A numerical approach to compute the topology of the Apparent Contour of a smooth mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Nicolas Delanoue, Sébastien Lagrange

LARIS, 62 avenue Notre Dame du Lac 49000 Angers, France

Abstract

A rigorous algorithm for computing the topology of the Apparent Contour of a generic smooth map is designed and studied in this paper. The source set is assumed to be a simply connected compact subset of the plane and the target space is the plane. Whitney proved that, generically, critical points of a smooth map are folds or cusps [9]. The Apparent Contour is the set of critical values, that is, the image of the critical points. Generically speaking, the Apparent Contour does not have triple points and double points are normal crossings (*i.e.* crossing without tangency). Each of those particular cases, cusp and normal crossing, is described in order to be rigorously handled by an interval analysis based scheme. The first step of the presented method provides an enclosure of those particular points. The second part of the designed method is a computation of a graph which is homeomorphic to the Apparent Contour. Edges of this graph are computed by testing connectivity of those particular points in the source space. This paper also defines a concept called *portrait*. Relations between this notion and the more classical notion of Apparent Contour are discussed.

Keywords: Rigorous numerics, Apparent contour, Interval analysis, Portrait

1. Introduction

Let X be a simply connected compact subset of $\mathbb{R}^2$, and f a smooth map from X to the plane. The boundary of X is assumed to be regular and denoted ∂X. Singular points of f are points at which the rank of the differential df is less than 2. There are only two generic local types of f at singular points : fold and cusp. In other words, if (x_0, y_0) is a singular point of a generic map f, there exists a neighborhood of (x_0, y_0) such that

☆

Email addresses: `nicolas.delanoue@univ-angers.fr` (Nicolas Delanoue), `sebastien.lagrange@univ-angers.fr` (Sébastien Lagrange)

URL: `http://perso-laris.univ-angers.fr/~delanoue` (Nicolas Delanoue)

f is locally equivalent to either of these following maps : $(x, y) \mapsto (x, y^2)$ or $(x, y) \mapsto (x, xy - y^3)$. Figures 1 and 2 show these two normal forms :

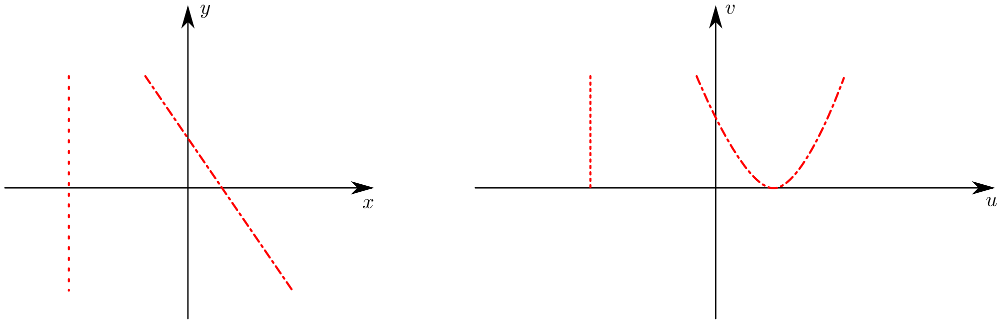


Figure 1: Normal form for a fold : $(x, y) \mapsto (x, y^2)$.

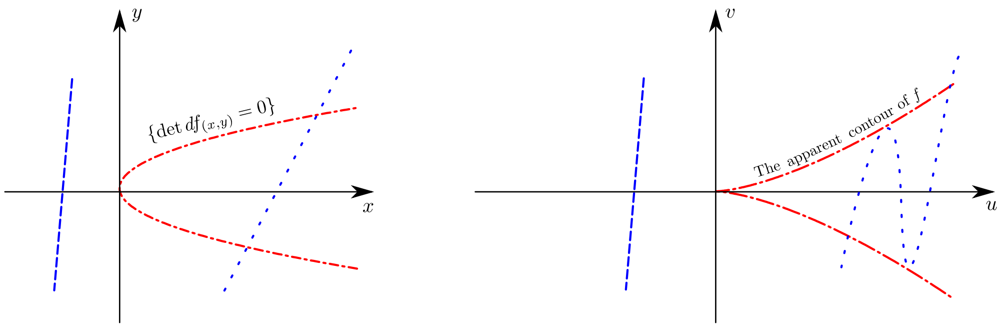


Figure 2: Normal form for a cusp : $f : (x, y) \mapsto (x, xy - y^3)$.

The image of the singular point set S of f is called the *Apparent Contour* of f. Form the mathematical point of view, the Apparent Contour is an important information about a given smooth map. Indeed, it is an invariant under diffeomophisms on the source set or on the target set. In computer graphics, the Apparent contour is called silhouette and plays an important role in shape recognition. It provides one of the main cues for figure-to-ground distinction, see [3] for a survey of algorithms generating contours. In robotics, a serial robot is described by equations that maps the joints parameters to the configuration of the robot system. The Apparent Contour gives information about the behaviour of the serial manipulator [1]. Interpretation and classification of singularities on general manipulators was developed by Zlatanov [2]. By the end of his work, Zlatanov himself pointed out the necessity to obtain general algorithms for computing and representing the singularity set. Until now, however, no general method has been given to this end yet. From the algebraic topology point of view, the Apparent Contour can inform on the topology of the source set. In [14], Paolini presents a software code to manipulate Apparent Contours.

In the following sections, we will provide an algorithm that divides the singular point set S and the boundary ∂X to build a graph homeomorphic to the Apparent Contour. This discrete output can be viewed as a persistent part of a discrete object that completely describes a given smooth map under a group of diffeomophisms. This object is called a *portrait* and is introduced in subsection 1.1.

It is well known that a generic map has only generic singular points (cusp and fold) and the self intersection of the Apparent Contour are transverse.

Subsection 1.2 generalises this result to take into account the boundary of the source set. It gives a list of cases that our algorithm has to be able to handle in order to terminate (in the gerenic case).

The rest of the paper is organised as follows. Section 2 contains technical results needed in the analysis of a given smooth map. This section can be omitted in first lecture. Section 3 gives algorithms based on interval analysis [5, 6, 7] to rigorously enclose cusps, self intersection of the Apparent Contour and intersections of the Apparent Contour with the image of the boundary set. Some of this cases are illustrated by the following figure :

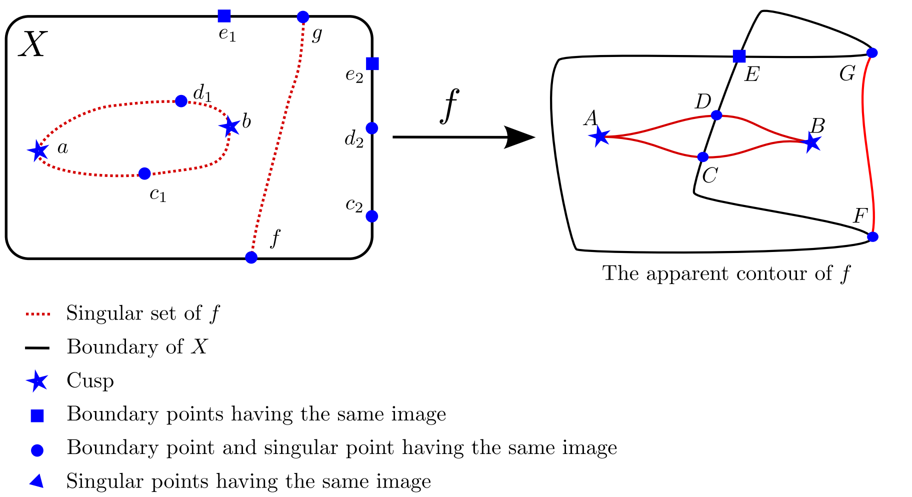


Figure 3: The Apparent Contour of f.

In each subsection, one proves that each case can be rewritten such that it can be handled by the interval Newton method. The analysis is synthesized in Section 4 where these particular points are connected to form a graph. Importance of this graph and relation with the Apparent contour are also discussed. Section 5 concludes the paper.

1.1. Simplicial representation of a smooth map : portrait

Definition 1. Let $f_1, f_2 : X \to Y$ be smooth maps. The map f_1 is equivalent to f_2 if there exist diffeomorphims $g : X \to X$ and $h : Y \to Y$ such that $h \circ f_1 = f_2 \circ g$, i.e. the following diagram commutes :

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y \\ \downarrow g & & \downarrow h \\ X & \xrightarrow{f_2} & Y \end{array}$$

We write

$$f_1 \sim f_2, \tag{1}$$

to denote that f_1 and f_2 are equivalent.

Note that this equivalence is an equivalence relation.

Example 1. Let us consider the two following real valued functions defined on the real line : $f_1 : x \mapsto x^2$ and $f_2 : x \mapsto ax^2+bx+c$ with $a, b, c \in \mathbb{R}, a \neq 0$. One has $f_1 \sim f_2$ since $h^{-1} \circ f_1 \circ g = f_2$ where g and h^{-1} are the following diffeomorphims $g : x \mapsto x + \frac{b}{2a}$ and $h^{-1} : y \mapsto ay - \frac{b^2}{4a} + c$.

Singularity theory is based on the obvious but fundamental theorem :

Theorem 1. *Suppose that f_1 and f_2 are equivalent smooth mappings with diffeomorphisms h and g such that $f_2 \circ g = h \circ f_1$. Let x_1, x_2 be elements of X, if $g(x_1) = x_2$ then $\operatorname{rank} df_1(x_1) = \operatorname{rank} df_2(x_2)$.*

Corollary 2. *Suppose f_1 and f_2 are equivalent smooth mappings, their singular sets S_{f_1} and S_{f_2} are homeomorphic. Moreover $f_1(S_{f_1})$ and $f_2(S_{f_2})$ are homeomorphic.*

In other words, the topology of the singular set (and the topology of its image) of a given smooth map f provides necessary information about which class f belongs modulo the equivalence relation (1).

Example 2. Let f_1 and f_2 be defined by

$$\begin{array}{cccc} f_1 : & [-3,3] & \to & \mathbb{R} \\ & x & \mapsto & x+1, \end{array}$$

$$\begin{array}{cccc} f_2 : & [-3,3] & \to & \mathbb{R} \\ & x & \mapsto & x^2+1, \end{array}$$

According to Theorem 1, singularities are preserved under the action of couples of diffeomorphims on X and Y.

$$\begin{array}{cccc} (g,h) : & \mathcal{C}^\infty(X,Y) & \to & \mathcal{C}^\infty(X,Y) \\ & f_1 & \mapsto & h^{-1} \circ f_1 \circ g \end{array} \tag{2}$$

Therefore, maps f_1 and f_2 are not equivalent since f_2 has a singularity whereas f_1 does not have any.

Modulo the equivalence relation (1), many mappings can be finitely described by a *portrait*. This global "picture" of the map does not depend on a choice of system of coordinates neither on the source set nor on the target set. The definition of a portrait relies on a formal object named an abstract simplicial complex [4]. The notion of abstract simplicial complex can be view as a generalization of the notion of graph. Indeed, a graph is an abstract simplicial complex with only singletons and pairs.

Definition 2. Let $\mathcal{N}$ be a finite set of symbols $\{a^0, a^1, \dots, a^n\}$. An abstract simplicial complex $\mathcal{K}$ is a subset of the powerset of $\mathcal{N}$ satisfying

$$\sigma \in \mathcal{K} \Rightarrow \forall \sigma_0 \subset \sigma, \sigma_0 \in \mathcal{K} \tag{3}$$

Example 3. The set

$$\begin{aligned} \mathcal{K} \quad = \quad & \{\emptyset, \{a^0\}, \{a^1\}, \{a^2\}, \{a^3\}, \{a^4\}, \{a^0, a^1\}, \\ & \{a^1, a^2\}, \{a^0, a^2\}, \{a^3, a^4\}, \{a^0, a^1, a^2\}\}\,. \end{aligned}$$

is an abstract simplicial complex.

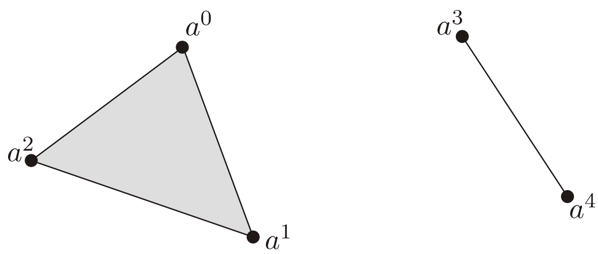


Figure 4: A geometric realization of $\mathcal{K}$.

The complete enumeration of elements of an abstract simplicial complex $\mathcal{K}$ is non necessary. It suffices to enumerate only facets (*i.e.* maximal dimensional simplices) of a complex. Indeed, suppose $\{a^0, a^1, a^2\} \in \mathcal{K}$, relation (3) implies that $\{a^0\}$, $\{a^1\}$, $\{a^2\}$, $\{a^0, a^1\}$, $\{a^1, a^2\}$, $\{a^0, a^2\}$ are also elements of $\mathcal{K}$. This remark suggests the following construction.

Notation 1. With $\mathcal{V}$ a finite collection of elements (abstract vertices) $\mathcal{V} = \{a^0, a^1, \dots, a^n\}$, and $2^{\mathcal{V}}$ the power set of $\mathcal{V}$, a simplicial complex is a subset of $2^{\mathcal{V}}$ satisfying (3). Letting $\{\sigma_1, \dots, \sigma_m\}$ be included in $2^{\mathcal{V}}$ (not necessarily an abstract simplicial complex), we denote by $\sigma_1 + \cdots + \sigma_m$ the following abstract simplicial complex[1] :

$$\sigma_1 + \cdots + \sigma_m := \bigcup_{i=1}^{i=m} 2^{\sigma_i}$$

[1]The reader can check that $\sigma_1 + \cdots + \sigma_m$ is the smallest, with inclusion defined on $2^{2^{\mathcal{V}}}$ as order relation, abstract simplicial complex that contains σ_1, …, σ_m, as simplices.

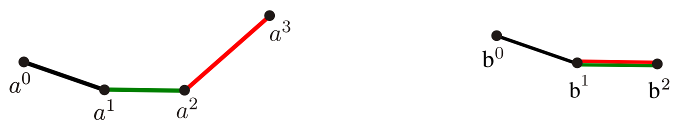
With this notation, the abstract simplicial complex of Example 3 can be denoted by $a^0a^1a^2 + a^3a^4$. Given an abstract simplicial complex $\mathcal{K}$, the subset of its singletons is called the 0-skeleton and denoted by $\mathcal{K}^0$. A map between the 0-skeleton of two abstract simplicial complexes $\mathcal{K}$ and $\mathcal{L}$ is said to be *simplicial* if the following property holds :

$$\{a^0, a^1, \ldots, a^n\} \in \mathcal{K} \Rightarrow \{F(a^0), F(a^1), \ldots, F(a^n)\} \in \mathcal{L}. \quad (4)$$

Example 4. Let $\mathcal{K}$ and $\mathcal{L}$ be the two abstract simplicial complexes defined by $\mathcal{K} = a_0a_1 + a_1a_2 + a_2a_3$ and $\mathcal{L} = b_0b_1 + b_1b_2$. The map F defined by

$$\begin{array}{rcll} F & : & a^0 & \mapsto b^0 \\ & & a^1 & \mapsto b^1 \\ & & a^2 & \mapsto b^2 \\ & & a^3 & \mapsto b^1 \end{array}$$

is a simplicial map.



A simplicial map F from $\mathcal{K}^0$ to $\mathcal{L}^0$ can be naturally extented to a map from $\mathcal{K}$ to $\mathcal{L}$ with :

$$F(\sigma) := \{F(a^i) | a^i \in \sigma\}$$

In this article, simplicial map will, most of the time, refer to its natural extension. A priori, comparing a simplicial map and a smooth map seems to be not possible since a simplicial map is an abstract object mapping elements from a finite set to a finite set and these finite sets are not supposed to be endowed with any topology. A good way to be able to compare a smooth map f and a simplicial map F is to perform a geometric realization of F.

Geometric realization is a classical construction [4] that geometrizes an abstract complex based on the Whitney embedding theorem. This theorem shows that any abstract simplicial complex can be embedded in $\mathbb{R}^n$ with n sufficiently large. Any embedding of a given abstract simplicial complex is called a *geometric realization*. A geometric realization of an abstract simplicial complex $\mathcal{K}$ is denoted by $|\mathcal{K}|$.

Similarly, a simplicial map $F : \mathcal{K} \to \mathcal{L}$ can be realized with affine maps between simplexes of $|\mathcal{K}|$ and $|\mathcal{L}|$. Condition (4) ensures that any geometric realization of F is continuous. A geometric realization is a piecewise smooth map since its restriction to each simplex is affine and we would like to employ the notion of simplicial map to encode a smooth map. Let us consider the following equivalence between continuous maps.

Definition 3. Let f_1 and f_2 be continuous maps. Then f_1 and f_2 are topologically conjugate if there exists *homeomorphisms* $g : X \to X'$ and $h : Y \to Y'$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "g"] & Y \arrow[d, "h"] \\ X' \arrow[r, "f_2"] & Y' \end{tikzcd}$$

commutes. We denote by $f_1 \sim_0 f_2$ this relation.

This relation of topological conjugation is weaker than equivalence in the sense that $f_1 \sim f_2 \Rightarrow f_1 \sim_0 f_2$ for any smooth maps f_1 and f_2. Nevertheless, this relation enables the comparison of a simplicial map and a smooth map.

Definition 4 (Portrait). Let f be a smooth map and F a simplicial map, F is a *portrait* of f if a geometric realization of F is topologically conjuguate with f.

Example 5. 1. The simplicial map given in example 4 is a portrait of the smooth map $f : [-4, 2] \ni x \mapsto x^2 - 1 \in \mathbb{R}$. Indeed, the simplicial map

$$\begin{array}{rcll} F & : & a^0 & \mapsto b^0 \\ & & a^1 & \mapsto b^1 \\ & & a^2 & \mapsto b^2 \\ & & a^3 & \mapsto b^1 \end{array}$$

can be geometrically realized with

$$\begin{array}{rccl} |F| : & [-2; 1] & \to & [0, 2] \\ & x & \mapsto & \begin{cases} -x, & \text{if } x \in [-2, -1] \\ -x, & \text{if } x \in [-1, 0] \\ x, & \text{if } x \in [0, 1]. \end{cases} \end{array}$$

Maps g and h^{-1} are homeomorphims and make the following diagram commutes :

$$\begin{tikzcd} {[-2, 1]} \arrow[r, "|F|"] \arrow[d, "g"] & {[0, 2]} \arrow[d, "h"] \\ {[-4, 2]} \arrow[r, "f"] & {[-1, 15]} \end{tikzcd}$$

with

$$\begin{array}{rccc} g : & [-2, 1] & \to & [-4, 2] \\ & x & \mapsto & 2x, \end{array}$$

and

$$\begin{array}{rccc} h^{-1} : & [-1, 15] & \to & [0, 2] \\ & x & \mapsto & \frac{1}{2}\sqrt{y + 1}. \end{array}$$

2. Let us consider the two following abstract simplicial complexes $\mathcal{K} = a_0(a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6 + a_6a_1)$, $\mathcal{L} = b_0(b_1b_2 + b_2b_3 + b_3b_4 + b_4b_1)$. The map $F : \mathcal{K}^0 \to \mathcal{L}^0$ defined by : $F(a_0) = b_0, F(a_1) = b_1, F(a_2) = b_2, F(a_3) = F(a_5) = b_3, F(a_4) = F(a_6) = b_4$ extends to a simplicial map from $\mathcal{K}$ to $\mathcal{L}$. Let p be a simple cusp point of a smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$, then there exists a closed neighborhood V of p such that F is a portrait of $f|V$.

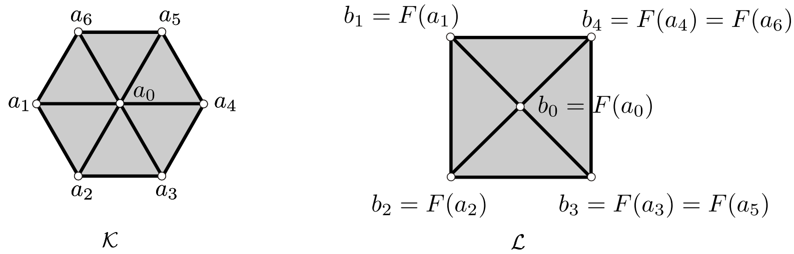


Figure 5: A portrait of a cusp from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Computing a portrait of given smooth map f is a challenging problem. To author knowledge, no algorithm has never been proposed to generate a portrait of a given smooth map f (even if f is a smooth map from a simply connected compact subset of the plane to the plane). To our point of view, this method is central in the classification of smooth maps. The present paper proposes a method to compute, from a given map, an invariant that all portrait has in commun : the topology of the Apparent Contour.

1.2. Classification of singularities

In this section, we enumerate cases happening generically. The main idea is to apply the Thom Transversality theorem and its multi-jet version extension.

Theorem 3 (Properties of a generic mapping). *Let X be a compact simply connected subset of $\mathbb{R}^2$ with smooth boundary ∂X. A generic smooth map f from X to $\mathbb{R}^2$ satisfies the following properties :*

i) *the set of singular points, denoted by $S = \{x \in X \mid \det df_x = 0\}$, is a regular curve of X, moreover elements of S are folds or cusps; the set of cusps is discrete,*
ii) *three different singular points do not have the same image; two singular points having the same image are fold points and have normal crossing.*
iii) *three boundary points do not have the same image; two boundary points having the same image cross normally,*

iv) *three different points belonging to the union the singularity curve and boundary do not have the same image. If a point on the singularity curve and a boundary have the same image, the singular point is a fold and they have normal crossing.*

v) *if the singularity curve intersects the boundary, then this point is a fold, moreover tangents to the singularity curve and boundary curve are different.*

PROOF. To prove this theorem, one has to prove that the set of maps satisfying conditions i), ii), iii), iv) and v) is a residual subset of $\mathcal{C}^\infty(X, \mathbb{R}^2)$. Since to be residual is stable under finite intersection, is suffices to prove that each set of maps (corresponding to each case) is a residual subset of $\mathcal{C}^\infty(X, \mathbb{R}^2)$ to conclude.

- Case i) is the classical application of the Thom tranversality Theorem which states the following : If X and Y are smooth manifolds and W a submanifold of the jet space $J^k(X, Y)$ then the set of smooth maps $T_W = \{f \in \mathcal{C}^\infty(X, Y) \mid j^k f \pitchfork W\}$ is a residual subset of $\mathcal{C}^\infty(X, Y)$ in the $\mathcal{C}^\infty$ topology. Case i is proved using the 1-jet of f

$$\begin{array}{rccc} j^1 f & : & X & \to & J^1(X, \mathbb{R}^2) \\ & & x & \mapsto & (x, f(x), df_x) \end{array}$$

  where $J^1(X, \mathbb{R}^2) = X \times \mathbb{R}^2 \times L(\mathbb{R}^2, \mathbb{R}^2)$ and $L(\mathbb{R}^2, \mathbb{R}^2)$ denotes the set of linear applications from $\mathbb{R}^2$ to $\mathbb{R}^2$. First, applying Thom transversality Theorem with the manifold $W_1 = X \times \mathbb{R}^2 \times \{0\}$, we can conclude that $T_{W_1} = \{f \in \mathcal{C}^\infty(X, \mathbb{R}^2) \mid j^1 f \pitchfork W_1\}$ is a residual subset of $\mathcal{C}^\infty(X, \mathbb{R}^2)$. Moreover, the following statement is also true : Suppose f and W are such that $j^k f \pitchfork W$ then $\operatorname{codim}\{x \mid j^k(x) \in W\} = \operatorname{codim}(W)$ [11]. Since W_1 is of codimension 4, one can conclude that if f is in the residual subset T_{W_1}, then $\{x \in X \mid j^1(x) \in W_1\} = \{x \in X \mid df_x = 0\}$ is empty.

  Let us now consider $W_1' = \{(x, y, u) \in J^1(X, \mathbb{R}^2) \mid \det(u) = 0\}$. W_1' is a submanifold of $X \times \mathbb{R}^2 \times (L(\mathbb{R}^2, \mathbb{R}^2) - \{0\})$. Then $T_{W_1'} = \{f \in \mathcal{C}^\infty(X, \mathbb{R}^2) \mid j^1 f \pitchfork W_1'\}$ is a residual subset of $\mathcal{C}^\infty(X, \mathbb{R}^2)$. Moreover, since W_1' is of codimension 1, if f is in the residual subset $T_{W_1'}$, then $\{x \in X \mid j^1(x) \in W_1'\} = \{x \in X \mid df_x = 0\}$ is a regular curve.

  Therefore, if f is in the residual subset $T_{W_1} \cap T_{W_1'}$ then $\{x \mid df_x = 0\}$ is a regular curve. In our terminology, with f a generic map, S_f is a regular curve. The proof that this regular curve is composed by folds and cusps is based on 2-jet and 3-jet of f and can be found in [11].

- Let us now consider case ii). One has to prove that the set of smooth maps satisfying condition ii) is a residual subset of $\mathcal{C}^\infty(X, \mathbb{R}^2)$. The proof is based on a generalization of the Thom transversality theorem named multijet transversality theorem. This theorem states the following : If X and Y are smooth manifolds and W a submanifold of the multi-jet space $J^k_{(n)}(X,Y)$ then the set of smooth maps $T_W = \{f \in \mathcal{C}^\infty(X,Y) \mid j^k_{(n)} f \pitchfork W\}$ is a residual subset of $\mathcal{C}^\infty(X,Y)$.

One needs to consider the triple 1-jet of f defined by

$$\begin{array}{cccc} j^1_{(3)}f & : & \Delta_{(3)}X & \to & J^1_{(3)}(X,\mathbb{R}^2) \\ & & (x_1,x_2,x_3) & \mapsto & (x_1,f(x_1),df_{x_1},x_2,f(x_2),df_{x_2},x_3,f(x_3),df_{x_3}) \end{array}$$

where $\Delta_{(3)}X$ is the subset of X^3 with pairwise distinct elements :

$$\Delta_{(3)}X = \{(x_1,x_2,x_3) \in X^3 \mid i \neq j \Rightarrow x_i \neq x_j\}$$

and

$$J^1_{(3)}(X,\mathbb{R}^2) = \{(x_i,y_i,u_i)_{1\le i\le 3} \in (J^1(X,\mathbb{R}^2))^3 \mid (x_1,x_2,x_3) \in \Delta_{(3)}\}$$

Let us consider the submanifold W_2 of $J^1_{(3)}(X,\mathbb{R}^2)$ defined by

$$\begin{array}{rcl} W_2 & = & \{(x_1,y_1,u_1,x_2,y_2,u_2,x_3,y_3,u_3) \mid \\ & & \det u_1 = 0 \wedge \det u_2 = 0 \wedge \det u_3 = 0 \wedge y_1 = y_2 \wedge y_2 = y_3\}\,. \end{array}$$

According to the multijet transversality Theorem, T_{W2} is residual in $\mathcal{C}^\infty(X,\mathbb{R}^2)$. Moreover, W_2 is of codimension 7, and $\Delta_{(3)}$ is of dimension 6. Therefore, with f in T_{W2}, $\{(x_i)_{1\le i\le 3} \in \Delta_{(3)}X \mid j^1_{(3)}f \pitchfork W_2\}$ is empty. In other words, three different singular points do not have the same image.

Let us consider the submanifold W_2' of $J^1_{(2)}$ defined by

$$W_2' = \{(x_1,y_1,u_1,x_2,y_2,u_2) \mid \det u_1 = 0 \wedge \det u_2 = 0 \wedge y_1 = y_2\}\,.$$

W_2' is of codimension 4, therefore if f is in $T_{W_2'}$ then the set of pairs of differents points of S are of dimension 0 (*i.e.* codimension 4 in $\Delta_{(2)}X$). Moreover $W_2' \cap \{\operatorname{im} u_1 = \operatorname{im} u_2\}$ is of codimension bigger than 4, then two different singular points having the same image have normal crossing.

- Other cases can be easily proved by the multi-jet version of the Thom transversality theorem using the same approach.

$\square$

Case i of Theorem 3 is illustrated by Figures 6 and 7.

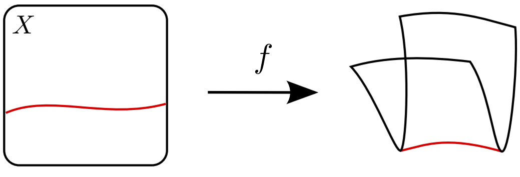


Figure 6: A fold curve.

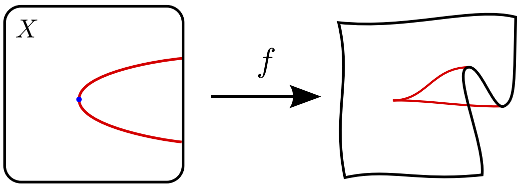


Figure 7: A fold curve with one cusp.

According to Theorem 3, our analysis needs generically to consider the 2 multi-jet of a given smooth map f. Cases ii, iii, iv and v are respectively illustrated by figures 8, 9, 10, 11.

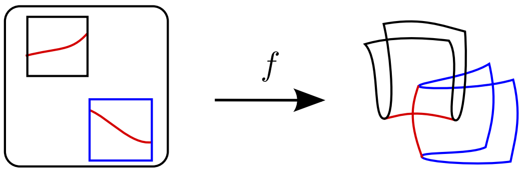


Figure 8: Two fold curves with normal crossing, case ii of Theorem 3.

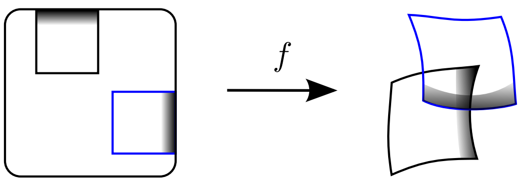


Figure 9: Two boundary curves with normal crossing, case iii of Theorem 3.

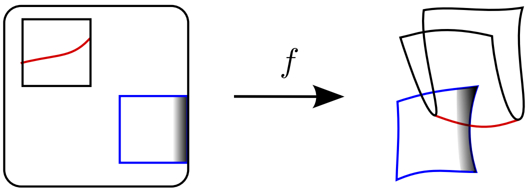


Figure 10: A boundary curve and a fold singular curve with normal crossing in the target space, case iv of Theorem 3.

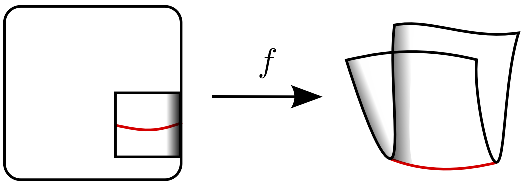


Figure 11: A boundary curve and a fold singular curve with normal crossing in the source space, case v of Theorem 3.

2. Preliminary results

2.1. A sufficient condition to injectivity

Inverse function theorem states that for a given smooth map f from an open subset X of $\mathbb{R}^n$ to $\mathbb{R}^n$ such that df is an isomorphism at $x_0 \in U$, there exits an open neighborhood V of x_0 such that $f|V$ is a diffeomorphism. This is a pure local result, indeed, there exist maps $f : X \to \mathbb{R}^n$ $(n > 1)$ such that df_x is an isomorphism for all $x \in X$ and $f|X$ is not a diffeomorphism. Lemma 4 gives a sufficient condition for a given map to be a diffeomorphism.

Definition 5. Let $f : X \to \mathbb{R}^p$ be smooth and let X be an open subset of $\mathbb{R}^n$. Let us recall to that the differential of f is a map from X to $L(\mathbb{R}^n, \mathbb{R}^p) = \bigoplus_{i=1}^p L(\mathbb{R}^n, \mathbb{R})$ defined by $df_x = (\sum_{j=1}^n \partial_j f_1(x) dx_j, \dots, \sum_{j=1}^n \partial_j f_p(x) dx_j)$. Let us denote by $\tilde{d}f(X)$ the subset of $L(\mathbb{R}^n, \mathbb{R}^p)$ defined by

$$\tilde{d}f_X = \{(\sum_{j=1}^n \partial_j f_1(\xi_1) dx_j, \dots, \sum_{j=1}^n \partial_j f_p(\xi_p) dx_j) \mid \xi_1, \dots, \xi_p \in X\}$$

Note that $df_X = \{df_x \mid x \in X\}$ is a subset of $\tilde{d}f_X$.

Lemma 4 (Injective map). *Let X be a convex compact subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}^p$ a smooth mapping with $n \leq p$. If $\forall J \in \tilde{d}f(X)$, J is an monomorphism then f is an embedding. In other words, f is equivalent to the map $i : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0)$.*

Proof. The proof can be found in [8]. □

2.2. Second result

One of the main problem that we will have to solve is to prove that a given a smooth map $f : X \to \mathbb{R}^2$ restricted to its singular curve is an embedding. It is not possible to directly apply Lemma 4 since the singular curve is not in general a convex subset of X. Lemma 5 shows that the singular curve can be implicitly parametrizated by an ordinary differential equation. Lemma 6 gives a sufficient condition to prove that a map f restricted to a contractible implicitly defined curve is injective.

Lemma 5 (Parametrization). *Let us suppose that X is a compact subset of $\mathbb{R}^2$ and that the real valued function $\Gamma : (x_1, x_2) \in X \mapsto \Gamma(x_1, x_2)$ is a submersion. If the subset $S = \{x \in X \mid \Gamma(x) = 0\}$ is contractible and $(x_1^0, x_2^0) \in S$, then the solution of the initial value problem*

$$\begin{cases} \dot{x}_1 &= \partial_2\Gamma(x_1, x_2) \\ \dot{x}_2 &= -\partial_1\Gamma(x_1, x_2) \\ x_1(0) &= x_1^0 \\ x_2(0) &= x_2^0 \end{cases} \tag{5}$$

is a coordinates system of the curve S.

Moreover, chain rule gives :

$$\begin{cases} \ddot{x}_1 &= -\partial_{12}^2\Gamma\partial_2\Gamma + \partial_{22}^2\Gamma\partial_1\Gamma \\ \ddot{x}_2 &= \partial_{11}^2\Gamma\partial_2\Gamma - \partial_{21}^2\Gamma\partial_1\Gamma \end{cases}$$

Proof. Let us denote by γ the solution of the initial value problem. Since Γ is a submersion, the subset S is regular curve. At $t = 0$, the solution belongs to S since $\gamma(0) = (x_1^0, x_2^0) \in S$. From

$$\begin{aligned} \tfrac{d}{dt}\Gamma(\gamma(t)) &= \partial_1\Gamma(\gamma(t))\dot{\gamma_1}(t) + \partial_2\Gamma(\gamma(t))\dot{\gamma_2}(t) \\ &= \partial_1\Gamma(\gamma(t))\partial_2\Gamma(\gamma(t)) - \partial_2\Gamma(\gamma(t))\partial_1\Gamma(\gamma(t)) \\ &= 0, \end{aligned}$$

one concludes that $\forall t, \gamma(t) \in S$. Since Γ is a submersion, $\forall(x_1, x_2) \in X, (\dot{x}_1, \dot{x}_2) \neq (0, 0)$. Moreover, since X is compact and S contractible there exists an interval I such that

$$\forall x \in S, \exists t \in I \mid \gamma(t) = x$$

So, $\gamma : I \to X$ is a parametrization of S and solution to the ordinary differential equation (5). $\square$

Lemma 6. *Let $f : X \to \mathbb{R}^n$ be a smooth map and X a compact subset of $\mathbb{R}^2$. Let $\Gamma : X \to \mathbb{R}$ be a submersion such that the curve $S = \{x \in X \mid \Gamma(x) = 0\}$ is contractible. If*

$$\forall J \in \tilde{d}f_X \cdot \begin{pmatrix} \partial_2\Gamma(X) \\ -\partial_1\Gamma(X) \end{pmatrix}, J \textit{ is an monomorphism}$$

then $f|S$ is an embedding.

PROOF. Since Γ is a submersion and S contractible, Lemma 5 implies that there exists a parametrization $t \to \gamma(t)$ of S. The assumption that $\tilde{d}f(X) \cdot \begin{pmatrix} \partial_2\Gamma(X) \\ -\partial_1\Gamma(X) \end{pmatrix}$ contains only monomorphism implies that $\tilde{d}(f \circ \gamma)_X$ contains only monomorphism. From Lemma 4, $f \circ \gamma$ is injective. Therefore, since γ is a parametrization of S, $f \mid S$ is an embedding. $\square$

2.3. Third result

Lemma 7 provides a sufficient condition to claim that two given curves intersecting tangentially do not intersect somewhere else. Before proving Lemma 7, one recalls some formulas. Suppose that $\gamma : t \mapsto (y_1, y_2) = (\gamma_1(t), \gamma_2(t))$ is a smooth curve of the plane. If $\dot{\gamma}_1$ is non vanishing, then

$$\frac{dy_2}{dy_1} = \frac{\dot{\gamma}_2}{\dot{\gamma}_1}.$$

By chain rule, one also has :

$$\frac{d^2y_2}{dy_1^2} = \frac{\partial_t \frac{\dot{\gamma}_2}{\dot{\gamma}_1}}{\dot{\gamma}_1} = \frac{1}{\dot{\gamma}_1} \frac{\ddot{\gamma}_2\dot{\gamma}_1 - \dot{\gamma}_2\ddot{\gamma}_1}{\dot{\gamma}_1^2} = \frac{\ddot{\gamma}_2\dot{\gamma}_1 - \dot{\gamma}_2\ddot{\gamma}_1}{\dot{\gamma}_1^3}.$$

Lemma 7 (Two tangential curves). *Let $\alpha : t \mapsto (\alpha_1(t), \alpha_2(t))$ and $\beta : t \mapsto (\beta_1(t), \beta_2(t))$ be two smooth curves such that*

$$\forall t, \begin{cases} \dot{\alpha}_1 & > & 0 \\ \dot{\beta}_1 & > & 0 \end{cases} \tag{6}$$

$$\exists t_\alpha \exists t_\beta \begin{cases} \alpha(t_\alpha) & = & \beta(t_\beta) \\ \frac{\dot{\alpha}_2}{\dot{\alpha}_1}(t_\alpha) & = & \frac{\dot{\beta}_2}{\dot{\beta}_1}(t_\beta) \end{cases} \tag{7}$$

$$\forall t_1 \forall t_2, \frac{\ddot{\alpha}_2\dot{\alpha}_1 - \dot{\alpha}_2\ddot{\alpha}_1}{\dot{\alpha}_1^3} > \frac{\ddot{\beta}_2\dot{\beta}_1 - \dot{\beta}_2\ddot{\beta}_1}{\dot{\beta}_1^3} \tag{8}$$

Then $\alpha(t_1) = \beta(t_2)$ implies $t_1 = t_\alpha$ and $t_2 = t_\beta$.

PROOF. The first assumption implies that one can simultaneously parametrize the two curves with the x-axis. That is to say that α and β are graphs of two functions $\tilde{\alpha}, \tilde{\beta} : \mathbb{R} \to \mathbb{R}$. In this coordinate system, the second and third assumptions respectively means

$$\exists x_0, \begin{cases} \tilde{\alpha}(x_0) & = & \tilde{\beta}(x_0) \\ \partial_x\tilde{\alpha}(x_0) & = & \partial_x\tilde{\beta}(x_0) \end{cases}$$

$$\forall x, \partial^2_{xx}\tilde{\alpha}(x) > \partial^2_{xx}\tilde{\beta}(x)$$

From Taylor formula, it exists ξ such that

$$\tilde{\alpha}(x) - \tilde{\beta}(x) \quad = \quad (\partial^2_{xx}\tilde{\alpha}(\xi) - \partial^2_{xx}\tilde{\beta}(\xi))(x - x_0)^2$$

Therefore, $\tilde{\alpha}(x) = \tilde{\beta}(x)$ implies $x = x_0$. $\square$

3. Cases analysis

3.1. Outline

As presented in Section 1.2, generically speaking, there exists a finite number of local cases. This section analyses each of them, and proposes a method to rigorously enclose these particular points. Subsection 3.2 focuses on cusps where a formulation equivalent to the classic one is proposed. Advantage of this rewriting is that existence and uniqueness of a simple cusp can be proved thanks to the interval Newton method. Moreover, this subsection also gives a sufficient condition that a box containing a unique cusp does not contain two fold points intersecting in the target. Subsection 3.3 provides a method to rigorously enclose couples of fold points intersecting in the target. The idea is again to carefully use the interval Newton method. A sufficient condition to prove that the fold curve does not self intersects in the target space is also given. These two results are combined into an algorithm which divides $X \times X$. Subsections 3.4 and 3.5 respectively analyse cases iii and iv of Theorem 3. These subsections are structurally equivalent to subsection 3.3 taking into account particularities.

3.2. Cusp

Let f be a generic map and p be in $S = \{x \in X \mid \det df_x = 0\}$. One of the following two situations can occur :

1. $T_pS(f) \oplus \ker df_p = T_pX$,
2. $T_pS(f) = \ker df_p$.

where T_pX denotes the tangent space at p of X.

Note that if p is a singularity satisfying condition 1., then p is called a fold point. The Whitney theorem states that there exists a neighborhood V of p such that $f|V$ is equivalent to the map

$$(x_1, x_2) \mapsto (x_1, x_2^2).$$

The case 2, corresponding to a cusp, is considerably more complicated. Let us choose a non vanishing vector field ξ, such that $\xi_x \in \ker df_x$. By assumption, ξ_p is tangent to S at p. The nature of the singularity at p depends on *what order of contact* ξ has with S at p. Let k be a smooth function on X, such that $k|S = 0$ and $dk_p \neq 0$ and consider the function $w : s \mapsto dk_s \cdot \xi_s$. By assumption, one has $w(p) = 0$. Note that the order of this zero does not depend of the choice of ξ or k. A point p is called simple cusp if this zero is a simple zero. The second theorem of Whitney states that if p is a simple cusp, there exists a neighborhood V of p such that $f|V$ is equivalent to the map

$$(x_1, x_2) \mapsto (x_1, x_1x_2 + x_2^3).$$

We use the following proposition and interval Newton method to prove that f restricted to a convex part of X contains an unique simple cusp.

Proposition 8. *Let f be a smooth generic map from X to $\mathbb{R}^2$, let us denote by c the map defined by :*

$$\begin{array}{cccc} c & : & X & \to & \mathbb{R}^2 \\ & & p & \mapsto & df_p \xi_p \end{array} \tag{9}$$

where ξ is the vector field defined by $\xi_p = \begin{pmatrix} \partial_2 \det df_p \\ -\partial_1 \det df_p \end{pmatrix}$.
If $c(p) = 0$ and dc_p is invertible then p is simple cusp. This sufficient condition is locally necessary.

Proof. Let us first prove the sufficiency : $\xi_p \neq 0$ and $c(p) = 0$ mean that $\ker df_p \neq \{0\}$, that is to say that $p \in S$. Since $\mathbb{R}\xi_p = T_pS$, the equation $c(p) = 0$ is equivalent to $T_pS = \ker df_p$.
The local necessity comes from the genericity of f. Let $p \in S$ be a cusp, then generically the smooth function $d \det df$ is non vanishing at p. Therefore, there exists a neighborhood V of p such that $\forall p' \in V, \xi_{p'} \neq 0$. Restricted to this neighborhood, $df_p \xi_p = 0$ implies that $\ker df_p = T_pS (= \mathbb{R}\xi_p)$. From the fact that p is simple, one can obviously deduce that dc is invertible at p. □

Next proposition gives a sufficient condition to prove that a given curve with singularities is injective. The main idea is to have a target space direction that the curve follows.

Proposition 9 (Injective). *Let δ be a smooth curve $\delta : [0,1] \ni t \mapsto \delta(t) \in \mathbb{R}^n$, and let us denote by w the function defined by $w : t \mapsto \alpha \cdot \delta(t)$ where α is a linear form of $\mathbb{R}^n$. If $\dot{w}(t) \geq 0$ and the set $\{t \mid \dot{w}(t) = 0\}$ is finite then δ is injective.*

Proof. The family of level set $D_\lambda = \{x \in \mathbb{R}^n \mid \alpha \cdot x = \lambda\}_{\lambda \in \mathbb{R}}$ is a foliation of the target space with hyperplanes. Condition $\dot{w}(t) \geq 0$ and the set $\{t \mid \dot{w}(t) = 0\}$ is finite simply means that $t \neq t'$ implies $\delta(t) \in D_\lambda$ and $\delta(t') \in D_{\lambda'}$ with $\lambda \neq \lambda'$. Since $\lambda \neq \lambda' \Rightarrow D_\lambda \cap D_{\lambda'} = \emptyset$, δ is injective. □

Last proposition cannot be directly applied to $f|S$ since, in general, we do not have a parametrization of S.

Proposition 10. *Let $f : X \to \mathbb{R}^2$ be a smooth generic map where X is a simply connected compact subset of $\mathbb{R}^2$. Suppose that there exists a unique simple cusp p_0 in the interior of X, let us denote by α a non vanishing covector normal to $\operatorname{im} df_{p_0}$, and ξ a non vanishing vector field such that $\forall p \in S, \xi_p \in T_pS$ (S contractible).*

If $g = \sum \alpha_i \xi^3 f_i : X \to \mathbb{R}$ is a non vanishing function then $f|S$ is injective. This condition is locally necessary.

Here the vector field ξ is seen as the derivation of $\mathcal{C}^\infty(X)$ defined by

$$\xi = \sum \xi_i \partial_{x_i}.$$

Proof. Let us consider the map $\delta : t \mapsto f \circ \gamma$ where γ is a parametrization of S such that $\dot\gamma = \xi$ and $\gamma(0) = p_0$. Let us denote by w the function $w : t \mapsto \alpha \cdot \delta(t)$. By definition, $f \circ \gamma$ admits a cusp point at $t = 0$, in other words $\dot\delta(0) = 0$ (and $\dot w(0) = 0$). One also has $\ddot w(0) = \alpha \cdot \ddot\delta(0) = 0$ since $\ddot\delta(0)$ belongs to $\operatorname{im} df_{p_0}$. Let us now suppose that the function g is non vanishing. One can remark that $g \circ \gamma = \dddot w$.

Finally, the followings assertions are true :

1. $\dot w(0) = 0$,
2. $\ddot w(0) = 0$,
3. $t \mapsto \dddot w(t)$ is non vanishing.

From 1. and 2., one can conclude that 0 is an extremum of $\dot w$. Let us suppose that $g > 0$, then 3. implies that 0 the unique minimum of $\dot w$. Therefore, from Proposition 9, one can conclude that $f|S$ is injective. In the other case ($g < 0$), it suffices to consider $-w$ to conclude. $\square$

Note that from the computational point of view, computing algebraically the expression of g is to expensive in time and space. The following example illustrates the approach and can be handled by hand. In practice, to prove that the restriction to S of a given map f with cusps is injective, one applies the previous proposition using Automatic Differentiation [12].

Example 6. Let us consider the map $(x, y) \mapsto (f_1, f_2) = (x, y^3 - xy)$. The point of coordinate $(0, 0)$ is a simple cusp. In this case, one has : $df = \begin{pmatrix} 1 & 0 \\ -y & 3y^2 - x \end{pmatrix}$, $d \det df = 3y^2 - x$ and $\xi = 6y\partial_x + 1\partial_y$. To prove that $f|S$ is injective, one only has to prove that the function g is non vanishing.

$$g = \alpha_1 \xi^3 f_1 + \alpha_2 \xi^3 f_2$$

with $\alpha = (\alpha_1, \alpha_2) = (0, 1)$ which is orthogonal to $\operatorname{im} df_{(0,0)}$. Finally, since $\xi^3 f_2 = -12$, $f|S$ is injective. Figure 12 illustrates this example.

3.3. Fold curves intersecting in the target

The main goal of this section is to provide an algorithm that rigorously isolates the set of couples composed by different fold points having the same image. That is to say situation illustrated by Figure 8. In other words, we want to find an enclosure of the set

$$S^{\Delta 2} = \{(\alpha, \beta) \in S \times S - \Delta S \mid f(\alpha) = f(\beta)\}/\sim$$

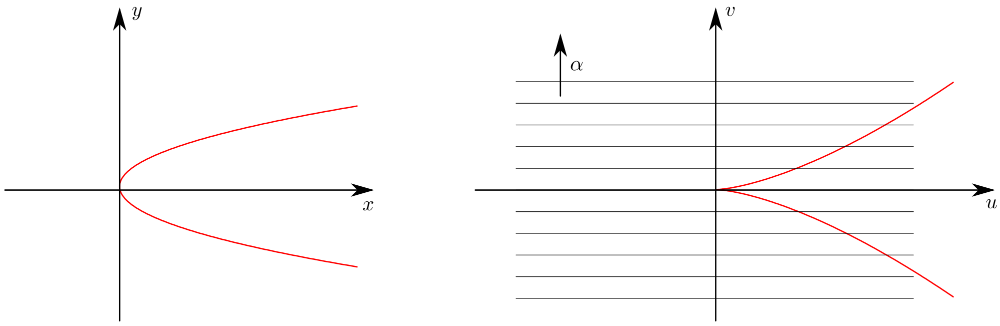


Figure 12: $f|S$ is an injective map and $\{y=\lambda\}_{\lambda\in\mathbb{R}}$ is a foliation of the target space.

where ΔX is the diagonal of given set X :

$$\Delta X = \{(x,x) \in X \times X \mid x \in X\}$$

and $\sim$ is the equivalence relation defined by $(\alpha_1,\beta_1) \sim (\alpha_2,\beta_2) \Leftrightarrow (\alpha_1,\beta_1) = (\beta_2,\alpha_2)$. The proposed approach is based on interval analysis and Lemma 6. The main idea is to decompose $X \times X$ into parts taking into account the relation $\sim$. Let us define the map $folds$ by

$$\begin{array}{cccc} folds & : & X \times X & \to & \mathbb{R}^4 \\ & & \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} & \mapsto & \begin{pmatrix} \det df(x_1,y_1) \\ \det df(x_2,y_2) \\ f_1(x_1,y_1) - f_1(x_2,y_2) \\ f_2(x_1,y_1) - f_2(x_2,y_2) \end{pmatrix} \end{array} \tag{10}$$

The set $S^{\Delta 2}$ is directly connected to the map $folds$ by the following relation

$$S^{\Delta 2} = folds^{-1}(\{0\}) - \Delta S / \sim .$$

The equivalence relation $\sim$ will be taken into account during the subdivision process. It only remains to enclose $folds^{-1}(\{0\}) - \Delta S$. The proposed approach is partially based on Interval Newton method. This method can be used to prove existence and uniqueness of a zero over a given box. It can also be employed to guarantee that a given box does not contain any zero. Interval Newton method is classically coupled with a subdivision scheme. If the Newton Interval method does neither success nor prove absence of a zero with a box b, this box is devided into boxes and the method is called recursively. Using roughly this method to enclose zeros of the map $folds$ will fail. Indeed, the invertibility of the differential at the zero is necessary condition to success. This condition is not fulfilled for points belonging to

ΔS. For any (α, α) in ΔS, the differential of $folds$ is conjugate to

$$\begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & a & b \\ a_{11} & a_{12} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{21} & a_{22} \end{pmatrix}$$

which is not invertible since $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \det df(\alpha) = 0$. In other words, as any box of the form $[\alpha] \times [\alpha]$ contains ΔS, the interval Newton method will fail. To get round this difficulty, one can remark that the assertion $\{(\alpha, \beta) \in S \times S - \Delta S \mid f(\alpha) = f(\beta)\} \cap [x] \times [x] = \emptyset$ is equivalent to $f|S \cap [x]$ is an embedding. Our algorithm checks whether or not the sufficient condition of Lemma 6 is satisfied with Γ being $\det df$, proving that $f|S \cap [x]$ is an embedding. This sufficient condition can not be fulfilled if the current set $[x]$ contains a cusp. In such a case, Proposition 10 is employed.

Algorithm 1 Calculate a rigorous enclosure of $S^{\Delta 2}$

Require: • a smooth map $f : X \to \mathbb{R}^2$ where X a simply connected compact subset of $\mathbb{R}^2$,

Ensure: A discrete set $P = \{s_j\}_j$ of pairs of 2-dimensional boxes such that

$$S^{\Delta 2} \subset \bigcup_j s_j, \text{ and } \forall j \exists ! x, x \in s_j \cap S^{\Delta 2}.$$

Initialisation : $P \leftarrow \emptyset$, $P' \leftarrow \{X \times X\}$.
while $P' \neq \emptyset$ **do**
  $[x_1] \times [x_2] \leftarrow s$ where $s \in P'$.
  $P' \leftarrow P' - \{s\}$.
  if $[x_1] \neq [x_2]$, **then**
    if Interval Newton algorithm for (10) on $[x_1] \times [x_2]$ succeed **then**

$$P \leftarrow P \cup \{[x_1] \times [x_2]\}.$$

    else
      Divide $[x_1]$ into $[x_1^a]$ and $[x_1^b]$.
      Divide $[x_2]$ into $[x_2^a]$ and $[x_2^b]$.
      $P' \leftarrow P' \cup \{[x_1^a] \times [x_2^a]\} \cup \{[x_1^a] \times [x_2^b]\} \cup \{[x_1^b] \times [x_2^a]\} \cup \{[x_1^b] \times [x_2^b]\}$;
    end if
  else
    if $f|S \cap [x_1]$ is not an embbeding, **then**
      Divide $[x_1]$ into $[x_1^a]$ and $[x_1^b]$.
      $P' \leftarrow P' \cup \{[x_1^a] \times [x_1^a]\} \cup \{[x_1^a] \times [x_1^b]\} \cup \{[x_1^b] \times [x_1^b]\}$;
    end if
  end if
end while

3.4. Boundary points intersecting in the target

This section concerns the case iii of Theorem 3 : two different points belonging to the boundary can have the same image. That is to say situation illustrated by Figure 9. The boundary is supposed to be defined by $\partial X = \{x \in X \mid \Gamma(x) = 0\}$ with Γ a submersion at least on ∂X. More precisely, one wants to rigorously enclose the set

$$\partial X^{\Delta 2} = \{(\alpha, \beta) \in \partial X \times \partial X - \Delta \partial X \mid f(\alpha) = f(\beta)\} / \sim$$

As before, the interval Newton method is used to find zeros of the map

$$\begin{array}{ccc} X \times X & \to & \mathbb{R}^4 \\ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} & \mapsto & \begin{pmatrix} \Gamma(x_1, y_1) \\ \Gamma(x_2, y_2) \\ f_1(x_1, y_1) - f_1(x_2, y_2) \\ f_2(x_1, y_1) - f_2(x_2, y_2) \end{pmatrix} \end{array} \tag{11}$$

The main steps of algorithm to enclose $\partial X^{\Delta 2}$ are almost the same as for $S^{\Delta 2}$. Generically, tangents at this points are mapped differently, this ensures that the differential of (11) is invertible at zero. As for the last case, interval Newton method fails on the diagonal. To guarantee that a given set of the form $A \times A$ does not contain any elements of $\partial X^{\Delta 2}$, the sufficient condition of Lemma 6 is checked.

3.5. A fold and a boundary intersecting

This section concerns the two last cases that can happen generically and proposes an algorithm to enclose the following set

$$A = \{(\alpha, \beta) \in S \times \partial X \mid f(\alpha) = f(\beta)\}.$$

We consider two subcases and see the set A as the disjoint union of sets defined by

$$A - \Delta A = \{(\alpha, \beta) \in A \mid \alpha \neq \beta\},$$
$$\Delta A = \{(\alpha, \beta) \in A \mid \alpha = \beta\}.$$

The set $A - \Delta A$ can be enclose thanks to the interval Newton method applied to the following map :

$$\begin{array}{ccc} X \times X & \to & \mathbb{R}^4 \\ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} & \mapsto & \begin{pmatrix} \det df(x_1, y_1) \\ \Gamma(x_2, y_2) \\ f_1(x_1, y_1) - f_1(x_2, y_2) \\ f_2(x_1, y_1) - f_2(x_2, y_2) \end{pmatrix}. \end{array} \tag{12}$$

This method cannot be used for points belonging to ΔA since the differential is not invertible. Indeed, the last two rows are identical. However, zeros of the following map

$$\begin{array}{ccc} X & \to & \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} & \mapsto & \begin{pmatrix} \det df(x_1, y_1) \\ \Gamma(x_1, y_1) \end{pmatrix} \end{array} \tag{13}$$

are intersections of folds curve with boundary in the source. Generically, tangents at this points are mapped differently, this ensures that the differential is invertible at this point. Suppose that the interval Newton method succeeds and proves that there a zero for a given subset $[x]$, this is not enough. Roughly speaking, one knows that $\Delta A \cap [x]$ is a point. The problem is similar that what happened for folds, one needs to prove that this point is the only one in A (*i.e.* $A \cap [x]$ is a point). To prove this, sufficient condition of Lemma 7 is checked with α and β respectively implicitly being defined by Γ and $\det df$.

Our method is summerized by the following algorithm :

Algorithm 2 A rigorous enclosure of A

Require:

- a smooth map $f : X \to \mathbb{R}^2$.

Ensure: A discrete set $P = \{s_j\}_j$ of 4-dimensional boxes such that

$$A \subset \bigcup_j s_j, \text{ and } \forall j \exists! x, x \in s_j \cap A.$$

Initialisation : $P \leftarrow \emptyset$, $P' \leftarrow \{X \times X\}$,
while $P' \neq \emptyset$ **do**
  $[x_1] \times [x_2] \leftarrow s$ where $s \in P'$.
  $P' \leftarrow P' - \{s\}$.
  if Interval Newton method for (12) on $[x_1] \times [x_2]$ succeed **then**

$$P \leftarrow P \cup \{[x_1] \times [x_2]\}.$$

  else
    if Interval Newton method for (13) on $[x_1] \times [x_2]$ succeed and condition of Lemma 7 is fullfilled **then**

$$P \leftarrow P \cup \{[x_1] \times [x_2]\}.$$

    else
      Divide $[x_1]$ into $[x_1^a]$ and $[x_1^b]$.
      Divide $[x_2]$ into $[x_2^a]$ and $[x_2^b]$.
      $P' \leftarrow P' \cup \{[x_1^a] \times [x_2^a]\} \cup \{[x_1^a] \times [x_2^b]\} \cup \{[x_1^b] \times [x_2^a]\} \cup \{[x_1^b] \times [x_2^b]\}$;
    end if
  end if
end while

---

4. Synthesis

The section proposes a method able to compute a graph homeomorphic to the Apparent Contour of a given generic smooth map. The main idea is to apply previous algorithms to rigorously enclose the critical points and then to connect them.

4.1. Connecting elements

Let f be a generic smooth map from a compact simply connected domain X of $\mathbb{R}^2$ to $\mathbb{R}^2$. Let us denote by X_0 the union of cusps and double points. Formally, one has :

$$X_0 = \left\{ x \in X \mid \begin{array}{ll} & x \text{ is a cusp} \\ \text{or} & \exists \tilde{x}, (x, \tilde{x}) \in S^{\Delta 2} \\ \text{or} & \exists \tilde{x}, (x, \tilde{x}) \in \partial X^{\Delta 2} \\ \text{or} & \exists \tilde{x}, (x, \tilde{x}) \in A \text{ or } (\tilde{x}, x) \in A \end{array} \right\}$$

and let Y_0 be the set $f(X_0)$. Let X_1 be the set $S \cup \partial X - X_0$ and $Y_1 = f(X_1)$. Generically, sets X_0 and Y_0 are discrete with $\#Y_0 \leq \#X_0$. Moreover (Y_0, Y_1) (respectively (X_0, X_1)) is a stratification of the Apparent Contour (respectively $S \cup \partial X$). By construction, $f|X_1$ is injective, i.e. X_1 are Y_1 are homeomorphic.

The mapping f induces a relation on X defined by $\alpha f \beta \Leftrightarrow f(\alpha) = f(\beta)$. This relation is an equivalence relation and the quotient X/f is well defined. Generically, the following statements are true :

- X_0/f is homeomorphic to Y_0,
- $X_0 \cup X_1/f$ is homeomorphic to the Appararent contour $Y_0 \cup Y_1$.

In pratice, to compute the topology of the Apparent Contour, our approach is to, first, devide the source set X with a covering $\{p_i\}$ and to glue the pieces together. The output of this first step is a simple graph which is homeomorphic to the singular set $S \cup \partial X = X_0 \cup X_1$. From this ouput and thanks to the equivalence relation f, a graph homeomorphic to the Apparent contour is computed. The following theorem summarises the principle of the method:

Theorem 11. *Let $P = \{p_i\}_{1 \leq i \leq n}$ be a covering such that*

i) $S \cup \partial X \subset \cup_i p_i$,
ii) $\forall (p, q), p \cap q \neq \emptyset \Rightarrow (S \cup \partial X) \cap p \cap q$ *is simply connected,*
iii) $\forall p, X \cap p$ *contains at most one element of* X_0,

Let $\mathcal{X}$ be the relation on $\{p_i\}_{1 \leq i \leq n}$ defined by

$$p \mathcal{X} q \Leftrightarrow (S \cup \partial X) \cap p \cap q \text{ is simply connected.}$$

Let us define an equivalence relation f on $\{p_i\}$ by

$$p f q \Leftrightarrow f(X_0 \cap p) = f(X_0 \cap q) \text{ and } X_0 \cap p \neq \emptyset,$$

then $\mathcal{X}/f$ is homeomorphic to the Apparent contour.

Proof. To prove that $\mathcal{X}/f$ is homeomorphic to the Apparent contour, let us first prove that $\mathcal{X}$ is homeomorphic to $S \cup \partial X$. The proof is based on the construction of a homeomorphism ϕ from a geometric realization of $\mathcal{X}$ to $\partial X \cup S$.

As shown in the introduction of this subsection, (X_0, X_1) is a stratification of $S \cup \partial X$. Let us denote by P_0 the set $\{p \in P \mid X \cap p \text{ is a singleton}\}$. By hypothesis, X_0 is homeomorphic to P_0 since the stata X_0 is the union of cusps and double points. Condition ii implies $\forall p \in P, (S \cup \partial X) \cap p$ is simply connected. In particular, for each $p \in P_0$, $(S \cup \partial X) \cap p$ is homeomorphic to one of the following sets :

Figure 13: P_0 contains either a cusp, an element of $S^{\Delta 2}$, an element of $\partial X^{\Delta 2}$ or an element of $A - \Delta A$ (left), P_0 contains an element of ΔA (right).

For any p of P_0, let us denote by $\hat{p} \in p$ the element of X_0 and by P_1 the set $P - P_0$. Condition ii also implies that for any $p \in P_1$, $(S \cup \partial X) \cap p$ is homeomorphic to a curve passing through the box p. For any $p \in P_1$, let us choose an arbitrary point $\hat{p}$ of this curve (note that $\#\{\hat{p}\} = \#P$). Condition ii implies that for any couple (p, q) in relation $p\mathcal{X}q$, there exists a homeomophism from $[0, 1]$ to the curve connecting $\hat{p}$ to $\hat{q}$ in $\partial X \cup S \cup p \cup q$ (i.e. $\phi_{p,q}(0) = \hat{p}$ and $\phi_{p,q}(1) = \hat{q}$). As a consequence, for any $(p, q) \in P^2$ such that $p\mathcal{X}q$, there exists a homeomorphism $\phi_{p,q}$ from $[p, q]$ to the curve connecting $\hat{p}$ to $\hat{q}$ in $\partial X \cup S \cup p \cup q$ (where $[p, q]$ is the geometric realization of $p\mathcal{X}q$). In conclusion, let us denote by ϕ the map from $\mathcal{X}$ to $\partial X \cup S$ defined by : $\phi(t) = \phi_{p,q}(t)$ if t belongs to $[p, q]$. Therefore, ϕ is a homeomorphism from a geometric realization of $\mathcal{X}$ to the Apparent Contour.

The end of the proof is simply based on the quotient defined by the equivalence relation f. $\square$

In practice, the method recursively divides X until the covering satisfies conditions of Theorem 11. The first condition and the extended relation f are obtained from enclosure computed in Section 3. Presented algorithm in [13] is applied to guarantee that a given curve is simply connected. Finally, the computed graph $\mathcal{X}/f$ is an important invariant of a given map since :

Theorem 12. *Let f be a generic smooth map from a compact simply connected domain X of $\mathbb{R}^2$ to $\mathbb{R}^2$. For every portrait F of f, the 1-skeleton of $\operatorname{im} F$ contains a subgraph that is an expansion of $\mathcal{X}/f$.*

4.2. Examples

This section is composed of 4 examples. Example 7 illustrates the principle of the approach introduced in section 4.1, whereas examples 8 and 9 are numerical examples.

Example 7. Let us consider the following map :

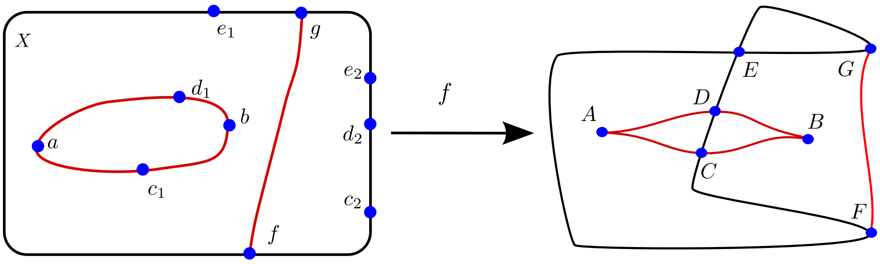


Figure 14: A stratification of the critical set and the Apparent Contour of f.

For this generic map, X_0 is the finite set $\{a, b, c_1, c_2, d_1, d_2, e_1, e_2, f, g\}$ and X_1 has 11 connected components. More precisely, a and b are cusps, $S^{\Delta 2}$ is empty, (c_1, c_2) and (d_1, d_2) are in A, (e_1, e_2) belongs to $\partial X^{\Delta 2}$, and (f, f) and (g, g) are in ΔA. Elements of the image set $Y_0 = f(X_0)$ are denoted by $\{A, B, C, D, E, F, G\}$ where each upper case is used as the image of the lower case ($A = f(a)$, $B = f(b)$, $\dots$). The only elements of X_0 that are in relation through the equivalence relation f are therefore : $c_1 f c_2, d_1 f d_2$ and $e_1 f e_2$. Furthermore, X_0/f can be identified with $\{a, b, c_1, d_1, e_1, f, g\}$ which is homeomorphic to $\{A, B, C, D, E, F\}$. Y_1 also has 11 connected components since $f|X_1$ is injective.

The following figure proposes a covering $\{p_i\}$ of $S \cup \partial X$ satisfying conditions of Theorem 11.

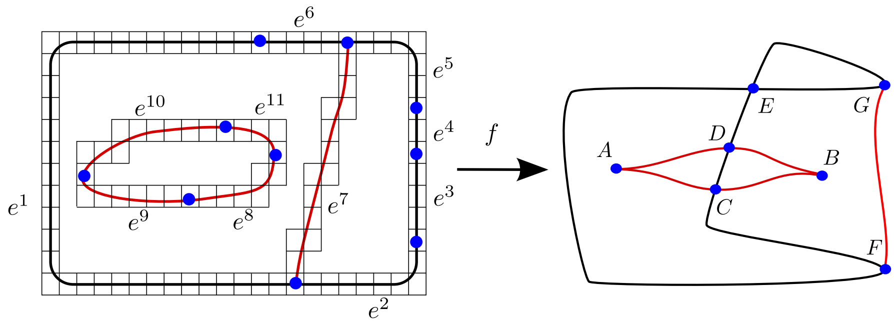


Figure 15: A stratification of critical set and the Apparent Contour of f.

From this covering, and the relation f, one can formally compute the following incidence matrix of the graph $\mathcal{X}$:

$$\mathcal{X} = \begin{array}{c} \\ a \\ b \\ c_1 \\ c_2 \\ d_1 \\ d_2 \\ e_1 \\ e_2 \\ f \\ g \end{array} \begin{array}{c} \begin{array}{ccccccccccc} e^1 & e^2 & e^3 & e^4 & e^5 & e^6 & e^7 & e^8 & e^9 & e^{10} & e^{11} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{14}$$

This graph was obtained by smoothing out the computed graph with respect to nodes that are in P_1. This operation reduces the number of nodes without changing its topology and is compatible with the relation f. It can be quotiented by the equivalence relation f :

$$\mathcal{X}/f = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} \begin{array}{c} \begin{array}{ccccccccccc} e^1 & e^2 & e^3 & e^4 & e^5 & e^6 & e^7 & e^8 & e^9 & e^{10} & e^{11} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{15}$$

This quotient can be obtained row by row. Each row Y of $\mathcal{X}/f$ is simply the addition of any rows y_i such that $f(y_i) = Y$, e.g. $C = c_1 + c_2$. According to

Theorem 11, $\mathcal{X}/f$ is homeomorphic to the Apparent contour of f.

Example 8. This example is purely academic, and illustrates the adaptivity of the proposed method :

$$\begin{array}{cccc} f: & [-1,4]\times[-0.4,0.6] & \to & \mathbb{R}^2 \\ & \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} 0.5x+(1-y^2)\cos(x+0.2y) \\ (1-y^2)\sin(x) \end{pmatrix} \end{array}$$

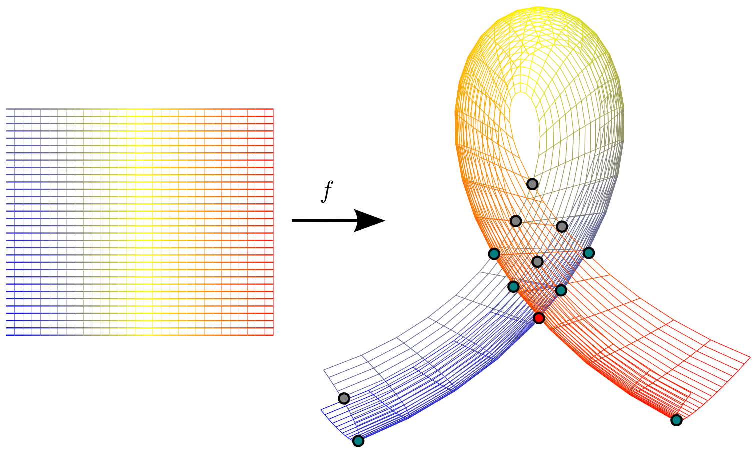


Figure 16: Illustration of the map of Example 8.

All computations were performed on a Intel Core i7, 64 bit computer with 3887 MB of RAM. The program was compiled with gcc, version 4.6.3. The software for interval arithmetic was provided by the filib++ Interval Library version 3.0.2 (See [15]). The solver proves X_0 has 22 elements. More precisely, the output of the solver was the following :

1. the number of cusp points is exactly 0,
2. the number of transversal intersection of fold curves in the target is exactly 1,
3. the number of transversal intersection of boundary curves in the target is exactly 5,
4. the number of intersection fold curves with boundary is exactly 6.

The topology of the Apparent Contour is the following :

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

The entire run time was 1 min 33 s. To compute the topology of $S \cup \partial X$, the source set has been divided with a regular covering composed by 27^2 cells. The solver, available on the web page of the author, is named Thom in honor of the great mathematician René Thom.

Example 9. To illustrate that the presented method also works for map with cusps, we consider the following map :

$$
\begin{array}{llll}
f: & [-0.2, 0.33] \times [-0.7, 0.61] & \to & \mathbb{R}^2 \\
 & \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} x + 0.2y \\ y^3 - xy \end{pmatrix}
\end{array}
$$

The solver proves that the number of intersection fold curves with boundary is exactly 3 and that this map has 1 cusp. The software is also able to generate Figure 17. The entire run time was 2 s including the generation of the Figure.

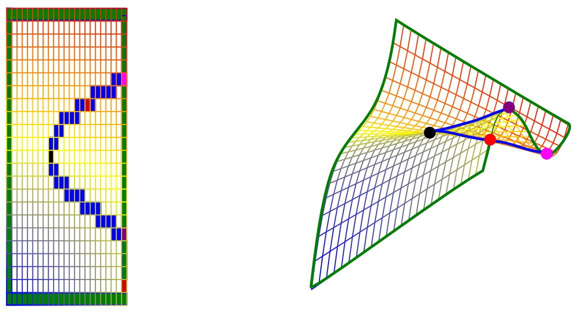


Figure 17: Generated Figure from Thom.

Note that Lemma 6 has been extented to curves that are only piecewise smooth to take into account the fact that the boundary of the domain is not smooth.

Example 10. As our final example, let us consider the map :

$$\begin{array}{rccc} f: & X = [-1,1] \times [-1,1] & \to & \mathbb{R}^2 \\ & \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \end{array}$$

The proposed method is not able to handle this map. Indeed, the singular set is $\{x = 0\} \cup \{y = 0\}$ and the origin is neither a fold nor a cusp. For this example, Algorithm 1 will not terminate. Indeed, the real map defined by $(x, y) \mapsto \det df_{(x,y)} = 4xy$ is not a submersion at the origin. For each box of the form $[x_1] \times [x_1]$ containing the origin, $f|S \cap [x_1]$ is not an embedding and therefore the algorithm will run forever.

5. Conclusions

In this paper, we presented a validated method for computing a graph homeomorphic to the Apparent Contour of a given map f. The method is based on constructing a graph homeomorphic to singular curve in the source set. The functionality of the designed algorithms, implemented in c++, was confirmed on some selected illustrating examples. As mentioned in the introduction, this output is of great importance since it is an invariant modulo the action of diffeomorphisms. It is not an enought stronger invariant to classify maps since two non equivalent maps can have two homeomorphic Apparent Contours. Nevertheless, one conjectures that

Conjecture 1. *From a graph homeomorphic to the Apparent Contour of a given smooth map f and its right embedding in $\mathbb{R}^2$, it is possible to create a portrait for f.*

[1] P. S. Donelan, Singularity-theoretic methods in robot kinematics, Robotica 25, Cambridge University Press (2007), 641-659.

[2] Zlatanov, D. Generalized Singularity Analysis of Mechanisms. PhD thesis, University of Toronto, (1998).

[3] T. Isenberg, B. Freudenberg, N. Halper , S. Schlechtweg and T. S Trothotte. A developers guide to silhouette algorithms for polygonal models, IEEE Comput. Graph. Appl. 23 (2003), 28-37.

[4] C. R. F. Maunder, Algebraic Topology, Courier Dover Publications, (1996).

[5] R.E. Moore, Interval Analysis, Prentice-Hall, Englewood Cliffs, NJ, (1966).

[6] R.E. Moore, Methods and Applications of Interval Analysis, SIAM Studies in Applied Mathematics, Philadelphia, (1979).

[7] A. Neumaier, Interval Methods for Systems of Equations, Encyclopedia of Mathematics and its Applications, vol. 37, Cambridge Univ. Press, Cambridge, (1990).

[8] S. Lagrange, N. Delanoue, L. Jaulin, Injectivity Analysis using Interval Analysis : Application to Structural Identifiability. Automatica, Volume 44, Issue 11, (2008), 2959-2962.

[9] H. Whitney. On singularities of mappings of Euclidean space. I. Mappings of the plane to the plane. Ann. Math. 62 (1955), 374-410.

[10] H. Edelsbrunner, D. Morozov and A. K. Patel. The stability of the apparent contour of an orientable 2-manifold, Topological Data Analysis and Visualization, 27-42, (2011).

[11] M. Golubitsky, V. Guillemin, Stable Mappings and Their Singularities. Springer Verlag, New York, (1973).

[12] A. Griewank, A. Walther, Evaluating Derivatives: Principles and Techniques of Algorithmic Differentiation, Other Titles in Applied Mathematics. 105 (2nd ed.) (2008)

[13] N. Delanoue, L. Jaulin, B. Cottenceau, Guaranteeing the Homotopy Type of a Set Defined by Non-Linear Inequalities, Reliable Computing, Volume 13, Issue 5, (2007) 381-398

[14] M. Paolini, A software code to manipulate apparent contours, Oberwolfach Reports, 4 (2007), 179-181. Available from `http://www.dmf.unicatt.it/~paolini/`

[15] Lerch, M., Tischler, G., Wolff von Gudenberg, J., Hofschuster, W., Krmer, W.: a filib++, a Fast Interval Library Supporting Containment Computations. ACM TOMS 32(2), 299 - 324 (2006) Available from `http://www2.math.uni-wuppertal.de/~xsc/software/filib.html`