Attraction domain of a nonlinear system using interval analysis and Lyapunov Theory

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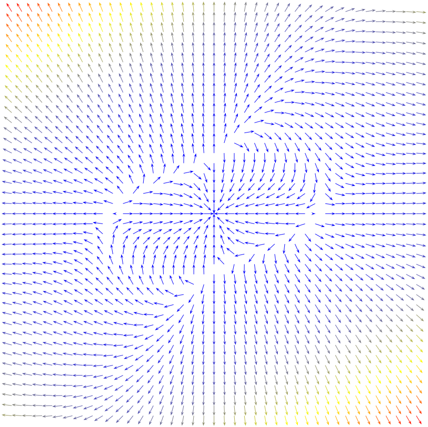
School and Conference on Computational Methods in Dynamics - Triestre

Outline

Let us consider :

$$\begin{cases} \dot{x} = f(x) \\ x \in \mathbb{R}^n \end{cases}, f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n).$$



Definition - The flow

Let us denote by $\{\varphi^t : \mathbb{R}^n \to \mathbb{R}^n\}_{t\in\mathbb{R}}$ the flow, i.e.

$$\left.\frac{d}{dt}\varphi^t(x)\right|_{t=0} = f(x) \text{ and } \varphi^0 = Id \quad (1)$$

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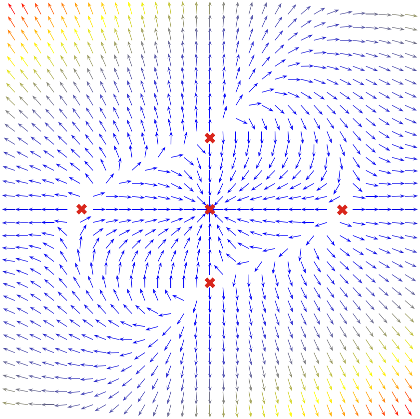
The map $t \mapsto \varphi^t x$ is the solution of the initial value problem :

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x. \end{cases}$$

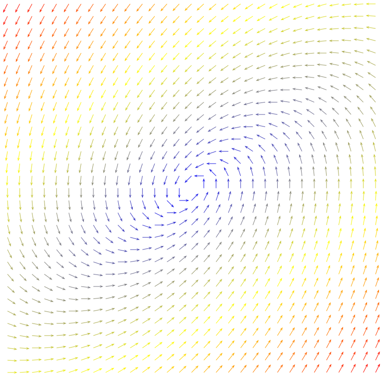
Definition

A point $x \in \mathbb{R}^n$ is an *equilibrium point* if $\varphi^t(x) = x, \forall t \in \mathbb{R}$ *i.e.* $f(x) = 0$.



Definition - Stable

The equilibrium point x_∞ is *stable* if
for any neighborhood E_0 of x_∞ there exists a neighborhood E of x_∞ such that $x \in E \Rightarrow \varphi^t x \in E_0, \forall t \geq 0.$



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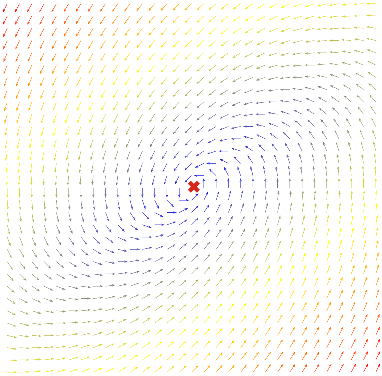

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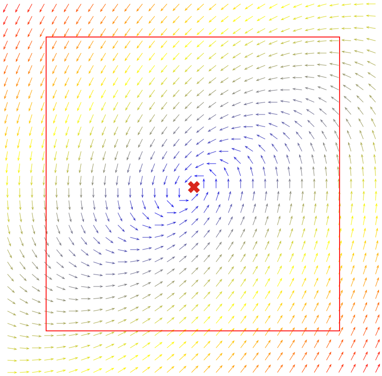

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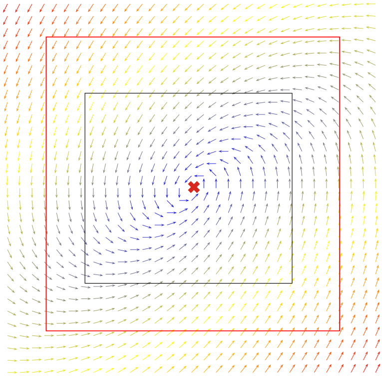

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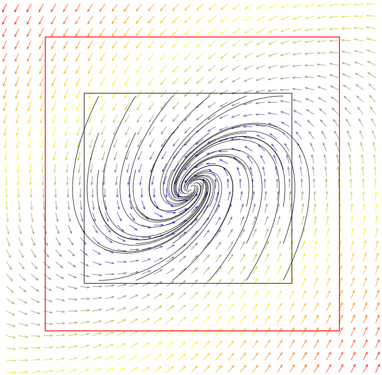

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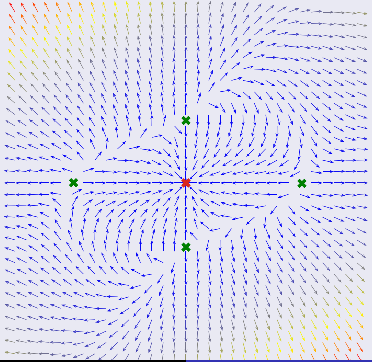

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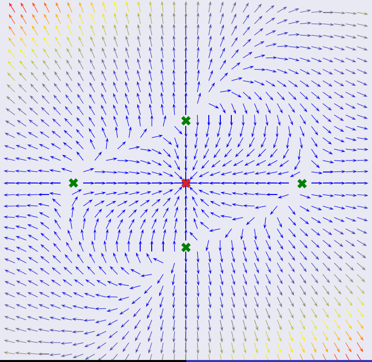

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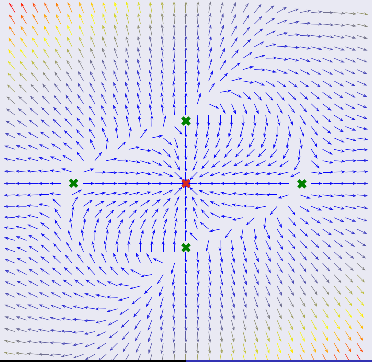
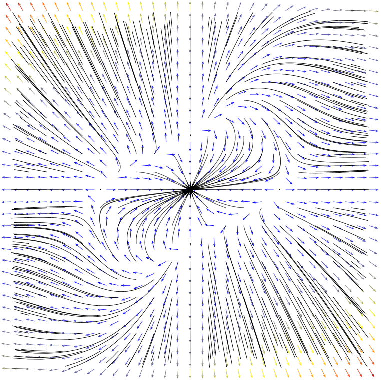


Definition - attraction domain

The attraction domain of x_∞ is the set

$$A_{x_\infty} = \{x \mid \lim_{t\to\infty} \varphi^t(x) = x_\infty\}.$$



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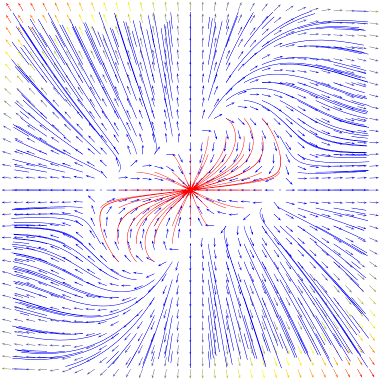
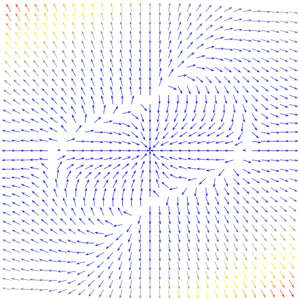


Compute the attraction domain A_{x_∞}.

1.
2.
3.



Compute the attraction domain A_{x_∞}.

1. Show that there exists an unique equilibrium point x_∞,
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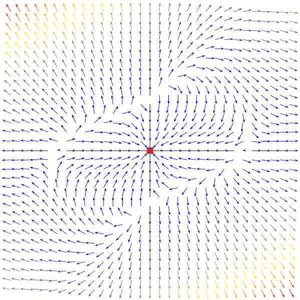

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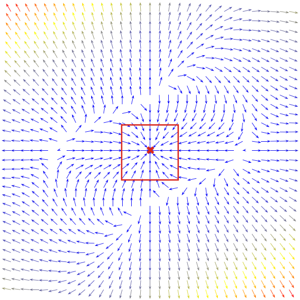

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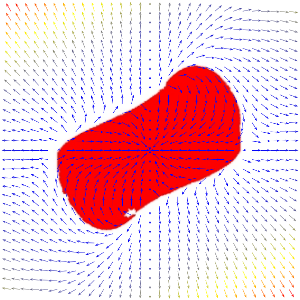
1. Show that there exists an unique equilibrium point x_∞,
2. Prove that x_∞ is asymptotically stable and compute a neighborhood of x_∞ included in the attraction domain.
3. Discretize the flow to compute a sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets such that $A_n \to_{n\to\infty} A_{x_\infty}$.



Outline

Theorem - Linear case

Let us consider $\dot{x} = Ax$.
The origin is asymptotically stable if and only if all eigenvalues λ of A have negative real parts.

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field. A equilibrium point x_∞ is *hyperbolic* if no eigenvalue of $Df(x_\infty)$ has its real part equal to 0.

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Hartman Grobman Theorem

If x_∞ is a hyperbolic equilibrium point then there exists a neighborhood E of x_∞ and a homeomorphism h such that the flow of f is topologically conjugate by h to the flow of its linearization $Df(x_\infty)$.

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Problem

How to compute such a neighborhood ?

Theorem

Let $x_0 \in E$ where E is a convex set of $\mathbb{R}^n$, and $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$. If

1. $\exists x_0 \in E$ such that $f(x_0) = 0$ and $Df(x_0) = 0$.
2. $\forall x \in E, D^2 f(x) \succeq 0$.

then $\forall x \in E, f(x) \geq 0$.

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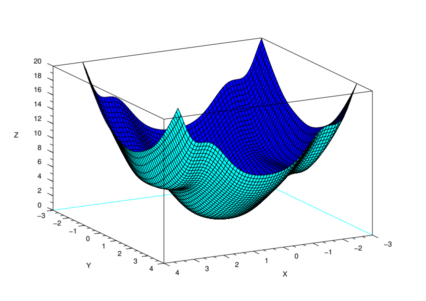
Definition

A symmetric matrix A is positive definite (denoted $A \succeq 0$) if

$$\forall x \in \mathbb{R}^n - \{0\}, x^T A x > 0$$

Example

To prove that $f(x) \geq 0, \forall x \in [-1/2, 1/2]^2$
where $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by
$f(x, y) = -\cos(x^2 + \sqrt{2}\sin^2 y) + x^2 + y^2 + 1.$



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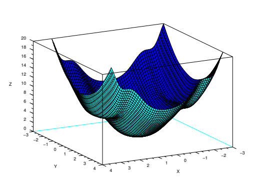
1. One has : $f(0,0) = 0$ and $Df(0,0) = (0,0)$

$$Df(x,y) = \begin{pmatrix} 2x(\sin(x^2 + \sqrt{2}\sin^2 y) + 1) \\ 2\sqrt{2}\cos y \sin y \sin\left(\sqrt{2}\sin^2 y + x^2\right) + 2y \end{pmatrix}.$$



$$D^2 f = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

$$\begin{array}{rcc} a_{1,1} & = & 2\sin\left(\sqrt{2}\sin^2 y + x^2\right) \\ & & +4x^2\cos\left(\sqrt{2}\sin^2 y + x^2\right) + 2. \\ & & \\ a_{2,2} & = & -2\sqrt{2}\sin^2 y \sin(\sqrt{2}\sin^2 y + x^2) \\ & & +2\sqrt{2}\cos^2 y \sin(\sqrt{2}\sin^2 y + x^2) \\ & & +8\cos^2 y \sin^2 y \cos(\sqrt{2}\sin^2 y + x^2) + 2. \\ & & \\ a_{1,2} = a_{2,1} & = & = 4\sqrt{2}x\cos y \sin y \\ & & \cos\left(\sqrt{2}\sin y^2 + x^2\right). \end{array}$$

Evaluation with interval analysis gives : $\forall x \in [-1/2, 1/2]^2$, $D^2 f(x) \subset [A]$

$$[A] = \begin{pmatrix} [1.9, 4.1] & [-1.3, 1.4] \\ [-1.3, 1.4] & [1.9, 5.4] \end{pmatrix}.$$

One only has to check that : $\forall A \in [A]$, A is positive definite.

Definition

A set of symmetric matrices $[A]$ is an interval of symmetric matrices if :

$$[A] = \{(a_{ij})_{ij}, a_{ij} = a_{ji}, a_{ij} \in [a]_{ij}\}$$

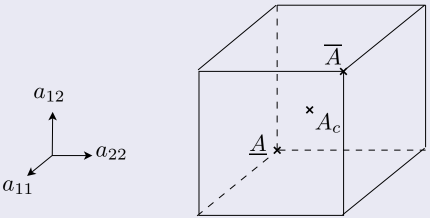
i.e.

$$[\underline{A}, \overline{A}] = \left\{A \text{ symmetric}, \underline{A} \leq A \leq \overline{A}\right\}.$$

Example

Using $\mathbb{R}^2$, a symmetric matrix A

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{1,2} & a_{2,2} \end{pmatrix}$$

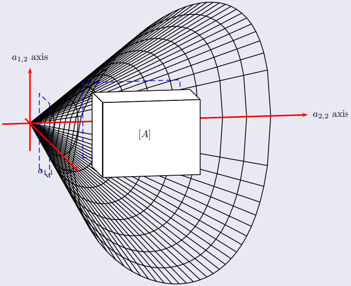


Remark - Rohn

Let $V([A])$ finite set of corners of $[A]$. Since $\{A \succeq 0\}$ and $[A]$ are convex subset of S^n :

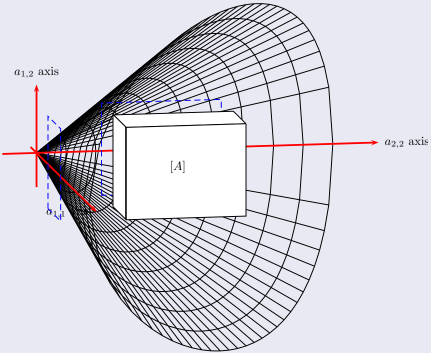
$$[A] \succeq 0 \Leftrightarrow V([A]) \succeq 0$$

$\{A \mid a_{ij} = a_{ji}\}$ is of dimension $\frac{n(n+1)}{2}$ then $\#V([A]) = 2^{\frac{n(n+1)}{2}}$.



Theorem- Adefeld

Let $[A]$ a symmetric interval matrix
and $C = \{z \in \mathbb{R}^n \text{ such that } |z_i| = 1\}$
If $\forall z \in C$, $A_z = A_c + \mathrm{Diag}(z)\Delta\mathrm{Diag}(z) \succeq 0$ then $[A] \succeq 0$.



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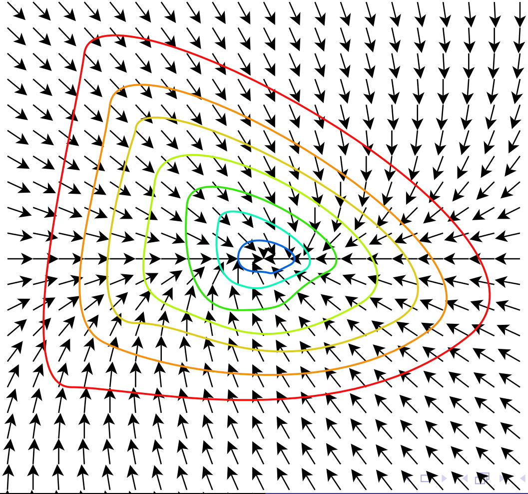
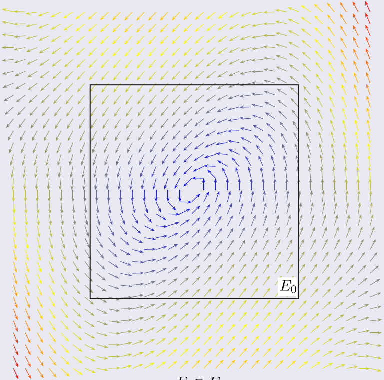
1. $L(x) = 0 \Leftrightarrow x = x_\infty$
2. $x \in E - \{x_\infty\} \Rightarrow L(x) > 0$
3. $DL(x) \cdot f(x) < 0, \ \forall x \in E - \{x_\infty\}$

$$\left.\frac{d}{dt}\right|_{t=0} L(x(t)) = DL(x(0)) \cdot \dot{x}(0) = DL(x) \cdot f(x)$$



Lyapunov theorem

Let E_0 a compact subset of $\mathbb{R}^n$ and $x_\infty \in E_0$.
If $L : E_0 \to \mathbb{R}$ is of Lyapunov (f) then x_∞ is asymptotically stable.
In particular, $\exists E \in \mathcal{V}(x_\infty), E \subset A_{x_\infty}$.



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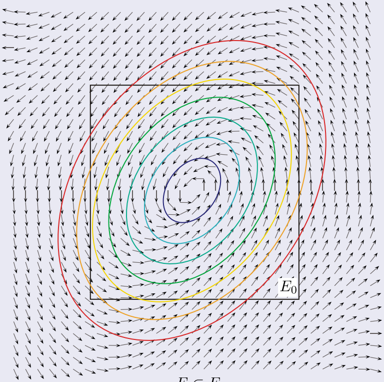

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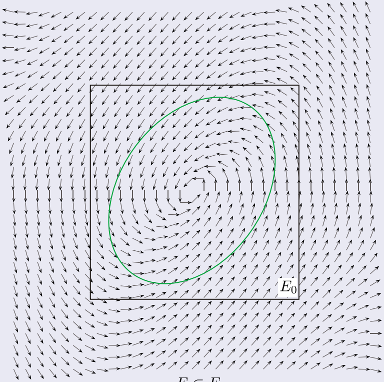

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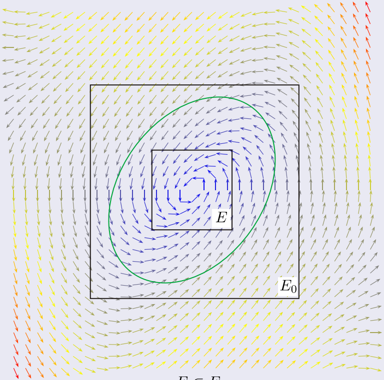

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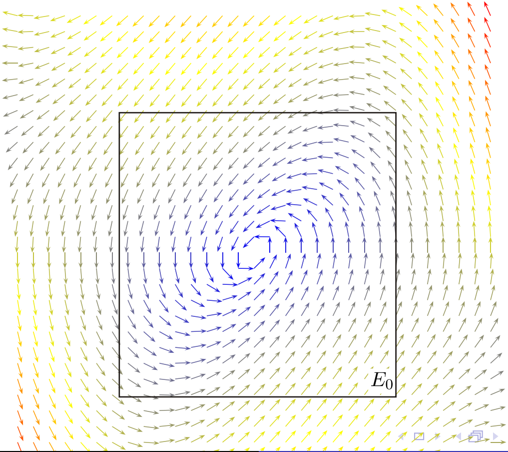
Theorem

Let $\dot{x} = Ax$, O is asymptotically stable if and only if the solution W of the equation $-(A^T W + WA) = I$ is positive definite.

Algorithm A

Step 1. Prove that E_0 contains a unique equilibrium point x_∞.
Step 2. Find $[x_\infty] \subset E_0$ such that $x_\infty \in [x_\infty]$.



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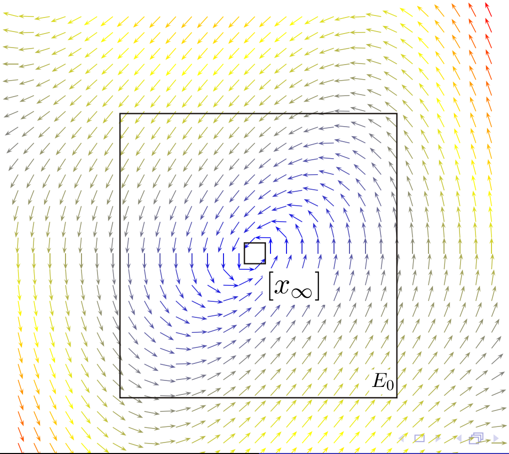
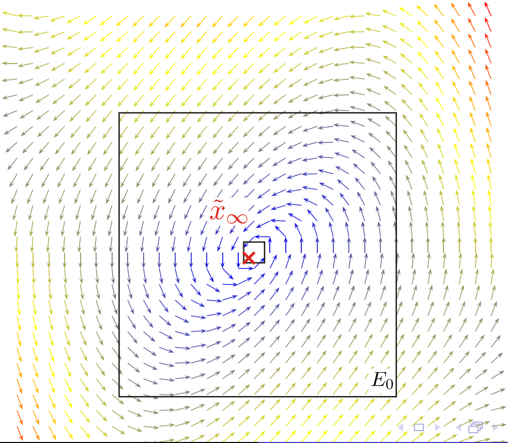


Algorithm A

Step 3. Linearize around x_∞ with $\tilde{x}_\infty$ ($\tilde{x}_\infty \in [x_\infty]$) :
$\dot{\overline{(x - x_\infty)}} = Df(\tilde{x}_\infty)(x - x_\infty)$.



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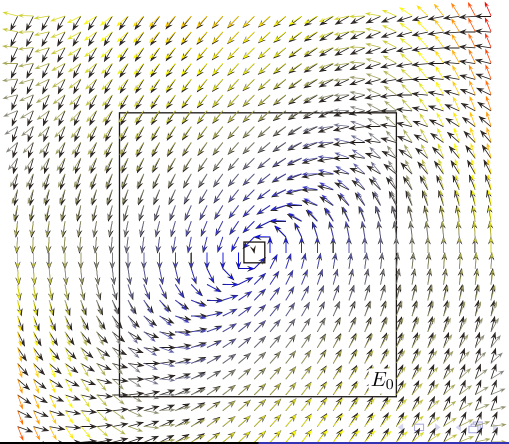

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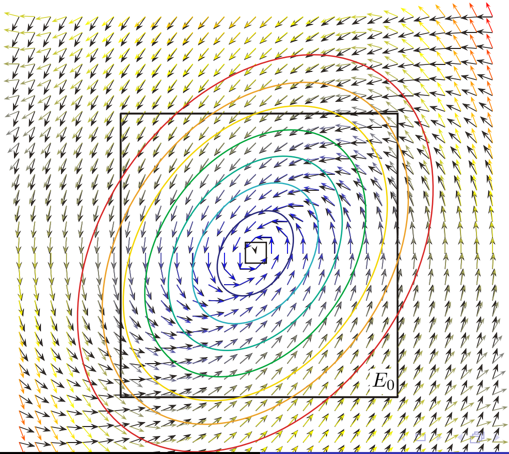
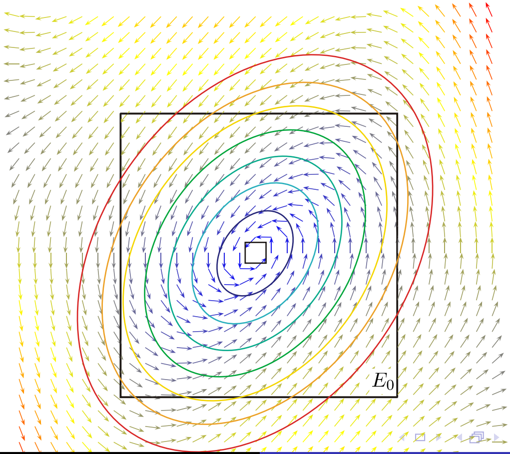
Step 4. Find a Lyapunov function L_{x_∞} for the linear system $Df(\tilde{x}_\infty)$.



Algorithm A

Step 5. Check that L_{x_∞} is of Lyapunov for the non linear system $\dot{x} = f(x)$.



About step 5 : $L_{x_\infty}(x) = (x - x_\infty)^T W_{\tilde{x}_\infty}(x - x_\infty)$
One has to check that

$$g_{x_\infty}(x) = -DL_{x_\infty}(x) \cdot f(x) \geq 0$$

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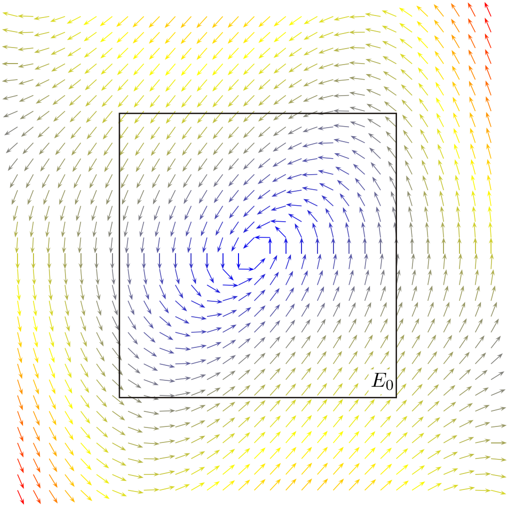
One has to check that

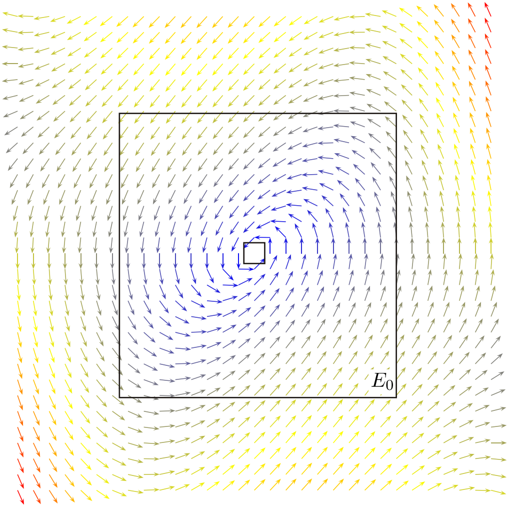
$$g_{x_\infty}(x) = -DL_{x_\infty}(x) \cdot f(x) \geq 0$$

- $g_{x_\infty}(x_\infty) = 0$
- $Dg_{x_\infty}(x_\infty) = 0$

According to theorem 2, we only have to check :

$$D^2 g_{x_\infty}(E_0) \succeq 0$$


E_0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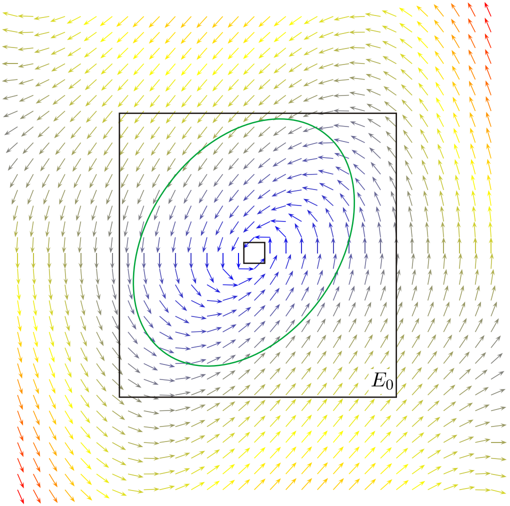
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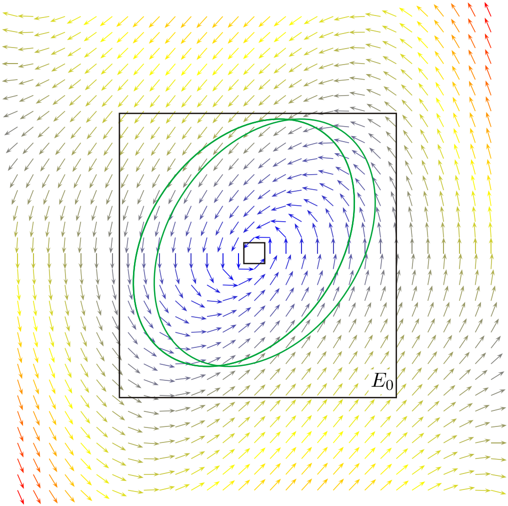
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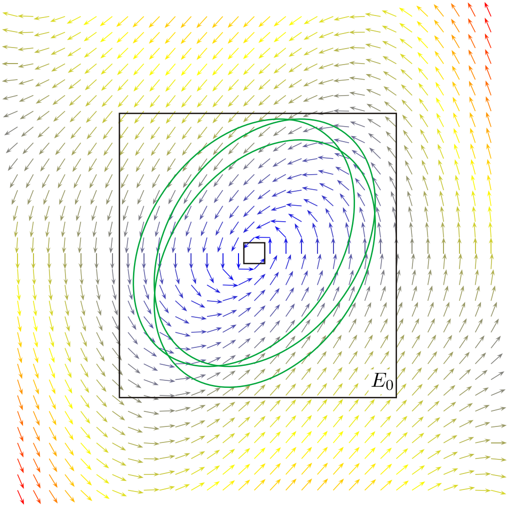
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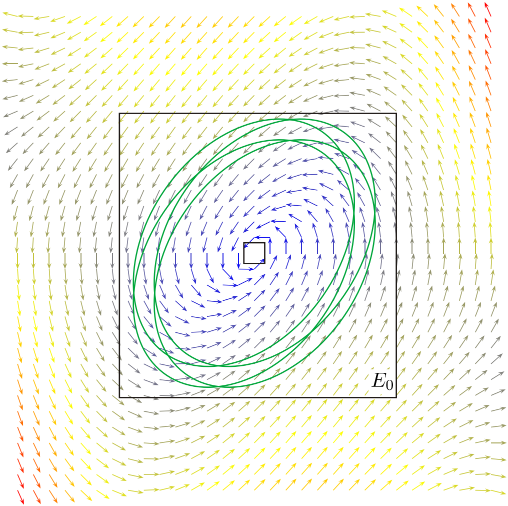
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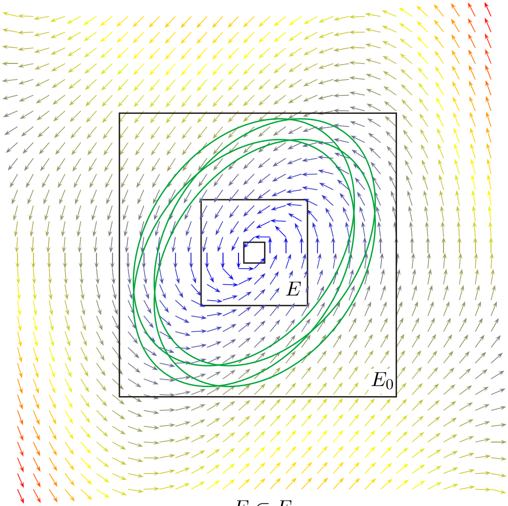
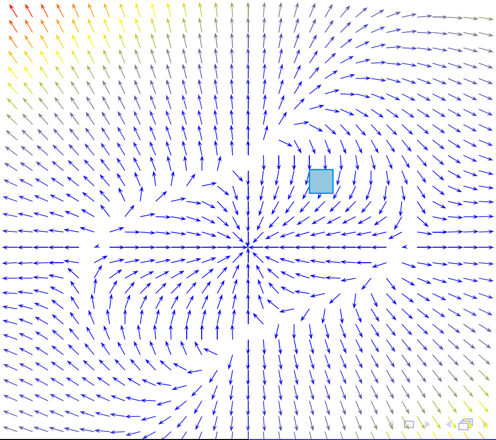


$E \subset E_0$

Outline

Picard-Linderlöf

Let $\dot{x} = f(x)$ and $t \in \mathbb{R}$, there exists guaranteed methods able to compute an inclusion function for the flow $\varphi^t : \mathbb{R}^n \to \mathbb{R}^n$.



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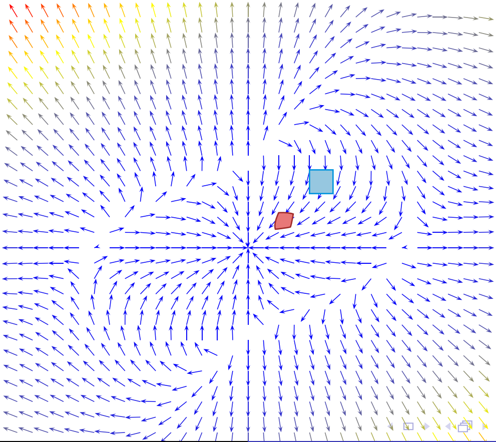

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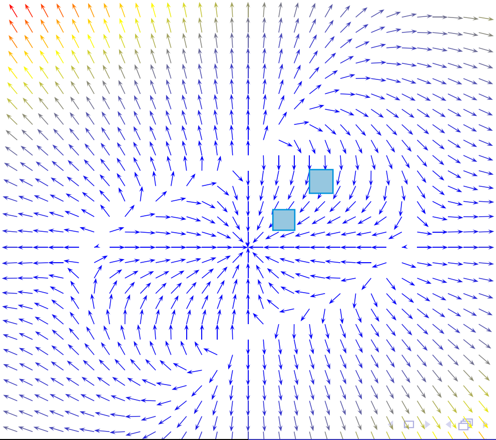
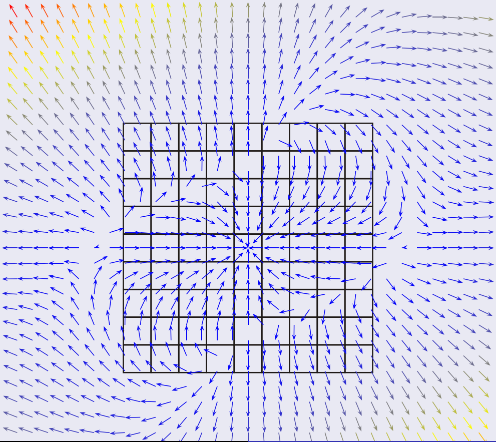


Definition

Let $t \in \mathbb{R}$, and $\{\mathbb{S}_i\}_{i \in I}$ a paving of $\mathbb{S}$, let us denote by $\mathcal{R}$ the relation on I defined by $i\mathcal{R}j \Leftrightarrow \varphi^t(\mathbb{S}_i) \cap \mathbb{S}_j \neq \emptyset$.



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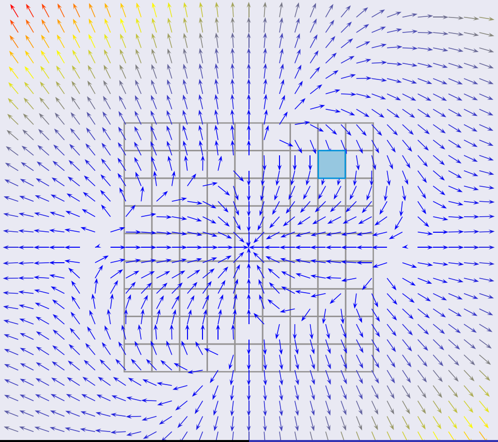

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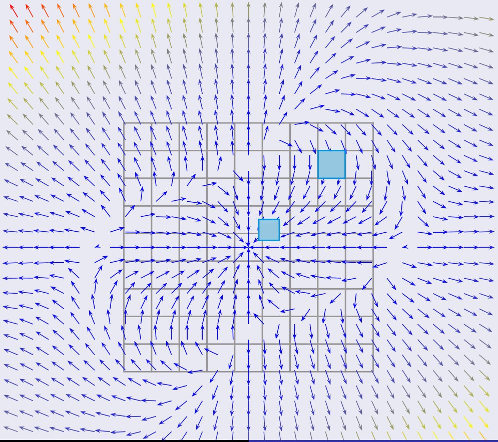

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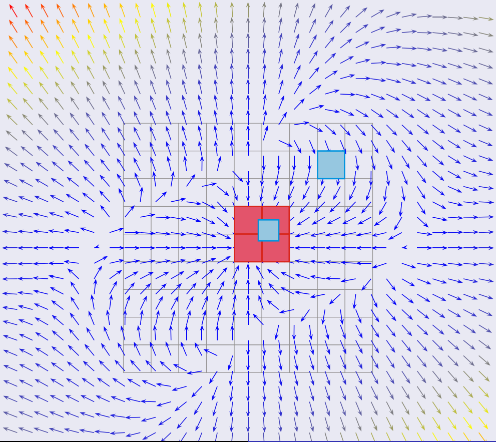

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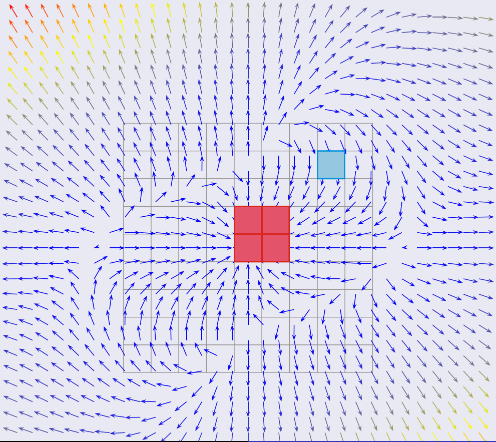

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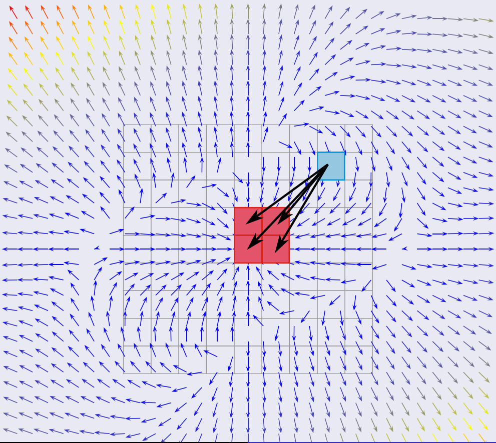
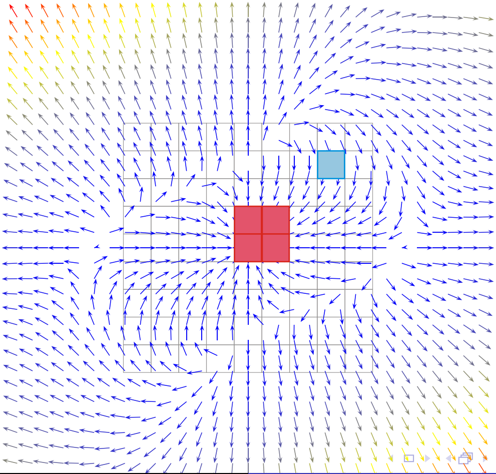


Proposition

If $\forall j \in I, i\mathcal{R}j \Rightarrow \mathbb{S}_j \subset A_{x_\infty}$ then $\mathbb{S}_i \subset A_{x_\infty}$.



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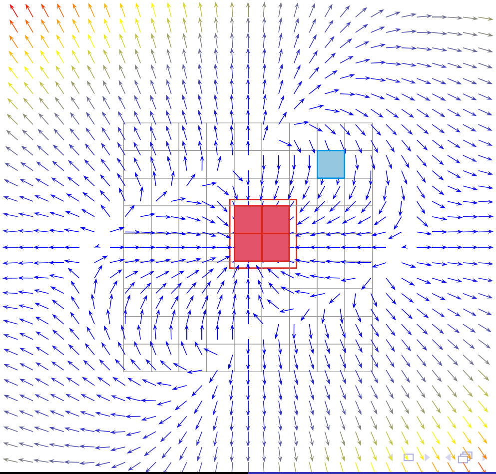

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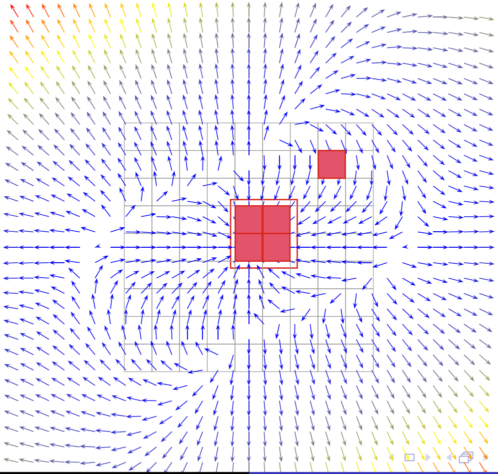

Algorithm B

Inputs :

- $\dot{x} = f(x)$,
- x_∞ an asymptotically stable point,
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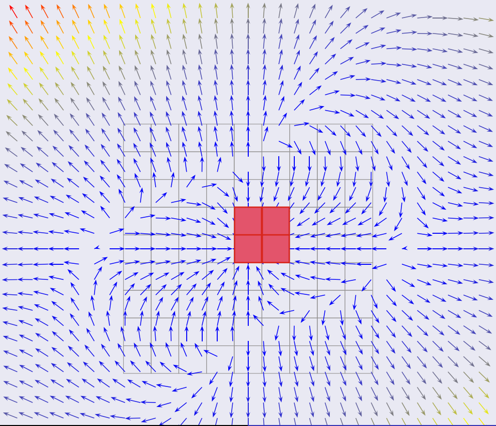
2. Compute the relation $\mathcal{R}$.
3. For each i of I, if

$$\forall j \in I, i\mathcal{R}j \Rightarrow \mathbb{S}_j \subset A$$

then $A := A \cup \mathbb{S}_i$, go to step 3 until a fixed point.

Example

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(x^2 - xy + 3y^2 - 1) \\ y(x^2 - 4yx + 3y^2 - 1) \end{pmatrix}$$



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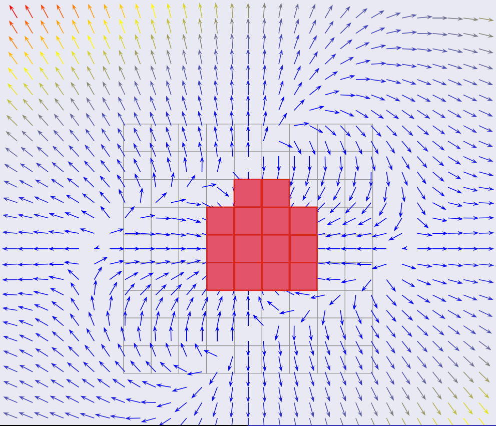

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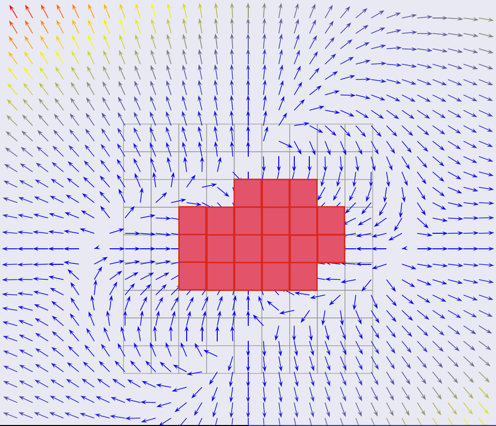

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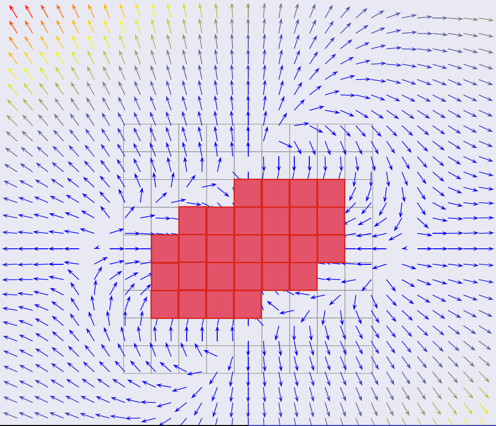

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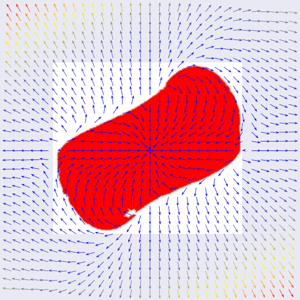

Conclusion

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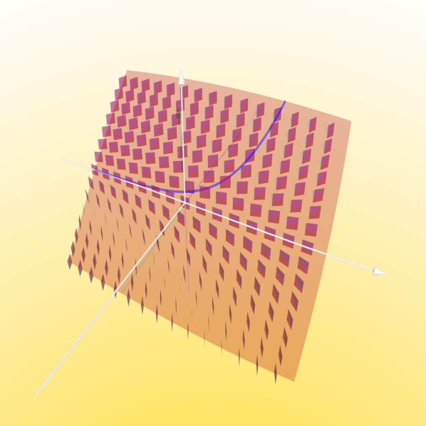
And now ? Jet space and interval analysis



- Grazie per la vostra attenzione !