Stabilility analysis of a nonlinear system using interval analysis.

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Abstract

Consider a given dynamical system, described by $\dot{x} = f(x)$ (where f is a nonlinear function) and $[x_0]$ a subset of $\mathbb{R}^n$. We present an algorithm, based on interval analysis, able to show that there exists a unique equilibrium state $x_\infty \in [x_0]$ which is asymptotically stable. The effective method also provides a set $[x]$ (subset of $[x_0]$) which is included in the attraction domain of x_∞.

Index Terms

interval computations, reliable algorithm, stability of nonlinear system.

I. Introduction

There is a considerable number of works devoted to the stability problem of dynamical systems $\dot{x} = f(x)$ using interval computations [13], [14], [15], [16]. We recall some definitions and notations related to stability.

Consider a dynamical system

$$\dot{x} = f(x) \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable function. Let $\{\varphi^t\}$ denotes the flow associated to the vector field $x \mapsto f(x)$.

Definition 1 *A subset D of $\mathbb{R}^n$ is stable if $\varphi^{\mathbb{R}^+}(D) \subset D$, where $\varphi^{\mathbb{R}^+}(D) = \{\varphi^t(x), x \in D, t \in \mathbb{R}^+\}$*

Definition 2 *Let D and D' be two subsets of $\mathbb{R}^n$ such that $D \subset D'$. A equilibrium point x_∞ is asymptotically (D, D')-stable if $\varphi^{\mathbb{R}^+}(D) \subset D'$ and $\varphi^\infty(D) = \{x_\infty\}$, where $\varphi^\infty(D)$ denotes the set $\{x_\infty \in \mathbb{R}^n \mid x_\infty = \lim_{t\to\infty} \varphi^t(x), x \in D\}$*

This notion is illustrated by Figure 1.

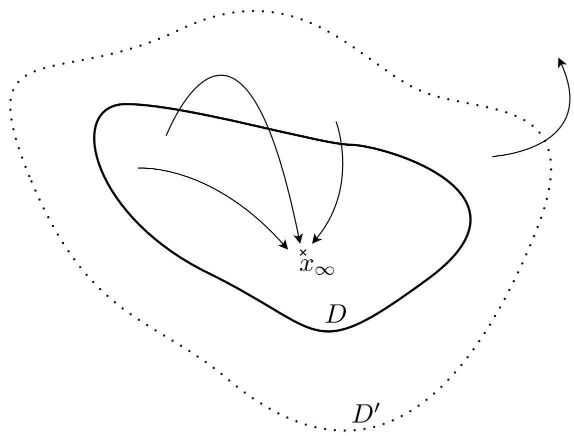

Fig. 1. The point x_∞ is asymptotically (D, D')-stable.

When f is sufficiently regular around an equilibrium state x_∞ and $\det Df(x_\infty) \neq 0$, the qualitative behavior of the dynamical system $\dot{x} = f(x)$ around x_∞ is the same that of $\dot{x} = Df(x_\infty)(x - x_\infty)$, the stability of which can be determined by counting the number of eigenvalues with negative real parts. Now, in practice, we are only able to compute an approximation $\tilde{x}_\infty$ of x_∞ and thus we cannot conclude to the local stability of (1) around x_∞.

Moreover, even if we were able to compute exactly x_∞ and to prove its local stability, to our knowledge, no general method seems to be available to compute a neighborhood D of x_∞ such that $\varphi^\infty(D) = \{x_\infty\}$.

The main contribution of this paper is a method to compute, from a set D', a domain D such that the system is (D, D')-stable.

The approach to be considered is based on interval analysis briefly presented in Section II. Interval analysis is used to prove uniqueness of an equilibrium state. Section III provides a method and a sufficient condition to check that a real valued function is positive.

Section IV contains an algorithm combining interval analysis and Lyapunov able to solve our stability problem. Finally, an example illustrates our approach in Section V.

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II. Interval arithmetic

This section introduce notations and definitions related to interval analysis.

An interval $[\underline{x}, \overline{x}]$ of $\mathbb{R}^n$ is a which can be written as $\{x \in \mathbb{R}^n, \underline{x} \leq x \leq \overline{x}\}$ with $\underline{x}$ and $\overline{x}$ in $\mathbb{R}^n$. Here $\leq$ has to be understood component-wise.

The set of bounded intervals is usually denoted by $\mathbb{IR}^n$.

Definition 3 *Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. A function $[f] : \mathbb{IR}^n \rightarrow \mathbb{IR}^m$ satisfying : $\forall [x] \in \mathbb{IR}^n, f([x]) \subset [f]([x])$ is an inclusion function of f.*

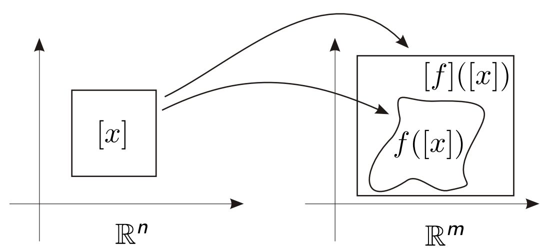

Fig. 2. Illustration of inclusion function.

Interval arithmetic [1] provides an effective method to build inclusion functions.

In [3], Neumaier proves that it is always possible to find an inclusion function $[f]$ when f is defined by an arithmetical expression. This possibility to enclose the range of an interval $[x]$ under f is powerful. Let us suppose that $0 \notin [f]([x])$, one can conclude that $\forall x \in [x], f(x) \neq 0$. On the other hand, if $0 \in [f]([x])$, this does not imply that $\exists x \in [x] \mid f(x) = 0$.
Since Moore's works [1] [2], a lot of algorithms have been developed in different areas (Optimization, Non-linear system, ...). Since they provide rigorous methods, algorithms based on interval analysis can prove mathematical assertion. For instance, in 2003, Hales launched the "Flyspeck project" ("Formal Proof of Kepler") in an attempt to use computers to automatically verify every step of the proof of the Kepler's conjecture.
Another important example is the generalized Newton method, used to find all zeros of function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. The interval Newton method creates a sequence of intervals containing zeros of f and has very interesting properties. Combined with Brouwer fixed point theorem, it can prove

existence and uniqueness of a zero of f [11], [4].

With $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the set of inclusion functions of f is partially ordered by the relation : $[f]_1 \leq' [f]_2 \Leftrightarrow \forall [x] \in \mathbb{IR}, [f]_1([x]) \subset [f]_2([x])$. Due to the fact that the available inclusion function is rarely minimal (related to $\leq'$), we will not be able to prove the assertion $f([x]) \geq 0$ when $\exists x_0 \in [x] \mid f(x_0) = 0$. The next section shows how such a proof can be done by combining interval computation with algebra calculus.

III. SUFFICIENT CONDITION TO CHECK $f \geq 0$.

This section presents a theorem which provides a sufficient condition to check that $f([x]) \geq 0$.

Definition 4 *A symmetric matrix* A *is positive definite if* $\forall x \in \mathbb{R}^n - \{0\}, x^T A x > 0$. *The set of positive definite symmetric* $n \times n$ *matrices is denoted by* S^{n+}.

Theorem 1 *Let* $f \in \mathcal{C}^\infty([x] \subset \mathbb{R}^n, \mathbb{R})$. *If*

- $\exists x_0 \in [x]$ *such that* $f(x_0) = 0$ *and* $\nabla f(x_0) = 0$.
- $\nabla^2 f([x]) \subset S^{n+}$

then $\forall x \in [x] - x_0,\ f(x) > 0$.

Proof: The assertion $\forall x \in [x], \nabla^2 f(x) \in S^{n+}$ implies that f is a strictly convex function defined on a convex set $[x]$. Since $\nabla f(x_0) = 0$, one can conclude that

$$\min_{x \in [x] - \{x_0\}} f(x) > f(x_0) = 0. \tag{2}$$

In other words : $\forall x \in [x] - \{x_0\}, f(x) > 0$. ■

This theorem induces an effective method to prove that $\forall x \in [x], f(x) \geq 0$. Indeed, if $f(x_0) = 0$ and $\nabla f(x_0) = 0$ for some $x_0 \in [x]$ can be checked by calculus algebras [5], one only has to check that $\nabla^2 f([x])$ is included in S^{n+}.

In practice, this last test is performed using interval analysis and interval symmetric matrices. With $\underline{A}$ and $\overline{A}$ two symmetric matrices such that $\underline{A} \leq \overline{A}$, an interval symmetric matrix [12] is a set $[A]$ of symmetric matrices of the form :

$$[A] = \{A \in \mathbb{R}^{n \times n}, \underline{A} \leq A \leq \overline{A}, A^T = A\} \tag{3}$$

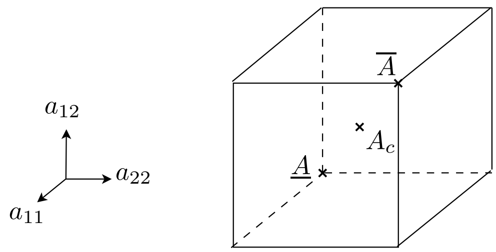

Fig. 3. With $n = 2$, an interval symmetric matrix $[\underline{A}, \overline{A}\ ]$.

Here the partial order relation $\leq$ between matrices is understood component-wise.

Definition 5 *A symmetric interval matrix $[A]$ is positive definite if $[A] \subset S^{n+}$.*

Remark 1 *Let $V([A])$ denotes the finite set of corners of $[A]$. Since S^{n+} and $[A]$ are convex subsets of the set of symmetric matrices, one has the following equivalence [6] :*

$$[A] \subset S^{n+} \Leftrightarrow V([A]) \subset S^{n+}$$

The set of symmetric $n \times n$-matrices is a vector space of dimension $\frac{n(n+1)}{2}$. Therefore, one has $\#V([A]) = 2^{\frac{n(n+1)}{2}}$. In [4], Alefeld proposes a method to check $[A] \subset S^{n+}$ by testing positive definiteness of only 2^{n-1} matrices.

Example 1 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = -\cos(x^2 + \sqrt{2}\sin^2 y) + x^2 + y^2 + 1$. This function satisfies $f(0, 0) = 0$ and $\nabla f(0, 0) = 0$ since*

$$\nabla f(x, y) = \begin{pmatrix} 2x(\sin(x^2 + \sqrt{2}\sin^2 y) + 1) \\ 2\sqrt{2}\cos y \sin y \sin\left(\sqrt{2}\sin^2 y + x^2\right) + 2y \end{pmatrix}. \quad (4)$$

One has

$$\nabla^2 f = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \quad (5)$$

where $a_{i,j}$ are given by the following formulas :

$$\begin{aligned} a_{1,1} \quad = \quad & 2\sin\left(\sqrt{2}\sin^2 y + x^2\right) \\ & +4x^2\cos\left(\sqrt{2}\sin^2 y + x^2\right) + 2. \end{aligned} \quad (6)$$

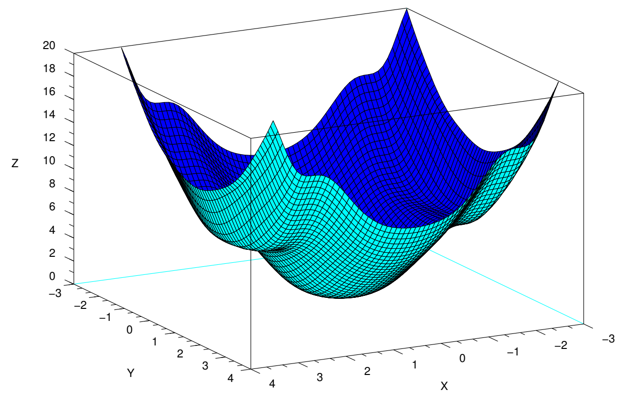

Fig. 4. Graph of f.

$$\begin{aligned} a_{2,2} \;=\; & -2\sqrt{2}\sin^2 y \sin(\sqrt{2}\sin^2 y + x^2) \\ + \; & 2\sqrt{2}\cos^2 y \sin(\sqrt{2}\sin^2 y + x^2) \\ + \; & 8\cos^2 y \sin^2 y \cos(\sqrt{2}\sin^2 y + x^2) + 2. \end{aligned} \tag{7}$$

$$a_{1,2} = a_{2,1} = 4\sqrt{2}\,x\,\cos y\,\sin y\,\cos\left(\sqrt{2}\,\sin^2 y + x^2\right). \tag{8}$$

Thanks to interval analysis, it is possible to guarantee that for all x in $[-1/2, 1/2]^2$, $\nabla^2 f(x)$ is in $[A]$ where $[A]$ is

$$[A] = \begin{pmatrix} [1.9, 4.1] & [-1.3, 1.4] \\ [-1.3, 1.4] & [1.9, 5.4] \end{pmatrix} \tag{9}$$

According to Remark 1, to prove that $f(x) \geq 0$ for all x in $[-\frac{1}{2}, \frac{1}{2}]^2$, one only has to check that the 2^{2-1} matrices :

$$A_1 = \begin{pmatrix} 1.9 & -1.3 \\ -1.3 & 1.9 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1.9 & 1.4 \\ 1.4 & 1.9 \end{pmatrix} \tag{10}$$

are definite positive [4] (which can be done in a guaranteed way).

IV. Algorithm proving stability

In this Section, an efficient method able to prove asymptotic stability is given. This section is divided into two subsections. The theorem presented in Subsection IV-A induces an algorithm given in Subsection IV-B.

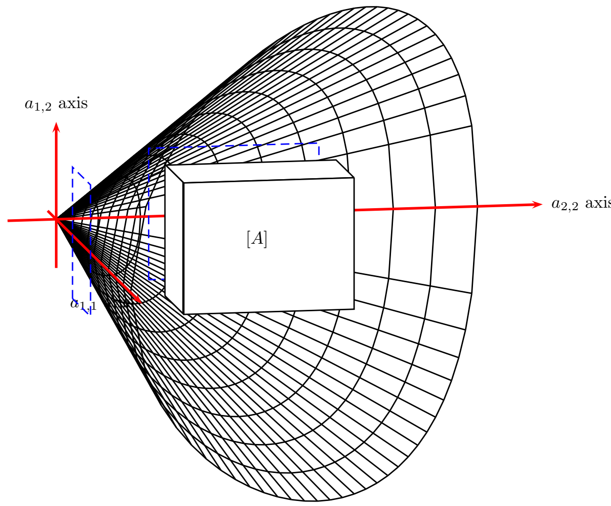

Fig. 5. The interval symmetric matrix $[A]$ is positive definite since it is included in S^{n+}.

A. A theorem

To prove stability, the most popular methods are based on Lyapunov theory. It consists in creating a real valued L function which is energy-like. Before introducing our algorithm, let us present some definitions and theorems related to stability.

Definition 6 *Let D' be a subset of $\mathbb{R}^n$ and x_∞ be in the interior of D'. A differentiable real valued function L is a Lyapunov function for $\dot{x} = f(x)$ if :*

1) $L(x) = 0 \Leftrightarrow x = x_\infty$.
2) $x \in D' - \{x_\infty\} \Rightarrow L(x) > 0$.
3) $\langle \nabla L(x), f(x) \rangle < 0, \ \forall x \in D' - \{x_\infty\}$.

This theory is motivated by the following theorem which gives a sufficient condition to stability.

Theorem 2 *If $L : D' \to \mathbb{R}$ is a Lyapunov function related to the dynamical system (1) then there exists a subset D of D' such that the point $x_\infty \in D$ (the unique one satisfying $L(x_\infty) = 0$) is asymptotically (D, D')-stable.*

The proof can be found in [9]. If one has to check the stability of a given system, one merely has to :

1) find a candidate for the Lyapunov function,
2) check that this candidate is Lyapunov.

In the first step, in practice, since $\mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$ is an infinite dimensional vector space, the candidate is searched in a finite dimensional vector subspace of $\mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$, for instance L is constrained to be a quadratic form, $L(x) = x^T W x$ where W is a symmetric square matrix.

It is well known [9] that, in the linear case ($\dot{x} = Ax$), the origin 0 is asymptotically stable if and only if there exist two matrices in S^{n+} such that

$$A^T W + WA = -I. \tag{11}$$

Solving this equation amounts to solving linear equations. When matrix W is positive definite, all conditions of Theorem 2 are combined, therefore, 0 is asymptotically stable. In other words, in the linear case, an effective method to prove stability exists. Our algorithm is partially based on this effective method.

Definition 7 *With $[x]$ a box of $\mathbb{R}^n$, we denote by $B(r, [x])$ the set $\{x \in \mathbb{R}^n, \min_{a\in[x]} \|a - x\| < r\}$. Let us denote by d the function defined on $\mathbb{IR}^n \times \mathbb{IR}^n$ by*

$$d : ([x], [y]) \mapsto \sup\{r \in \mathbb{R} \mid B(r, [x]) \subset [y]\}.$$

Theorem 3 *Consider the dynamical system (1) and a matrix $W \in S^{n+}$ whose maximum and minimum eigenvalues are respectively λ_{max} and λ_{min}. Define $g_a(x) = -\langle W(x-a), f(x)\rangle$. If $[x_\infty]$ is a box included in the box $[x_0]$ and $[x]$ a box with center in $[x_\infty]$ and radius $\sqrt{n\frac{\lambda_{min}}{\lambda_{max}}} d([x_\infty], [x_0])$ then we have the following implication:*

1) *there exists a single $x_\infty \in [x]$ strictly inside $[x_\infty]$, such that $f(x_\infty) = 0$.*
2) $\nabla^2 g_{[x_\infty]}([x_0]) \subset S^{n+}$.

implies that x_∞ is asymptotically $([x_0], [x])$-stable.

Proof: Let L_{x_∞} be a quadratic form defined by

$$\begin{array}{rccl} L_{x_\infty}: & D & \to & \mathbb{R} \\ & x & \mapsto & (x - x_\infty)^T W (x - x_\infty) \end{array} \tag{12}$$

Since $W \in S^{n+}$, one has :

1) $L_{x_\infty}(x) = 0 \Leftrightarrow x = x_\infty$

2) $x \in D - \{x_\infty\} \Rightarrow L_{x_\infty}(x) > 0$

With $h(x) = -\langle \nabla L_{x_\infty}(x), f(x) \rangle$, to prove that L_{x_∞} is Lyapunov, it remains to show that : $h([x_0]) \leq 0$. By construction, we have:

1) $h(x_\infty) = 0$ and $\nabla h(x_\infty) = 0$.
2) $\nabla^2 h([x_0]) \subset S^{n+}$ since $\nabla^2 h([x_0]) \subset 2\nabla^2 g_{[x_\infty]}([x_0])$.

Applying Theorem 1 to h, one concludes that L_{x_∞} is a Lyapunov function for the dynamical system (1). Therefore, there exists a subset $[x]$ of $[x_0]$ and $x_\infty \in [x]$ such that :

$$\begin{cases} \varphi^{+\infty}([x]) = \{x_\infty\}. \\ \varphi^t([x]) \subset [x_0], \forall t \in \mathbb{R}^+. \end{cases}$$

Let $\mathcal{E}$ be the ellipsoid oriented by W, with center x_∞, and long axe $\sqrt{\lambda_{min}} d([x_\infty],[x_0])$. Obviously, the set $\mathcal{E}$ is included in $[x_0]$ and is stable. Therefore, a box $[x]$ whose center is in $[x_\infty]$ and whose radius is $\sqrt{n \frac{\lambda_{min}}{\lambda_{max}}} d([x_\infty],[x_0])$ is, by construction, included in the ellipsoid $\mathcal{E}$. Therefore, x_∞ is asymptocally $([x_0],[x])$-stable. ■

From a dynamical system $\dot{x} = f(x)$ and a set $D' = [x_0]$, our algorithm proves that there exists an unique equilibrium point x_∞, in a computed set $D = [x]$, which is asymptotically (D, D')-stable. The set $D = [x]$ is therefore included in the attraction domain of x_∞.

B. *Algorithm*

The main idea, of this algorithm, is first to linearize the given system using a point close to the equilibrium state. In a second time, one checks that a Lyapunov function for the linearized system is also a Lyapunov function for the nonlinear one according to results obtained in Section II. This can be summarized in Algorithm 1.

Steps 1 can be performed using the interval Newton method previously cited. In Step 4, linear algebra is used to solve linear equations. In Step 5, interval analysis is used to prove that :

$$\nabla^2 g_{[x_\infty]}([x_0]) \subset S^{n+} \tag{15}$$

Alg. 1 Algorithm

Require: A box $[x_0]$ of $\mathbb{R}^n$ and a dynamical system

$$\dot{x} = f(x) \tag{13}$$

where $f \in \mathcal{C}^\infty(D, \mathbb{R}^n)$.

Ensure: A proof that the system (13) is asymptoticly stable over $[x]$ included in $[x_0]$.

1: $[x_\infty] :=$ Newton Interval Algorithm for $f(x) = 0$, $x \in [x_0]$.

2: $\tilde{x}_\infty :=$ center of $[x_\infty]$.

3:
$$A := \left(\frac{df}{dx}_{|x=\tilde{x}_\infty} \right) \tag{14}$$

4: Solve $A^T W + WA = -I$.

5: **if** $W \in S^{n+}$ and $\nabla^2 g_{[x_\infty]}([x_0]) \subset S^{n+}$ **then**

6: Return "The system is asymptotically stable over the box $[x]$ whose center is $\tilde{x}_\infty$ and width is $\sqrt{n}\sqrt{\frac{\lambda_{min}}{\lambda_{max}}} d([x_0], [x_\infty])$."

7: **end if**

V. Illustrative example

In this Section, our method is discussed via the example :

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 - (1 - x_1^2)x_2 \end{pmatrix} \tag{16}$$

where $[x_0] = [-0.6, 0.6]^2$.

First, interval Newton method is used to prove that the box $[x_0]$ contains a unique x_∞ equilibrium state. Moreover, this fixed point of the flow is proven to lie in $[x_\infty] = [-0.02, 0.02]^2$. Then, the dynamical system is linearized around $\tilde{x}_\infty = (0.01, 0.01)$.

The vector field associated to this dynamical system is represented on Figure 6. The figure also shows the linearized one around $\tilde{x}_\infty$. In this case, the Lyapunov function created is :

$$L_{x_\infty}(x) = (x - x_\infty)^T \begin{pmatrix} -1,51 & 0,49 \\ 0,49 & -1,01 \end{pmatrix} (x - x_\infty) \tag{17}$$

Some level curves of L_{x_∞} are represented on Figure 7. In a neighborhood of $[x_\infty]$, the function L_{x_∞} seems to be a Lyapunov function since vectors $f(x)$ cross the level curves form outside to

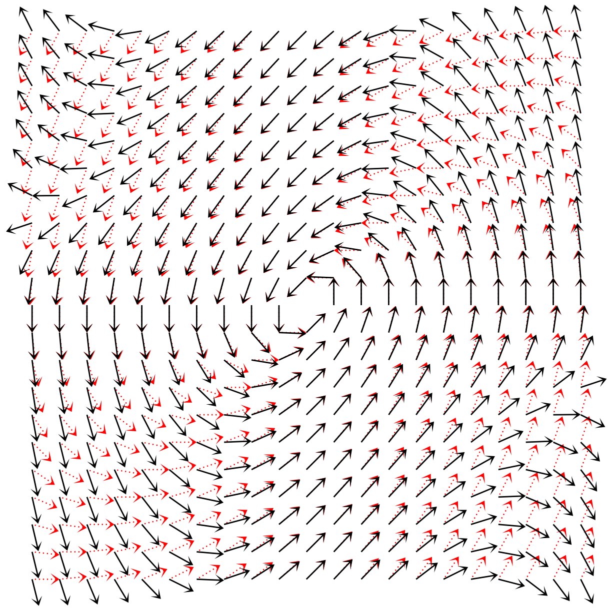

Fig. 6. Normalized vector field and its linearization around x_∞. The linearized one is represented by dotted lines.

inside. As L_{x_∞} is Lyapunov for the linearized system, the last geometrical assertion is equivalent to $g_{x_\infty}(x) > 0, \forall x \in [x_0]^2 - x_\infty$. This last assertion is true since :

- $g_{x_\infty}(x_\infty) = 0$
- $\nabla g_{x_\infty}(x_\infty) = 0$
- $\nabla^2 g_{[x_\infty]}([x_0]) \subset S^{n+}$ since $\nabla^2 g_{[x_\infty]}([x_0]) \subset [A]$ where $[A] = \begin{pmatrix} [-1.78, 5.78] & [-4.14, 4.15] \\ [-4.14, 4.15] & [0.56, 3.45] \end{pmatrix}$ is positive definite.

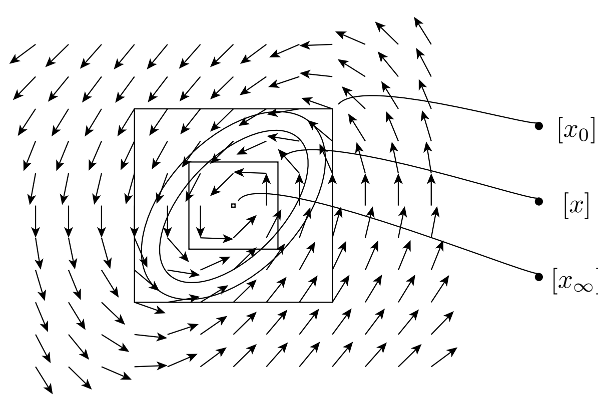

Fig. 7. Lyapunov function level curves and a box $[x_\infty]$ which contains a unique equilibrium state.

VI. Conclusion

This paper provides an effective method able, from a dynamical system described by $\dot{x} = f(x)$ (where f is a nonlinear function), to prove that a set D' contains a subset D with an unique point x_∞ which is asymptotically (D, D')-stable.

To fill out this work, different perspectives appear. It could be interesting to have a sufficient condition on f to guarantee that our algorithm terminates. This method could be combined with graph theory and guaranteed numerical integration of O.D.E. [7], [8] to compute a guaranteed approximation of the attraction domain of x_∞.

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