# Proving that a set is connected via interval analysis Minisymposium, Interval methods, Para' 04

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# OutLine

## Motivation

2 Topology recall

### Proving that a set is star-shaped

- A sufficient condition for proving that a set is star-shaped
- An example

### Discretization

- The idea
- Theorem
- An example and the solver CIA

## 5 Conclusion

- What have we done?
- Future work

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### Coordinates of A

$$\begin{cases} x_A = 2\cos(\alpha) \\ y_A = 2\sin(\alpha) \end{cases}$$



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### Coordinates of A

$$\begin{cases} x_A = 2\cos(\alpha) \\ y_A = 2\sin(\alpha) \end{cases}$$



### Coordinates of B

$$\begin{cases} x_B = 2\cos(\alpha) + 1.5\cos(\alpha + \beta) \\ y_B = 2\sin(\alpha) + 1.5\sin(\alpha + \beta) \end{cases}$$

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### Constraints on A

$$y_A \in [0,y_0]$$



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$$y_A \in [0,y_0]$$

### Constraints on B

$$y_B \in ]-\infty, y_0$$

Constraints on  $\alpha$  and  $\beta$ 

$$\alpha \in ]-\pi,\pi[,\beta \in ]-\pi,\pi[$$





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### Feasible configuration set

$$\mathbb{S} = \left\{ (\alpha, \beta) \in ] - \pi, \pi [^2 / \left\{ \begin{array}{l} 2\sin(\alpha) & \in & [0, y_0] \\ 2\sin(\alpha) + 1.5\sin(\alpha + \beta) & \in & ] - \infty, y_0] \end{array} \right\}$$

Path connected set Star Star shaped set

### Definition (path-connected set)

A topological space  $\mathbb{S}$  is *path-connected* if and only if for every two points  $x, y \in \mathbb{S}$ , there is a continuous function f from [0,1] to  $\mathbb{S}$  such that f(0) = x and f(1) = y.



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### Definition (star)

The point  $v^*$  is a *star* for a subset X of an Euclidean space if  $\forall x \in X$ , the segment  $[x, v^*]$  is include in X.





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### Definition (*star-shaped set*)

If there exists  $v^* \in X$  such that  $v^*$  is a star for X, then we say that X is *star-shaped* or  $v^*$ -*star-shaped*.





Path connected set Star Star shaped set

### Proposition 1

A star-shaped set is a path-connected set.

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Path connected set Star Star shaped set

### Proposition 1

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### Proposition 2

Let X and Y two v\*-star-shaped set, then  $X \cap Y$  and  $X \cup Y$  are also v\*-star-shaped.



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Path connected set Star Star shaped set

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A sufficient condition for proving that a set is star-shaped An example

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### Theorem

If  $\mathbb{S} = \{x \in D \subset \mathbb{R}^n / f(x) \le 0\}$  where f is a  $C^1$  function from D to  $\mathbb{R}$ , D a convex set,  $v^*$  be in  $\mathbb{S}$  and if

$$f(x) = 0, Df(x).(x - v^*) \le 0, x \in D$$

is inconsistent then  $v^*$  is star a for  $\mathbb{S}$ .



Proof idea - Geometric interpretation

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A sufficient condition for proving that a set is star-shaped An example

Let us prove that 
$$v^* = (0.6, -0.5)$$
 is a star for the set defined by

$$\mathbb{S} = \{(x,y) \in \mathbb{R}^2, \text{ such that } f(x,y) = x^2 + y^2 + xy - 2 \le 0\}$$

A sufficient condition for proving that a set is star-shaped An example

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 is inconsistent

$$\Leftrightarrow \left\{ \begin{array}{l} x^2 + y^2 + xy - 2 = 0\\ \partial_x f(x, y).(x - 0.6) + \partial_y f(x, y).(y + 0.5) \le 0 \end{array} \right. \text{ is inconsistent}$$

A sufficient condition for proving that a set is star-shaped An example

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$$\Leftrightarrow \begin{cases} x^2 + y^2 + xy - 2 = 0\\ (2x + y)(x - 0.6) + (2y + x)(y + 0.5) \le 0 \end{cases}$$
 is inconsistent

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The idea Theorem An example and the solver CIA



The idea Theorem An example and the solver CIA

## The idea

To divide it with a paving  $\mathcal{P}$  such that, on each part  $p, \mathbb{S} \cap p$  is star-shaped.



The idea Theorem An example and the solver CIA

## The idea

Let us define the notion of *star-spangled* graph with the relation :  $\mathbb{S} \cap p \cap q \neq \emptyset$ .



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### Definition

A star-spangled graph of a set S, noted by  $\mathcal{G}_S$ , is a relation  $\mathcal{R}$  on a paving  $\mathcal{P}$  where

- $\mathcal{P} = (p_i)_{i \in I}$ , for all p of  $\mathcal{P}, \mathbb{S} \cap p$  is star-shaped. And  $\mathbb{S} \subset \bigcup_{i \in I} p_i$
- $\mathcal{R}$  is the reflexive and symmetric relation on  $\mathcal{P}$  defined by  $p \mathcal{R} q \Leftrightarrow \mathbb{S} \cap p \cap q \neq \emptyset$ .



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### Theorem





Proof idea

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### Corollary

Let  $\mathcal{G}_{\mathbb{S}}$  be a star-spangled graph of a set  $\mathbb{S}$ .  $\mathcal{G}_{\mathbb{S}}$  has the same number of connected components than  $\mathbb{S}$ . i.e.  $\pi_0(\mathbb{S}) = \pi_0(\mathcal{G}_{\mathbb{S}})$ .



#### Proof idea

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### Feasible configuration set, $y_0 = 2.3$



$$\mathbb{S} = \left\{ (\alpha, \beta) \in ] - \pi, \pi [^2 / \left\{ \begin{array}{ll} 2\sin(\alpha) & \in & [0, y_0] \\ 2\sin(\alpha) + 1.5\sin(\alpha + \beta) & \in & ] - \infty, y_0] \end{array} \right\}$$

An example and the solver CIA

The solver CIA (path-Connected via Interval Analysis)

### Feasible configuration set, $y_0 = 2.3$

$$\mathbb{S} = \left\{ (\alpha, \beta) \in ] - \pi, \pi [^2 / \left\{ \begin{array}{ll} 2\sin(\alpha) & \in & [0, y_0] \\ 2\sin(\alpha) + 1.5\sin(\alpha + \beta) & \in & ] - \infty, y_0] \end{array} \right. \right\}$$

### <u>Feasible</u> configuration set, $y_0 = 2.3$

$$\begin{aligned} (\alpha,\beta) \in &] - \pi,\pi \begin{bmatrix} 2 \\ / \\ &\begin{cases} f_1(\alpha,\beta) = 2\sin(\alpha) - y_0 &\leq 0\\ f_2(\alpha,\beta) = -2\sin(\alpha) &\leq 0\\ f_3(\alpha,\beta) = 2\sin(\alpha) + 1.5\sin(\alpha+\beta) - y_0 &\leq 0 \end{aligned}$$

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What have we done? Future work

# Conclusion



What have we done? Future work

### Future work

- Build a triangulation to guarantee more topology properties of a set, e.g. to be able to compute its :
  - homotopy type, Fundamental Group  $(\pi_1(\mathbb{S}))$
  - homology groups  $(H_1(\mathbb{S}), H_2(\mathbb{S}), \dots)$
  - Betti number



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If  $\mathbb{S} = \{x \in D \subset \mathbb{R}^n / f(x) \le 0\}$  where f is a  $C^1$  function from D to  $\mathbb{R}$ , D a convex set,  $v^*$  be in  $\mathbb{S}$  and if

$$f(x) = 0, Df(x).(x - v^*) \le 0, x \in D$$

is inconsistent then  $v^*$  is star a for  $\mathbb{S}$ .

(1) is inconsitent  $\Leftrightarrow \forall x \in D, f(x) = 0 \Rightarrow Df(x).(x - v^*) > 0$ 



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