepsilon & 0 \\ \varepsilon & \varepsilon \end{pmatrix}$ $A_0^{+} = \begin{pmatrix} \varepsilon & -3 \\ \varepsilon & \varepsilon \end{pmatrix}$ $A_1^{+} = \begin{pmatrix} \varepsilon & \varepsilon \\ -10 & \varepsilon \end{pmatrix}$

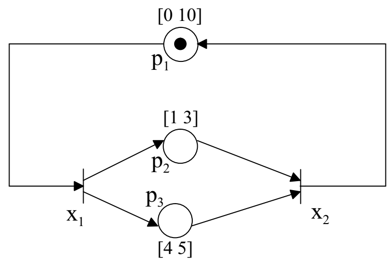

Fig. 1. p-time event graphs not live

$(A_0^-)^* \otimes A_1^- = \begin{pmatrix} \varepsilon & 0 \\ \varepsilon & 4 \end{pmatrix}$ and $(A_0^+)^* \otimes A_1^+ = \begin{pmatrix} -13 & \varepsilon \\ -10 & \varepsilon \end{pmatrix}$ but $A_0^- \oplus A_0^+ = \begin{pmatrix} \varepsilon & -3 \\ 4 & \varepsilon \end{pmatrix}$ and $(A_0^- \oplus A_0^+)^* = \begin{pmatrix} T & T \\ T & T \end{pmatrix}$

Consequently, the following result can be deduced in a natural manner.

Proposition 4.1 A necessary condition of a state evolution in $\mathbb{R}_{\max}$ is the convergence of $(A_0^- \oplus A_i^+)^*$ in $\mathbb{R}_{\max}$

In other words, there is no strictly positive circuit in $A_0^- \oplus A_0^+$.

In short, the model can be written as follows.

Dynamic part

$$x \geq (A_0^- \oplus A_0^+)^*[\bigoplus_{1\leq i\leq M} A_i^- \otimes \gamma^i x \oplus \bigoplus_{1\leq i\leq M} A_i^+ \otimes \gamma^{-i} x] \tag{8}$$

Static part

$$\begin{cases} x \geq (A_0^- \oplus A_0^+) \otimes x \\ x \geq B^- \otimes u \text{ and } x \geq C^+ \otimes y \\ y \geq C^- \otimes x \text{ and } u \geq B^+ \otimes x \end{cases} \tag{9}$$

This set of inequations contains a lot of loops which can produce inconsistency in the model. They are in $x \geq (A_0^- \oplus A_0^+) \otimes x$ but more generally, in the static part, in the dynamic part and in their interconnexions. In the aim of reduction of the complexity of the problem, we assume that additional input and output places has been added. A consequence is that the loops of the initial P-time Event Graph will only be contained in the two first inequations and not in the input and output inequations which correspond to null loops.

The two forms 8 and 9 express the system completely and can be simplified by increasing the vector state.

Dynamic part

$$\mathcal{X} \geq (\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+)\mathcal{X} \tag{10}$$

Static part

$$\begin{cases} \mathcal{X} \geq \mathcal{A}^- \otimes \mathcal{X} \\ \mathcal{X} \geq \mathcal{B}^- \otimes u \text{ and } \mathcal{X} \geq \mathcal{C}^+ \otimes y \\ y \geq \mathcal{C}^- \otimes \mathcal{X} \text{ and } u \geq \mathcal{B}^+ \otimes \mathcal{X} \end{cases} \tag{11}$$

Remark. This form generalizes the classical state equation of the Timed Event Graphs: if $\mathcal{A}^+ = \varepsilon$, $B^+ = \varepsilon$ and $C^+ = \varepsilon$, the system becomes $\begin{cases} \mathcal{X} \geq \gamma^1.\mathcal{A}^-\mathcal{X} \oplus B^- \otimes u \\ y \geq C^- \otimes \mathcal{X} \end{cases}$

These expressions describe the "lower" constraints on $\mathcal{X}$ produced by the model which can maximize it. Symmetrically, as $(\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+)$ is residuated, the following form expresses every "upper" constraint on $\mathcal{X}$ which can minimize it.

$\mathcal{X} \leq (\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+)\backslash\mathcal{X}$

$u \leq B^-\backslash\mathcal{X}$ and $\mathcal{X} \leq B^+\backslash u$

$\mathcal{X} \leq C^-\backslash y$ and $y \leq C^+\backslash\mathcal{X}$

The two models show a dualism if we remark that $(\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+)\mathcal{X} = \gamma^1.\mathcal{A}^-\mathcal{X} \oplus \gamma^{-1}.\mathcal{A}^+\mathcal{X}$ and $(\gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+)\backslash\mathcal{X} = (\gamma^{-1}.\mathcal{A}^+)\backslash\mathcal{X} \wedge (\gamma^1.\mathcal{A}^-)\backslash\mathcal{X}$ (property f3 in [2] part 4.4.4)

Symbols $\geq$, $\oplus$ and $\otimes$,correspond respectively to $\leq$, $\wedge$ and $\backslash$. Symbol γ^1 is replaced by γ^{-1} and reciprocally. Each lower (upper) matrix correspond respectively to upper (lower) matrix with the same notation.

V. Extremal acceptable trajectories

An acceptable functioning of a system can be defined by any functioning which guarantees the liveness of tokens and which does not lead to any deadlock situation, consequently. As this behavior can be represented by a state trajectory which verifies the algebraic model, an aim is to study the existence of a state trajectory. The resolution of the following problem will give an approach.

Let us assume that an initial condition is given by $\mathcal{X}(0) \in [\mathcal{X}_0^-, \mathcal{X}_0^+]$. Another aim is the determination of the lowest (respectively, greatest) acceptable trajectories $(\mathcal{X}, u, y)$ satisfying this initial condition. In other words, the problem is the determination if there is an acceptable trajectory starting from the interval $[\mathcal{X}_0^-, \mathcal{X}_0^+]$.

We consider a finite horizon which introduces a new dificulty on the initial and final values. A realistic assumption is that the model, operates on the same horizon. Therefore, the process starts at $k = 0$ and the constraints before zero cannot be considered. So, the only constraint on $\mathcal{X}(k)$ for $k = 0$ is $\mathcal{X}(0) \geq \mathcal{A}^+ \otimes \mathcal{X}(1) \oplus \mathcal{X}_0^-$. Symmetrically, as the process can stop after h , the only constraint on $\mathcal{X}(k)$ for $k = h$ is $\mathcal{X}(h) \geq \mathcal{A}^- \otimes \mathcal{X}(h-1)$. Let us notice that the hypothesis of an initial condition $x(-1) = \varepsilon$ is usually taken in Timed Event Graphs and a final condition $x(h+1) = T$ is usually taken for the corresponding classical "backward" equations (part 5.6.2 in [2]) .

A. Lowest state trajectory

Theorem 5.1 If the process operates on the horizon h and if the matrices w_k defined below have no positive circuit, the lowest state trajectory checking $\mathcal{X}(0) \geq \mathcal{X}_0^-$ is given by the following forward/backward algorithm.

Forward/backward algorithm

a) Coefficients by forward iteration

Initialization: $w_0 = \varepsilon$ and $\beta_0^- = \mathcal{X}_0^-$
for $k = 1$ to h, $\quad w_k = \mathcal{A}^- \otimes (w_{k-1})^* \otimes A^+$ and $\beta_k^- = \mathcal{A}^- \otimes (w_{k-1})^* \otimes \beta_{k-1}^-$,
b) Trajectory $\mathcal{X}^+$ by backward iteration
$\mathcal{X}^-(h) = (w_h)^* \otimes \beta_h^-$
for $k = h - 1$ to 0, $\mathcal{X}^-(k) = (w_k)^* \otimes [\mathcal{A}^+ \otimes \mathcal{X}(k+1) \oplus \beta_k^-]$

B. Greatest state trajectory

Theorem 5.2 If the process operates on the horizon h and if the matrices w_k defined below have no positive circuit, the greatest state trajectory checking $\mathcal{X}(0) \leq \mathcal{X}_0^+$ is given by the following forward/backward algorithm.

Forward/backward algorithm
Coefficients by forward iteration
a) Initialization: $w_0 = \varepsilon$ and $\beta_0^+ = \mathcal{X}_0^+$
for $k = 1$ to h, $\quad w_k = \mathcal{A}^- \otimes (w_{k-1})^* \otimes A^+$ and $\beta_k^+ = ((w_{k-1})^* \otimes \mathcal{A}^+) \backslash \beta_{k-1}^+$
b) Trajectory $\mathcal{X}^+$ by backward iteration
$\mathcal{X}^+(h) = (w_h)^* \backslash \beta_h^+$
for $k = h - 1$ to 0, $\mathcal{X}^+(k) = (w_k)^* \backslash [\mathcal{A}^- \backslash \mathcal{X}(k+1) \wedge \beta_k^+]$

In short, the two algorithms allow us to determine lowest (greatest, respectively) acceptable trajectories verifying $\mathcal{X}(0) \geq \mathcal{X}_0^-$ (respectively, $\mathcal{X}(0) \leq \mathcal{X}_0^+$). They make it possible to check the existence of a trajectory verifying $\mathcal{X}(0) \in [\mathcal{X}_0^-, \mathcal{X}_0^+]$ if the constraints $\mathcal{X}(0) \leq \mathcal{X}_0^+$ and $\mathcal{X}(0) \geq \mathcal{X}_0^-$ are respectively added in the corresponding algorithms.

Remark
Defined on an interval $[\mathcal{X}_0^-, \mathcal{X}_0^+]$, the initial condition is less restrictive that the more usual $\mathcal{X}(0) = \mathcal{X}_0$ and generalizes it. In a natural manner, the checking of this last case is realized as follows. The determination of the lowest trajectory such as $\mathcal{X}(0) \in [\mathcal{X}_0, \mathcal{X}_0^+]$, makes it possible to check the admissibility of $\mathcal{X}_0$ or in other words, if $\mathcal{X}(0) = \mathcal{X}_0$ is possible. The determination of the greatest trajectory such as $\mathcal{X}(0) \in [\mathcal{X}_0^-, \mathcal{X}_0]$ gives the same result.

The following theorem gathers the results of this paper.

Theorem 5.3
A necessary and sufficient condition of existence in $\mathbb{R}$ of a state trajectory on an infinite horizon starting from a finite initial condition is:
- logical liveness : no circuit is without token
- temporal liveness : the following matrices have only negative or null circuits: the matrix $A_0^- \oplus A_0^+$ (static case); the matrices w_k (dynamic case)

Proof:
Representing the logical point of view, the first condition is well known. The introduction of time introduces new conditions and the propositions 4.1 and 5.1 relatively to the static and dynamic cases gives conditions of existence in $\mathbb{R}_{\max}$. Therefore, the components of a possible state trajectory belongs to $\mathbb{R} \cup \{\varepsilon\}$. The Forward/backward algorithm shows that we can express a least finite trajectory verifying the model if we choose a finite $\mathcal{X}_0^-$. In this case, $\mathcal{X}(0) \geq \mathcal{X}_0^-$ and each component is different from ε because the trajectory is non decreasing. ■

VI. Computational complexity

The following curve gives indications on the possible CPU times needed to compute the different matrices w_k, and the lowest and greatest trajectories on an ordinary $Pentium$ 1.3 GHz for a horizon h= 1000. Computation tests are made using maxplus toolboxes under Scilab. The matrices $\mathcal{A}^-$ and $\mathcal{A}^+$ are completely full: there is a place containing a token between each couple of transitions. For instance, if $n = 50$, the relevant Petri net contains 50 transitions and 2500 places. The matrix $\mathcal{A}^-$ is generated randomly and $\mathcal{A}^+$ is deduced from $\mathcal{A}^-$ such that the system is temporally live on the desired horizon: the complete calculations are made, therefore. In that objective, we also take $\mathcal{A}^= = \varepsilon$ which do not effect significantly the time. At the moment, the code is not completely optimized and contains redundant operations.

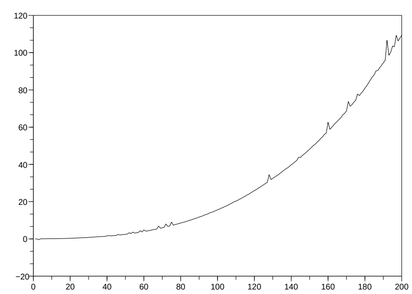

Fig. 2. CPU time for different dimensions from 3 to 200 and h=1000

The algorithms use elementary operations on matrices as $\otimes$, $\oplus$, $\backslash$, $\wedge$ and the more complex operation Klenne Star $*$. The last one determines the computational complexity of each step and the complexities of the different known algorithms are polynomial. Therefore, the complexity of calculation of the greatest trajectory is about $O(h.n^2)$ with h the horizon and n the dimension of the matrices. The space needed for the matrices w_k is $l.n^2$ with l the minimum of the horizon h and the length of the transient period. In short, the algorithm can consider important sizes of Event Graphs and horizon of calculation. Future papers will also consider sparse matrices.

VII. Example

The following example allows us to illustrate and apply the results about liveness and extremal trajectories. The horizon of calculation considered here is $h = 9$. Computation tests are made using max-plus toolbox under Scilab.

The state $\mathcal{X}(k)$ is defined by:
$\begin{pmatrix} x_1(k) & x_2(k) & x_3(k) & x_4(k) \end{pmatrix}^t$ (t transposed)

The modelling of the event graph of figure 3 enabled us to deduce the following matrices :

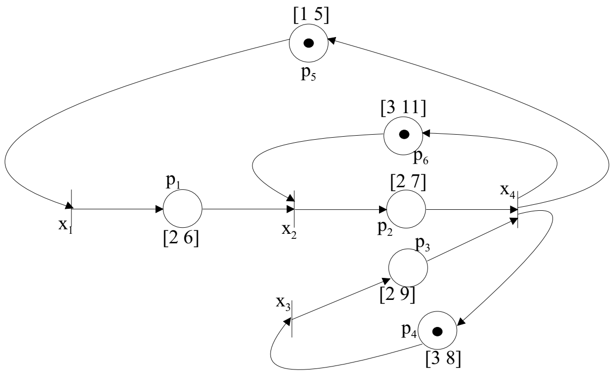

Fig. 3. Example of a live p-time event graph

$$A_0^- \oplus A_0^+ = \begin{pmatrix} \varepsilon & -6 & \varepsilon & \varepsilon \\ 2 & \varepsilon & \varepsilon & -7 \\ \varepsilon & \varepsilon & \varepsilon & -9 \\ \varepsilon & 2 & 2 & \varepsilon \end{pmatrix} \quad A_1^- = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & 1 \\ \varepsilon & \varepsilon & \varepsilon & 3 \\ \varepsilon & \varepsilon & \varepsilon & 3 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \quad A_1^+ = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ -5 & -11 & -8 & \varepsilon \end{pmatrix}$$

$$(A_0^- \oplus A_0^+)^* = \begin{pmatrix} 0 & -6 & -11 & -13 \\ 2 & 0 & -5 & -7 \\ -5 & -7 & 0 & -9 \\ 4 & 2 & 2 & 0 \end{pmatrix} \quad \mathcal{A}^- = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & 1 \\ \varepsilon & \varepsilon & \varepsilon & 3 \\ \varepsilon & \varepsilon & \varepsilon & 3 \\ \varepsilon & \varepsilon & \varepsilon & 5 \end{pmatrix} \mathcal{A}^+ = \begin{pmatrix} -18 & -24 & -21 & \varepsilon \\ -12 & -18 & -15 & \varepsilon \\ -14 & -20 & -17 & \varepsilon \\ -5 & -11 & -8 & \varepsilon \end{pmatrix}$$

with $\varepsilon = -\infty$ in the usual algebra.

The calculation of the matrices w_k shows that they are constant : $w_k = w_2$ for $w \geq 2$.

$$w_k = \begin{pmatrix} -4 & -10 & -7 & \varepsilon \\ -2 & -8 & -5 & \varepsilon \\ -2 & -8 & -5 & \varepsilon \\ 0 & -6 & -3 & \varepsilon \end{pmatrix}$$

This convergence implies the existence of an acceptable trajectory on the horizon $h = 9$. Given the following initial conditions, $\mathcal{X}^-(0) = (1,\ 3,\ 3,\ 5)^t$ and $\mathcal{X}^+(0) = (5,\ 11,\ 8,\ 17)^t$, we obtain the following tables which give respectively the lowest and greatest state trajectories :

k	1	2	3	4	5	6	7	8	9
x_1^-	1	6	11	16	21	26	31	36	41
x_2^-	3	8	13	18	23	28	33	38	43
x_3^-	3	8	13	18	23	28	33	38	43
x_4^-	5	10	15	20	25	30	35	40	45

k	1	2	3	4	5	6	7	8	9
x_1^+	5	22	39	56	73	90	107	124	141
x_2^+	11	28	45	62	79	96	113	130	147
x_3^+	8	25	42	59	76	93	110	127	144
x_4^+	17	38	55	72	89	106	123	140	157

VIII. Conclusion

P-time Event Graphs presents a nondeterministic behavior defined by lower and upper limits. In this paper, we have shown that it can be modelled under the special form of a model which uses a "noncausal" matrix (exponents can be negative. See definition 5.35 in [2]). This fact entails that the trajectories cannot easily be deduced by a simple forward iteration like in the state equation in Timed Event Graphs but must be expressed by a forward/backward iteration. In reality, a p-time Event Graph naturally contains many circuits in the dynamic part but also in the static part which increases the complexity of the resolution. The introduction of a nondecreasing serie of matrices makes it possible to determine the extremal state trajectories satisfying an initial condition defined on an interval. It is important to notice that each extremal trajectory depends on the lower and upper bounds of the model and not only, one limit. Its convergence determines the existence of a trajectory without deaths of tokens and introduces natural conditions of existence for the static and dynamic parts. As the size of the matrices corresponds to the size of the forward/backward model which depends on the number of transitions and the initial marking, this serie gives an efficient way to calculate the circuit weights of the dynamic induced graph and to solve the temporal liveness problem.

References

[1] P. Aygalinc, S. Calvez, W. Khansa and S. Collart-Dutilleul, *Using P-time Petri Nets for robust control of manufacturing systems*, 1st IFAC-Workshop on Manufacturing Systems (MIM'97), Wien, Austria, February 1997, pp. 75-80.

[2] F. Baccelli, G. Cohen, G.J. Olsder and J.P. Quadrat, *Synchronization and linearity. An Algebra for Discrete Event Systems*, New York, Wiley, 1992.

[3] J. Cochet-Terrasson, S. Gaubert and J. Gunawardena. *A constructive fixed point theorem for min-max functions*, Dynamics and Stability of Systems, Vol. 14 N°4, 1999, pp. 407-433.

[4] S. Collart-Dutilleul and P. Yim, *Time window specification and validation with Petri nets*, ETFA 2003, IEEE Conference on Emerging Technologies and Factory Automation, September 16-19, 2003, Lisbon, Portugal, pp. 232-237.

[5] J.P. Courtiat, M. Diaz, R.C. De Oliviera and P. Sénac. *Formal models for the description of timed behaviors of multimedia and hyper media distributed systems*, Computer Communications 19, 1996, page 1134-1150.

[6] P. Declerck and M.K. Didi Alaoui, *Optimal Control synthesis in Interval Descriptor Systems, Application to Time Stream Event Graphs*, IFAC Congress Praha 2005.

[7] P. Declerck and M.K. Didi Alaoui, *Extremal trajectories in P-time Event Graphs: application to control synthesis with specifications.* 44th IEEE Conference on Decision and Control and European Control Conference ECC 2005, CDC-ECC'05, Seville, Spain, 12-15 December 2005.

[8] M. Diaz and P. Owezarski, *From multimedia models to multimedia transport protocols*, Computer Networks and ISDN Systems 29, 1997 pp 745-758.

[9] M.K. Didi Alaoui, *Etude et supervision des graphes d'événements temporisés et temporels : vivacité, estimation et commande*, Thesis, ISTIA, Université d'Angers, 7 december 2005.

[10] S. Gaubert and J. Gunawardena, *the duality theorem for min-max functions*, CRAS, t. 326, Série I, 1998, pp. 43-48.

[11] R. Kumar and V.K. Garg, *Extremal solutions of inequations over lattices with applications to supervisory control*, Theoret. Comp. Sci., 148 (1995), pp. 67-92.

[12] L. Libeaut, *Sur l'utilisation des dioïdes pour la commande des systèmes à événements discrets*, Thesis, Ecole Centrale de Nantes, 1996