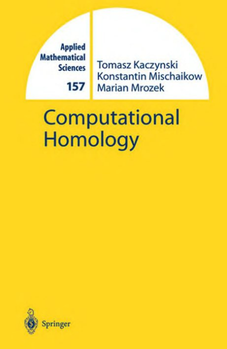
Computational Homology

Nicolas Delanoue

Lisa - Angers

Février 2009

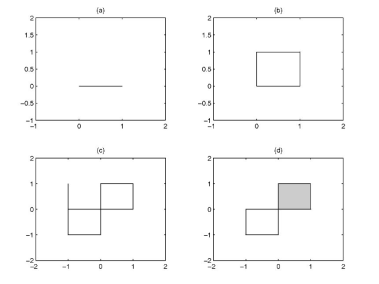


Computational Homology
Tomasz Kaczynski, Konstantin Mischaikow, Marian Mrozek.

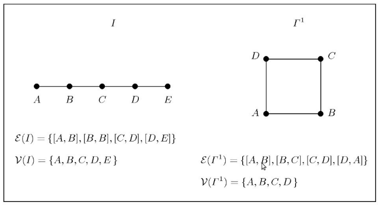
Plan

1. Les graphes et l'homologie calculatoire
   - Introduction, formalisme
   - Groupe d'homologie H_i
   - Cas particulier : le groupe d'homologie H_0
   - Exemples
2. Homologie d'une fonction
   - Introduction
   - Théorie du degré
   - Application, résolution d'équation $f(x) = 0$
   - Calcul pratique de l'homologie d'une fonction
3. Applications
   - Traitement d'images
   - Systèmes dynamiques

Exemples de graphe du plan



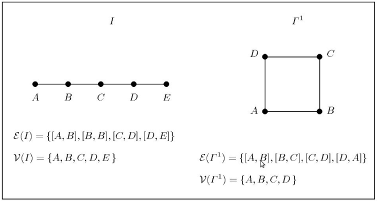
Graphes et graphes combinatoires



Définition

Soit G un graphe du plan, on note par $\mathcal{E}(G)$ la liste de ses noeuds et par $\mathcal{V}(G)$ la liste de ses arcs.

Graphes et graphes combinatoires



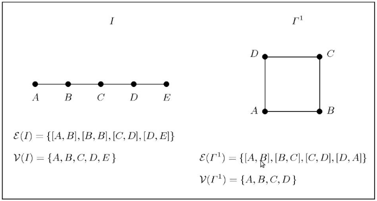
Définition - Chaînes

Soit G un graphe du plan, on note par

- C_1 l'ensemble des sommes finies d'éléments de $\mathcal{V}(G)$, et par
- C_0 les sommes d'élements de $\mathcal{E}(G)$.

i.e. le groupe libre abélien ...

Graphes et graphes combinatoires



Exemples

1. $[A, B] + [B, C] \in C_1$
2. $A + B + B + C \in C_0$

Définition - opérateur de bord

On note par ∂_1 la fonction linéaire

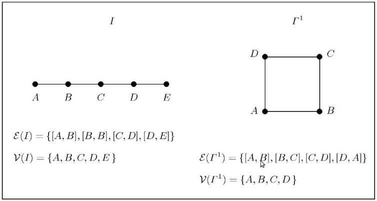
$$\partial_1 : \quad C_1 \quad \rightarrow \quad C_0$$

tel que $\partial_1([X, Y]) = X + Y$.

Exemple

$\partial_1([A, B] + [D, E]) = A + B + D + E$

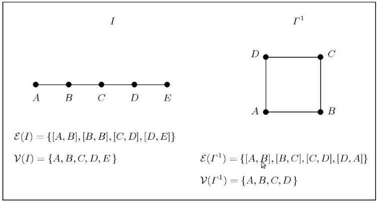
Graphes et graphes combinatoires



Bord topologique et bord algébrique

Topology	Algebra
$\mathrm{bd}\,[A,B] = \{A\} \cup \{B\}$	$\partial[\widehat{A,B}] = \widehat{A} + \widehat{B}$
$\mathrm{bd}\,[B,C] = \{B\} \cup \{C\}$	$\partial[\widehat{B,C}] = \widehat{B} + \widehat{C}$
$\mathrm{bd}\,[C,D] = \{C\} \cup \{D\}$	$\partial[\widehat{C,D}] = \widehat{C} + \widehat{D}$
$\mathrm{bd}\,[D,E] = \{D\} \cup \{E\}$	$\partial[\widehat{D,E}] = \widehat{D} + \widehat{E}$

Graphes et graphes combinatoires



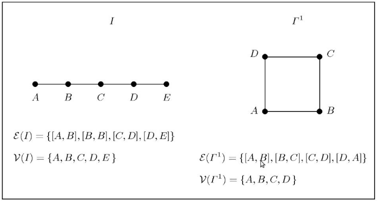
Exemple

$$\partial_1([A,B]+[B,C]) = A+B+B+C = A+2B+C$$

On peut considérer que les coefficients sont tous définies modulo 2 (i.e. dans $\mathbb{Z}/2\mathbb{Z}$). Dans ce cas, on a :

$$\partial_1([A,B]+[B,C]) = A+C$$

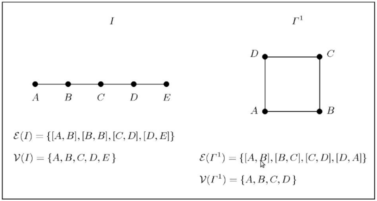
Remarque : ∂_1 est linéaire



$([A, B], [B, C], [C, D], [D, E])$ est une base de C_1.

$$[A, B] + [C, D] = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

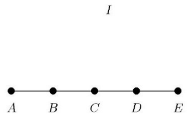
Remarque : ∂_1 est linéaire



(A, B, C, D, E) est une base de C_0.

$$A + 2B + C = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Remarque : ∂_1 est linéaire



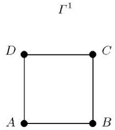
$\mathcal{E}(I) = \{[A, B], [B, B], [C, D], [D, E]\}$

$\mathcal{V}(I) = \{A, B, C, D, E\}$

L'opérateur ∂_1 a pour matrice :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Remarque : ∂_1 est linéaire



$\mathcal{E}(\Gamma^1) = \{[A, B], [B, C], [C, D], [D, A]\}$

$\mathcal{V}(\Gamma^1) = \{A, B, C, D\}$

L'opérateur ∂_1 a pour matrice :

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Définition - opérateur de bord

On note par ∂_0 la fonction linéaire

$$\begin{array}{rccl} \partial_0 : & C_0 & \longrightarrow & C_{-1} = \{0\} \\ & c & \longmapsto & 0 \end{array}$$

Exemple

$\partial_0(A + B) = 0$

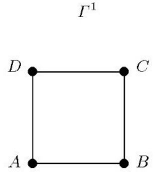
Définition - cycle

Un cycle est un élément c_i de C_i tel que $\partial_i c_i = 0$,
On note :

$$Z_0 = \{c_0 \in C_0 \mid \partial_0 c_0 = 0\} = \ker \partial_0$$

$$Z_1 = \{c_1 \in C_1 \mid \partial_1 c_1 = 0\} = \ker \partial_1$$

Exemple



$\mathcal{E}(\Gamma^1) = \{[A, B], [B, C], [C, D], [D, A]\}$

$\mathcal{V}(\Gamma^1) = \{A, B, C, D\}$

- $A + B \in Z_0$, $A \in Z_0$
- $[A, B] + [B, C] \notin Z_1$ car $\partial_1([A, B] + [B, C]) \neq 0$
- $[A, B] + [B, C] + [C, D] + [D, A] \in Z_1$

Définition - bord

On note par

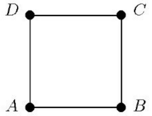
$$B_0 = \{c_0 \in C_0 \mid \exists c_1 \in C_1 \text{ tel que } \partial_1 c_1 = c_0\} = \mathrm{im}\partial_1$$

$$B_1 = \{c_1 \in C_1 \mid \exists c_2 \in C_2 \text{ tel que } \partial_2 c_2 = c_1\} = \mathrm{im}\partial_2$$

Les éléments de B_i sont appelés les *bords*.

Exemple

Γ^1



$$\mathcal{E}(\Gamma^1) = \{[A,B],[B,C],[C,D],[D,A]\}$$

$$\mathcal{V}(\Gamma^1) = \{A,B,C,D\}$$

- $A + C \in B_0$ car $\partial_1([A,B]+[B,C]) = A + C$
- $A \notin B_0$

Remarque

On a $B_i \subset Z_i \subset C_i$.

Définition

On définit le groupe d'homologie H_i comme étant le quotient

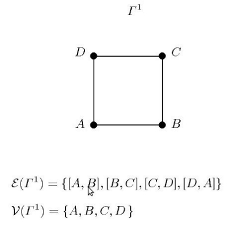
$$H_i = Z_i / B_i$$

i.e.

$$H_i = \ker \partial_i / \mathrm{im} \partial_{i+1}$$

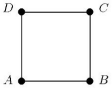
$$H_0 = \ker \partial_0 / \mathrm{im}\partial_1$$

$$\ker \partial_0 = C_0, \mathrm{im}\partial_1 = \text{"les bords"}.$$



$$\partial_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Γ^1

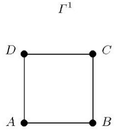


$\mathcal{E}(\Gamma^1) = \{[A,B],[B,C],[C,D],[D,A]\}$

$\mathcal{V}(\Gamma^1) = \{A,B,C,D\}$

$$\ker \partial_0 = \langle A, B, C, D \rangle$$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

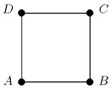


$\mathcal{E}(\Gamma^1) = \{[A,B],[B,C],[C,D],[D,A]\}$

$\mathcal{V}(\Gamma^1) = \{A,B,C,D\}$

$$\begin{aligned}
\mathrm{im}\partial_1 &= \langle A+B, B+C, C+D, D+A \rangle \\
&= \langle A-B, B-C, C-D, D-A \rangle \\
&= \langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle
\end{aligned}$$

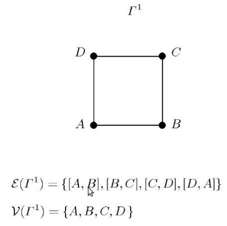
Γ^1



$\mathcal{E}(\Gamma^1) = \{[A,B],[B,C],[C,D],[D,A]\}$

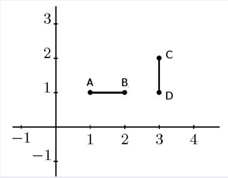
$\mathcal{V}(\Gamma^1) = \{A,B,C,D\}$

$$H_0 = \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\rangle / \left\langle \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix} \right\rangle$$



$$H_0 = \ker \partial_0 / \mathrm{im} \partial_1$$

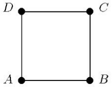
- $H_0 = \langle A, B, C, D \rangle / \langle A - B, B - C, C - D, D - A \rangle$
- $A \sim B \mod \mathrm{im}\partial_1$
- $A \sim B \sim C \sim D \mod \mathrm{im}\partial_1$
- $H_0 = \langle [A] \rangle$



$$H_0 = \ker \partial_0 / \mathrm{im}\partial_1$$

- $H_0 = \langle A, B, C, D\rangle / \langle A - B, C - D\rangle$,
- $A \sim B \mod \mathrm{im}\partial_1$,
- $C \sim D \mod \mathrm{im}\partial_1$,
- $A \not\sim C \mod \mathrm{im}\partial_1$,
- $H_0 = \langle [A], [C]\rangle$,

Γ^1

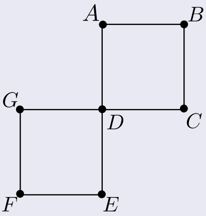


$\mathcal{E}(\Gamma^1) = \{[A,B],[B,C],[C,D],[D,A]\}$

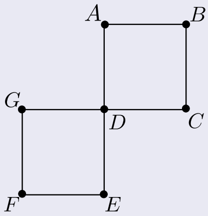
$\mathcal{V}(\Gamma^1) = \{A,B,C,D\}$

$$H_1 = \ker \partial_1 / \mathrm{im}\partial_2$$

- $\mathrm{im}\partial_2 = \{0\} \subset C_1$
- $$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

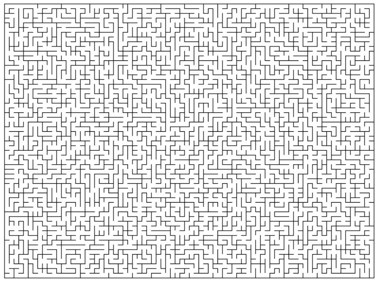


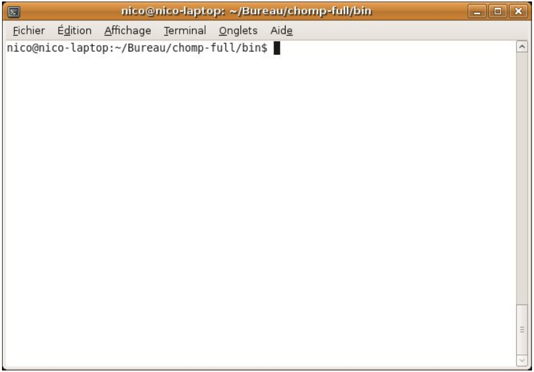
$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in \ker \partial_1$$



$$\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle = \ker \partial_1$$

Labyrinthe

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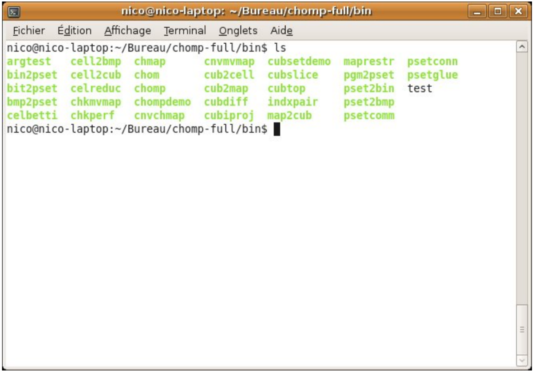
nico@nico-laptop:~/Bureau/chomp-full/bin$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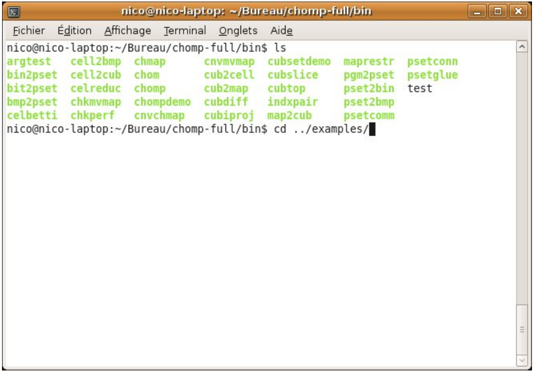
argtest cell2bmp chmap cnvmvmap cubsetdemo maprestr psetconn
bin2pset cell2cub chom cub2cell cubslice pgm2pset psetglue
bit2pset celreduc chomp cub2map cubtop pset2bin test
bmp2pset chkmvmap chompdemo cubdiff indxpair pset2bmp
celbetti chkperf cnvchmap cubiproj map2cub psetcomm
nico@nico-laptop:~/Bureau/chomp-full/bin$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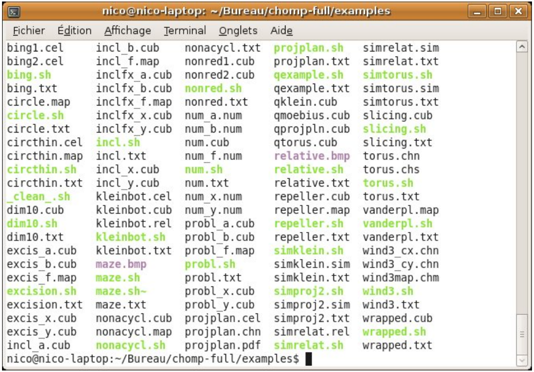
nico@nico-laptop:~/Bureau/chomp-full/bin$ ls
argtest cell2bmp chmap cnvmvmap cubsetdemo maprestr psetconn
bin2pset cell2cub chom cub2cell cubslice pgm2pset psetglue
bit2pset celreduc chomp cub2map cubtop pset2bin test
bmp2pset chkmvmap chompdemo cubdiff indxpair pset2bmp
celbetti chkperf cnvchmap cubiproj map2cub psetcomm
nico@nico-laptop:~/Bureau/chomp-full/bin$ cd ../examples/


nico@nico-laptop: ~/Bureau/chomp-full/examples
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bing1.cel incl_b.cub nonacycl.txt projplan.sh simrelat.sim
bing2.cel incl_f.map nonred1.cub projplan.txt simrelat.txt
bing.sh inclfx_a.cub nonred2.cub qexample.sh simtorus.sh
bing.txt inclfx_b.cub nonred.sh qexample.txt simtorus.sim
circle.map inclfx_f.map nonred.txt qklein.cub simtorus.txt
circle.sh inclfx_x.cub num_a.num qmoebius.cub slicing.cub
circle.txt inclfx_y.cub num_b.num qprojpln.cub slicing.sh
circthin.cel incl.sh num.cub qtorus.cub slicing.txt
circthin.map incl.txt num_f.num relative.bmp torus.chn
circthin.sh incl_x.cub num.sh relative.sh torus.chs
circthin.txt incl_y.cub num.txt relative.txt torus.sh
clean.sh kleinbot.cel num_x.num repeller.cub torus.txt
dim10.cub kleinbot.cub num_y.num repeller.map vanderpl.map
dim10.sh kleinbot.rel probl_a.cub repeller.sh vanderpl.sh
dim10.txt kleinbot.sh probl_b.cub repeller.txt vanderpl.txt
excis_a.cub kleinbot.txt probl_f.map simklein.sh wind3_cx.chn
excis_b.cub maze.bmp probl.sh simklein.sim wind3_cy.chn
excis_f.map maze.sh probl.txt simklein.txt wind3map.chm
excision.sh maze.sh~ probl_x.cub simproj2.sh wind3.sh
excision.txt maze.txt probl_y.cub simproj2.sim wind3.txt
excis_x.cub nonacycl.cub projplan.cel simproj2.txt wrapped.cub
excis_y.cub nonacycl.map projplan.chn simrelat.rel wrapped.sh
incl_a.cub nonacycl.sh projplan.pdf simrelat.sh wrapped.txt
nico@nico-laptop:~/Bureau/chomp-full/examples$

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nico@nico-laptop: ~/Bureau/chomp-full/examples
Fichier  Édition  Affichage  Terminal  Onglets  Aide
bing1.cel      incl_b.cub     nonacycl.txt  projplan.sh   simrelat.sim
bing2.cel      incl_f.map     nonred1.cub   projplan.txt  simrelat.txt
bing.sh        inclfx_a.cub   nonred2.cub   qexample.sh   simtorus.sh
bing.txt       inclfx_b.cub   nonred.sh     qexample.txt  simtorus.sim
circle.map     inclfx_f.map   nonred.txt    qklein.cub    simtorus.txt
circle.sh      inclfx_x.cub   num_a.num     qmoebius.cub  slicing.cub
circle.txt     inclfx_y.cub   num_b.num     qprojpln.cub  slicing.sh
circthin.cel   incl.sh        num.cub       qtorus.cub    slicing.txt
circthin.map   incl.txt       num_f.num     relative.bmp  torus.chn
circthin.sh    incl_x.cub     num.sh        relative.sh   torus.chs
circthin.txt   incl_y.cub     num.txt       relative.txt  torus.sh
_clean_.sh     kleinbot.cel   num_x.num     repeller.cub  torus.txt
dim10.cub      kleinbot.cub   num_y.num     repeller.map  vanderpl.map
dim10.sh       kleinbot.rel   probl_a.cub   repeller.sh   vanderpl.sh
dim10.txt      kleinbot.sh    probl_b.cub   repeller.txt  vanderpl.txt
excis_a.cub    kleinbot.txt   probl_f.map   simklein.sh   wind3_cx.chn
excis_b.cub    maze.bmp       probl.sh      simklein.sim  wind3_cy.chn
excis_f.map    maze.sh        probl.txt     simklein.txt  wind3map.chm
excision.sh    maze.sh~       probl_x.cub   simproj2.sh   wind3.sh
excision.txt   maze.txt       probl_y.cub   simproj2.sim  wind3.txt
excis_x.cub    nonacycl.cub   projplan.cel  simproj2.txt  wrapped.cub
excis_y.cub    nonacycl.map   projplan.chn  simrelat.rel  wrapped.sh
incl_a.cub     nonacycl.sh    projplan.pdf  simrelat.sh   wrapped.txt
nico@nico-laptop:~/Bureau/chomp-full/examples$ ./maze.sh
```

nico@nico-laptop: ~/Bureau/chomp-full/examples

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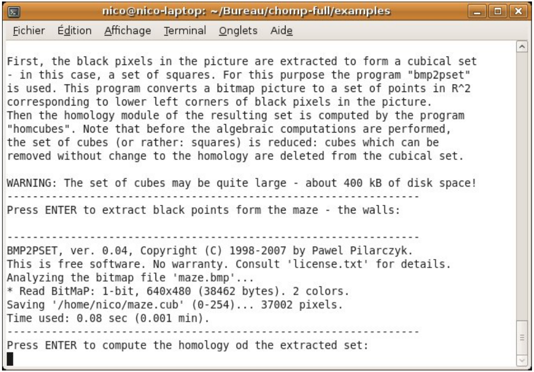
```
*** maze ***

In this example the homology of a closed maze stored in a bitmap file
is computed. The external wall forms the circle in which the maze is
contained. If the maze picture has the homology of the circle,
then this means that all the internal walls can be homotopically
contracted towards the external wall and, as a consequence, from
every place inside the maze to every other place there exists
a path between the walls and this path is unique.

First, the black pixels in the picture are extracted to form a cubical set
- in this case, a set of squares. For this purpose the program "bmp2pset"
is used. This program converts a bitmap picture to a set of points in R^2
corresponding to lower left corners of black pixels in the picture.
Then the homology module of the resulting set is computed by the program
"homcubes". Note that before the algebraic computations are performed,
the set of cubes (or rather: squares) is reduced: cubes which can be
removed without change to the homology are deleted from the cubical set.

WARNING: The set of cubes may be quite large - about 400 kB of disk space!
-----------------------------------------------------------------
Press ENTER to extract black points form the maze - the walls:
```


nico@nico-laptop: ~/Bureau/chomp-full/examples
Fichier Édition Affichage Terminal Onglets Aide
First, the black pixels in the picture are extracted to form a cubical set
- in this case, a set of squares. For this purpose the program "bmp2pset"
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removed without change to the homology are deleted from the cubical set.
WARNING: The set of cubes may be quite large - about 400 kB of disk space!
Press ENTER to extract black points form the maze - the walls:
BMP2PSET, ver. 0.04, Copyright (C) 1998-2007 by Pawel Pilarczyk.
This is free software. No warranty. Consult 'license.txt' for details.
Analyzing the bitmap file 'maze.bmp'...
* Read BitMaP: 1-bit, 640x480 (38462 bytes). 2 colors.
Saving '/home/nico/maze.cub' (0-254)... 37002 pixels.
Time used: 0.08 sec (0.001 min).
Press ENTER to compute the homology od the extracted set:

nico@nico-laptop: ~/Bureau/chomp-full/examples

Fichier Édition Affichage Terminal Onglets Aide

```
-----------------------------------------------------------------
Press ENTER to compute the homology od the extracted set:

-----------------------------------------------------------------
HOMCUBES, ver. 3.07, 05/02/08. Copyright (C) 1997-2008 by Pawel Pilarczyk.
This is free software. No warranty. Consult 'license.txt' for details.
Reading cubes to X from '/home/nico/maze.cub'... 37002 cubes read.
300 bit fields allocated (0 MB) to speed up 2-dimensional reduction.
Reducing full-dim cubes from X... 9977 removed, 27025 left.
Transforming X into cells... 27025 cells added.
Collapsing faces in X... .. 161188 removed, 5996 left.
Note: The dimension of X decreased from 2 to 1.
Creating the chain complex of X... . Done.
Vertices used: 64356 of dim 2.
Time used so far: 1.46 sec (0.024 min).
Computing the homology of X over the ring of integers...
Reducing D_1: 0 + 2997 reductions made.
H_0 = Z
H_1 = Z
Total time used: 1.47 sec (0.025 min).
Thank you for using this software. We appreciate your business.
-----------------------------------------------------------------
Press ENTER to finish:
```

nico@nico-laptop: ~/Bureau/chomp-full/examples

Fichier Édition Affichage Terminal Onglets Aide

```
Press ENTER to compute the homology od the extracted set:

-----------------------------------------------------------------
HOMCUBES, ver. 3.07, 05/02/08. Copyright (C) 1997-2008 by Pawel Pilarczyk.
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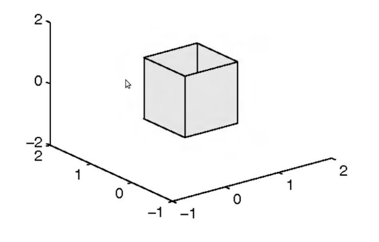
-----------------------------------------------------------------
Press ENTER to finish:

nico@nico-laptop:~/Bureau/chomp-full/examples$
```

Généralisation



$$H_0 = \mathbb{Z}$$

$$H_1 = \mathbb{Z}$$

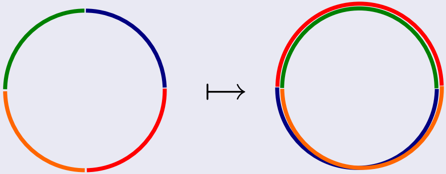
Chapitre 2 et 3.

Proposition

Soit $f : X \longrightarrow X'$ une fonction continue, cette fonction f induit une fonction linéaire de $f_* : H_*(X) \longrightarrow H_*(X')$.

Exemple

$$f : \quad S^1 \quad \longrightarrow \quad S^1$$



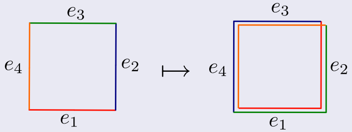
$$\longmapsto$$

$$f : \theta \mapsto 2\theta$$

Proposition

Soit $f : X \to X'$ une fonction continue, cette fonction f induit une fonction $f_* : H_*(X) \to H_*(X')$.

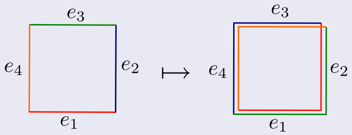
$$f : \quad S^1 \longrightarrow S^1$$



Proposition

Soit $f : X \to X'$ une fonction continue, cette fonction f induit une fonction $f_* : H_*(X) \to H_*(X')$.

$$f : \quad S^1 \quad \to \quad S^1$$



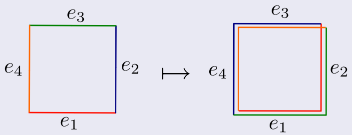
$$\varphi_1 : C_1 \to C_1'$$

$$\varphi_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Proposition

Soit $f : X \longrightarrow X'$ une fonction continue, cette fonction f induit une fonction $f_* : H_*(X) \longrightarrow H_*(X')$.

$$f: \quad S^1 \quad \longrightarrow \quad S^1$$

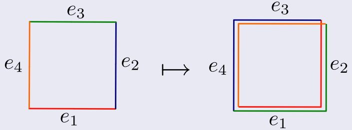


$$\begin{array}{cccc} f_{*1}: & H_1 & \longrightarrow & H_1' \\ & [c] & \longmapsto & [\varphi_1(c)] \end{array}$$

Proposition

Soit $f : X \to X'$ une fonction continue, cette fonction f induit une fonction $f_* : H_*(X) \to H_*(X')$.

$$f : \quad S^1 \quad \longrightarrow \quad S^1$$



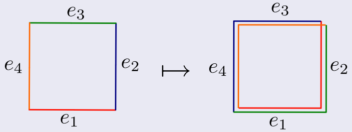
$$f_{*1}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right) = \left[\varphi_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right] = \begin{bmatrix}2\\2\\2\\2\end{bmatrix}$$

$$\Rightarrow f_{*1} = 2$$

Proposition

Soit $f : X \to X'$ une fonction continue, cette fonction f induit une fonction $f_* : H_*(X) \to H_*(X')$.

$$f : \quad S^1 \quad \longrightarrow \quad S^1$$



$$\partial'_k \varphi_k = \varphi_{k-1} \partial_k$$

Proposition

Soit S^d la sphère de dimension $d > 0$.

$$H_k(S^d) = \begin{cases} \mathbb{Z} = \langle e \rangle & \text{si } k = d \\ 0 & \text{sinon} \end{cases}$$

Définition

Soit $f : S^d \to S^d$ continue, le degré de f est l'entier n vérifiant $f_{*d}([e]) = n[e]$.

Proposition

Le degré vérifie les propriétés suivantes :

- $\deg(id) = 1$,
- si f est une fonction constante, $\deg(f) = 0$,
- $\deg(f \circ g) = \deg(f)\deg(g)$,
- $f \sim g \Rightarrow \deg(f) = \deg(g)$,
- si f peut être prolongée continuement en $\bar{f} : \bar{B}^{d+1} \to S^d$ alors $\deg(f) = 0$.

Soit $U = [-m, m]^{d+1}$,

Définition

Soit $f : U \to \mathbb{R}^{d+1}$ tel que $0 \notin f(bdU)$, on définit $\tilde{f} : S_0^d \to S_0^d$

$$\tilde{f}(x) := \frac{f(mx)}{\|f(mx)\|_0}$$

Proposition

Soit $f : U \to \mathbb{R}^{d+1}$ tel que $0 \notin f(bdU)$.
Si $\deg \tilde{f} \neq 0$ alors $f(x) = 0$ admet une solution dans U.

Exemple

Soit $f : [-1, 1]^2 \to \mathbb{R}^2$ définie par

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$$

On a $\deg(\tilde{f}) = 2$ donc $\exists (x, y) \in [-1, 1]^2 \mid f(x, y) = (0, 0)$.

Remarque

1. *C'est la fonction* $\mathbb{C} \ni z \mapsto z^2 \in \mathbb{C}$
2. La méthode "classique" de Newton par intervalle ne peut conclure car $Df(0) = 0$.

Contre exemple de la réciproque

Exemple

On idenfifie les points du cercle S^1 avec les points du segment $[0, 1]$ où $0 \sim 1$.
Soit $f : S^1 \to S^1$ définie par $f(t) = 3t(1 - t)$.

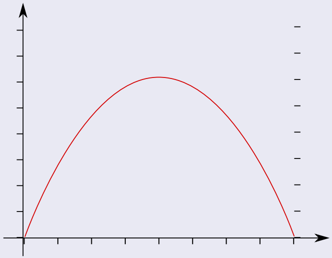


Fig.: Graphe de la fonction.

Exemple

On idenfifie les points du cercle S^1 avec les points du segment $[0, 1]$ où $0 \sim 1$.
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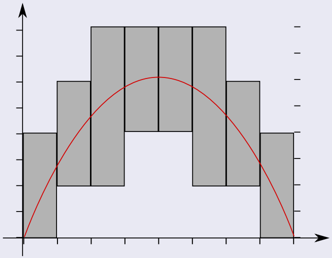

FIG.: Calcul par intervalles.

Exemple

On idenfifie les points du cercle S^1 avec les points du segment $[0, 1]$ où $0 \sim 1$.
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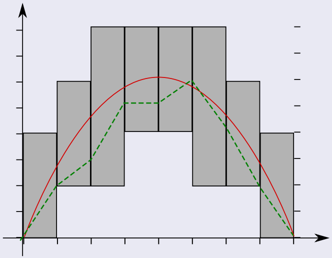

FIG.: $f_{*1} = \varphi_{*1}$, $(\partial\varphi = \varphi\partial)$.

Exemple

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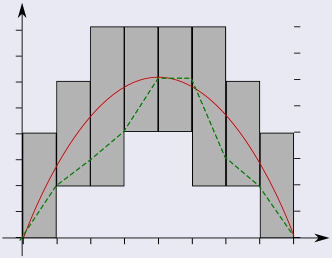

FIG.: $f_{*1} = \varphi'_{*1}$, $(\partial\varphi' = \varphi'\partial)$.

Exemple

On idenfifie les points du cercle S^1 avec les points du segment $[0, 1]$ où $0 \sim 1$.
Soit $f : S^1 \to S^1$ définie par $f(t) = 3t(1 - t)$.

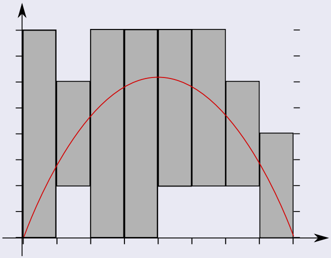

FIG.: Calcul par intervalles.

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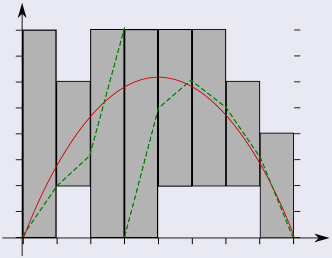


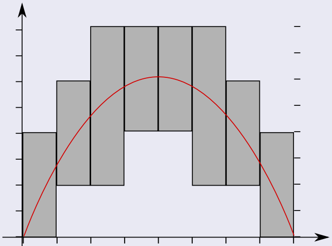
FIG.: $f_{*1} \neq \varphi_{*1}$, ($\varphi_1(e_3)$ contient un cycle.)

Théorème

Soit $F : X \rightrightarrows Y$ une fonction d'inclusion semi-continue inférieurement. Si pour tout $x \in X$, $F(x)$ est acyclique, alors il existe une fonction chaine $\varphi : C(X) \to C(Y)$ tel que

- $|\varphi(E)| \subset F(E), \forall E \in \mathcal{K}(X)$
- $\varphi(c) \in \mathcal{K}_0(F(c)), \forall c \in \mathcal{K}_0(X)$

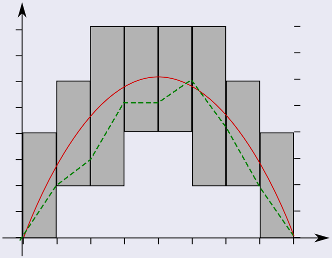
Dans ce cas, on dit que φ est une sélection de F.

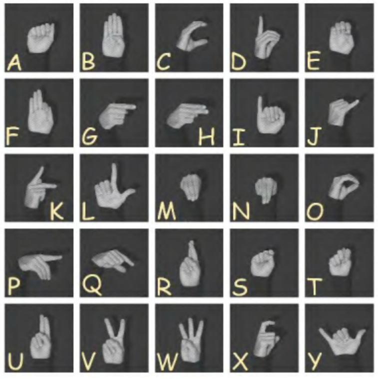


Théorème

Soient $F : X \rightrightarrows Y$ une fonction d'inclusion semi-continue inférieurement. On suppose que pour tout $x \in X$, $F(x)$ est acyclique, et que φ est une sélection.

$$f_* = \varphi_*$$

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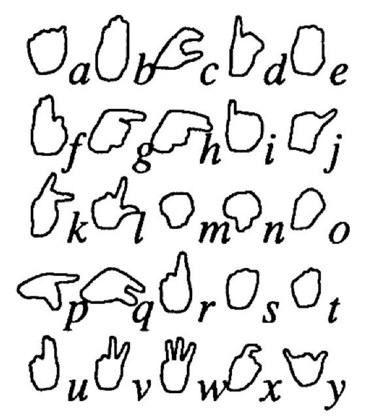
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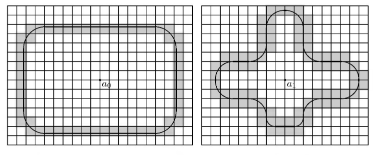
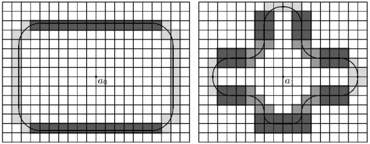

a_0
a_1



(a) (b)

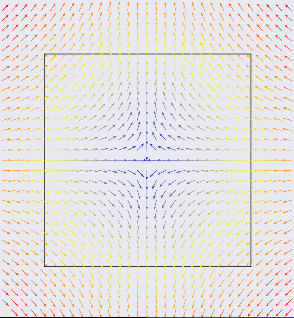
$\beta \setminus \alpha$	6	7	8	9
6	2			
7	2	2		
8	2	2	2	
9	1	1	1	1

$\beta \setminus \alpha$	3	4	5	6	7	8	9
3	4						
4	4	4					
5	3	3	3				
6	3	3	3	3			
7	3	3	3	3	3		
8	1	1	1	1	1	1	
9	1	1	1	1	1	1	1

Soit X un espace topologique, et $\varphi : \mathbb{R} \times X \to X$ le flot associé à l'équation différentielle $\dot{x} = f(x)$.

Soit W un sous ensemble de X, on définit

1. $W^0 = \{x \in W \mid \text{ il existe } t > 0 \text{ tel que } \varphi(t, x) \notin W\}$
2. $W^- = \{x \in W \mid \varphi([0, t], x) \not\subset W, \forall t > 0\}$



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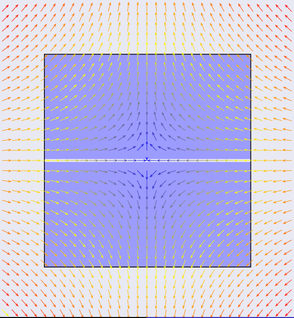

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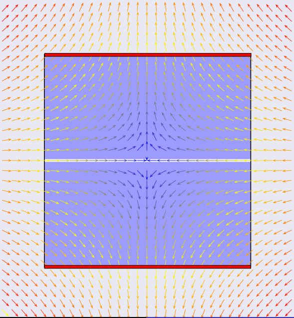
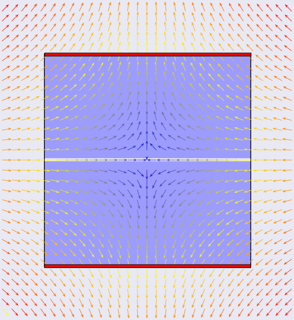


Définition

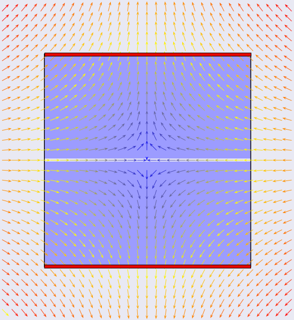
Un ensemble W est un ensemble de *Ważewski* si

- $x \in W \wedge \varphi([0,t],x) \subset \bar{W} \Rightarrow \varphi([0,t],x) \subset W$,
- W^- est fermé dans W^0.



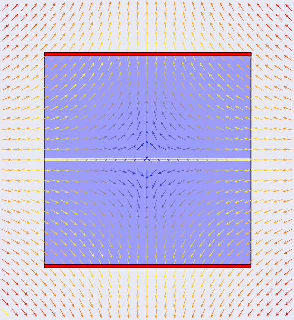
Théorème

Soit W un ensemble de Ważewski. W^- est une déformation par rétraction de W^0 et W^0 est ouvert dans W.



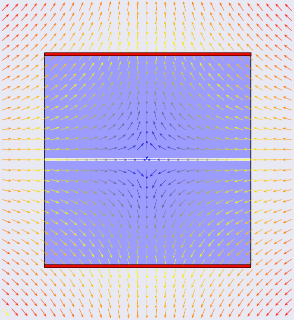
Corollaire

Soit W un ensemble de Ważewski. $H_*(W^-) = H_*(W^0)$

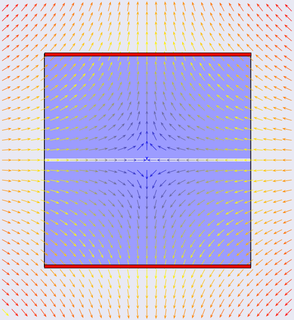


Corollaire

Soit W un ensemble de Ważewski tel que $H_*(W) \neq H_*(W^0)$ alors $W \backslash W^0 \neq \emptyset$.
i.e. il existe des solutions qui, dans le futur, restent dans W.

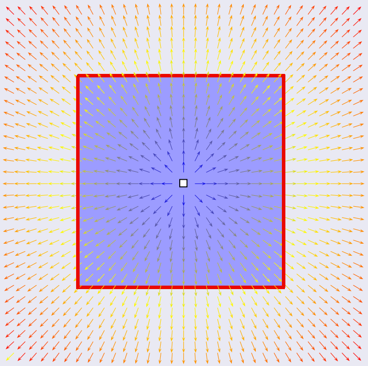


Exemple



$$H_0(W^-) = \mathbb{Z}^2 \neq \mathbb{Z} = H_0(W)$$

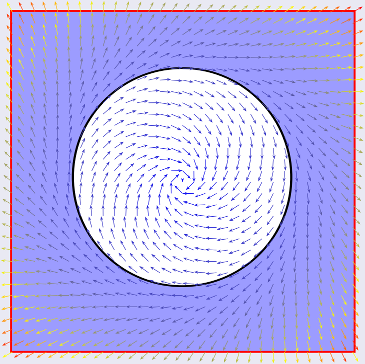
Exemple



$$H_0(W^-) = \mathbb{Z} = H_0(W)$$

$$H_1(W^-) = \mathbb{Z} \neq \{0\} = H_1(W)$$

Exemple

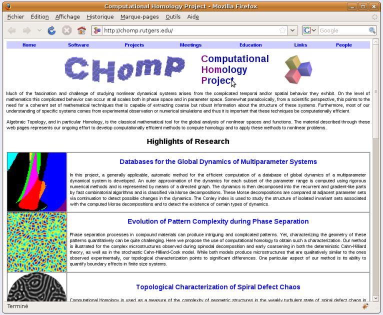


$$H_0(W^-) = \mathbb{Z} = H_0(W)$$

$$H_1(W^-) = \mathbb{Z} \neq \{0\} = H_1(W)$$

Systèmes dynamiques $x_{n+1} = f(x_n)$.

- la théorie des *indices de Conley*,
- système dynamiques symboliques



http://chomp.rutgers.edu/