Classification of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$

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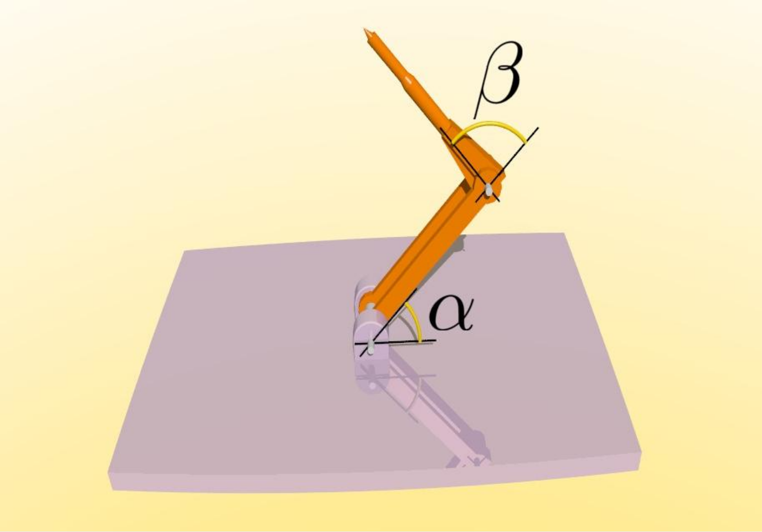
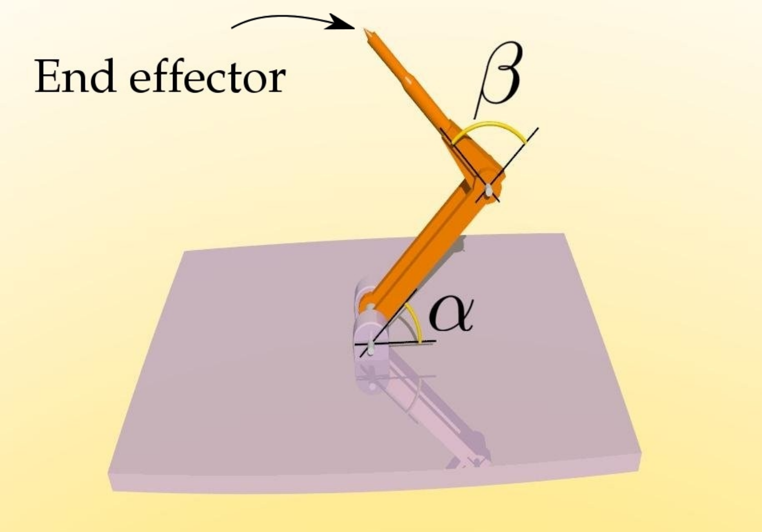
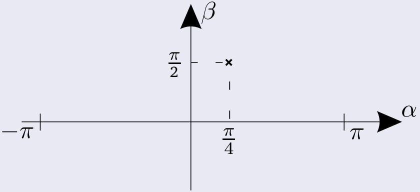
Outline


β
α

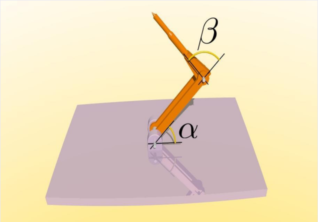

End effector
β
α

Position of the end effector depends on α and β

$$f : \begin{array}{ccc} X & \to & \mathbb{R}^2 \\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} & \mapsto & \begin{pmatrix} 2\cos(\alpha) + cos(\alpha+\beta) \\ 2\sin(\beta) + sin(\alpha+\beta) \end{pmatrix} \end{array}$$



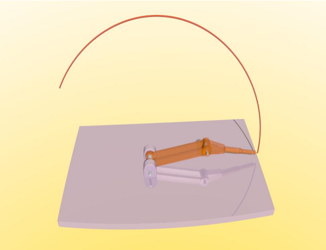
Configuration space



Working space

Given a path δ for the end-effector in the working space, find a curve γ in the configuration space such that

$$f \circ \gamma = \delta$$



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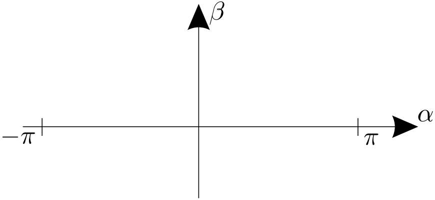

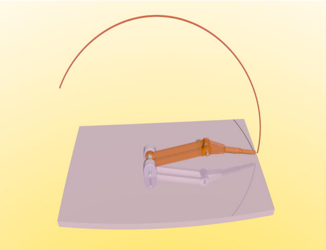

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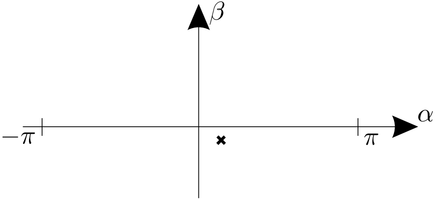

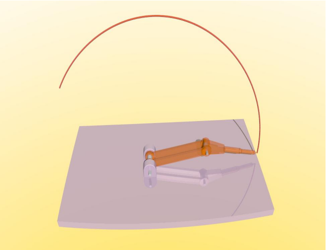

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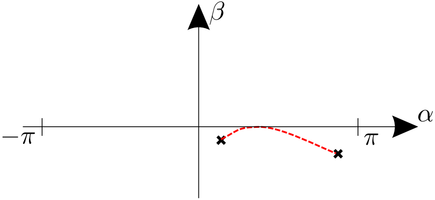

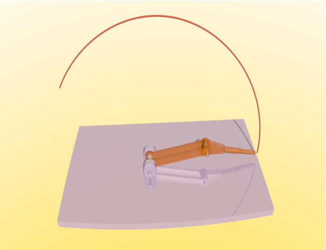

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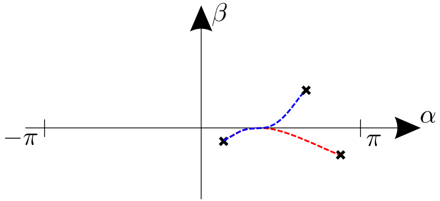
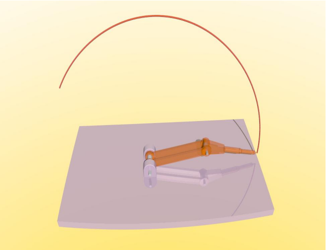
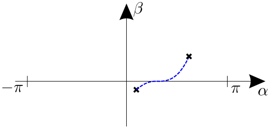
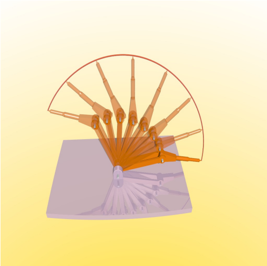
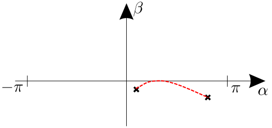





β
$-\pi$
π
α

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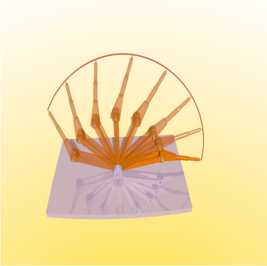
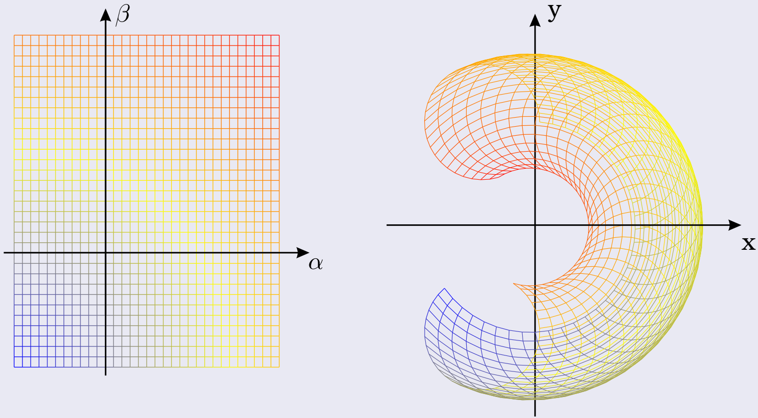


"Graph" of f



Global picture

One wants a global “picture” of the map which does not depend on a choice of system of coordinates neither on the configuration space nor on the working space.

Equivalence

Let f and f' be two smooth maps. Then $f \sim f'$ if there exists diffeomorphisms $g : X \to X'$ and $h : Y' \to Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \uparrow h \\ X' & \xrightarrow{f'} & Y' \end{array}$$

commutes.

Examples

1. $f_1(x) = 2x + 1,\ f_2(x) = -x + 2,$
$$f_1 \sim f_2$$
2. $f_1(x) = x^2,\ f_2(x) = ax^2 + bx + c,\ a \neq 0$
$$f_1 \sim f_2$$
3. $f_1(x) = x^2 + 1,\ f_2(x) = x + 1,$
$$f_1 \not\sim f_2$$

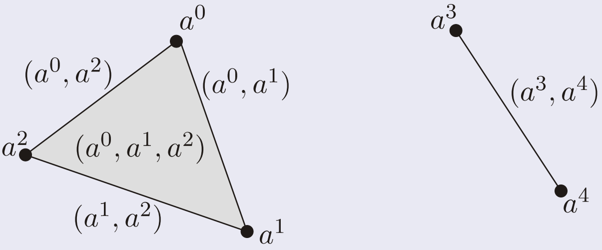
Definition - Abstract simplicial complex

Let $\mathcal{N}$ be a finite set of symbols $\{(a^0), (a^1), \ldots, (a^n)\}$
An abstract simplicial complex $\mathcal{K}$ is a subset of the powerset of $\mathcal{N}$ satisfying : $\sigma \in \mathcal{K} \Rightarrow \forall \sigma_0 \subset \sigma, \sigma_0 \in \mathcal{K}$

Example

$$\mathcal{K} = \{(a^0),(a^1),(a^2),(a^3),(a^4),\\ (a^0,a^1),(a^1,a^2),(a^0,a^2),(a^3,a^4),\\ (a^0,a^1,a^2)\}$$

This will be denoted by $a^0a^1a^2 + a^3a^4$

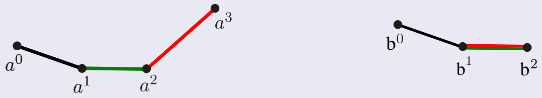


Definition - Simplicial map

Given abstract simplicial complexes $\mathcal{K}$ and $\mathcal{L}$, a simplicial map $F : \mathcal{K}^0 \to \mathcal{L}^0$ is a map with the following property :
If $(a^0, a^1, \dots, a^n)$ is an element of $\mathcal{K}$ then $F(a^0), F(a^1), \dots, F(a^n)$ span a simplex of $\mathcal{L}$.

Example - Simplicial map

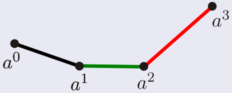
$$\mathcal{K} = a_0 a_1 + a_1 a_2 + a_2 a_3, \quad \mathcal{L} = b_0 b_1 + b_1 b_2$$

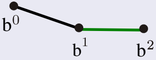


$$F \quad : \quad \begin{array}{ll} a^0 & \mapsto b^0 \\ a^1 & \mapsto b^1 \\ a^2 & \mapsto b^2 \\ a^3 & \mapsto b^1 \end{array}$$

Example - NOT a Simplicial map

$$\mathcal{K} = a_0a_1 + a_1a_2 + a_2a_3, \quad \mathcal{L} = b_0b_1 + b_1b_2$$

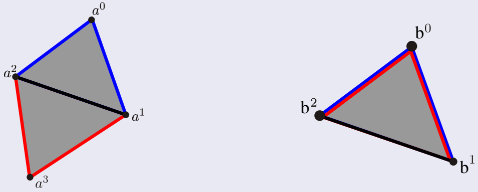




$$F : \begin{array}{lcl} a^0 & \mapsto & b^0 \\ a^1 & \mapsto & b^1 \\ a^2 & \mapsto & b^2 \\ a^3 & \mapsto & b^0 \end{array}$$

Example - Simplicial map

$$\mathcal{K} = a_0 a_1 a_2 + a_1 a_2 a_3, \quad \mathcal{L} = b_0 b_1 b_2$$



$$F \quad : \quad \begin{array}{lll} a^0 & \mapsto & b^0 \\ a^1 & \mapsto & b^1 \\ a^2 & \mapsto & b^2 \\ a^3 & \mapsto & b^0 \end{array}$$

Definition - Topologically conjugate

Let f and f' be continous maps. Then f and f' are topologically conjugate if there exists *homeomorphism* $g : X \to X'$ and $h : Y \to Y'$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \uparrow h \\ X' & \xrightarrow{f'} & Y' \end{array}$$

commutes.

Proposition

$$f \sim f' \Rightarrow f \sim_0 f'$$

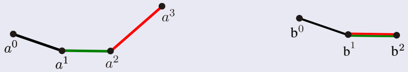
Definition - Portrait

Let f be a smooth map and F a simplicial map, F is a portrait of f if

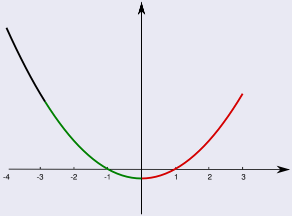
$$f \sim_0 F$$

Example - Simplicial map

The simplicial map



is a portrait of $[-4,3] \ni x \mapsto x^2 - 1 \in \mathbb{R}$



Proposition

Suppose that $f \sim f'$ with

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & Y_1 \\ \downarrow g & & \uparrow h \\ X_2 & \xrightarrow{f'} & Y_2 \end{array}$$

then $f^{-1}(\{y_1\})$ is homeomorphic to $f'^{-1}(\{y_2\})$.

Proposition

For every closed subset A of $\mathbb{R}^n$, there exists a smooth real valued function f such that

$$A = f^{-1}(\{0\})$$

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We are not going to consider all cases ...

Definition - Stable mapping

Let f be a smooth map, f is stable if their exists a nbrd N_f such that

$$\forall f' \in N_f, f' \sim f$$

Examples

1. $g : x \mapsto x^2$ is stable,
2. $f_0 : x \mapsto x^3$ is not stable, since with $f_\epsilon : x \mapsto x(x^2 - \epsilon)$,

$$\epsilon \neq 0 \Rightarrow f_\epsilon \not\sim f_0.$$

Proposition

Suppose that $f \sim f'$ with

$$\begin{tikzcd} x_1 \arrow[r, "f"] \arrow[d, "g"] & y_1 \\ x_2 \arrow[r, "f'"] & y_2 \arrow[u, "h"'] \end{tikzcd}$$

then $\operatorname{rank} df_{x_1} = \operatorname{rank} df'_{x_2}$,

Inverse function theorem

For a differentiable map $f : X \to Y$ with $\dim X = \dim Y = n$, if the rank $\mathrm{d} f_p = n$ then there exists an open nbhd U_p of p such that

$$f|U_p : U_p \to f(U_p)$$

is a diffeomorphism.

Inverse function theorem

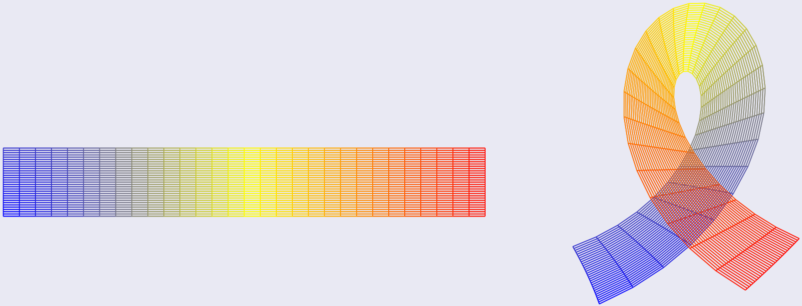
For a differentiable map $f : X \to Y$ with $\dim X = \dim Y = n$, if the $\operatorname{rank} \mathrm{d} f_p = n$ then there exists an open nbhd U_p of p such that

$$f|U_p : U_p \to f(U_p)$$

is a diffeomorphism.

Globalisation

Does $\forall p \in X, \operatorname{rank} \mathrm{d} f_p = n$ imply that $f : X \to Y$ is a diffeomorphism ?



Definition - Differential

$$df(x) = \begin{pmatrix} \partial_1 f_1(x) & \dots & \partial_n f_1(x) \\ \vdots & \ddots & \vdots \\ \partial_1 f_p(x) & \dots & \partial_n f_p(x) \end{pmatrix}$$

Definition - $\tilde{d}f(X)$

$$\tilde{d}f(X) = \left\{ \begin{pmatrix} \partial_1 f_1(\xi^1) & \dots & \partial_n f_1(\xi^1) \\ \vdots & \ddots & \vdots \\ \partial_1 f_p(\xi^p) & \dots & \partial_n f_p(\xi^p) \end{pmatrix} | \xi^1, \dots, \xi^p \in X \right\}$$

Remark

$$df(X) \subset \tilde{d}f(X) \subset \text{ natural extension of } df \text{ with } X.$$

Lemma

Let X be a convex compact subset of $\mathbb{R}^n$,
$f : X \to \mathbb{R}^p$ a smooth mapping with $n \leq p$.
If $\forall J \in \tilde{d}f(X)$, $\operatorname{rank} J = n$ then f is an embedding.

Lemma

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In other words, $f \sim i$ where $i : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$.

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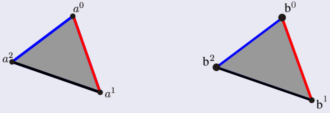
In other words, I is portrait of f, where I is the abstract simplicial identity map.

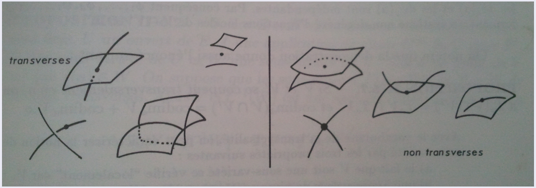


Transversality - Definition

Two submanifolds of M, L_1 and L_2 are said to intersect transversally if

$$\forall p \in L_1 \cap L_2, \mathrm{T}_p M = \mathrm{T}_p L_1 + \mathrm{T}_p L_2.$$

One denotes this by $L_1 \pitchfork L_2$



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Transversality - Definition

Let $f : X \to Y$ be a smooth map between manifolds, and let Z be a submanifold of Y. We say that f is transversal to Z, denoted as $f \pitchfork Z$, if

$$x \in f^{-1}(Z) \Rightarrow df_x T_x X + T_{f(x)} Z = T_{f(x)} Y$$

Proposition

Let k be the codimension of Z in Y.
If $f \pitchfork Z$, then $f^{-1}(Z)$ is a regular submanifold (possibly empty) of X of codimension k.

Thom transversality Theorem

Let Z be submanifold of Y,

$$\{f \in \mathcal{C}^\infty(X, Y) \mid f \pitchfork Z\} \text{ is residual.}$$

In this case, one says that f is generic.

Thom transversality Theorem

Let Z be submanifold of Y,

$$\{f \in \mathcal{C}^\infty(X,Y) \mid f \pitchfork Z\} \text{ is residual.}$$

In this case, one says that f is generic.

Example

Generically, for a smooth map from $f : \mathbb{R}^n \to \mathbb{R}^n$, one has $f \pitchfork \{0\}$. Therefore $\{x \in X \mid f(x) = 0\}$ is a 0-dimensional manifold.

Thom transversality Theorem

Let Z be submanifold of $J^r(X,Y)$,

$$\{f \in \mathcal{C}^\infty(X,Y) \mid j^r f \pitchfork Z\} \text{ is residual.}$$

In this case, one says that f is generic.

Example

For a generic smooth map from $f : \mathbb{R} \to \mathbb{R}$, one has $j^1 f \pitchfork \{y = 0, p = 0\}$. Therefore

$$\{x \mid f(x) = 0 \wedge f'(x) = 0\} = \emptyset$$

Withney theorem

Let X and Y be 2-dimentional manifolds and $f : X \to Y$ be generic. The set $S(f) = \{x \in X \mid \det df_x = 0\}$ is a regular curve. Let $p \in S(f)$, $f(p) = q$. One of the following two situations can occur :

$T_pS(f) \oplus \ker df_p = T_pX$ or $T_pS(f) = \ker df_p$

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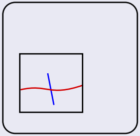
$T_pS(f) \oplus \ker df_p = T_pX$ or $T_pS(f) = \ker df_p$

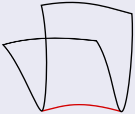
Normal forms

1. if $T_pS(f) \oplus \ker df_p = T_pX$, then there exits nbrds N_p and N_q such that
$$f|N_p \sim (x, y) \mapsto (x, y^2)$$
2. if $T_pS(f) = \ker df_p$, then there exists nbrds N_p and N_q such that
$$f|N_p \sim (x, y) \mapsto (x, xy + y^3)$$

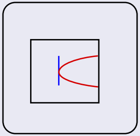
Geometric representation

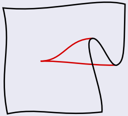
1. if $T_pS(f) \oplus \ker df_p = T_pX$,





2. if $T_pS(f) = \ker df_p$,

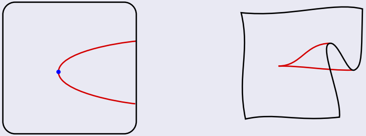




Theorem (Properties of generic maps)

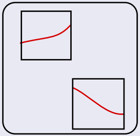
Let X be a compact simply connected domain of $\mathbb{R}^2$ with $\partial X = \Gamma^{-1}(\{0\})$. A generic smooth map f from X to $\mathbb{R}^2$ has the following properties :

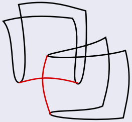
1. $S = \{p \in X | \det df_p = 0\}$ *is regular curve. Moreover, elements of S are folds and cusp. The set of cusp is discrete.*



Theorem

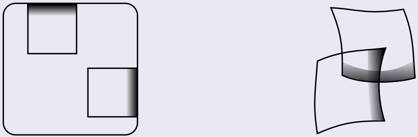
3. *3 singular points do not have the same image,*
4. *2 singular points having the same image are folds points and they have normal crossing.*





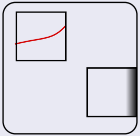
Theorem

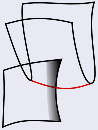
5. *3 boundary points do not have the same image,*
6. *2 boundary points having the same image cross normally.*



Theorem

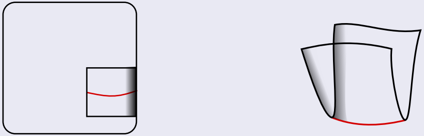
7. *3 different points belonging to the union the singularity curve and boundary do not have the same image,*
8. *If a point on the singularity curve and a boundary have the same image, the singular point is a fold and they have normal crossing.*

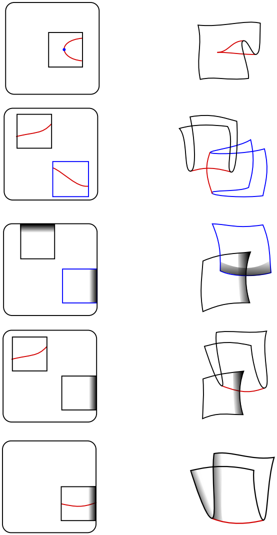


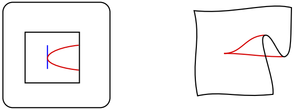


Theorem

9. *if the singularity curve intersects the boundary, then this point is a fold,*
10. *moreover tangents to the singularity curve and boundary curve are different.*







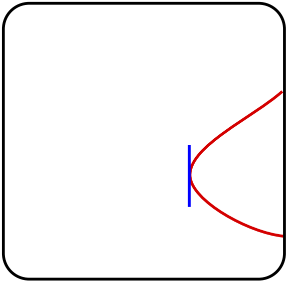
Proposition

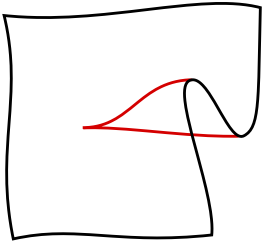
Let f be a smooth generic map from X to $\mathbb{R}^2$, let us denote by c the map defined by :

$$\begin{array}{cccc} c & : & X & \rightarrow & \mathbb{R}^2 \\ & & p & \mapsto & df_p\xi_p \end{array} \quad (1)$$

where ξ is the vector field defined by $\xi_p = \begin{pmatrix} \partial_2 \det df_p \\ -\partial_1 \det df_p \end{pmatrix}$.

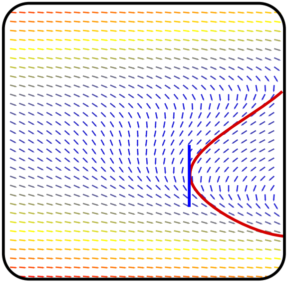
If $c(p) = 0$ and dc_p is invertible then p is a simple cusp. This sufficient condition is locally necessary.

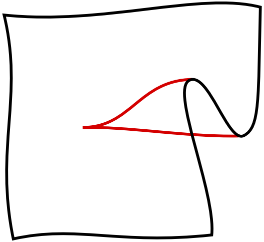




Interval Newton method

$$\begin{array}{cccc} c : & X & \to & \mathbb{R}^2 \\ & p & \mapsto & df_p \xi_p \end{array} \qquad (2)$$

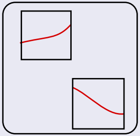


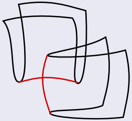


Interval Newton method

$$\begin{array}{cccc} c : & X & \rightarrow & \mathbb{R}^2 \\ & p & \mapsto & df_p \xi_p \end{array} \quad (3)$$

2 different folds



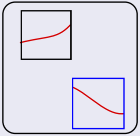


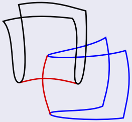
$$S^{\Delta 2} = \{(x_1, x_2) \in S \times S - \Delta(S) \mid f(x_1) = f(x_2)\} / \simeq$$

where $\simeq$ is the relation defined by
$(x_1, x_2) \simeq (x_1', x_2') \Leftrightarrow (x_1, x_2) = (x_2', x_1')$.

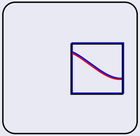
Method

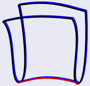
Adaptive bisection scheme on $X \times X$.





$$[x_1] \neq [x_2]$$





$$[x_1] = [x_2]$$

Let us define the map *folds* by

$$\begin{array}{cccc} folds & : & X \times X & \rightarrow & \mathbb{R}^4 \\ & & \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} & \mapsto & \begin{pmatrix} \det df(x_1, y_1) \\ \det df(x_2, y_2) \\ f_1(x_1, y_1) - f_1(x_2, y_2) \\ f_2(x_1, y_1) - f_2(x_2, y_2) \end{pmatrix} \end{array}$$

One has

$$S^{\Delta 2} = folds^{-1}(\{0\}) - \Delta S / \simeq .$$

For any (α, α) in ΔS, the d *folds* is conjugate to

$$\begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & a & b \\ a_{11} & a_{12} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{21} & a_{22} \end{pmatrix}$$

which is not invertible since $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \det df(\alpha) = 0$. In other words, as any box of the form $[x_1] \times [x_1]$ contains ΔS, the interval Newton method will fail.

One needs a method to prove that $f|S \cap [x_1]$ is an embedding.

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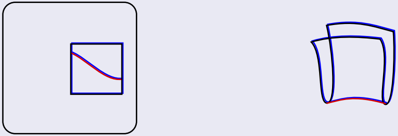

$$[x_1] = [x_2]$$

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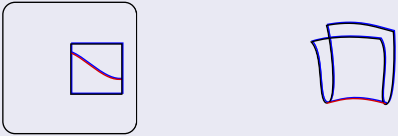
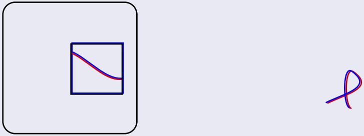


$$[x_1] = [x_2]$$

Not in this case ...



Corollary

Let $f : X \to \mathbb{R}^2$ be a smooth map and X a compact subset of $\mathbb{R}^2$. Let $\Gamma : X \to \mathbb{R}$ be a submersion such that the curve $S = \{x \in X \mid \Gamma(x) = 0\}$ is contractible. If

$$\forall J \in \tilde{d}f(X) \cdot \begin{pmatrix} \partial_2 \Gamma(X) \\ -\partial_1 \Gamma(X) \end{pmatrix}, \text{rank } J = 1$$

then $f|S$ is an embedding.

The last condition is not satisfiable if $[x_1]$ contains a cusp ...

Proposition

Suppose that there exists a unique simple cusp p_0 in the interior of X. Let $\alpha \in \mathbb{R}^{2*}$, s.t. $\alpha \cdot \mathrm{Im}\, df_{p_0} = 0$, and ξ a non vanashing vector field such that $\forall p \in S, \xi_p \in T_pS$ (S contractible).
If $g = \sum \alpha_i \xi^3 f_i : X \to \mathbb{R}$ is a nonvanishing function then $f|S$ is injective. This condition is locally necessary.

Here the vector field ξ is seen as the derivation of $\mathcal{C}^\infty(X)$ defined by

$$\xi = \sum \xi_i \frac{\partial}{\partial x_i}.$$

Initialisation : $P \leftarrow \emptyset$, $P' \leftarrow \{X \times X\}$,
while $P' \neq \emptyset$ **do**
    $[x_1] \times [x_2] \leftarrow s$ where $s \in P'$.
    $P' \leftarrow P' - \{[x_1] \times [x_2]\}$.
    if $[x_1] = [x_2]$ **then**
        if $f|S \cap [x_1]$ is an embedding **then**

$$\text{Print } ([x_1] \times [x_1]) \cap S^{\Delta 2} = \emptyset$$

        else
            Divide $[x_1]$ into $[x_1^a]$ and $[x_1^b]$
            $P' \leftarrow P' \cup \{[x_1^a] \times [x_1^a]\} \cup \{[x_1^a] \times [x_1^b]\} \cup \{[x_1^b] \times [x_1^b]\}$;
        end if
    else
        if Interval Newton algorithm with *folds* on $[x_1] \times [x_2]$ succeed **then**
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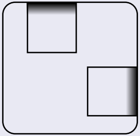
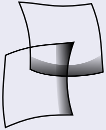
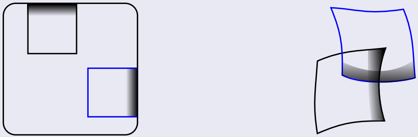




$$\partial X^{\Delta 2} = \{(x_1, x_2) \in \partial X \times \partial X - \Delta(\partial X) \mid f(x_1) = f(x_2)\}/ \simeq$$

$[x_1] \neq [x_2]$



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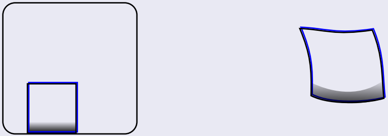


Let us define the map *boundaries* by

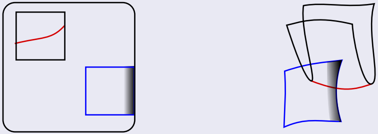
$$\begin{array}{llll} \textit{boundaries} & : \quad X \times X & \rightarrow & \mathbb{R}^4 \\ & \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} & \mapsto & \begin{pmatrix} \Gamma(x_1, y_1) \\ \Gamma(x_2, y_2) \\ f_1(x_1, y_1) - f_1(x_2, y_2) \\ f_2(x_1, y_1) - f_2(x_2, y_2) \end{pmatrix} \end{array}$$

One has

$$\partial X^{\Delta 2} = \textit{boundaries}^{-1}(\{0\}) - \Delta \partial X / \simeq .$$

$$BF = \{(x_1, x_2) \in \partial X \times S \mid f(x_1) = f(x_2)\}$$

$[x_1] \neq [x_2]$



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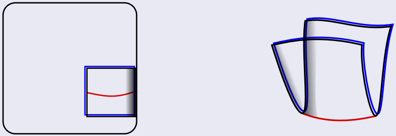

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$$\begin{array}{ccc} X \times X & \to & \mathbb{R}^4 \\ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} & \mapsto & \begin{pmatrix} \det df(x_1, y_1) \\ \gamma(x_2, y_2) \\ f_1(x_1, y_1) - f_1(x_2, y_2) \\ f_2(x_1, y_1) - f_2(x_2, y_2) \end{pmatrix} \end{array}$$

$[x_1] = [x_2]$

$$\begin{array}{ccc} X & \to & \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} & \mapsto & \begin{pmatrix} \det df(x_1, y_1) \\ \gamma(x_1, y_1) \end{pmatrix} \end{array}$$

Lemma

Let $\alpha : t \mapsto (\alpha_1(t), \alpha_2(t))$ and $\beta : t \mapsto (\beta_1(t), \beta_2(t))$ be two smooth curves such that

$$\forall t, \begin{cases} \dot{\alpha}_1 & > & 0 \\ \dot{\beta}_1 & > & 0 \end{cases} \tag{4}$$

$$\exists t_\alpha \exists t_\beta \begin{cases} \alpha(t_\alpha) & = & \beta(t_\beta) \\ \frac{\dot{\alpha}_2}{\dot{\alpha}_1}(t_\alpha) & = & \frac{\dot{\beta}_2}{\dot{\beta}_1}(t_\beta) \end{cases} \tag{5}$$

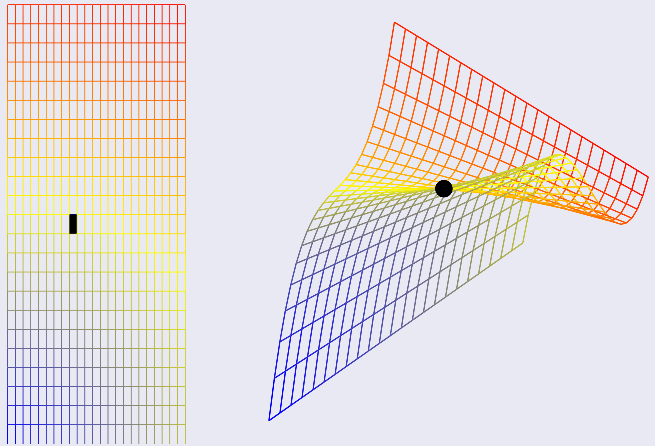
$$\forall t_1 \forall t_2, \frac{\ddot{\alpha}_2\dot{\alpha}_1 - \dot{\alpha}_2\ddot{\alpha}_1}{\dot{\alpha}_1^3} > \frac{\ddot{\beta}_2\dot{\beta}_1 - \dot{\beta}_2\ddot{\beta}_1}{\dot{\beta}_1^3} \tag{6}$$

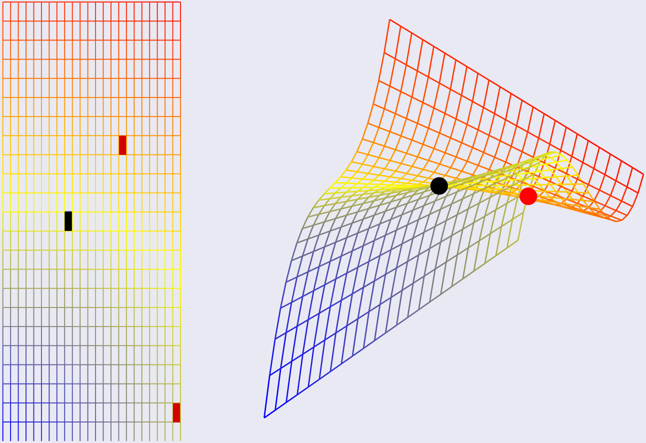
Then $\alpha(t_1) = \beta(t_2)$ implies $t_1 = t_\alpha$ and $t_2 = t_\beta$.

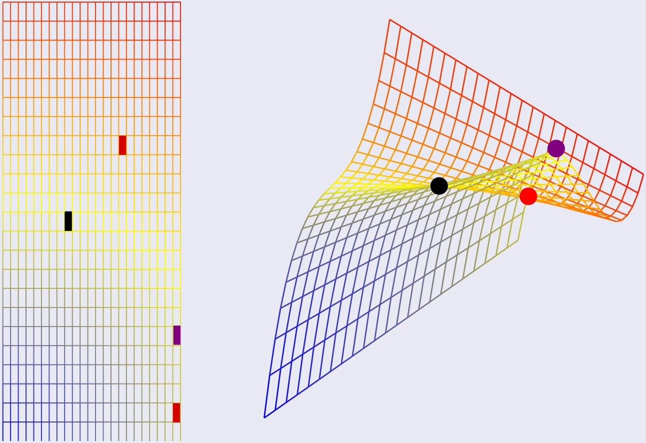
Definition

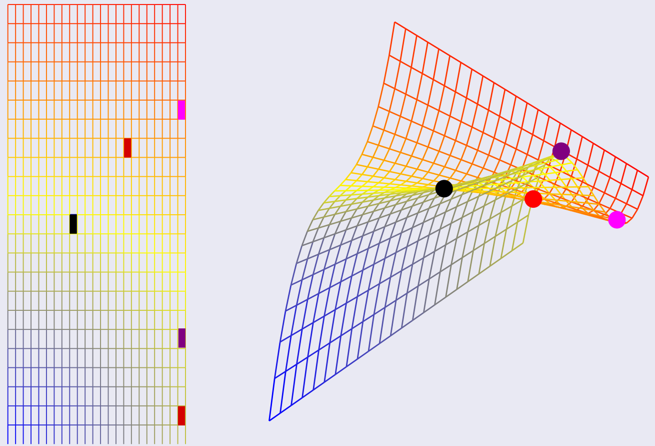
Let f be a smooth map from a compact simply connected domain X of $\mathbb{R}^2$ to $\mathbb{R}^2$. Let us denote by X_0 the subset of X defined by

$$X_0 = \left\{ x \in X \;\middle|\; \begin{array}{ll} & x \text{ is a cusp} \\ \text{or} & \exists y, (x,y) \in S^{\Delta 2} \\ \text{or} & \exists y, (x,y) \in \partial X^{\Delta 2} \\ \text{or} & \exists y, (x,y) \in BF \\ \text{or} & \exists y, (y,x) \in BF \end{array} \right\}$$









Theorem

Let $P = \{p_i\}_{1 \leq i \leq n}$ be a paving such that

i) $S \cup \partial X \subset \cup_i p_i$,

ii) $\forall (p_i, p_j), p_i \cap p_j \neq \emptyset \Rightarrow (S \cup \partial X) \cap p_i \cap p_j$ is simply connected,

iii) $\forall p_i, X \cap p_i$ contains at most one element of X_0,

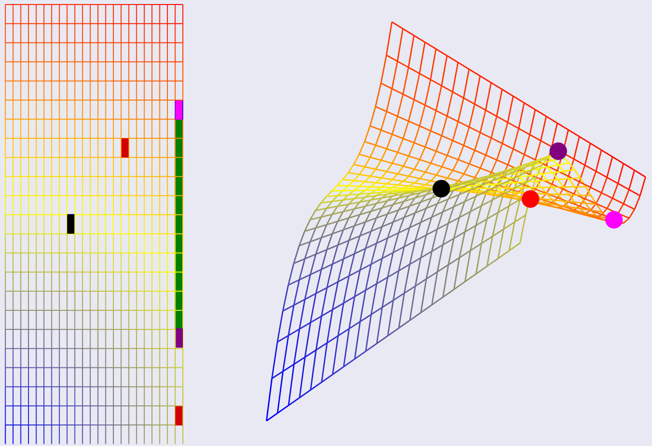
Let $\mathcal{X}$ be the relation on $\{p_i\}_{1 \leq i \leq n}$ defined by

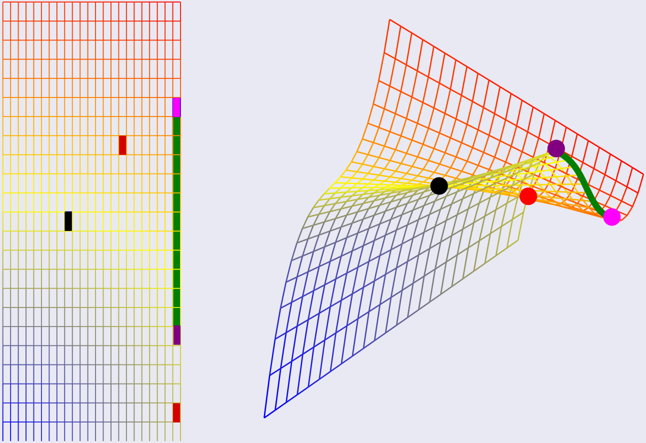
$$p \mathcal{X} q \Leftrightarrow (S \cup \partial X) \cap p \cap q \text{ is simply connected.}$$

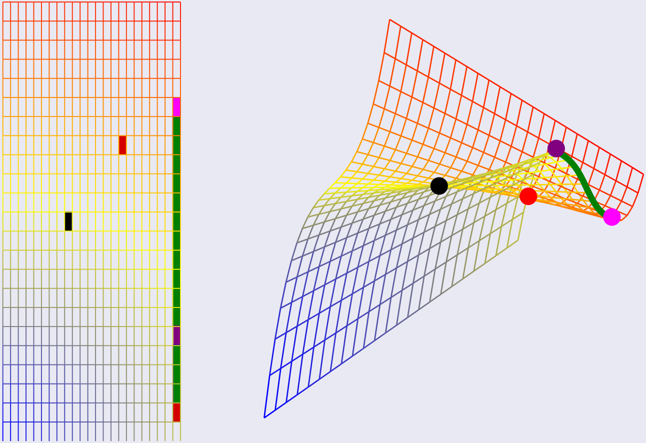
Let us define an equivalence relation f on $\{p_i\}$ by

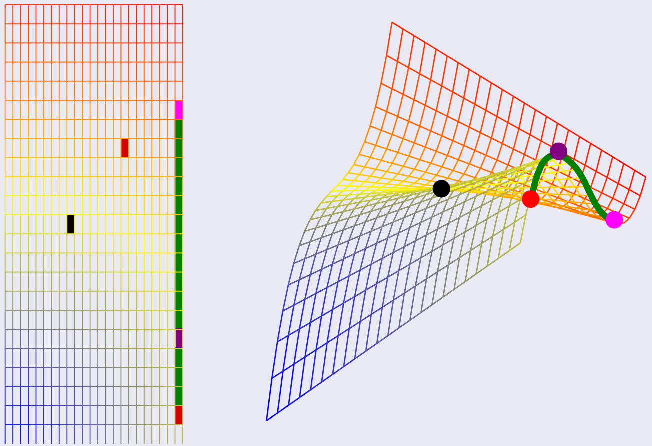
$$p f q \Leftrightarrow f(X_0 \cap p) = f(X_0 \cap q) \text{ and } X_0 \cap p \neq \emptyset,$$

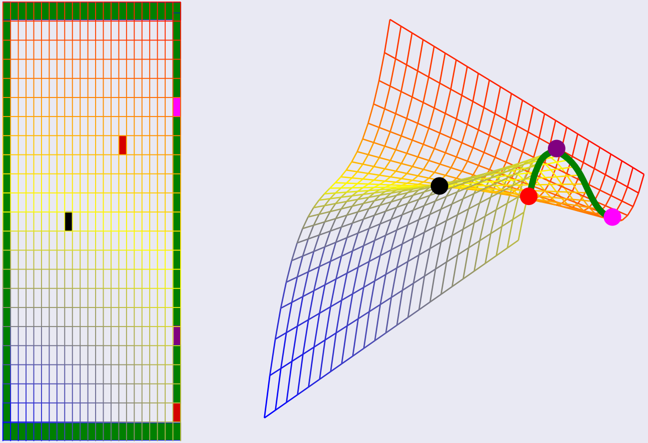
then $\mathcal{X}/f$ is homeomorphic to the Apparent contour of f.

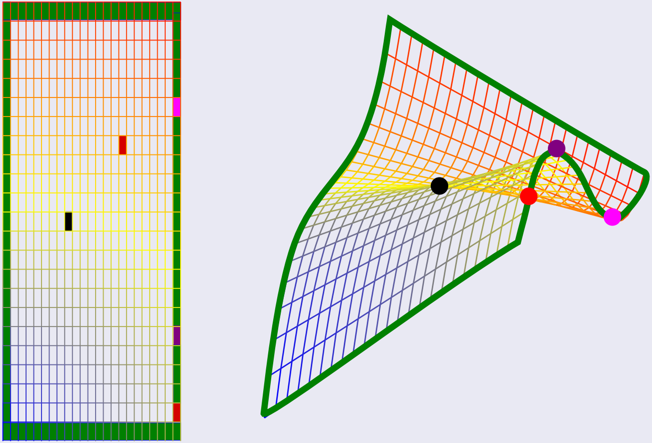


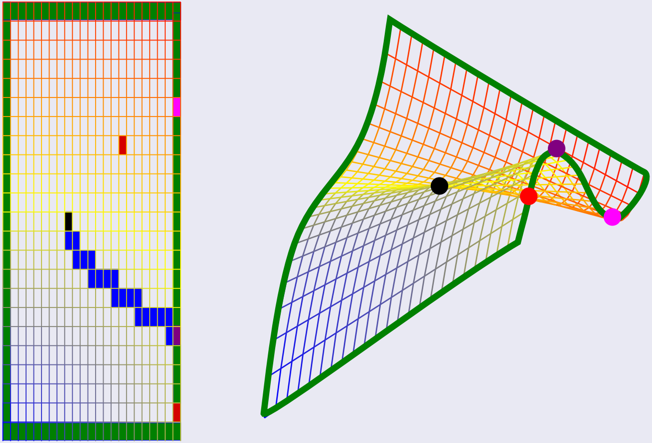


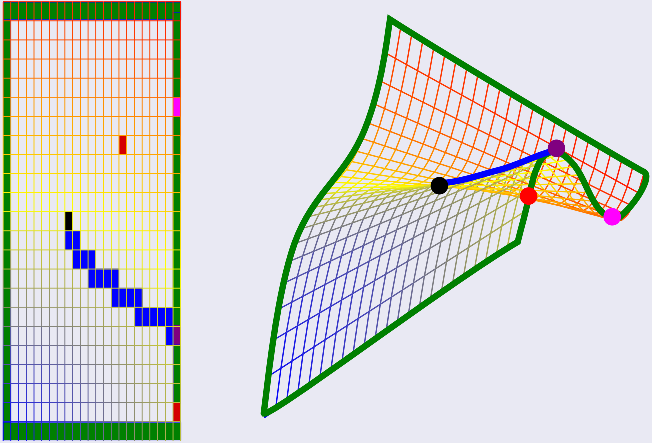


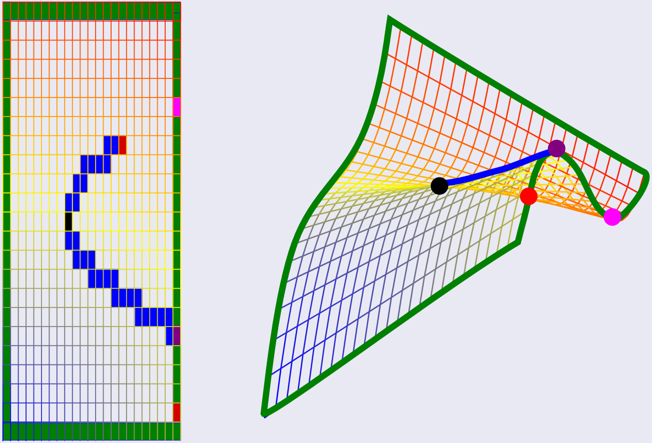


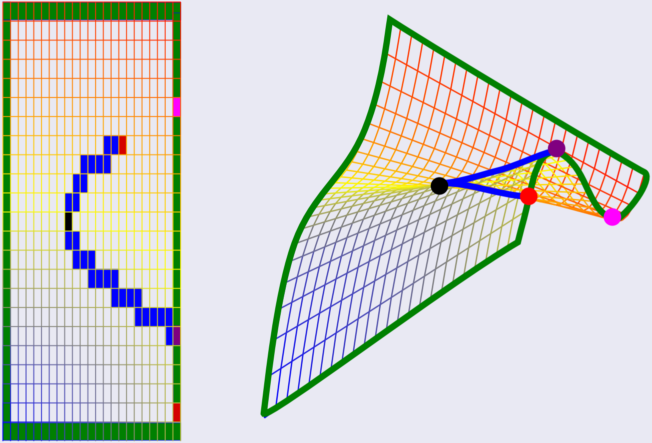


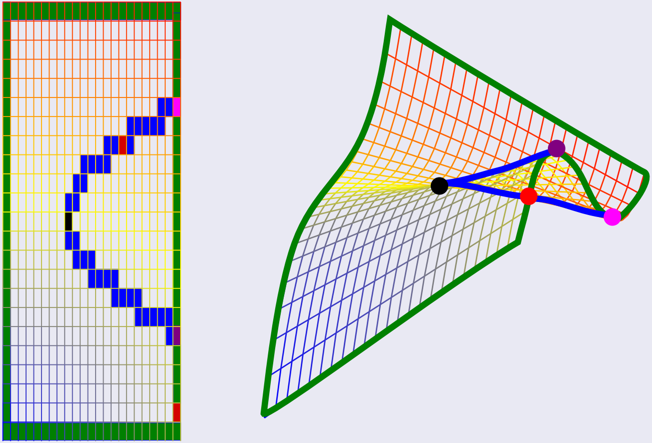


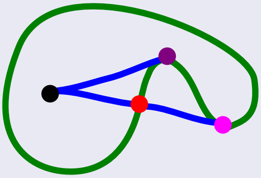










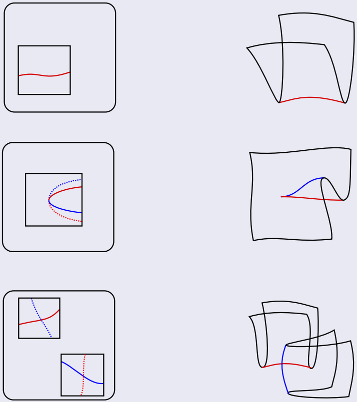


Theorem

Let f be a smooth map from a compact simply connected domain X of $\mathbb{R}^2$ to $\mathbb{R}^2$. For every portrait F of f, the 1-skeleton of ImF contains a subgraph that is an expansion of G_f.

Conjecture

From G_f and its right embedding in $\mathbb{R}^2$ it is possible to create a portrait for f.



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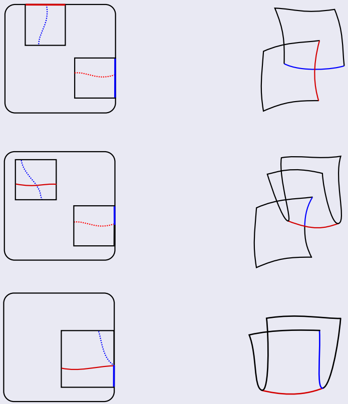

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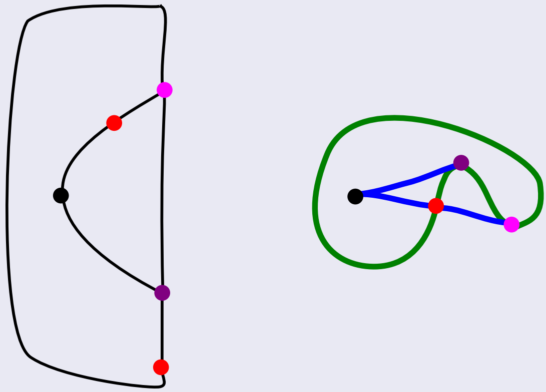

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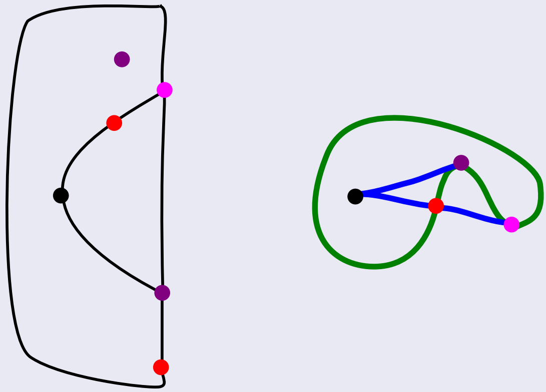

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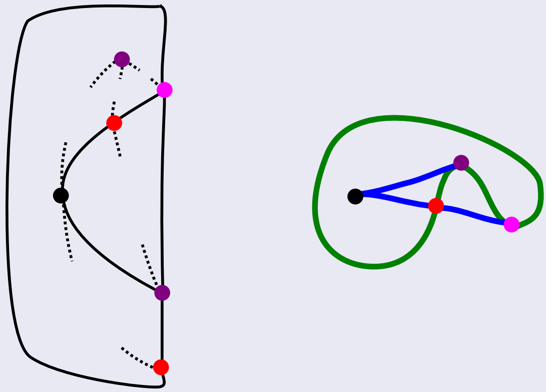

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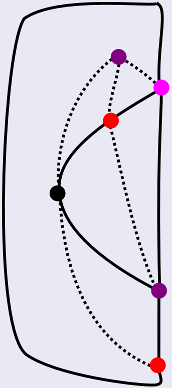

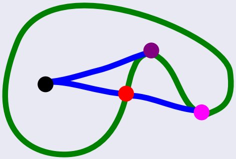

Tack för din uppmärksamhet !