Classification of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$

Nicolas Delanoue - Sébastien Lagrange

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Outline

1. Robotics - Introduction
   - Motion planning
   - Discretization - Portrait of a map

2. Stable mappings and their singularities
   - Stable maps
   - (Genericity and Thom transversality theorem)
   - Withney theorem
   - Compact simply connected with boundary

3. Interval analysis and mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$.

4. Algorithm computing an invariant

5. Conjecture and conclusion
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Stable mappings and their singularities
Interval analysis and mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$.
Algorithm computing an invariant
Conjecture and conclusion

Geometric model
Motion planning
Equivalence
Discretization - Portrait of a map

Classification of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$
End effector
Position of the end effector depends on $\alpha$ and $\beta$

$$f : \mathbb{X} \rightarrow \mathbb{R}^2$$

$$(\alpha, \beta) \mapsto \left( 2 \cos(\alpha) + \cos(\alpha + \beta), 2 \sin(\beta) + \sin(\alpha + \beta) \right)$$

Configuration space  
Working space
Given a path $\delta$ for the end-effector in the working space, find a curve $\gamma$ in the configuration space such that

$$f \circ \gamma = \delta$$
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Classification of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$
"Graph" of $f$

Classification of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$
Global picture

One wants a global “picture” of the map which does not depend on a choice of system of coordinates neither on the configuration space nor on the working space.

Equivalence

Let \( f \) and \( f' \) be two smooth maps. Then \( f \sim f' \) if there exists diffeomorphisms \( g : X \rightarrow X' \) and \( h : Y' \rightarrow Y \) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \uparrow{h} \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

commutes.
Examples

1. \( f_1(x) = 2x + 1, \ f_2(x) = -x + 2, \)
   \[ f_1 \sim f_2 \]

2. \( f_1(x) = x^2, \ f_2(x) = ax^2 + bx + c, \ a \neq 0 \)
   \[ f_1 \sim f_2 \]

3. \( f_1(x) = x^2 + 1, \ f_2(x) = x + 1, \)
   \[ f_1 \not\sim f_2 \]
Definition - Abstract simplicial complex

Let $\mathcal{N}$ be a finite set of symbols $\{(a^0), (a^1), \ldots, (a^n)\}$
An abstract simplicial complex $\mathcal{K}$ is a subset of the powerset of $\mathcal{N}$
satisfying : $\sigma \in \mathcal{K} \Rightarrow \forall \sigma_0 \subset \sigma, \sigma_0 \in \mathcal{K}$
Example

\[ \mathcal{K} = \{(a^0), (a^1), (a^2), (a^3), (a^4), \\
(a^0, a^1), (a^1, a^2), (a^0, a^2), (a^3, a^4), \\
(a^0, a^1, a^2)\} \]

This will be denoted by \( a^0 a^1 a^2 + a^3 a^4 \)
Definition - Simplicial map

Given abstract simplicial complexes $\mathcal{K}$ and $\mathcal{L}$, a simplicial map $F : \mathcal{K}^0 \rightarrow \mathcal{L}^0$ is a map with the following property:
If $(a^0, a^1, \ldots, a^n)$ is an element of $\mathcal{K}$ then $F(a^0), F(a^1), \ldots, F(a^n)$ span a simplex of $\mathcal{L}$. 
Example - Simplicial map

\[ \mathcal{K} = a_0 a_1 + a_1 a_2 + a_2 a_3, \quad \mathcal{L} = b_0 b_1 + b_1 b_2 \]

\[ F : \begin{array}{c} a^0 \mapsto b^0 \\ a^1 \mapsto b^1 \\ a^2 \mapsto b^2 \\ a^3 \mapsto b^1 \end{array} \]
Example - NOT a Simplicial map

\[ K = a_0a_1 + a_1a_2 + a_2a_3, \quad L = b_0b_1 + b_1b_2 \]

\[ F : \begin{align*}
    a^0 & \mapsto b^0 \\
    a^1 & \mapsto b^1 \\
    a^2 & \mapsto b^2 \\
    a^3 & \mapsto b^0
\end{align*} \]
Example - Simplicial map

\[ \mathcal{K} = a_0 a_1 a_2 + a_1 a_2 a_3, \quad \mathcal{L} = b_0 b_1 b_2 \]

\[ F : \begin{align*}
    a^0 & \mapsto b^0 \\
    a^1 & \mapsto b^1 \\
    a^2 & \mapsto b^2 \\
    a^3 & \mapsto b^0 
\end{align*} \]
**Definition - Topologically conjugate**

Let $f$ and $f'$ be continuous maps. Then $f$ and $f'$ are topologically conjugate if there exists a homeomorphism $g : X \rightarrow X'$ and $h : Y \rightarrow Y'$ such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow g & & \uparrow h \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

commutes.

**Proposition**

\[f \sim f' \Rightarrow f \sim_0 f'\]
Definition - Portrait

Let \( f \) be a smooth map and \( F \) a simplicial map, \( F \) is a portrait of \( f \) if

\[
f \sim_0 F
\]
Example - Simplicial map

The simplicial map

\[ a^0 \rightarrow a^1 \rightarrow a^2 \rightarrow a^3 \]
\[ b^0 \rightarrow b^1 \rightarrow b^2 \]

is a portrait of \([-4, 3] \ni x \mapsto x^2 - 1 \in \mathbb{R}\]
Proposition

Suppose that $f \sim f'$ with

\[ x_1 \xrightarrow{f} y_1 \] \[ \downarrow g \] \[ \uparrow h \] \[ \downarrow \] \[ x_2 \xrightarrow{f'} y_2 \]

then $f^{-1}({y_1})$ is homeomorphic to $f'^{-1}({y_2})$.
Proposition

For every closed subset $A$ of $\mathbb{R}^n$, there exists a smooth real valued function $f$ such that

$$A = f^{-1}(\{0\})$$
Proposition

For every closed subset $A$ of $\mathbb{R}^n$, there exists a smooth real valued function $f$ such that

$$A = f^{-1}(\{0\})$$

We are not going to consider all cases ...
Definition - Stable mapping

Let $f$ be a smooth map, $f$ is stable if there exists a nbhd $N_f$ such that

$$\forall f' \in N_f, f' \sim f$$

Examples

1. $g : x \mapsto x^2$ is stable,
2. $f_0 : x \mapsto x^3$ is not stable, since with $f_\epsilon : x \mapsto x(x^2 - \epsilon)$,

$$\epsilon \neq 0 \Rightarrow f_\epsilon \not\sim f_0.$$
Proposition

Suppose that $f \sim f'$ with

\[ \begin{array}{ccc}
  x_1 & \xrightarrow{f} & y_1 \\
  \downarrow{g} & & \uparrow{h} \\
  x_2 & \xrightarrow{f'} & y_2 
\end{array} \]

then $\text{rank } df_{x_1} = \text{rank } df'_{x_2}$.
Inverse function theorem

For a differentiable map $f : X \rightarrow Y$ with $\dim X = \dim Y = n$, if the rank $d f_p = n$ then there exists an open nbhd $U_p$ of $p$ such that

$$f|U_p : U_p \rightarrow f(U_p)$$

is a diffeomorphism.
Inverse function theorem

For a differentiable map \( f : X \rightarrow Y \) with \( \dim X = \dim Y = n \), if the rank of \( f_p = n \) then there exists an open nbhd \( U_p \) of \( p \) such that

\[
f|_{U_p} : U_p \rightarrow f(U_p)
\]

is a diffeomorphism.

Globalisation

Does \( \forall p \in X, \text{rank } d f_p = n \) imply that \( f : X \rightarrow Y \) is a diffeomorphism?
Robotics - Introduction
Stable mappings and their singularities
Interval analysis and mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$.
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Introduction
Stable maps
(Genericity and Thom transversality theorem)
Whitney theorem
Compact simply connected with boundary

Classification of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$
**Definition - Differential**

\[ df(x) = \begin{pmatrix} \partial_1 f_1(x) & \ldots & \partial_n f_1(x) \\ \vdots & \ddots & \vdots \\ \partial_1 f_p(x) & \ldots & \partial_n f_p(x) \end{pmatrix} \]

**Definition - \( \tilde{df}(X) \)**

\[ \tilde{df}(X) = \left\{ \begin{pmatrix} \partial_1 f_1(\xi^1) & \ldots & \partial_n f_1(\xi^1) \\ \vdots & \ddots & \vdots \\ \partial_1 f_p(\xi^p) & \ldots & \partial_n f_p(\xi^p) \end{pmatrix} \mid \xi^1, \ldots, \xi^p \in X \right\} \]

**Remark**

\[ df(X) \subset \tilde{df}(X) \subset \text{natural extension of } df \text{ with } X. \]
Lemma

Let $X$ be a convex compact subset of $\mathbb{R}^{n}$, $f : X \to \mathbb{R}^{p}$ a smooth mapping with $n \leq p$. If $\forall J \in \tilde{df}(X)$, rank $J = n$ then $f$ is an embedding.
Lemma

Let $X$ be a convex compact subset of $\mathbb{R}^n$, $f : X \to \mathbb{R}^p$ a smooth mapping with $n \leq p$. If $\forall J \in \tilde{df}(X)$, $\text{rank } J = n$ then $f$ is an embedding.

In other words, $f \sim i$ where $i : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$. 
Lemma

Let $X$ be a convex compact subset of $\mathbb{R}^n$, $f : X \rightarrow \mathbb{R}^p$ a smooth mapping with $n \leq p$. If $\forall J \in \tilde{df}(X)$, rank $J = n$ then $f$ is an embedding.

In other words, $f \sim i$ where $i : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$.

In other words, $I$ is portrait of $f$, where $I$ is the abstract simplicial identity map.
Transversality - Definition

Two submanifolds of $M$, $L_1$ and $L_2$ are said to intersect transversally if

$$\forall p \in L_1 \cap L_2, T_p M = T_p L_1 + T_p L_2.$$ 

One denotes this by $L_1 \pitchfork L_2$

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a. Catastrophes et bifurcations - Michel Demazure

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Classification of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$
Transversality - Definition

Let \( f : X \rightarrow Y \) be a smooth map between manifolds, and let \( Z \) be a submanifold of \( Y \). We say that \( f \) is transversal to \( Z \), denoted as \( f \pitchfork Z \), if

\[
x \in f^{-1}(Z) \Rightarrow df_x T_x X + T_{f(x)} Z = T_{f(x)} Y
\]
Proposition

Let $k$ be the codimension of $Z$ in $Y$. If $f \pitchfork Z$, then $f^{-1}(Z)$ is a regular submanifold (possibly empty) of $X$ of codimension $k$. 
Thom transversality Theorem

Let $Z$ be submanifold of $Y$,

$$\{ f \in C^\infty(X, Y) \mid f \pitchfork Z \}$$

is residual.

In this case, one says that $f$ is generic.
Thom transversality Theorem

Let $Z$ be submanifold of $Y$, 

$$\{ f \in \mathcal{C}^\infty(X, Y) \mid f \pitchfork Z \} \text{ is residual.}$$

In this case, one says that $f$ is generic.

Example

Generically, for a smooth map from $f : \mathbb{R}^n \to \mathbb{R}^n$, one has $f \pitchfork \{0\}$. Therefore $\{ x \in X \mid f(x) = 0 \}$ is a 0-dimensional manifold.
Thom transversality Theorem

Let $Z$ be submanifold of $J^r(X, Y)$,

$$\{ f \in C^\infty(X, Y) \mid j^r f \pitchfork Z \}$$

is residual.

In this case, one says that $f$ is generic.

Example

For a generic smooth map from $f : \mathbb{R} \rightarrow \mathbb{R}$, one has $j^1 f \pitchfork \{ y = 0, p = 0 \}$. Therefore

$$\{ x \mid f(x) = 0 \land f'(x) = 0 \} = \emptyset$$
**Withney theorem**

Let $X$ and $Y$ be 2-dimensional manifolds and $f : X \to Y$ be generic. The set $S(f) = \{ x \in X \mid \det df_x = 0 \}$ is a regular curve. Let $p \in S(f)$, $f(p) = q$. One of the following two situations can occur:

- $T_p S(f) \oplus \ker df_p = T_p X$ or $T_p S(f) = \ker df_p$

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**Classification of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$**
**Withney theorem**

Let $X$ and $Y$ be 2-dimensional manifolds and $f : X \rightarrow Y$ be generic. The set $S(f) = \{x \in X \mid \det df_x = 0\}$ is a regular curve. Let $p \in S(f)$, $f(p) = q$. One of the following two situations can occur:

$$T_p S(f) \oplus \ker df_p = T_p X \text{ or } T_p S(f) = \ker df_p$$

**Normal forms**

1. If $T_p S(f) \oplus \ker df_p = T_p X$, then there exists nbrds $N_p$ and $N_q$ such that

$$f|N_p \sim (x, y) \mapsto (x, y^2)$$

2. If $T_p S(f) = \ker df_p$, then there exists nbrds $N_p$ and $N_q$ such that

$$f|N_p \sim (x, y) \mapsto (x, xy + y^3)$$
**Geometric representation**

1. If $T_pS(f) \oplus \ker df_p = T_pX$,
   
   ![Diagram 1](null)

2. If $T_pS(f) = \ker df_p$,
   
   ![Diagram 2](null)
Let $X$ be a compact simply connected domain of $\mathbb{R}^2$ with $\partial X = \Gamma^{-1} \{0\}$. A generic smooth map $f$ from $X$ to $\mathbb{R}^2$ has the following properties:

1. $S = \{ p \in X \mid \det df_p = 0 \}$ is regular curve. Moreover, elements of $S$ are folds and cusp. The set of cusp is discrete.
3 singular points do not have the same image,

2 singular points having the same image are folds points and they have normal crossing.
Theorem

5. 3 boundary points do not have the same image,
6. 2 boundary points having the same image cross normally.
Theorem

7. 3 different points belonging to the union the singularity curve and boundary do not have the same image,

8. If a point on the singularity curve and a boundary have the same image, the singular point is a fold and they have normal crossing.
Theorem

9. if the singularity curve intersects the boundary, then this point is a fold,

10. moreover tangents to the singularity curve and boundary curve are different.
Stable mappings and their singularities
Interval analysis and mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$.
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Conjecture and conclusion

Classification of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$
Proposition

Let $f$ be a smooth generic map from $X$ to $\mathbb{R}^2$, let us denote by $c$ the map defined by:

$$c : X \rightarrow \mathbb{R}^2$$

$$p \mapsto df_p \xi_p$$

where $\xi$ is the vector field defined by $\xi_p = \begin{pmatrix} \partial_2 \det df_p \\ -\partial_1 \det df_p \end{pmatrix}$.

If $c(p) = 0$ and $dc_p$ is invertible then $p$ is a simple cusp. This sufficient condition is locally necessary.
Interval analysis and mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Algorithm computing an invariant

Conjecture and conclusion

Cusp
Fold - Fold
Boundary - Boundary
Boundary - Fold

Interval Newton method

$$c : X \rightarrow \mathbb{R}^2$$
$$p \mapsto df_p \xi_p$$

Classification of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$
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Classification of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$
2 different folds

\[ S^{\Delta^2} = \{(x_1, x_2) \in S \times S - \Delta(S) \mid f(x_1) = f(x_2)\} / \simeq \]

where \( \simeq \) is the relation defined by
\[(x_1, x_2) \simeq (x'_1, x'_2) \iff (x_1, x_2) = (x'_2, x'_1).\]

**Method**

Adaptive bisection scheme on \( X \times X.\)
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Classification of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$

$[x_1] \neq [x_2]$

$[x_1] = [x_2]$
Let us define the map $folds$ by

$$folds : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^4$$

$$(x_1, y_1, x_2, y_2) \mapsto \left(\begin{array}{c} \det df(x_1, y_1) \\ \det df(x_2, y_2) \\ f_1(x_1, y_1) - f_1(x_2, y_2) \\ f_2(x_1, y_1) - f_2(x_2, y_2) \end{array}\right)$$

One has

$$S^{\Delta^2} = folds^{-1}(\{0\}) - \Delta S / \simeq .$$
For any \((\alpha, \alpha)\) in \(\Delta S\), the d folds is conjugate to
\[
\begin{pmatrix}
 a & b & 0 & 0 \\
 0 & 0 & a & b \\
 a_{11} & a_{12} & a_{11} & a_{12} \\
 a_{21} & a_{22} & a_{21} & a_{22}
\end{pmatrix}
\]
which is not invertible since \(\det\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \det df(\alpha) = 0\). In other words, as any box of the form \([x_1] \times [x_1]\) contains \(\Delta S\), the interval Newton method will fail.

One needs a method to prove that \(f|S \cap [x_1]\) is an embedding.
One needs a method to prove that $f|S \cap [x_1]$ is an embedding.

$[x_1] = [x_2]$
One needs a method to prove that $f|S \cap [x_1]$ is an embedding.

$[x_1] = [x_2]$
Corollary

Let $f : X \to \mathbb{R}^2$ be a smooth map and $X$ a compact subset of $\mathbb{R}^2$. Let $\Gamma : X \to \mathbb{R}$ be a submersion such that the curve $S = \{ x \in X \mid \Gamma(x) = 0 \}$ is contractible. If

$$\forall J \in \tilde{df}(X) \cdot \begin{pmatrix} \partial_2 \Gamma(X) \\ -\partial_1 \Gamma(X) \end{pmatrix}, \text{rank } J = 1$$

then $f|S$ is an embedding.
The last condition is not satisfiable if \([x_1]\) contains a cusp . . .

**Proposition**

Suppose that there exists a unique simple cusp \(p_0\) in the interior of \(X\). Let \(\alpha \in \mathbb{R}^2^*\), s.t. \(\alpha \cdot \text{Im } df_{p_0} = 0\), and \(\xi\) a non vanishing vector field such that \(\forall p \in S, \xi_p \in T_p S\) (\(S\) contractible).

If \(g = \sum \alpha_i \xi^3 f_i : X \to \mathbb{R}\) is a nonvanishing function then \(f|S\) is injective. This condition is locally necessary.

Here the vector field \(\xi\) is seen as the derivation of \(C^\infty(X)\) defined by

\[\xi = \sum \xi_i \frac{\partial}{\partial x_i}.\]
Initialisation : $P \leftarrow \emptyset$, $P' \leftarrow \{X \times X\}$,

while $P' \neq \emptyset$ do

$[x_1] \times [x_2] \leftarrow s$ where $s \in P'$.

$P' \leftarrow P' - \{[x_1] \times [x_2]\}$.

if $[x_1] = [x_2]$ then

if $f|S \cap [x_1]$ is an embedding then

Print ($[x_1] \times [x_1]$) $\cap S^{\Delta 2} = \emptyset$

else

Divide $[x_1]$ into $[x_1^a]$ and $[x_1^b]$

$P' \leftarrow P' \cup \{[x_1^a] \times [x_1^a]\} \cup \{[x_1^a] \times [x_1^b]\} \cup \{[x_1^b] \times [x_1^a]\}$

end if

else

Divide $[x_1] \times [x_2]$ $\leftarrow$ $[x_1^a] \times [x_2^a]$ $\cup \{[x_1^a] \times [x_2^b]\} \cup \{[x_1^b] \times [x_2^a]\} \cup \{[x_1^b] \times [x_2^b]\}$

end if

end if

end while
Initialisation : $P \leftarrow \emptyset$, $P' \leftarrow \{X \times X\}$,

while $P' \neq \emptyset$ do

$[x_1] \times [x_2] \leftarrow s$ where $s \in P'$.

$P' \leftarrow P' - \{[x_1] \times [x_2]\}$.

if $[x_1] = [x_2]$ then

if $f|S \cap [x_1]$ is an embedding then

Print $( [x_1] \times [x_1] ) \cap S^{\Delta_2} = \emptyset$

else

Divide $[x_1]$ into $[x_1]^a$ and $[x_1]^b$

$P' \leftarrow P' \cup \{[x_1]^a \times [x_1]^a\} \cup \{[x_1]^a \times [x_1]^b\} \cup \{[x_1]^b \times [x_1]^b\}$;

end if

else

Divide $[x_1]$ into $[x_1]^a$ and $[x_1]^b$

Divide $[x_2]$ into $[x_2]^a$ and $[x_2]^b$

$P' \leftarrow P' \cup \{[x_1]^a \times [x_2]^a\} \cup \{[x_1]^a \times [x_2]^b\} \cup \{[x_1]^b \times [x_2]^a\} \cup \{[x_1]^b \times [x_2]^b\}$;

end if

end if

end while
Initialisation: $P \leftarrow \emptyset$, $P' \leftarrow \{X \times X\}$,
while $P' \neq \emptyset$ do
    $[x_1] \times [x_2] \leftarrow s$ where $s \in P'$.
    $P' \leftarrow P' - \{[x_1] \times [x_2]\}$.
    if $[x_1] = [x_2]$ then
        if $f|S \cap [x_1]$ is an embedding then
            Print $([x_1] \times [x_1]) \cap S^{\Delta 2} = \emptyset$
        else
            Divide $[x_1]$ into $[x^a_1]$ and $[x^b_1]$
            $P' \leftarrow P' \cup \{[x^a_1] \times [x^a_1]\} \cup \{[x^a_1] \times [x^b_1]\} \cup \{[x^b_1] \times [x^a_1]\}$
        end if
    else
        Divide $[x_1]$ into $[x^a_1]$ and $[x^b_1]$
        $P' \leftarrow P' \cup \{[x^a_1] \times [x^a_1]\} \cup \{[x^a_1] \times [x^b_1]\} \cup \{[x^b_1] \times [x^b_1]\}$
    end if
end if
end while
Initialisation: \( P \leftarrow \emptyset, P' \leftarrow \{X \times X\}, \)

while \( P' \neq \emptyset \) do

\([x_1] \times [x_2] \leftarrow s \) where \( s \in P' \).

\( P' \leftarrow P' - \{[x_1] \times [x_2]\} \).

if \([x_1] = [x_2]\) then

if \( f|S \cap [x_1] \) is an embedding then

\[ ([x_1] \times [x_1]) \cap S^{\Delta^2} = \emptyset \]

else

Divide \([x_1]\) into \([x_1^a]\) and \([x_1^b]\)

\( P' \leftarrow P' \cup \{[x_1^a] \times [x_1^a]\} \cup \{[x_1^a] \times [x_1^b]\} \cup \{[x_1^b] \times [x_1]\} \)

end if

else

Divide \([x_2]\) into \([x_2^a]\) and \([x_2^b]\)

Divide \([x_1]\) into \([x_1^a]\) and \([x_1^b]\)

\( P' \leftarrow P' \cup \{[x_1^a] \times [x_2^a]\} \cup \{[x_1^a] \times [x_2^b]\} \cup \{[x_1^b] \times [x_2]\} \cup \{[x_1^b] \times [x_2^b]\} \)

end if

end if

end while

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Classification of mappings from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \)
Initialisation : $P \leftarrow \emptyset, P' \leftarrow \{X \times X\}$,
while $P' \neq \emptyset$ do
  $[x_1] \times [x_2] \leftarrow s$ where $s \in P'$.
  $P' \leftarrow P' - \{[x_1] \times [x_2]\}$.
  if $[x_1] = [x_2]$ then
    if $f|S \cap [x_1]$ is an embedding then
      Print $([x_1] \times [x_1]) \cap S^\Delta_2 = \emptyset$
    else
      Divide $[x_1]$ into $[x^a_1]$ and $[x^b_1]$
      $P' \leftarrow P' \cup \{[x^a_1] \times [x^a_1]\} \cup \{[x^b_1] \times [x^b_1]\}$
    end if
  else
    if Interval Newton algorithm with folds on $[x_1] \times [x_2]$ succeed then
      $P \leftarrow P \cup \{[x_1] \times [x_2]\}$
    else
      Divide $[x_1]$ into $[x^a_1]$ and $[x^b_1]$
      Divide $[x_2]$ into $[x^a_2]$ and $[x^b_2]$
      $P' \leftarrow P' \cup \{[x^a_1] \times [x^a_2]\} \cup \{[x^a_1] \times [x^b_2]\} \cup \{[x^b_1] \times [x^a_2]\} \cup \{[x^b_1] \times [x^b_2]\}$
    end if
  end if
end while

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\[ \partial X^{\Delta^2} = \left\{ (x_1, x_2) \in \partial X \times \partial X - \Delta(\partial X) \mid f(x_1) = f(x_2) \right\} / \simeq \]
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Let us define the map \( \text{boundaries} \) by

\[
\text{boundaries} : \quad X \times X \quad \rightarrow \quad \mathbb{R}^4
\]

\[
\begin{pmatrix}
  x_1 \\
y_1
\end{pmatrix}, \quad \begin{pmatrix}
  x_2 \\
y_2
\end{pmatrix} \quad \mapsto \quad \begin{pmatrix}
  \Gamma(x_1, y_1) \\
  \Gamma(x_2, y_2) \\
f_1(x_1, y_1) - f_1(x_2, y_2) \\
f_2(x_1, y_1) - f_2(x_2, y_2)
\end{pmatrix}
\]

One has

\[
\partial X^{\Delta^2} = \text{boundaries}^{-1}(\{0\}) - \Delta \partial X / \simeq.
\]
\[ BF = \{ (x_1, x_2) \in \partial X \times S \mid f(x_1) = f(x_2) \} \]
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\[
\begin{align*}
\text{Cusp} & \\
\text{Fold} - \text{Fold} & \\
\text{Boundary} - \text{Boundary} & \\
\text{Boundary} - \text{Fold} & \\
\end{align*}
\]
Lemma

Let $\alpha : t \mapsto (\alpha_1(t), \alpha_2(t))$ and $\beta : t \mapsto (\beta_1(t), \beta_2(t))$ be two smooth curves such that

$$\forall t, \begin{cases} \dot{\alpha}_1 > 0 \\ \dot{\beta}_1 > 0 \end{cases}$$ (4)

$$\exists t_\alpha \exists t_\beta \left\{ \begin{array}{c} \alpha(t_\alpha) = \beta(t_\beta) \\ \frac{\dot{\alpha}_2}{\dot{\alpha}_1}(t_\alpha) = \frac{\dot{\beta}_2}{\dot{\beta}_1}(t_\beta) \end{array} \right. $$ (5)

$$\forall t_1, t_2, \frac{\ddot{\alpha}_2 \dot{\alpha}_1 - \dot{\alpha}_2 \dddot{\alpha}_1}{\dot{\alpha}_1^3} > \frac{\ddot{\beta}_2 \dot{\beta}_1 - \dot{\beta}_2 \dddot{\beta}_1}{\dot{\beta}_1^3} $$ (6)

Then $\alpha(t_1) = \beta(t_2)$ implies $t_1 = t_\alpha$ and $t_2 = t_\beta$. 
Definition

Let $f$ be a smooth map from a compact simply connected domain $X$ of $\mathbb{R}^2$ to $\mathbb{R}^2$. Let us denote by $X_0$ the subset of $X$ defined by

$$X_0 = \begin{cases} 
  x \in X & | \begin{array}{ll}
  x \text{ is a cusp} & \\
  \text{or } & \exists y, (x, y) \in S^{\Delta^2} \\\n  \text{or } & \exists y, (x, y) \in \partial X^{\Delta^2} \\\n  \text{or } & \exists y, (x, y) \in BF \\\n  \text{or } & \exists y, (y, x) \in BF 
\end{array} 
\end{cases}$$
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Theorem

Let \( P = \{p_i\}_{1 \leq i \leq n} \) be a paving such that

i) \( S \cup \partial X \subset \bigcup_i p_i \),

ii) \( \forall (p_i, p_j), p_i \cap p_j \neq \emptyset \Rightarrow (S \cup \partial X) \cap p_i \cap p_j \) is simply connected,

iii) \( \forall p_i, X \cap p_i \) contains at most one element of \( X_0 \),

Let \( \mathcal{X} \) be the relation on \( \{p_i\}_{1 \leq i \leq n} \) defined by

\[
    p \mathcal{X} q \iff (S \cup \partial X) \cap p \cap q \text{ is simply connected.}
\]

Let us define an equivalence relation \( f \) on \( \{p_i\} \) by

\[
    pfq \iff f(X_0 \cap p) = f(X_0 \cap q) \text{ and } X_0 \cap p \neq \emptyset,
\]

then \( \mathcal{X}/f \) is homeomorphic to the Apparent contour of \( f \).
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Theorem

Let $f$ be a smooth map from a compact simply connected domain $X$ of $\mathbb{R}^2$ to $\mathbb{R}^2$. For every portrait $F$ of $f$, the 1-skeleton of $ImF$ contains a subgraph that is an expansion of $G_f$. 
Conjecture

From $G_f$ and its right embedding in $\mathbb{R}^2$ it is possible to create a portrait for $f$. 
Conjecture

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Tack för din uppmärksamhet!