Tropical algebra
From shortest path algorithms to Hamilton-Jacobi-Bellman Equation

Nicolas Delanoue
LARIS - Université d’Angers - France
http://perso-laris.univ-angers.fr/~delanoue/

Medellin - EAFIT http://www.eafit.edu.co/

November 2019
Outline

1. Graph theory
   - Bellman Ford Algorithm
   - An example

2. Tropical linear algebra
   - Semi ring
   - Bellman-Ford algorithm with tropical algebra

3. Optimal control - Hamilton Jacobi Bellman
**Definition - Graph**

A *directed graph* is an ordered pair $G = (V, E)$ where
- $V$ is a set whose elements are called vertices,
- $E$ is a set of ordered pairs of vertices, called directed edges.

**Example**

Here, $V = \{v_1, v_2, v_3, v_4\}$ and
$E = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_3, v_4)\}$
**Weighted graph**

A *weighted directed graph* is a directed graph with weights assigned to their edges, i.e. one has function $h : E \to \mathbb{R}$.

**Example**

Here, $h(v_1, v_2) = 2$, $h(v_1, v_3) = 5, \ldots$
Shortest path problem

The **shortest path problem** is the problem of finding a path between two vertices in a graph such that the sum of the weights of its constituent edges is minimized.

Algorithms

- Dijkstra’s algorithm solves the single-source shortest path problem with non-negative edge weight.
- Bellman Ford algorithm solves the single-source problem if edge weights may be negative.
- ...
Bellman-Ford Algorithm

**Input**: A weighted directed graph \((V, E, h)\), a source vertex \(s\)
**Bellman-Ford Algorithm**

**Input**: A weighted directed graph \((V, E, h)\), a source vertex \(s\)

**Output**: The cost of the shortest path from \(s\) to all other nodes:

\[
V \ni c \mapsto J(c) \in \mathbb{R}
\]
Bellman-Ford Algorithm

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\[
V \ni c \mapsto J(c) \in \mathbb{R}
\]

\[
\text{for } c \in V - \{s\} \text{ do}
\]
\[
\quad J(c, 0) \leftarrow +\infty ;
\]

\[
\text{end}
\]

\[
J(s, 0) \leftarrow 0;
\]
Bellman-Ford Algorithm

**Input**: A weighted directed graph \((V, E, h)\), a source vertex \(s\)

**Output**: The cost of the shortest path from \(s\) to all other nodes:

\[ V \ni c \mapsto J(c) \in \mathbb{R} \]

for \(c \in V - \{s\}\) do
  \(J(c, 0) \leftarrow +\infty\);
end

\(J(s, 0) \leftarrow 0;\)

for \(k \leftarrow 1\) to \(#V - 1\) do
  for \(c \in V\) do
    \(J' \leftarrow +\infty;\)
    for \((u, c) \in E\) do
      \(J' \leftarrow \min(J', J(u, k - 1) + h(u, c))\);
    end
    \(J(c, k) \leftarrow J';\)
  end
end

\(J(\cdot) = \min_k J(\cdot, k)\)
An example

Graph theory
Tropical linear algebra
Optimal control - Hamilton Jacobi Bellman

A graph with nodes $v_1$, $v_2$, $v_3$, and $v_4$. The edges are labeled with the following weights: $v_1$ to $v_2$ with 0, $v_2$ to $v_3$ with 1, $v_3$ to $v_4$ with 2, $v_1$ to $v_4$ with 9, $v_2$ to $v_1$ with 2, and $v_3$ to $v_2$ with 5.
**Initialisation steps:**

- $J(v_1) = (0, \ldots)$ since $v_1$ is the source,
An example

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- $J(v_i) = (\infty, \ldots)$ for all other vertices.
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- $J(v_i) = (\infty, \ldots)$ for all other vertices.
Iteration $k = 1$:

- $v_1$ has only $v_1$ as predecessors, therefore $J(v_1) = (0, 0, \ldots)$
An example

\[ J = (\infty, \ldots) \]

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Iteration \( k = 1 \):
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An example

\[ J = (\infty, \ldots) \]

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Iteration \( k = 1 \):

\[ J = (0, 0, \ldots) \]

\[ J = (\infty, \ldots) \]
An example

Iteration $k = 1$:

- $v_2$ has two predecessors: $v_1$ and $v_2$, therefore

$J = (\infty, \ldots)$

$J = (\infty, \ldots)$

$J = (0, 0, \ldots)$

$J = (\infty, \ldots)$
An example

\[ J = (\infty, \ldots) \quad J = (\infty, \ldots) \]

\[ J = (0, 0, \ldots) \quad J = (\infty, \ldots) \]

**Iteration k = 1:**

- \( v_2 \) has two predecessors: \( v_1 \) and \( v_2 \), therefore
  \[ J(v_2, k) = \min \{ J(v_1, 0) + h(v_1, v_2), J(v_2, 0) + h(v_2, v_2) \} \]
An example

\[ J = (\infty, \ldots) \quad J = (\infty, \ldots) \]

\[ J = (0, 0, \ldots) \quad J = (\infty, \ldots) \]

**Iteration k = 1:**

- \( v_2 \) has two predecessors: \( v_1 \) and \( v_2 \), therefore

\[
J(v_2, k) = \min\{ J(v_1, 0) + h(v_1, v_2), J(v_2, 0) + h(v_2, v_2) \} = \min\{ 0 + 2, \infty + 0 \} = 2
\]
Iteration $k = 1$:

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  $$J(v_2, k) = \min \{J(v_1, 0) + h(v_1, v_2), J(v_2, 0) + h(v_2, v_2)\}$$

  $$= \min \{0 + 2, \infty + 0\}$$

  $$= 2$$
An example

\[ J = (\infty, 2, \ldots) \]

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\[ J = (0, 0, \ldots) \]

\[ J = (\infty, \ldots) \]

Iteration \( k = 1 \):

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  \[
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  \]
  \[
  = \min\{ 0 + 2, \infty + 0 \}
  \]
  \[
  = 2
  \]
An example

\[ J = (\infty, 2, \ldots) \quad \text{and} \quad J = (\infty, \ldots) \]

\[ J = (0, 0, \ldots) \quad \text{and} \quad J = (\infty, \ldots) \]

\textbf{Iteration } k = 1 : 

- \( J(v_3, k) = 5 \)
An example

Graph theory

Tropical linear algebra

Optimal control - Hamilton Jacobi Bellman

Iteration $k = 1$

- $J(v_3, k) = 5$
An example

\[
J = (\infty, 2, \ldots) \quad J = (\infty, 5, \ldots)
\]

\[
J = (0, 0, \ldots) \quad J = (\infty, \ldots)
\]

Iteration \(k = 1\) :

- \(J(v_4, k) = 9\)
An example

\[ J = (\infty, 2, \ldots) \quad J = (\infty, 5, \ldots) \]

\[ J = (0, 0, \ldots) \quad J = (\infty, 9, \ldots) \]

**Iteration k = 1:**

- \( J(v_4, k) = 9 \)
An example

Initialisation

\[ J = (\infty, \ldots) \quad J = (\infty, \ldots) \]

\[ J = (0, \ldots) \quad J = (\infty, \ldots) \]
An example

$k = 1$

\[ J = (\infty, 2, \ldots) \quad \quad J = (\infty, 5, \ldots) \]

\[ J = (0, 0, \ldots) \quad \quad J = (\infty, 9, \ldots) \]
An example

\( k = 2 \)

\[ J = (\infty, 2, 2, \ldots) \quad J = (\infty, 5, 3, \ldots) \]

\[ J = (0, 0, 0, \ldots) \quad J = (\infty, 9, 7, \ldots) \]
An example

\[ k = 3 \]

\[ J = (\infty, 2, 2, 2) \quad J = (\infty, 5, 3, 3) \]

\[
\begin{array}{c}
\text{\(v_1\)} \\
0 \\
\text{\(v_2\)} \\
0 \\
\text{\(v_3\)} \\
0 \\
\text{\(v_4\)} \\
0
\end{array}
\]

\[
\begin{array}{cccc}
0 & 2 & 5 & 9 \\
1 & 1 & 2 & 9 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}
\]
Initialisation

| $\nu_1$ | 0 |
| $\nu_2$ |
| $\nu_3$ |
| $\nu_4$ |
An example

<table>
<thead>
<tr>
<th>$\nu_1$</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_2$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$\nu_3$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$\nu_4$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>
**An example**

**$k = 1$**

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$\infty$</td>
<td>2</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$\infty$</td>
<td>5</td>
</tr>
<tr>
<td>$v_4$</td>
<td>$\infty$</td>
<td>9</td>
</tr>
</tbody>
</table>
An example:

\( k = 2 \)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( v_2 )</td>
<td>( \infty )</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( v_3 )</td>
<td>( \infty )</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>( v_4 )</td>
<td>( \infty )</td>
<td>9</td>
<td>7</td>
</tr>
</tbody>
</table>
An example

$k = 3$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
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<tbody>
<tr>
<td>$v_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$\infty$</td>
<td>2</td>
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</tr>
<tr>
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<td>7</td>
<td>5</td>
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An example

\[ k = 3 \]

<table>
<thead>
<tr>
<th>( \nu_1 )</th>
<th>0</th>
<th>0</th>
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<th>0</th>
</tr>
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<tbody>
<tr>
<td>( \nu_2 )</td>
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</table>

\[ J_0 = \begin{pmatrix} 0 \\ \infty \\ \infty \\ \infty \end{pmatrix}, \text{ and } J_{k+1} = f(J_k). \]
Definition

Let $(V, E, h)$ be a weighted directed graph with $V = \{v_1, \ldots, v_n\}$, we define the square matrix $A = (a_{ij})_{i,j \in 1, \ldots, n}$ with

$$a_{ij} = h(v_i, v_j).$$

I call the matrix $A$ the HJB matrix.
Example 2

\begin{align*}
A = \begin{pmatrix}
0 & 2 & 5 & 9 \\
2 & 0 & 1 & \infty \\
5 & 1 & 0 & 2 \\
9 & \infty & 2 & 0
\end{pmatrix}
\end{align*}

Remark

The coefficient $i,j$ of $A$ is the cost from node $v_i$ to $v_j$ using one edge.
Recall - Matrix multiplication

Let $A = (a_{ij})_{i,j \in 1,...,n}$ be a square matrix then the $i,j$ coefficient of $A^2$, $c_{ij}$ is given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}$$

Example

$$c_{23} = a_{21} a_{13} + a_{22} a_{23} + a_{23} a_{33} + a_{24} a_{43}$$
Interpretation

The coefficient $c_{ij}$ is composed of all paths from $i$ to $j$ with two edges.

$$c_{23} = a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33} + a_{24}a_{43}$$
Interpretation

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\[ c_{23} = a_{21} \]
Interpretation

The coefficient $c_{ij}$ is composed of all paths from $i$ to $j$ with two edges.

$$c_{23} = a_{21}a_{13}$$
Interpretation

The coefficient \( c_{ij} \) is composed of all paths from \( i \) to \( j \) with two edges.

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Definition

The min tropical semiring is the semiring \((\mathbb{R} \cup \{\infty\}, \oplus, \otimes)\), with the operations:

1. \(x \oplus y = \min\{x, y\}\),
2. \(x \otimes y = x + y\).
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**Example**

- \(2 \oplus 3 = 2\),
**Definition**

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- \(x \otimes y = x + y\).

**Example**

- \(2 \oplus 3 = 2\),
- \(2 \otimes 3 = 5\).

**Remarks**

- The operations \(\oplus\) and \(\otimes\) are referred to as tropical addition and tropical multiplication respectively,
- The unit for \(\oplus\) is \(\infty\),
- the unit for \(\otimes\) is 0.
Linear tropical algebra

Let $A = (a_{ij})_{i,j \in 1,\ldots,n}$ be a square matrix then the $i,j$ coefficient of $A^2$, $c_{ij}$ is given by

$$c_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes a_{kj}$$

Example with HJB matrix

$$c_{2,3} = (a_{2,1} \otimes a_{1,3}) \oplus (a_{2,2} \otimes a_{2,3}) \oplus (a_{2,3} \otimes a_{3,3}) \oplus (a_{2,4} \otimes a_{4,3})$$

Lemma

The real value $c_{ij}$ is the smallest cost of paths from $v_i$ to $v_j$ following by two edges.
Proposition

Let $k \in \mathbb{N}$, with the min tropical semi ring, coefficient $i, j$ of the matrix $A^k$ contains the smallest cost of all paths from $v_i$ to $v_j$ using $k$ edges.
Proposition

Let \( k \in \mathbb{N} \), with the min tropical semi ring, coefficient \( i, j \) of the matrix \( A^k \) contains the smallest cost of all paths from \( v_i \) to \( v_j \) using \( k \) edges.

Bellman-Ford algorithm from the tropical point of view :

\[
\begin{cases}
    J_0 & = & H, \\
    J_{n+1} & = & AJ_n,
\end{cases}
\]

(1)

with \( H = (\infty, \ldots, \infty, 0, \infty, \ldots, \infty)^T \).
Proposition

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Solution:

\[J_n = A^n J_0\]
**Remarks**

Due to tropical linearity, i.e. superposition property:

\[
A^k(H_1 \oplus H_2) = A^k H_1 \oplus A^k H_2.
\]

- \(A^k(0, 0, \infty, \ldots, \infty)^T\) is the smallest cost to reach any nodes from one of the two sources \(v_1\) and \(v_2\).
- \((0, \infty, \ldots, \infty)A^k\) is the cost from any nodes to the target \(v_1\).
### Controlled dynamical system

\[
\begin{cases}
    x(0) = x_0 \\
    \dot{x}(\tau) = f(x(\tau), u(\tau)), \forall \tau \in [0, T],
\end{cases}
\]

where

- $\tau$ is the time,
- $x$ is the state,
- $f$ is a vector field (the dynamics),
- $u$ is the control.
### Optimal control problem

\[
J^* = \min_{u: [0, T] \rightarrow U} \int_0^T h(\tau, x(\tau), u(\tau)) d\tau + H(x(T))
\]

subject to

\[
\begin{align*}
  x(0) &= x_0 \\
  \dot{x}(\tau) &= f(x(\tau), u(\tau)), \forall \tau \in [0, T], \\
  x(\tau) &\in X, \forall \tau \in [0, T], \\
  x(T) &\in K.
\end{align*}
\]

where

- \( U \) is the set of admissible control,
- \( h \) and \( H \) are real valued functions.
Definition - Optimal cost

\[ J^*(x, t) = \min_{u: [t, T] \rightarrow U} \int_t^T h(x(\tau), u(\tau))\,d\tau + H(x(T)) \]

such that \( x : t \mapsto X \) satisfies

\[ \begin{aligned}
\dot{x}(\tau) &= f(x(\tau), u(\tau)) \\
\quad \\
x(t) &= x
\end{aligned} \]
The value function \((t, x) \mapsto J^*(t, x)\) satisfies the partial differential equation:

\[
\frac{\partial J^*}{\partial t} = - \min_{u(t) \in U} \left\{ h(x, u(t)) + \frac{\partial J^*}{\partial x} f(x, u(t)) \right\}
\]  

with final condition \(J^*(T, x) = H(x)\).
Hamilton Jacobi Bellman Theorem

The value function \((t, x) \mapsto J^*(t, x)\) satisfies the partial differential equation:

\[
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\]

(2)

with final condition \(J^*(T, x) = H(x)\).

Remark

- Equation (2) is an infinite dimensional dynamical system, indeed, the state space is the set of real value function \(\varphi : X \to \mathbb{R}\).
Proposition

Let us denote by $S^T(H)$ the solution of optimal control problem with final cost $H$. One has:

- $S^0 = \text{Id},$
- $S^{t_1+t_2} = S^{t_1}S^{t_2},$
- $S^t(\alpha \otimes H) = \alpha \otimes S^tH_1.$
- $S^t(H_1 \oplus H_2) = S^tH_1 \oplus S^tH_2.$
To finish

Suppose the function $J$ is solution of the following Hamilton-Jacobi equation

$$\frac{\partial J}{\partial t} = H(x, \frac{\partial J}{\partial x}) \text{ and } J(0, \cdot) = \varphi(\cdot)$$

with

$$H(x, p) = \min_u (h(u, x) + p \cdot u)$$

Hopf formula gives:

$$J(t, x) = \min_y t \cdot h\left(\frac{x - y}{t}\right) + \varphi(y)$$
To finish

Suppose the function $J$ is solution of the following Hamilton-Jacobi equation

$$\frac{\partial J}{\partial t} = H(x, \frac{\partial J}{\partial x})$$

and $J(0, \cdot) = \varphi(\cdot)$

with

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Hopf formula gives:

$$J(t, x) = \min_y t \cdot h\left(\frac{x - y}{t}\right) + \varphi(y) = \bigoplus_y t \cdot h_t(x - y)\varphi(y)dy$$

Note that in this case $h$ is the Legendre transform of $H$. 
To finish

Suppose the function $J$ is solution of the following Hamilton-Jacobi equation

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2. Max-plus approximations: from optimal control to template methods, Gaubert 2014

3. Dower, P.M. and McEneaney, W.M. A max-plus based fundamental solution for a class of infinite dimensional Riccati equations.

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Gracias por su atención.