# An algorithm for computing a neighborhood included in the attraction domain of an asymptotically stable point.

Nicolas Delanoue[a], Luc Jaulin[b], Bertrand Cottenceau[c]

[a] *LARIS, Université d'Angers, 62 avenue Notre Dame du Lac 49000 Angers - France*
[b] *ENSTA-Bretagne, 2 rue Franois Verny, 29806 Brest Cedex 09*
[c] *LARIS, Université d'Angers, 62 avenue Notre Dame du Lac 49000 Angers - France*

**Abstract**

Many methods exist to detect stable equilibrium points $x^*$ of nonlinear dynamical systems $\dot{x} = f(x)$. Most of them also prove the existence of a neighborhood $\mathcal{N}$ of $x^*$ such that all trajectories initialized in $\mathcal{N}$ converge to $x^*$. This paper provides a numerical method combining Lyapunov theory with interval analysis which makes to find a set $\mathcal{N}$ which is included in the attraction domain of $x^*$.

*Keywords:* interval computations, reliable algorithm, nonlinear stability theory.

## 1. Introduction

Consider a nonlinear dynamical system described by a differential equation $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field. The point $x^*$ is an equilibrium point if $f(x^*) = 0$. To find the equilibrium points it suffices to solve $n$ nonlinear equations with $n$ unknowns. This can be solved using elimination theory-based methods [18], or any local numerical algorithm [20]. A point $x^*$ is asymptotically stable if for all neighborhood $\mathcal{M}$ of $x^*$, there exists a neighborhood $\mathcal{N}$ of $x^*$ such that all trajectories initialized in $\mathcal{N}$ converge to $x^*$ and remain inside $\mathcal{M}$.

From the theoretical point of view, the Hartman-Grobman theorem states that if $f$ is sufficiently regular around a hyperbolic equilibrium state $x^*$ then there exists a local homeomorphism between the solutions of the $\dot{x} = f(x)$ and its linearization $\dot{x} = Df(x^*)(x - x^*)$. In other words, the qualitative behavior of the dynamical system $f$ around $x^*$ is the same that of $Df(x^*)$. Therefore, the existence of $\mathcal{N}$ is usually provided by studying the eigenvalues of the Jacobian matrix of $f$ at $x^*$. Interval based methods have already been used to study the stability of dynamical systems. In the case of linear system, a classical result from control theory states that the origin (which is always an equilibrium

*Email addresses:* `nicolas.delanoue@univ-angers.fr` (Nicolas Delanoue), `luc.jaulin@ensta-bretagne.fr` (Luc Jaulin), `bertrand.cottenceau@univ-angers.fr` (Bertrand Cottenceau)

state) is stable if and only if all roots of the characteristic polynomial of $f$ have a negative real part. Such a polynomial is said to be Hurwitz stable. In [16], Khraritonov gives a necessary and sufficient effective condition to the Hurwitz stability of a polynomial with interval coefficients. When $f$ is linear with unknown bounded coefficients (i.e. $f$ can be represented by a matrix whose entries are intervals), the Khraritonov's condition only offers a sufficient condition to check that the origin is stable. More recently, Wang and al [17] determine a necessary and sufficient effective condition to the Hurwitz stability of an interval matrix.

The present paper deals with nonlinear dynamical system. Contrary to the linear case, the stability of an equilibrium state is, most of the time, only local : the trajectories must be initialized sufficiently close to the equilibrium state $x^*$ to converge to $x^*$. The set of initial states for which the trajectory converges to $x^*$ is the *attraction domain* of $x^*$. The main contribution of this paper is an algorithm which provides a neighborhood $\mathcal{N}$ of $x^*$ included in the attraction domain of $x^*$.

Given an equilibrium for a dynamical system, we have the well-known connection with the linearization near the stationary point. By studying this linearization it is more or less straightforward to construct such neighborhoods $\mathcal{N}$, see for example [3], [4]. The approach to be considered, based on Lyapunov theory and interval analysis, also proves existence and uniqueness of an asymptotically stable equilibrium state $x^*$ even if we only have a rigorous enclosure of $x^*$.

The paper is organized as follows. Interval analysis is briefly presented in Section 2. Section 3 provides a method and a sufficient condition to check that a real valued function is positive. In Section 4, we combine interval analysis and Lyapunov analysis in an algorithm that is able to solve our stability problem. Finally, an example illustrates our approach in Section 5.

## 2. Interval arithmetic

This section introduces notations and definitions related to interval analysis. An interval $[\underline{x}, \overline{x}]$ of $\mathbb{R}^n$ is a set which can be written as $\{x \in \mathbb{R}^n, \underline{x} \leq x \leq \overline{x}\}$ with $\underline{x}$ and $\overline{x}$ in $\mathbb{R}^n$. Here the relation $\leq$ has to be understood component-wise. Note that this definition implies that intervals are bounded. The set intervals is usually denoted by $\mathbb{IR}^n$.

**Definition 1.** A map $[f] : \mathbb{IR}^n \to \mathbb{IR}^m$ is said to be an inclusion map of $f : \mathbb{R}^n \to \mathbb{R}^m$ if $\forall [x] \in \mathbb{IR}^n, f([x]) \subset [f]([x])$ (where $f([x]) = \{f(x) | x \in [x]\}$).

Interval arithmetic [1] provides an effective method to build inclusion maps. In [5], Neumaier proves that it is always possible to find an inclusion map $[f]$ when $f$ is defined by an arithmetical expression. This possibility to enclose the image of an interval $[x]$ under $f$ is powerful. Indeed, let us suppose that $0 \notin [f]([x])$, one can conclude that $\forall x \in [x], f(x) \neq 0$. On the other hand, if $0 \in [f]([x])$, this does not imply that $\exists x \in [x] \mid f(x) = 0$.

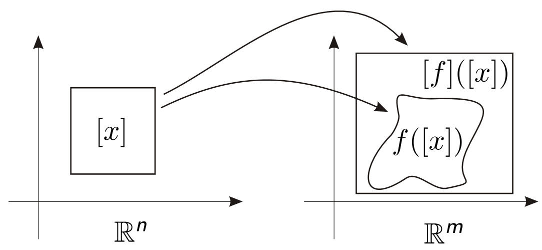


Figure 1: Illustration of inclusion function.

Since Moores works [1] [2] that introduced interval arithmetic, many algorithms have been developed in different areas, for example in global optimization [7], non-linear dynamical systems, etc. As interval analysis provides rigorous methods, these algorithms can prove mathematical assertion. For instance, in 2003, Hales launched the "Flyspeck project" ("Formal Proof of Kepler") in an attempt to use computers to automatically verify every step of the proof (partially based on interval analysis) of the Kepler's conjecture. Another important example is a generalization of the Newton method called Interval Newton method. This method can be applied to find all zeros of a given differentiable map $f : \mathbb{R}^n \to \mathbb{R}^n$. The interval Newton method creates a sequence of intervals containing zeros of $f$ and has very interesting properties : combined with Brouwer fixed point theorem, it can prove existence and uniqueness of a zero of $f$ [6, 14].

Note that the set of inclusion maps of a given $f : \mathbb{R}^m \to \mathbb{R}^n$ can be partially ordered by the relation : $[f]_1 \leq' [f]_2 \Leftrightarrow \forall [x] \in \mathbb{IR}^n, [f]_1([x]) \subset [f]_2([x])$. Due to the fact that the available inclusion map is rarely minimal (related to $\leq'$), interval analysis cannot basically be used to prove the assertion $\forall x \in [x], f(x) \geq 0$ in the case of existence of $x_0 \in [x]$ such that $f(x_0) = 0$. The next section shows how such a proof can be done by combining interval computation with algebra calculus.

## 3. Sufficient condition to check $f \geq 0$.

This section proposes a theorem which provides a sufficient condition to check the following assertion for a given differentiable real valued function $f$ : $\forall x \in [x], f(x) \geq 0$. The main idea is close to the second derivative criterion classically used in optimization. Then, an algorithm based on the proposed theorem and interval analysis is presented. Let us recall that a symmetric real matrix $A$ is positive definite if $\forall x \in \mathbb{R}^n - \{0\}, x^T A x > 0$. In this paper, the set of positive definite symmetric $n \times n$ matrices is denoted by $S^{n+}$.

**Theorem 1.** *Let $f \in C^\infty([x] \subset \mathbb{R}^n, \mathbb{R})$. If there exists $x^* \in [x]$ such that $f(x^*) = 0$ and $Df(x^*) = 0$, and $\forall x \in [x], D^2 f(x) \in S^{n+}$, then $\forall x \in [x], f(x) \geq 0$ and $f(x) = 0 \Rightarrow x = x^*$.*

PROOF. The assertion $\forall x \in [x], D^2 f(x) \in S^{n+}$ implies that $f$ is a strictly convex function defined on a convex set $[x]$. Since $Df(x^*) = 0$, one can conclude that $\inf_{x \in [x]} f(x) \geq f(x^*) = 0$. The proof of uniqueness is by reduction to a contradiction. Suppose that there exists $x^{**} \in [x]$ such that $f(x^{**}) = 0$ and $x^{**} \neq x^*$. As $f$ is strictly convex, one has

$$f\left(\frac{x^* + x^{**}}{2}\right) < \frac{1}{2} f(x^*) + \frac{1}{2} f(x^{**}) = 0$$

Therefore, since $[x]$ is convex, we have $m = \frac{x^* + x^{**}}{2} \in [x]$ such that $f(m) < 0$.

□

This theorem induces an effective method to prove that $\forall x \in [x], f(x) \geq 0$. Indeed, if $f(x^*) = 0$ and $Df(x^*) = 0$ for some $x^* \in [x]$ can be proved by calculus algebras [8], one only has to check that $D^2 f([x])$ is included in $S^{n+}$ to conclude.

In practice, this inclusion is performed using results based on interval symmetric matrices. With $\underline{A}$ and $\overline{A}$ two symmetric matrices such that $\underline{A} \leq \overline{A}$, an interval symmetric matrix is a set $[A]$ of symmetric matrices of the form : $[A] = \{A \in \mathbb{R}^{n \times n}, \underline{A} \leq A \leq \overline{A}, A^T = A\}$ [15]. Here the partial order relation $\leq$ between matrices is understood component-wise.

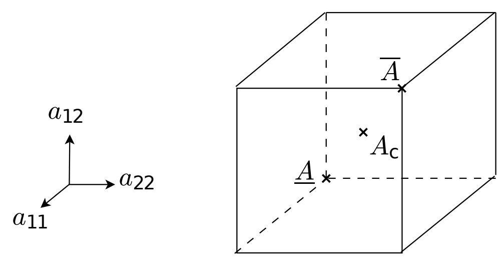


Figure 2: With $n = 2$, an interval symmetric matrix $[\underline{A}, \overline{A}\ ]$.

**Definition 2.** A symmetric interval matrix $[A]$ is positive definite if $[A] \subset S^{n+}$.

**Remark 1.** Let $V([A])$ denote the finite set of corners of $[A]$. Since the cone $S^{n+}$ and $[A]$ are convex subsets of the set of symmetric matrices, one has the following equivalence :

$$[A] \subset S^{n+} \Leftrightarrow V([A]) \subset S^{n+}$$

The set of symmetric $n \times n$-matrices is a vector space of dimension $\frac{n(n+1)}{2}$. Therefore $V([A])$ has a cardinality of $2^{\frac{n(n+1)}{2}}$. In [9], Rohn proposes a method to check $[A] \subset S^{n+}$ by testing positive definiteness of only $2^{n-1}$ matrices. The procedure is the following : with $[A]$ a interval symmetric matrix, one can create two symmetric matrices $A_c$ and $\Delta$ such that $[A] = \{A, A_c - \Delta \leq A \leq A_c + \Delta\}$ where $A_c = \frac{1}{2}(\underline{A} + \overline{A})$ and $\Delta = \frac{1}{2}(\overline{A} - \underline{A})$. Let us denote by $C$ the finite set : $C = \{x \in \mathbb{R}^n, |x_i| = 1, \forall i \in \{1, \ldots, n\}\}$. One has $\#C = 2^n$. For each $z$ in $C$, let us denote by $T_z$ the diagonal matrix defined by $z$, *i.e.* $T_z = diag(z)$ and by $A_z$ the matrix $A_c - T_z \Delta T_z$. Each $A_z$, with $z \in C$, is obviously in $[A]$, and since $A_{-z} = A_z$, the set $\{A_z, z \in C\}$ is finite and of cardinal $2^{n-1}$. In [9], Rohn proves that

$$[A] \subset S^{n+} \Leftrightarrow \{A_z, z \in C\} \subset S^{n+}.$$

**Example 1.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = -\cos(x^2 + \sqrt{2}\sin^2 y) + x^2 + y^2 + 1$. This function satisfies $f(0, 0) = 0$ and $Df(0, 0) = 0$ since

$$Df(x, y) = \begin{pmatrix} 2x(\sin(x^2 + \sqrt{2}\sin^2 y) + 1) \\ 2\sqrt{2}\cos y \sin y \sin\left(\sqrt{2}\sin^2 y + x^2\right) + 2y \end{pmatrix}^T . \tag{1}$$

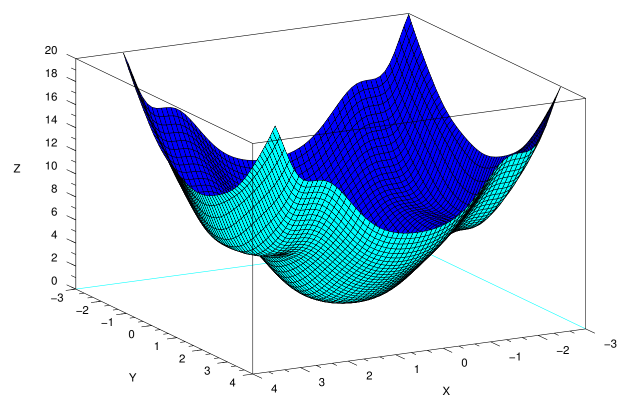


Figure 3: Graph of $f$.

The Hessian matrix is given by $D^2 f = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$ where $a_{i,j}$ are given by the following formulas :

$$\begin{aligned} a_{1,1} &= 2\sin\left(\sqrt{2}\sin^2 y + x^2\right) + 4x^2\cos\left(\sqrt{2}\sin^2 y + x^2\right) + 2, \\ a_{2,2} &= -2\sqrt{2}\sin^2 y \sin(\sqrt{2}\sin^2 y + x^2) + 2\sqrt{2}\cos^2 y \sin(\sqrt{2}\sin^2 y + x^2) \\ &\quad +8\cos^2 y \sin^2 y \cos(\sqrt{2}\sin^2 y + x^2) + 2, \\ a_{1,2} = a_{2,1} &= 4\sqrt{2}\,x\,\cos y\,\sin y\,\cos\left(\sqrt{2}\,\sin^2 y + x^2\right). \end{aligned}$$

Thanks to interval analysis, it is possible to guarantee that $\forall x \in [-0.5, 0.5]^2$, $D^2 f(x) \subset [A]$ where $[A]$ is the following interval symmetric matrix :

$$[A] = \begin{pmatrix} [1.9, 4.1] & [-1.3, 1.4] \\ [-1.3, 1.4] & [1.9, 5.4] \end{pmatrix}$$

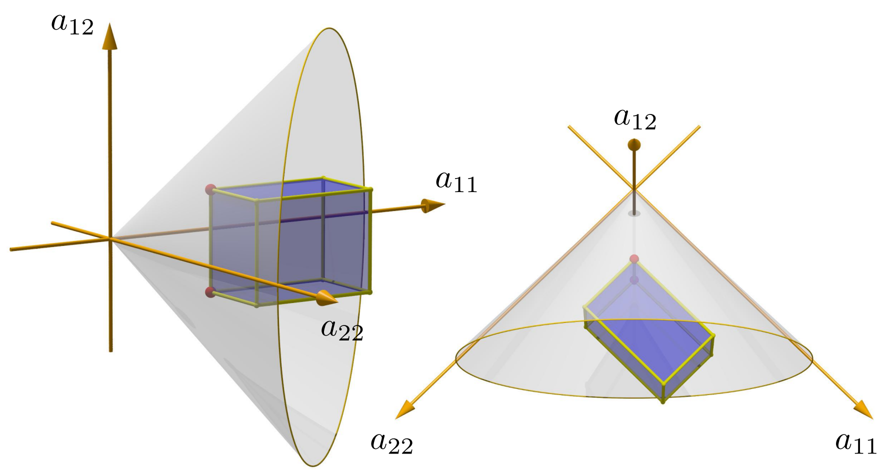


Figure 4: The interval symmetric matrix $[A]$ is represented in blue. The two matrices used in the Rohn sufficient condition are red corners. This interval symmetric is positive definite since it is included in the cone $S^{n+}$.

According to Remark 1, to prove that $f(x) \geq 0$ for all $x$ in $[-\frac{1}{2}, \frac{1}{2}]^2$, one only has to check that the 2 matrices :

$$A_1 = \begin{pmatrix} 1.9 & -1.3 \\ -1.3 & 1.9 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1.9 & 1.4 \\ 1.4 & 1.9 \end{pmatrix}$$

are definite positive [6] (this can be shown using rigorous computations). Interval symmetric matrix $[A]$ is represented in Figure 4 with a blue box. The two matrices $A_1$ and $A_2$ used in the Rohn [9] sufficient condition are red corners.

## 4. Algorithm for proving stability

In this section, an efficient method able to prove asymptotic stability is given. The theorem presented in 4.1 combines results of the previous section with Lyapunov theory and induces an algorithm given in 4.2. This algorithm also generates a subset $\mathcal{N}$ of the attraction domain of the asymptotically stable point.

### *4.1. Theorem*

To prove stability, most of the methods are based on Lyapunov theory. It consists in creating a real valued $L$ function which is energy-like. Before introducing our algorithm, let us present some definitions and theorems related to stability. Let $\{g^t : \mathbb{R}^n \to \mathbb{R}^n\}_{t\in\mathbb{R}}$ denotes the flow associated to the vector field $x \mapsto f(x)$, *i.e.* the 1-parameter family of functions $\{g^t : \mathbb{R}^n \to \mathbb{R}^n\}_{t\in\mathbb{R}}$ satisfying

:

$$
\begin{aligned}
\frac{d}{dt}g^t(x) &= f(g^t(x)) \text{ for all t } \in \mathbb{R} \\
g^0(x) &= x
\end{aligned}
$$

**Definition 3.** A subset $\mathcal{N}$ of $\mathbb{R}^n$ is stable (according to $f$) if $\forall t \geq 0, g^t(\mathcal{N}) \subset N$,

**Definition 4.** Let $\mathcal{M}$ and $\mathcal{N}$ be two subsets of $\mathbb{R}^n$ such that $\mathcal{N} \subset \mathcal{M}$. An equilibrium point $x^*$ is asymptotically $(\mathcal{N}, \mathcal{M})$-stable (according to $f$) if

$$
\begin{cases}
\forall t \geq 0, g^t(\mathcal{N}) \subset \mathcal{M} \\
\forall x \in \mathcal{N}, \lim_{t\to\infty} g^t(x) = x^*
\end{cases}
$$

This notion is illustrated by Figure 5.

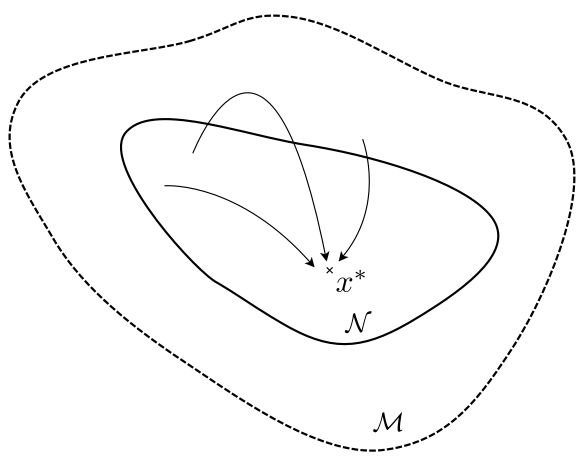


Figure 5: The point $x^*$ is asymptotically $(\mathcal{N}, \mathcal{M})$-stable.

**Definition 5.** Let $\mathcal{M}$ be a subset of $\mathbb{R}^n$ and $x^*$ be in the interior of $\mathcal{M}$. A differentiable real valued function $L$ is a Lyapunov function for the dynamical system $\dot{x} = f(x)$ if :

1. $L(x) = 0 \Leftrightarrow x = x^*$,
2. $\forall x \in \mathcal{M} - \{x^*\}, L(x) > 0$,
3. $\forall x \in \mathcal{M} - \{x^*\}, \langle DL(x), f(x)\rangle < 0$.

This theory is motivated by the following theorem which gives a sufficient condition to asymptotic stability.

**Theorem 2.** *If $L : \mathcal{M} \to \mathbb{R}$ is a Lyapunov function related to the dynamical system $\dot{x} = f(x)$ then there exists a subset $\mathcal{N}$ of $\mathcal{M}$ such that the point $x^* \in \mathcal{M}$ (the unique one satisfying $L(x^*) = 0$) is asymptotically $(\mathcal{N}, \mathcal{M})$-stable.*

The proof can be found in [12]. To check stability, one merely has to :

1. find a candidate for the Lyapunov function,
2. check that this candidate is of Lyapunov.

For the first step, since the set of differentiable functions is infinite dimensional, one prefers to limit the search for the candidate to a finite dimensional subspace. For instance, we may suppose that $L$ is a quadratic form $L(x) = x^T W x$ where $W$ is a symmetric square matrix. It is well known [12] that, in the linear case ($\dot{x} = Ax$ where $A$ is a square matrix), the origin is asymptotically stable if and only if there exists a matrix $W$ in $S^{n+}$ such that

$$A^T W + WA = -I. \tag{2}$$

Solving this equation whose unknown is $W$ amounts to solving linear equations. If W is positive definite, then all conditions of Theorem 2 are fulfilled, thus the origin is asymptotically stable. In other words, in the linear case, an effective method to prove stability exists. Our idea is partially based on this effective method. The main idea is to construct a quadratic Lyapunov function on $\mathcal{M}$ for a linear equation. Then we show that it is a Lyapunov function for the original equation as well. We conclude the existence of the locally asymptotically stable fixed point $x^*$ and construct the final neighborhood based on the eigenvalues of $W$ taking into account that we only have a rigorous enclosure of $x^*$.

**Definition 6.** With $[x]$ a box of $\mathbb{R}^n$, we denote by $B(r, [x])$ the set

$$B(r, [x]) = \{y \in \mathbb{R}^n, \exists x \in [x], \|x - y\| < r\}.$$

Let us denote by $l$ the real valued function defined on $\mathbb{IR}^n \times \mathbb{IR}^n$ by

$$l : ([x], [y]) \mapsto \sup\{r \in \mathbb{R} \mid B(r, [x]) \subset [y]\}.$$

**Theorem 3.** *Consider the dynamical system $\dot{x} = f(x)$ and a matrix $W \in S^{n+}$ whose maximum and minimum eigenvalues are $\lambda_{max}$ and $\lambda_{min}$ respectively. Let $L_{\xi^*}$ be a quadratic form defined by*

$$\begin{array}{rccl} L_{\xi^*} : & \mathcal{M} & \to & \mathbb{R} \\ & x & \mapsto & (x - \xi^*)^T W (x - \xi^*), \end{array} \tag{3}$$

*with $\xi^* \in \mathcal{M}$. Let $[x^*]$ and $\mathcal{N}$ be boxes such that*

- *$[x^*] \subset \mathcal{N} \subset \mathcal{M}$,*
- *the center of $\mathcal{N}$ is in $[x^*]$,*
- *the radius of $\mathcal{N}$ is smaller than $\sqrt{n \frac{\lambda_{min}}{\lambda_{max}}} l([x^*], \mathcal{M})$.*

*We have the following implication:*
*If there exists a single $x^* \in \mathcal{M}$ strictly inside $[x^*]$, such that $f(x^*) = 0$, and*

$$\forall x \in \mathcal{M}, \forall \xi^* \in [x^*], D^2 \langle DL_{\xi^*}(x), -f(x) \rangle \in S^{n+},$$

*then $x^*$ is asymptotically $(\mathcal{N}, \mathcal{M})$-stable.*

PROOF. Since $W \in S^{n+}$, one has :

1. $L_{x^*}(x) = 0 \Leftrightarrow x = x^*$
2. $\forall x \in \mathcal{M} - \{x^*\}, L_{x^*}(x) > 0$

Let $h$ be the real valued function defined by $h(x) = \langle DL_{x^*}(x), -f(x)\rangle$. By construction, we have $h(x^*) = 0$ and $Dh(x^*) = 0$.

Moreover, since $\forall x \in \mathcal{M}, \forall \xi^* \in [x^*], D^2\langle DL_{\xi^*}(x), -f(x)\rangle \in S^{n+}$, supposing $\xi^* = x^*$, one can conclude that $\forall x \in \mathcal{M}, D^2 h(x) \in S^{n+}$.

Applying Theorem 1 to $h$, one has $\forall x \in \mathcal{M}, h(x) \geq 0$. Therefore, $L_{x^*}$ is a Lyapunov function for the dynamical system $\dot{x} = f(x)$. In other words, there exists a subset $\mathcal{N}$ of $\mathcal{M}$ and $x^* \in \mathcal{N}$ such that :

$$\begin{cases} \forall t \in \mathbb{R}^+, g^t(\mathcal{N}) \subset \mathcal{M}, \\ \forall x \in \mathcal{N}, \lim\limits_{t \to +\infty} g^t(x) = x^*. \end{cases}$$

Let $\mathcal{E}$ be the ellipsoid oriented by $W$, with center $x^*$, and long axe $\sqrt{\lambda_{min}} l([x^*], \mathcal{M})$. Obviously, the set $\mathcal{E}$ is included in $\mathcal{M}$ and is stable. Thus, any boxes $\mathcal{N}$ whose center is in $[x^*]$ and whose radius is smaller than $\sqrt{n \frac{\lambda_{min}}{\lambda_{max}}} l([x^*], \mathcal{M})$ is, by construction, included in the ellipsoid $\mathcal{E}$. Therefore, $x^*$ is asymptotically $(\mathcal{N}, \mathcal{M})$-stable. □

From a dynamical system $\dot{x} = f(x)$ and a set $\mathcal{M}$, the following algorithm computes a set $\mathcal{N}$ and proves that there exists an unique equilibrium point $x^*$ which is asymptotically $(\mathcal{N}, \mathcal{M})$-stable. The computed set $\mathcal{N}$ is therefore included in the attraction domain of $x^*$.

### *4.2. Algorithm*

The main idea, of this algorithm, is first to linearize the given system using a point close to the equilibrium state. In a second step, one checks that a Lyapunov function for the linearized one is also a Lyapunov function for the nonlinear one according to results obtained in Section 3. This can be summarized in Algorithm 1.

Step 1 can be performed using the interval Newton method previously cited. Note that at step 3 the matrix $A$ chosen is not the exact linearization of $f$ with $x^*$ but is a linearization with an approximation of $x^*$ denoted $\tilde{x}^*$. This is important because, in practice, it is often impossible to compute the exact zero of a given function. The exact position of the equilibrium state is not needed for the rest of the procedure. An step 4, linear algebra is used to solve linear equations. This linear system does not need to be solved in a rigorous way to ensure the correctness of the general method. At step 5, interval analysis is used to prove that : $D^2\langle DL_{[x^*]}(\mathcal{M}), -f(\mathcal{M})\rangle \subset S^{n+}$,

**Alg. 1** Algorithm

**Require:** a box $\mathcal{M}$ of $\mathbb{R}^n$,

**Require:** a dynamical system $\dot{x} = f(x)$ where $f \in C^\infty(\mathcal{M}, \mathbb{R}^n)$,

**Ensure:** a box $\mathcal{N}$, and a proof of existence and uniqueness of an equilibrium state $x^*$ which is asymptotically $(\mathcal{N}, \mathcal{M})$ stable,

1: $[x^*] :=$ Newton Interval Algorithm for $f(x) = 0$, $x \in \mathcal{M}$.

2: $\tilde{x}^* :=$ an element of $[x^*]$.

3:
$$A := \left(\frac{df}{dx}_{|x=\tilde{x}^*}\right) \tag{4}$$

4: Solve $A^T W + WA = -I$.

5: **if** $W \in S^{n+}$ and $D^2\langle DL_{[x^*]}(\mathcal{M}), -f(\mathcal{M})\rangle \subset S^{n+}$, **then**

6: **Return** the box $\mathcal{N}$ whose center is $\tilde{x}^*$ and radius is $\sqrt{n\frac{\lambda_{min}}{\lambda_{max}}}l([x^*], \mathcal{M})$.

7: **else**

8: **Return** "Failure".

9: **end if**

## 5. Illustrative example

In this section, the proposed method is discussed via the example :

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 - (1 - x_1^2)x_2 \end{pmatrix} = f(x) \tag{5}$$

where $\mathcal{M} = [-0.6, 0.6]^2$. The vector field associated to this dynamical system is represented on Figure 6.

First, interval Newton method is used to prove that the box $\mathcal{M}$ contains a unique $x^*$ equilibrium state. Moreover, this fixed point of the flow is proven to lie in $[x^*] = [-0.02, 0.02]^2$. Then, the dynamical system is linearized with $\tilde{x}^* = (0.01, 0.01)$. Figure 6 also shows the linearized one with $\tilde{x}^*$ in red dotted lines. In this case, the Lyapunov function created is :

$$L_{\xi^*}(x) = (x - \xi^*)^T \begin{pmatrix} -1,51 & 0,49 \\ 0,49 & -1,01 \end{pmatrix} (x - \xi^*) \tag{6}$$

Some level curves of $L_{\xi^*}$ for some values $\xi^* \in [x^*]$ are represented on Figure 7. In a neighborhood of $[x^*]$, the function $L_{\xi^*}$ seems to be a Lyapunov function since vectors $f(x)$ cross the level curves form outside to inside. As $L_{\xi^*}$ is of Lyapunov for the linearized system, the last geometrical interpretation is equivalent to $\forall x \in \mathcal{M} - \{x^*\}, \langle DL(x), f(x)\rangle < 0$. This last assertion is true since $h(x^*) = 0$, $Dh(x^*) = 0$, and $D^2h_{[x^*]}(\mathcal{M}) \subset S^{n+}$ as $D^2h_{[x^*]}(\mathcal{M}) \subset [A]$ with $[A] = \begin{pmatrix} [-1.78, 5.78] & [-4.14, 4.15] \\ [-4.14, 4.15] & [0.56, 3.45] \end{pmatrix}$ positive definite.

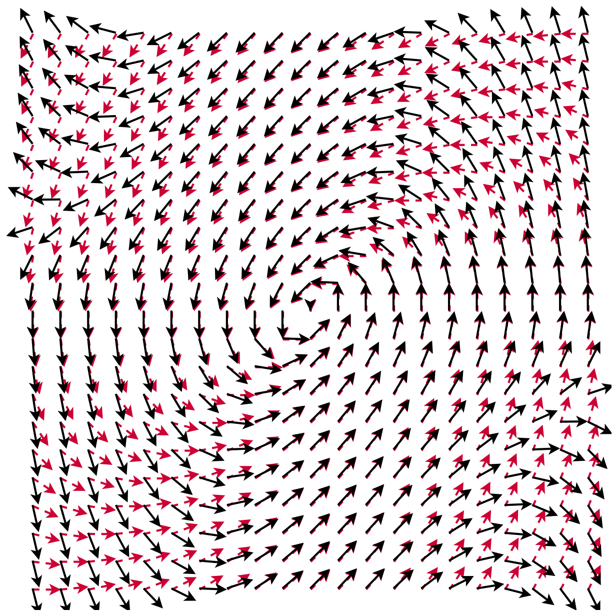

Figure 6: Normalized vector field and its linearization around $x^*$. The linearized one is represented by red dotted lines.

## 6. Conclusion

This paper provides an effective rigorous method able, from a given dynamical system described by $\dot{x} = f(x)$ and a given box $\mathcal{M}$ to compute a box $\mathcal{N}$ such that $\mathcal{N}$ contains an unique asymptotically $(\mathcal{N}, \mathcal{M})$-stable equilibrium state $x^*$. These ideas has already been employed by A. Rauh [22] to verify stability analysis of continuous-time control systems with bounded parameter uncertainties.

Our point of view is that the marriage of Lyapunov theory and interval analysis works because of genericity. Indeed, interval based method succeeds in generic cases and Lyapunov functions are stable in the sense that any positive definite function in a sufficiently small neighborhood containing a Lyapunov function for a dynamical system is also of Lyapunov.

To fill out this work, different perspectives appear. It could be interesting to prove that the proposed algorithm terminates in the generic case. This method could also be combined with graph theory and guaranteed numerical integration of O.D.E. [10, 11, 21] to compute a rigorous approximation of the attraction domain of $x^*$. The computed box $\mathcal{N}$ (with non empty interior) could be a good first approximation of the attraction domain for an iterative scheme.


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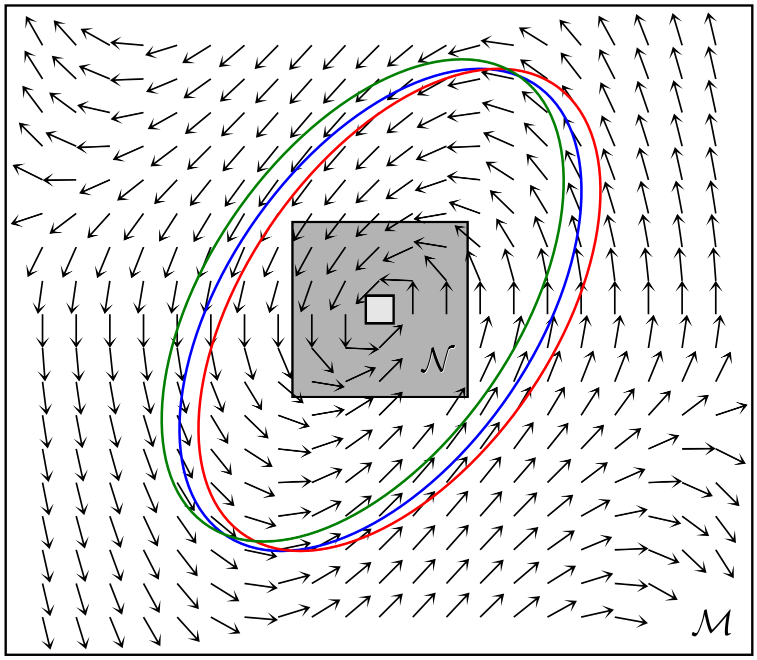


Figure 7: Lyapunov functions level curves and a box $[x^*]$ which contains a unique equilibrium state.


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