Stabilility analysis of a nonlinear system using interval analysis.

Groupe de Travail - Méthode ensembliste

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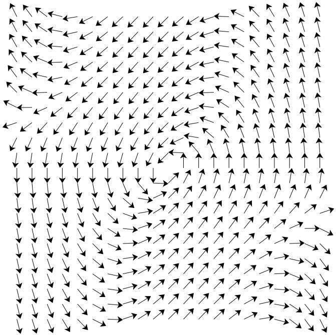
16 mars 2006

Let us consider the following dynamical system :

$$\begin{cases} \dot{x} = f(x) \\ x \in \mathbb{R}^n \end{cases} \text{ where } f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n). \quad (1)$$



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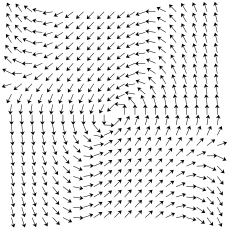
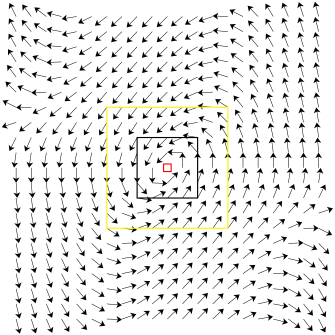


Notation

$$\begin{array}{cccc} \phi^{\cdot}(x) & : & \mathbb{R} & \rightarrow & \mathbb{R}^n \\ & & t & \mapsto & \phi^t(x) \end{array}$$

denotes the solution of (1) satisfying $\phi^0(x) = x$.

With $[x_0] \subset \mathbb{R}^n$,
Find $[x_\infty] \subset [x] \subset [x_0]$ such that $\exists x_\infty \in [x_\infty]$:



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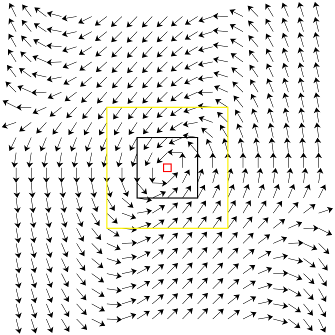

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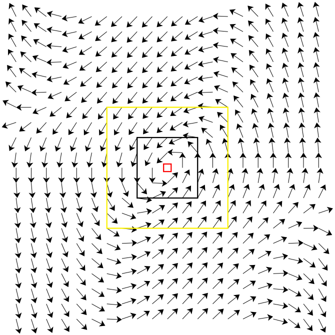

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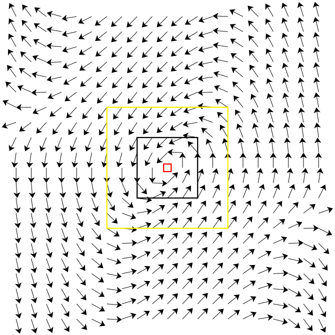

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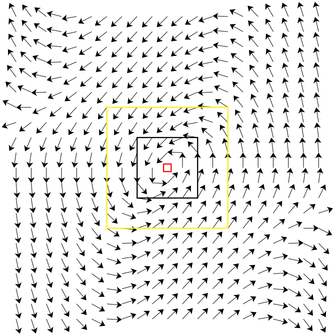
- $f(x_\infty) = 0$.
- $\forall x \in [x], \forall t \in \mathbb{R}^+, \phi^t(x) \in [x_0] \Leftrightarrow \phi^{\mathbb{R}^+}([x]) \subset [x_0]$
- $\forall x \in [x], \phi^t(x) \underset{t \to +\infty}{\longrightarrow} x_\infty. \Leftrightarrow \phi^\infty([x]) = \{x_\infty\}$



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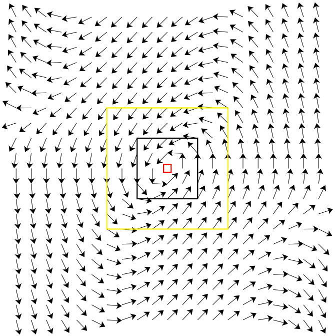

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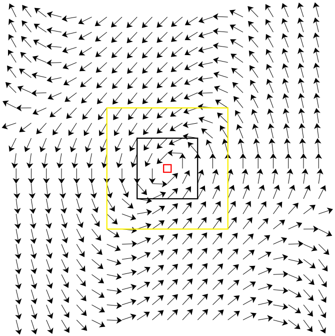

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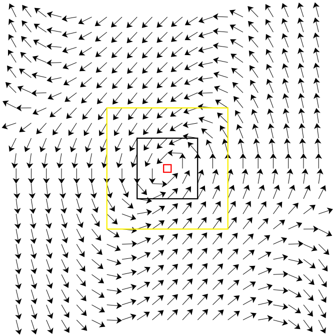
- $\phi^{\mathbb{R}^+}([x]) \subset [x_0]$ (Stability - Lyapunov)



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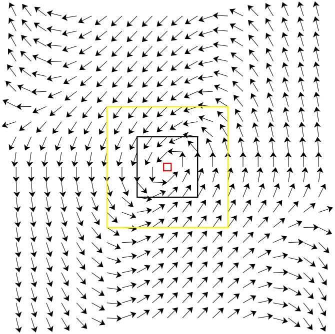

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Outline

1. Tools
   - Interval Newton method
   - Positivity
2. Lyapunov theory
   - Definitions of stability
   - Lyapunov function
   - The linear case
3. Algorithm to prove stability
   - Algorithm
   - Example
4. Future work

Newton-Raphson Method

Output

An approximation of a solution of $f(x) = 0$ where $x \in [x]$ where $f \in \mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R}^n)$.

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Algorithm

- Initialization x_0
- $x_{n+1} = x_n - Df^{-1}(x_n)f(x_n)$

Interval Newton Method

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- A box enclosing the solution of $f(x) = 0$.

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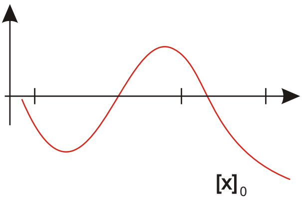
- A box enclosing the solution of $f(x) = 0$.
- A flag indicating the uniqueness and the existence of the solution.

Algorithm

- $[x]_0 = [x]$
- $[x]_{n+1} = [x]_n \cap \rho_{x_1}([x]_n)$

where $\rho_{x_1} : [x] \to \mathbb{R}^n$ with $\rho_{x_1}(x) = x_1 - Df^{-1}(x)f(x_1)$ and $x_1 \in [x]_n$



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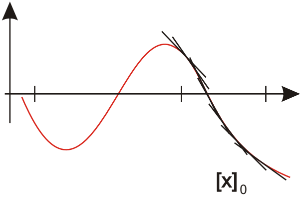

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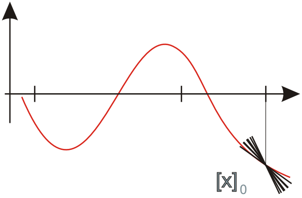

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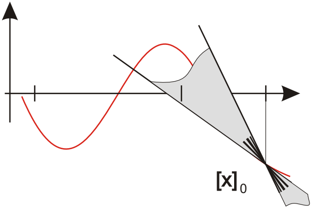

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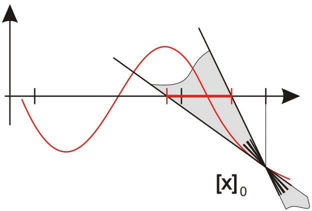

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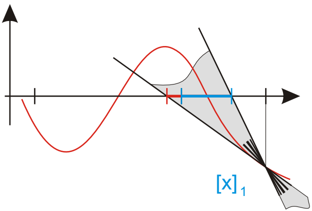
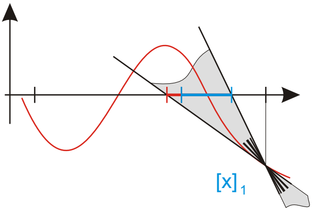


Properties

Let $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$, $x_1 \in [x]$. If $0 \notin det(Df([x]))$ then

1. $x^* \in [x], f(x^*) = 0 \Rightarrow x^* \in \rho_{x_1}([x])$
2. $\rho_{x_1}([x]) \subset [x] \Rightarrow \exists! x^* \in [x], f(x^*) = 0$



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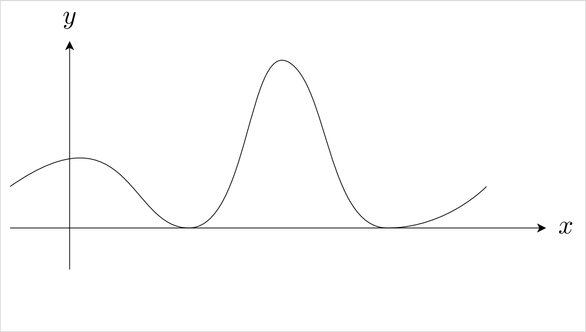
- Otherwise.


y
x

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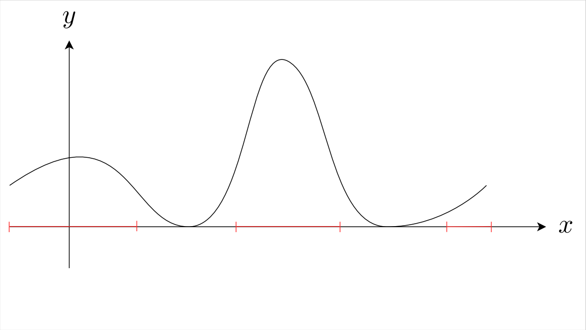
- multiple occurrence of variables.
- outward rounding.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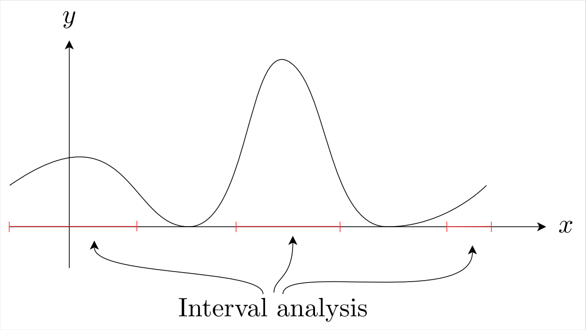
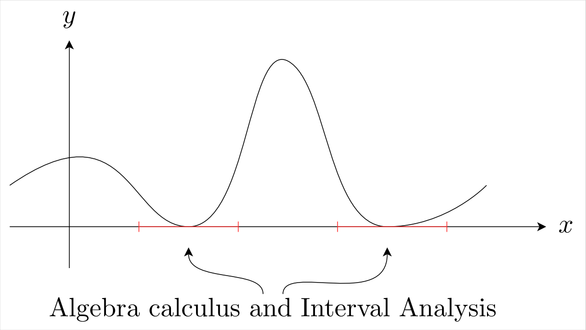

y
x
Interval analysis


y
x
Algebra calculus and Interval Analysis

Theorem

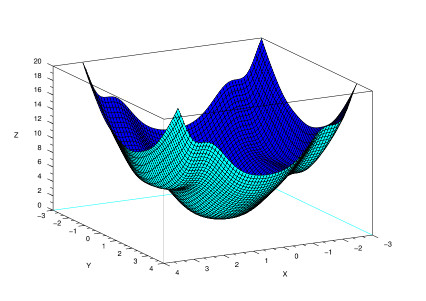
Let $x_0 \in E$ where E is a convex set of $\mathbb{R}^n$, and $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$. We have the following implication :

1. $\exists x_0$ such that $f(x_0) = 0$ and $Df(x_0) = 0$.
2. $\forall x \in E, D^2 f(x) > 0$.

then $\forall x \in E, f(x) \geq 0$.

Example

Let us prove that $f(x) \geq 0, \forall x \in [-1/2, 1/2]^2$
where $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by
$f(x, y) = -\cos(x^2 + \sqrt{2}\sin^2 y) + x^2 + y^2 + 1.$



Example

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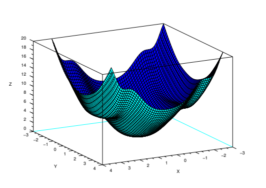
$f(x,y) = -\cos(x^2 + \sqrt{2}\sin^2 y) + x^2 + y^2 + 1.$

1. One has : $f(0,0) = 0$ and $\nabla f(0,0) = 0$

$$\nabla f(x,y) = \left(\begin{array}{c} 2x(\sin(x^2 + \sqrt{2}\sin^2 y) + 1) \\ 2\sqrt{2}\cos y \sin y \sin\left(\sqrt{2}\sin^2 y + x^2\right) + 2y \end{array} \right).$$



$$\nabla^2 f = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

$$\begin{aligned}
a_{1,1} &= 2\sin\left(\sqrt{2}\sin^2 y + x^2\right) \\
&\quad +4x^2\cos\left(\sqrt{2}\sin^2 y + x^2\right) + 2. \\
a_{2,2} &= -2\sqrt{2}\sin^2 y \sin(\sqrt{2}\sin^2 y + x^2) \\
&\quad +2\sqrt{2}\cos^2 y \sin(\sqrt{2}\sin^2 y + x^2) \\
&\quad +8\cos^2 y \sin^2 y \cos(\sqrt{2}\sin^2 y + x^2) + 2. \\
a_{1,2} = a_{2,1} &= = 4\sqrt{2}x\cos y \sin y \\
&\quad \cos\left(\sqrt{2}\sin y^2 + x^2\right).
\end{aligned}$$

Interval analysis : $\forall x \in [-1/2, 1/2]^2$, $\nabla^2 f(x) \subset [A]$

$$[A] = \begin{pmatrix} [1.9, 4.1] & [-1.3, 1.4] \\ [-1.3, 1.4] & [1.9, 5.4] \end{pmatrix}.$$

It remains to check : $\forall A \in [A]$, A is positive.

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Definition

A symmetric matrix A is positive definite if

$$\forall x \in \mathbb{R}^n - \{0\}, x^T A x > 0$$

The set of positive definite symmetric $n \times n$ matrices is denoted by S^{n+}.

Definition

A set of symmetric matrices $[A]$ is an interval symmetric matrix if :

$$[A] = \{(a_{ij})_{ij}, a_{ij} = a_{ji}, a_{ij} \in [a]_{ij}\}$$

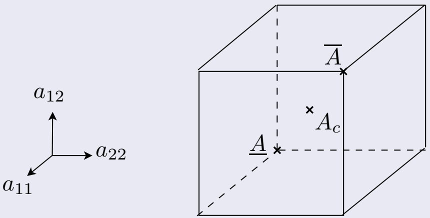
i.e.

$$[\underline{A}, \overline{A}] = \left\{A \text{ symmetric}, \ \underline{A} \leq A \leq \overline{A}\right\}.$$

Example

Working in $\mathbb{R}^2$, a symmetric matrix A

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{1,2} & a_{2,2} \end{pmatrix}$$

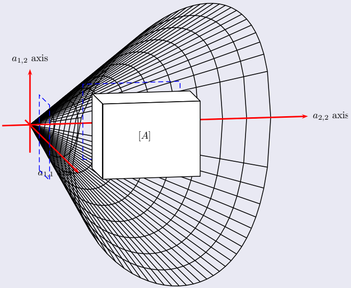


Remark - Rohn

Let $V([A])$ denotes the finite set of corners of $[A]$. S^{n+} and $[A]$ are convex subsets of S^n :

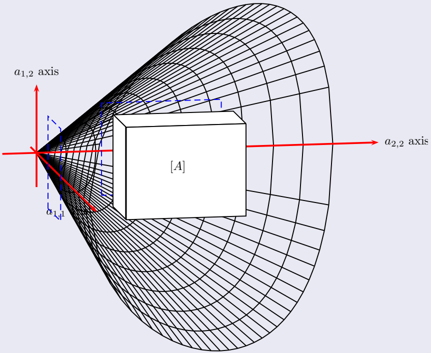
$$[A] \subset S^{n+} \Leftrightarrow V([A]) \subset S^{n+}$$

S^n is a vector space of dimension $\frac{n(n+1)}{2}$. $\#V([A]) = 2^{\frac{n(n+1)}{2}}$.



Theorem - Adefeld

Let $[A]$ be an interval symmetric matrix.
and $C = \{z \in \mathbb{R}^n \text{ tel que } |z_i| = 1\}$
If $\forall z \in C$, $A_z = A_c + \mathrm{Diag}(z)\Delta\mathrm{Diag}(z)$ is positive definite.
then $[A]$ is positive definite.



Let us consider the following dynamical system :

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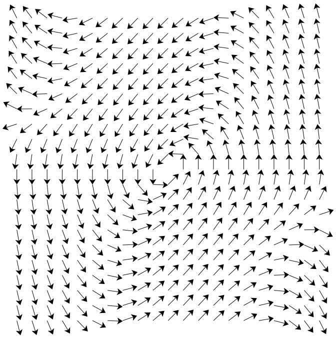
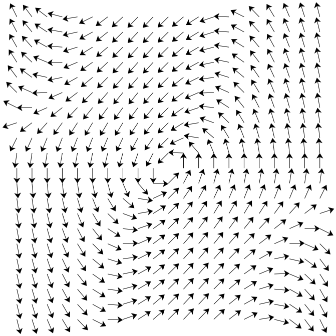


Definition

Let $x \in \mathbb{R}$, x is an *equilibrium state* if :

$$f(x) = 0$$



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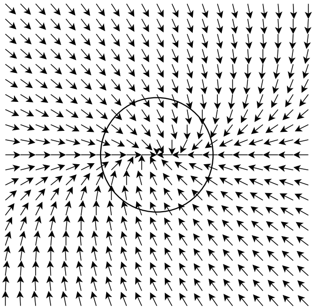
A set D is *stable* if :

$$\phi^{\mathbb{R}^+}(D) \subset D$$



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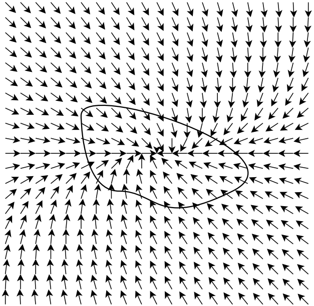
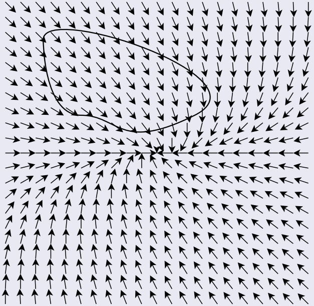


Non stable example



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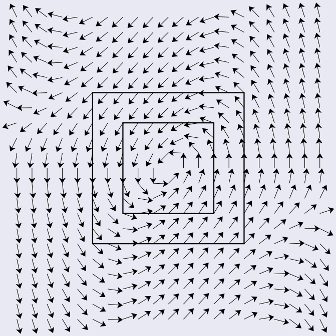
An equilibrium state x_∞ is *asymptotically* (D, D_0)*-stable* if



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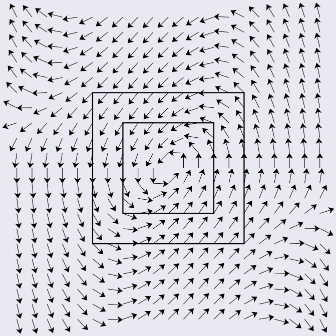

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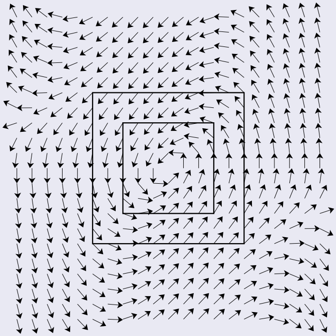
- $\phi^{\mathbb{R}^+}(D) \subset D_0$
- $\phi^{\infty}(D) = \{x_\infty\}$



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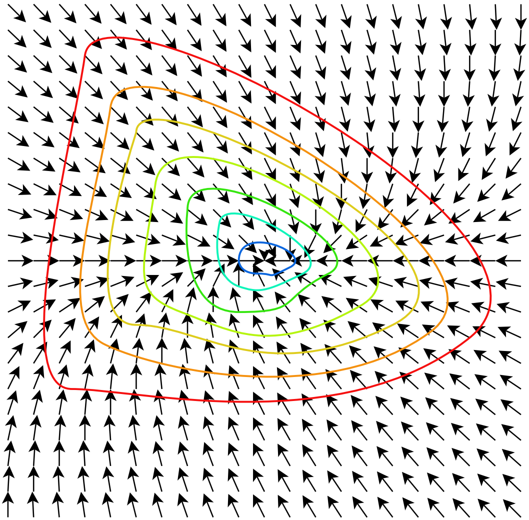
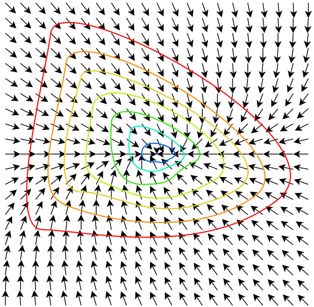
With $V : t \mapsto L(x(t))$, one has :

$$\begin{array}{rcl} \frac{d}{dt}V(t) & = & \frac{d}{dt}(L(x(t))) \\ & = & \frac{d}{dx}L \cdot \frac{d}{dt}x(t) \\ & = & \langle \nabla L(x), f(x(t)) \rangle < 0 \end{array}$$



Lyapunov's Theorem

Let D' be a compact subset of $\mathbb{R}^n$ and x_∞ in the interior of D'.
If $L : D' \to \mathbb{R}$ is Lyapunov for (1) then
there exists a subset D of D' such that the equilibrium state x_∞ is asymptotically D, D'-stable.



For linear systems :

$$\dot{x} = Ax \tag{2}$$

One usually takes $L = x^T W x$ where $W \in S^n$.
and $\langle \nabla L(x), f(x) \rangle = x^T (A^T W + WA) x$.

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Lyapunov conditions translate into

1. $W \in S^{n+}$.
2. $-(A^T W + WA) \in S^{n+}$.

For $\dot{x} = Ax$, to find a Lyapunov function for 2, one solves the Lyapunov equation of unknown

$$A^T W + WA = -I$$

and we check that $W \in S^{n+}$.

Theorem

The system $\dot{x} = Ax$ is asymptotically stable is equivalent to for all $Q \in S^{n+}$, the matrix W solution of

$$A^T W + WA = -Q$$

is positive definite.

Example

The system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

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5. Check that L_{x_∞} is also Lyapunov for $\dot{x} = f(x)$.

Explanations

Step 4 : $L_{x_\infty}(x) = (x - x_\infty)^T W_{\tilde{x}_\infty}(x - x_\infty)$
Step 5 : It remains to check :

$$g_{x_\infty}(x) = -\langle \nabla L_{x_\infty}(x), f(x) \rangle \geq 0$$

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One has :

- $g(x_\infty) = 0$
- $\nabla g_{x_\infty}(x_\infty) = 0$

According to theorem of positivity, one only has to check that

$$\nabla^2 g_{x_\infty}([x_0]) \subset S^{n+}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 - (1 - x_1^2)x_2 \end{pmatrix}$$

where $[x_0] = [-0.6, 0.6]^2$.

Future work

To combine :

- These results.
- Guaranteed integration of ODE.
- Graph theory.

to compute a guaranteed approximation of the attraction domain of x_∞.