Interval analysis and Optimal Transport

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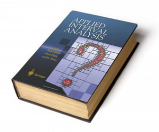
Constraints & Geometry
Mine de Nantes
http://www.ibex-lib.org/workshop-23-06-14

23th June 2014

Outline

Example with books



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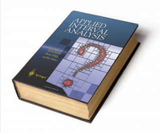
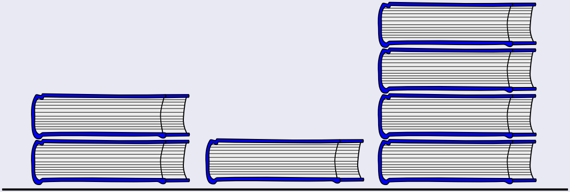


Example in the discrete case



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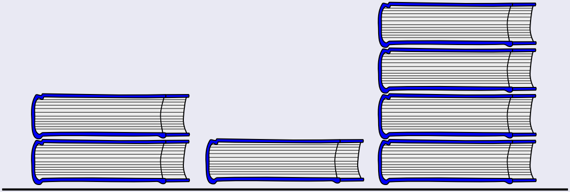

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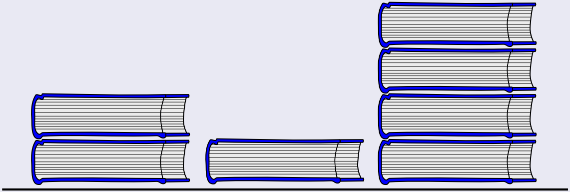

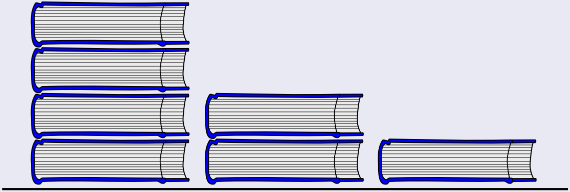

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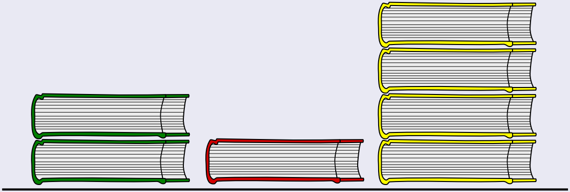

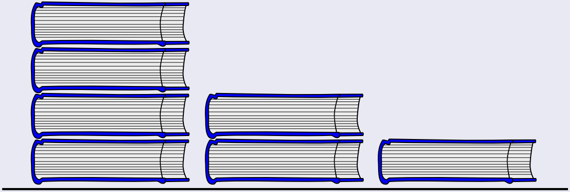

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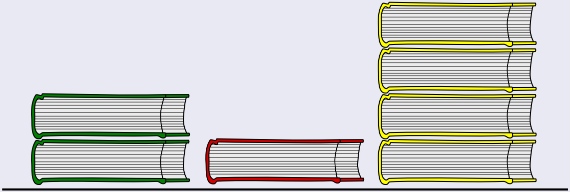

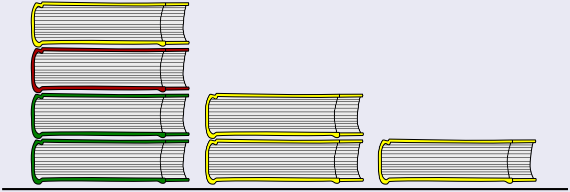

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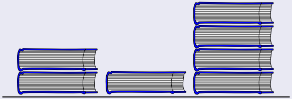
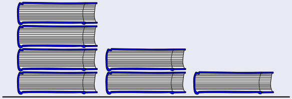


$$\mu = (2, 1, 4)$$

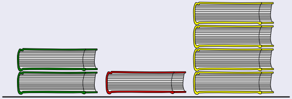


$$\nu = (4, 2, 1)$$

Transportation

	4	2	1
2			
1			
4			

Example in the discrete case



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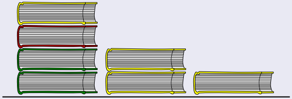

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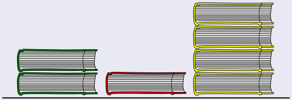

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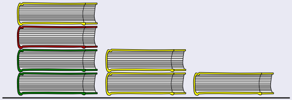

$$\nu = (4, 2, 1)$$

A plan transference π

	4	2	1
2	2	0	0
1	1	0	0
4	1	2	1

,

$$\pi = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

Plan transference problem

	4	2	1
2	.	.	.
1	.	.	.
4	.	.	.

Plan transference problem

$$\begin{array}{c|ccc} & 4 & 2 & 1 \\ \hline 2 & . & . & . \\ 1 & . & . & . \\ 4 & . & . & . \end{array}$$

Solutions

$$\pi = \begin{array}{c|ccc} & 4 & 2 & 1 \\ \hline 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 1 \end{array}, \quad \tilde{\pi} = \begin{array}{c|ccc} & 4 & 2 & 1 \\ \hline 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 4 & 2 & 2 & 0 \end{array}.$$

Definition - Transference plan

A transference plan (or a transportation) π is a measure on the product space $X \times Y$ such that

$$\begin{cases} \pi(A \times Y) = \mu(A), \\ \pi(X \times B) = \nu(B). \end{cases}$$

all measurable subsets A of X and B of Y.

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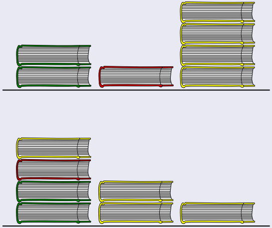
In the discrete case

$$\begin{cases} \forall i,\ \sum_j \pi_{ij} = \mu_i, \\ \forall j,\ \sum_i \pi_{ij} = \nu_j. \end{cases}$$

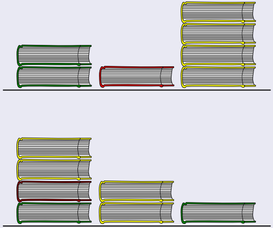
Comparing two plan transferences

Transportation π



$I(\pi) = 0 + 0 + 1 + 2 + 1 + 1 + 0 = 5$

Transportation $\tilde{\pi}$



$I(\tilde{\pi}) = 0 + 2 + 1 + 2 + 2 + 1 + 1 = 9$

In the discrete case

$$\begin{aligned} \min_{\pi \in \mathbb{R}^n \otimes \mathbb{R}^m} \quad & \sum_{i,j} c_{ij}\pi_{ij} \\ \text{subject to} \quad & \forall i,\ \sum_j \pi_{ij} = \mu_i, \\ & \forall j,\ \sum_j \pi_{ij} = \nu_j. \end{aligned} \tag{1}$$

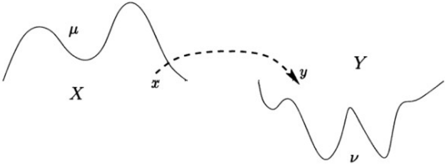
where c_{ij} are non negative real numbers which tells how much it costs to transport one unit of mass from location i to location j.

Kantorovich formulation

The *optimal transportation cost* between μ and ν is the value :

$$\begin{aligned} \mathcal{T}_c(\mu,\nu) = \inf_{\pi \in \mathcal{B}(X\times Y)} \quad & \int_{X\times Y} c(x,y)\mathrm{d}\pi(x,y) \\ \text{subject to} \quad & \pi_X = \mu, \\ & \pi_Y = \nu \end{aligned} \tag{2}$$

The optimal π's, i.e. those such that $I(\pi) = \mathcal{T}_c(\mu,\nu)$, if they exist, will be called *optimal transference plans*.



Remark

The *optimal transportation problem* is an infinite dimensional linear programming problem.
i.e. I is a linear cost function, and constraints are linear.

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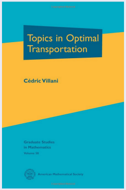
3. many others in



Topics in Optimal Transportation, Cédric Villani, AMS (2003)

Proposition - Relaxation

Let μ and ν (with support X and Y) be absolutely continuous measures with respect to Lebesgue measure. If $\{X_i\}_i$ and $\{Y_i\}_i$ be finite pavings of X and Y. Suppose that $\mu(X_i) \in [\underline{\mu}_i, \overline{\mu}_i]$, $\nu(Y_j) \in [\underline{\nu}_j, \overline{\nu}_j]$, and $\forall x, y \in X_i \times Y_j, \underline{c}_{ij} \leq c(x, y)$,

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$$
\begin{aligned}
\underline{\mathcal{T}} = \min_{\pi_{ij} \in \mathbb{R}^n \otimes \mathbb{R}^m} \quad & \sum_{i,j} \underline{c}_{ij} \pi_{ij} \\
\text{subject to} \quad & \forall i,\ \underline{\mu}_i \leq \sum_j \pi_{ij} \leq \overline{\mu}_i, \\
& \forall j,\ \underline{\nu}_j \leq \sum_i \pi_{ij} \leq \overline{\nu}_j, \\
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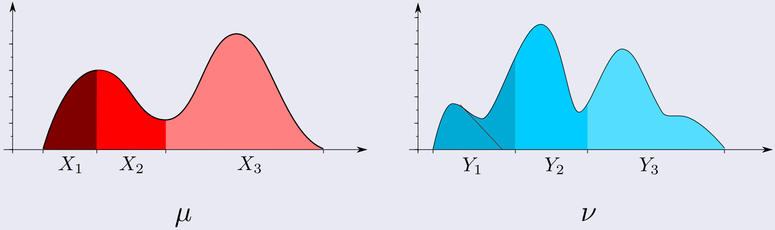
$$\text{then} \qquad \underline{\mathcal{T}} \leq \mathcal{T}_c(\mu, \nu).$$

Spatial discretization



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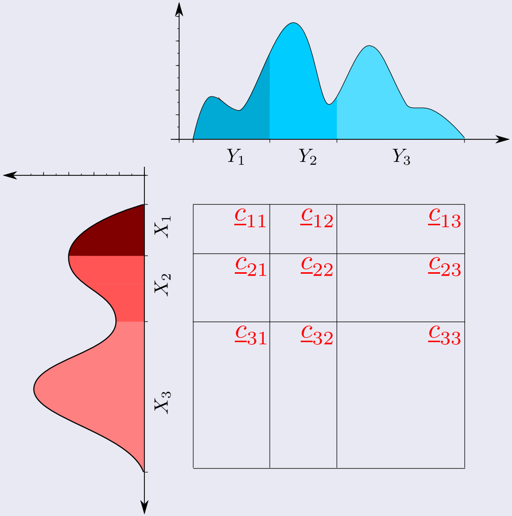
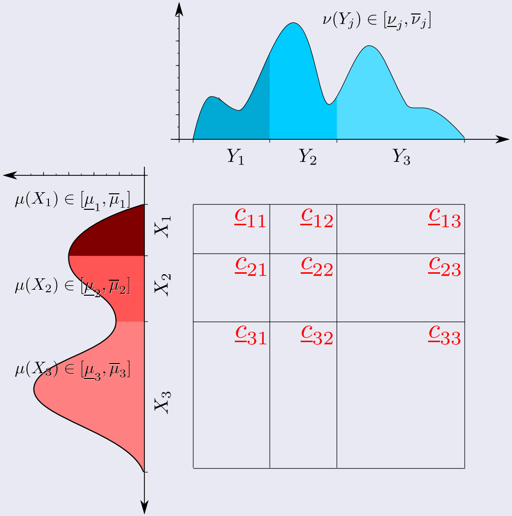


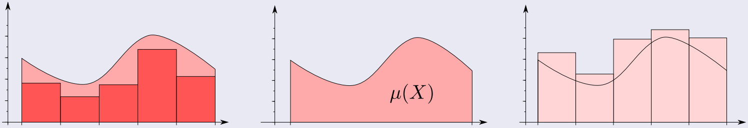
Spatial discretization



Enclosing

If $\mu = f(x)\mathrm{d}x$, and $[f]$ an inclusion function for f then

$$\int_X f(x)\mathrm{d}x \in \sum_i [f](X_i)\lambda(X_i)$$



$\underline{\mu}_i$ $\quad$ $\mu(X)$ $\quad$ $\overline{\mu}_i$

$$\sum_i \underline{f}(X_i)\lambda(X_i) \leq \int_X f(x)\mathrm{d}x \leq \sum_i \overline{f}(X_i)\lambda(X_i)$$

Proof

Let $\{X_i\}$, $\{Y_j\}$ be a pavings, let $\pi_{ij} = \pi(X_i \times Y_j)$ then $\forall \pi, \exists \xi_{ij} \in X_i \times Y_j$,

$$\sum_{i,j} c(\xi_{ij})\pi_{ij} = \int_{X \times Y} c(x,y)\mathrm{d}\pi(x,y) \tag{3}$$

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Since $\underline{c}_{ij} \leq c(\xi_{ij})$ and $\pi_{ij} \geq 0$, then
$\forall \pi$,

$$\sum_{i,j} \underline{c}_{ij}\pi_{ij} \leq \int_{X\times Y} c(x,y)\mathrm{d}\pi(x,y) \tag{4}$$

Proof

Let μ and ν (with support X and Y) be absolutely continuous measures with respect to Lebesgue measure. If $\{X_i\}_i$ and $\{Y_i\}_i$ be finite pavings of X and Y. Suppose that $\mu(X_i) \in [\underline{\mu}_i, \overline{\mu}_i]$, $\nu(Y_j) \in [\underline{\nu}_j, \overline{\nu}_j]$, and $\forall x, y \in X_i \times Y_j, \underline{c}_{ij} \leq c(x, y)$,

$$\begin{aligned} \mathcal{K} = \min_{\pi_{ij} \in \mathbb{R}^n \otimes \mathbb{R}^m} \quad & \sum_{i,j} \underline{c}_{ij} \pi_{ij} \\ \text{subject to} \quad & \forall i,\ \underline{\mu}_i = \sum_j \pi_{ij} = \overline{\mu}_i, \\ & \forall j,\ \underline{\nu}_j = \sum_i \pi_{ij} = \overline{\nu}_j, \\ & \forall i, \forall j,\ \pi_{ij} \geq 0. \end{aligned}$$

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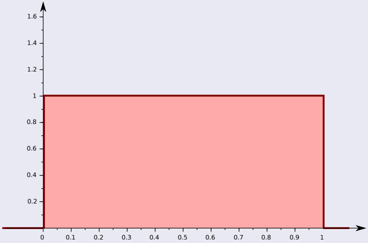
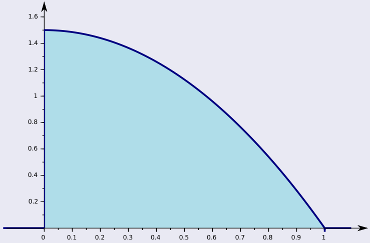
$$\text{then} \qquad \underline{\mathcal{T}} \leq \mathcal{T}_c(\mu, \nu).$$

Example

$X = Y = [0, 1]$





$\mu = 1\mathrm{d}x$ $\qquad$ $\nu = 1.5(1 - y^2)\mathrm{d}y$

$$c(x, y) = \|x - y\|^2$$

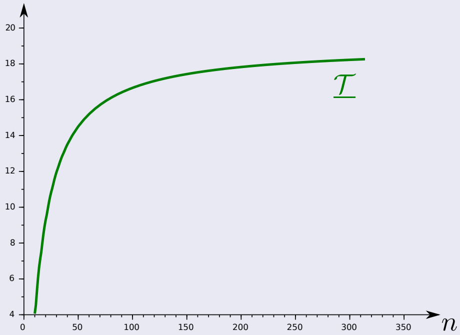


Figure : Guaranteed lower bounds of $\mathcal{T}_c(\mu,\nu)$ where $n = \mathrm{Card}\{X_i\}_i$

Outline

1. Introduction to Optimal Transport
   - Transportation
   - Optimal Transport
   - Some known results
2. A lower bound of the optimal value
   - Finite dimensional relaxation
3. An upper bound of the optimal value
   - Duality
   - Finite dimensional relaxation
4. Conclusion - Future work

Linear programming - Duality

Primal problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{subject to} \quad & Ax = b, \\ & x \geq 0. \end{aligned}$$

Linear programming - Duality

Primal problem

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & c^T x \\ \text{subject to} \quad & Ax = b, \\ & x \geq 0. \end{aligned}$$

Dual problem

$$\begin{aligned} \max_{y\in\mathbb{R}^m} \quad & b^T y \\ \text{subject to} \quad & y_i \in \mathbb{R}, \\ & A^T y \leq c. \end{aligned} \tag{5}$$

Duality

$$\inf_{\pi \in \mathcal{B}(X \times Y)} \int_{X \times Y} c(x,y) \mathrm{d}\pi(x,y)$$

subject to $\pi_X = \mu,$

$\pi_Y = \nu$

$$\begin{aligned} \sup_{\phi,\psi \in \mathcal{C}_b(X,Y)} \quad & \int_X \varphi(x)\,\mathrm{d}\mu(x) + \int_Y \psi(y)\,\mathrm{d}\nu(y) \\ \text{subject to} \quad & \varphi(x) + \psi(y) \le c(x,y). \end{aligned} \tag{6}$$

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where $\mathcal{C}_b(X,Y)$ denotes the set of all pairs of bounded and continuous functions $\phi : X \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$.

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where $\mathcal{C}_b(X, Y)$ denotes the set of all pairs of bounded and continuous functions $\phi : X \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$.

If X is compact and Haussdorff, $\mathcal{C}_b(X)^* = \{\text{Radon measure}\}$

Kantorovich Duality

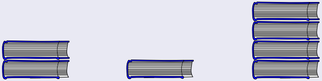
The minimum of the Kantorovich problem is equal to

$$\mathcal{T}_c(\mu,\nu) = \sup_{\phi,\psi \in \mathcal{C}_b(X,Y)} \int_X \varphi(x)\,\mathrm{d}\mu(x) + \int_Y \psi(y)\,\mathrm{d}\nu(y) \quad \text{subject to} \quad \varphi(x)+\psi(y) \leq c(x,y). \tag{7}$$

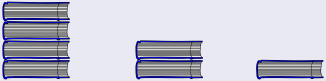
Interpretation in the discrete case

$$\sup_{(\phi_i)\in\mathbb{R}^n,(\psi_j)\in\mathbb{R}^m} \quad \sum_i \phi_i \mu_i + \sum_j \psi_j \nu_i \quad \text{subject to} \quad \phi_i + \psi_j \leq c_{ij} \tag{8}$$

$\mu =$



$\nu =$



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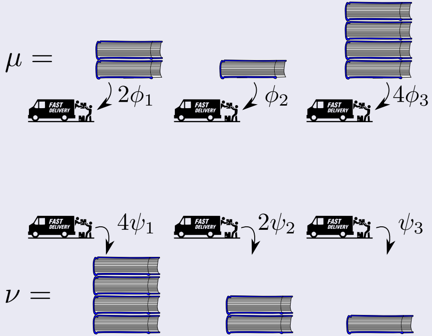

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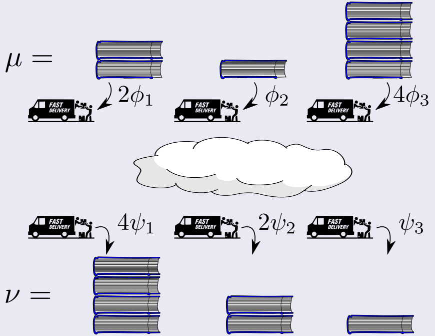

Proposition - Relaxation

Let μ and ν (with support X and Y) be absolutely continuous measures with respect to Lebesgue measure. If $\{X_i\}_i$ and $\{Y_i\}_i$ be finite pavings of X and Y. Suppose that $\mu(X_i) \in [\underline{\mu}_i, \overline{\mu}_i]$, $\nu(Y_j) \in [\underline{\nu}_j, \overline{\nu}_j]$, and $\forall x, y \in X_i \times Y_j, c(x, y) \leq \overline{c}_{ij}$,

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$$\begin{aligned} \overline{\mathcal{T}} = \sup_{(\phi_i)\in\mathbb{R}^n, (\psi_j)\in\mathbb{R}^m} \quad & \sum_i \phi_i \overline{\mu}_i + \sum_j \psi_j \overline{\nu}_i \\ \text{subject to} \quad & \phi_i + \psi_j \leq \overline{c}_{ij} \end{aligned} \tag{11}$$

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$$\text{then} \qquad \mathcal{T}_c(\mu, \nu) \leq \overline{\mathcal{T}}.$$

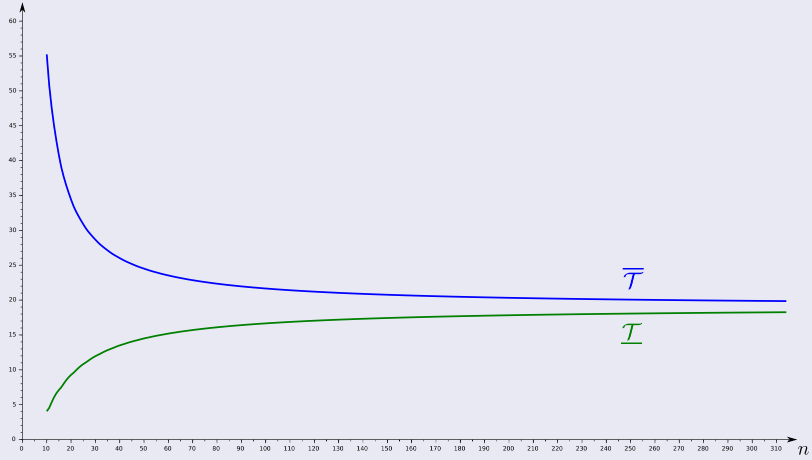

Figure : Guaranteed upper bounds of $\mathcal{T}_c(\mu,\nu)$ where $n = \text{Card}\{X_i\}_i$

Software

- filib - FI_LIB - A fast interval library,

  http://www2.math.uni-wuppertal.de/~xsc/software/filib.html

- GLPK - GNU Linear Programming Kit (GLPK),

  http://www.gnu.org/software/glpk/

- GMP - GNU Multiple Precision Arithmetic Library,

  https://gmplib.org/

- Source code is available on my webpage.

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