Proving that a set is connected via interval analysis

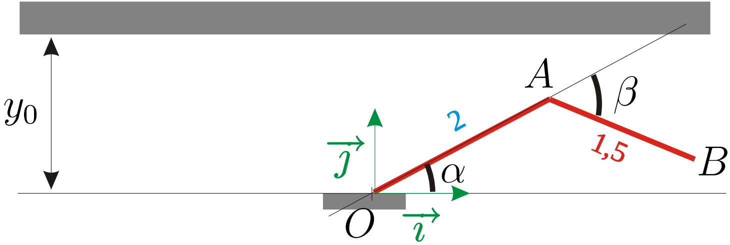
Minisymposium, Interval methods, Para' 04

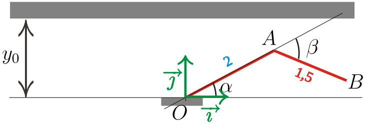
N. Delanoue, L. Jaulin, B. Cottenceau

University of Angers - LISA - France

Wednesday, June 23, 2004

OutLine


A
β
y_0
2
1,5
B
$\vec{j}$
α
O
$\vec{i}$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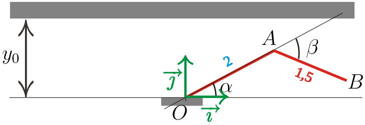
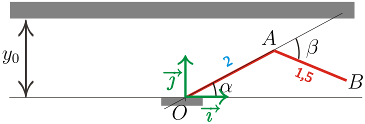


Coordinates of A

$$\begin{cases} x_A = 2\cos(\alpha) \\ y_A = 2\sin(\alpha) \end{cases}$$

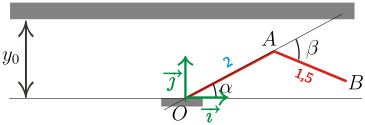


Coordinates of A

$$\begin{cases} x_A = 2\cos(\alpha) \\ y_A = 2\sin(\alpha) \end{cases}$$

Coordinates of B

$$\begin{cases} x_B = 2\cos(\alpha) + 1.5\cos(\alpha+\beta) \\ y_B = 2\sin(\alpha) + 1.5\sin(\alpha+\beta) \end{cases}$$


y_0
A
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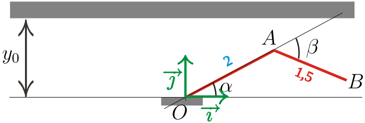
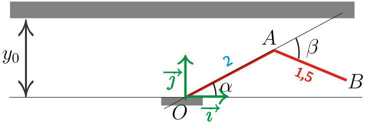


Constraints on A

$$y_A \in [0, y_0]$$



Constraints on A

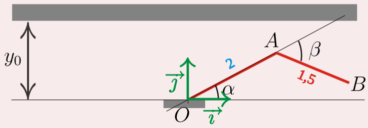
$$y_A \in [0, y_0]$$

Constraints on B

$$y_B \in]-\infty, y_0]$$

Constraints on α and β

$$\alpha \in]-\pi,\pi[, \beta \in]-\pi,\pi[$$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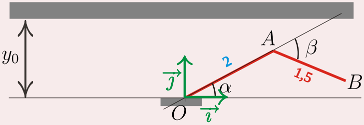
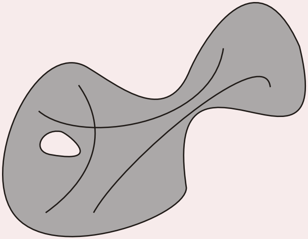


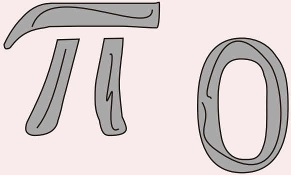
Feasible configuration set

$$\mathbb{S} = \left\{ (\alpha,\beta) \in]-\pi,\pi[^2 / \left\{ \begin{array}{lcl} 2\sin(\alpha) & \in & [0,y_0] \\ 2\sin(\alpha) + 1.5\sin(\alpha+\beta) & \in &]-\infty,y_0] \end{array} \right. \right\}$$

Definition (*path-connected set*)

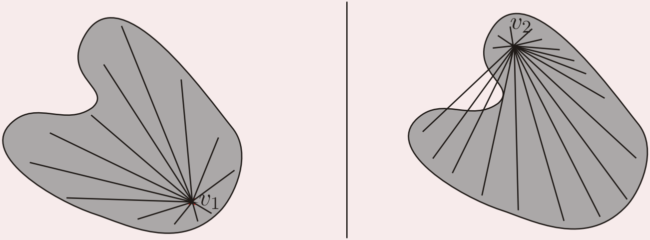
A topological space $\mathbb{S}$ is *path-connected* if and only if for every two points $x,y \in \mathbb{S}$, there is a continuous function f from $[0,1]$ to $\mathbb{S}$ such that $f(0) = x$ and $f(1) = y$.





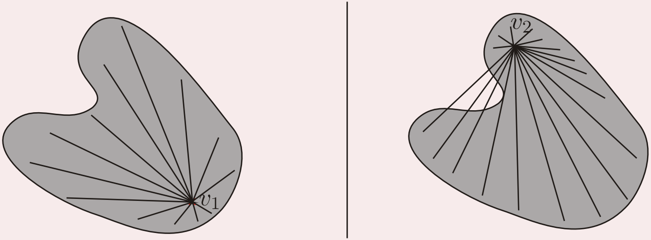
Definition (*star*)

The point v^* is a *star* for a subset X of an Euclidean space if $\forall x \in X$, the segment $[x,v^*]$ is include in X.



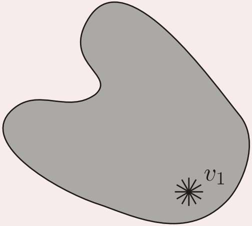
Definition (*star-shaped set*)

If there exists $v^* \in X$ such that v^* is a star for X, then we say that X is *star-shaped* or v^**-star-shaped*.



Proposition 1

A star-shaped set is a path-connected set.



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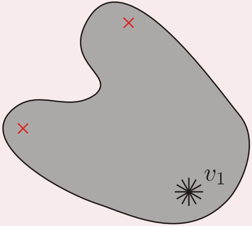

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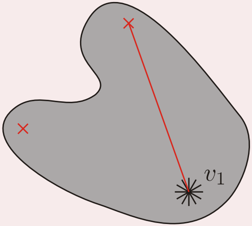

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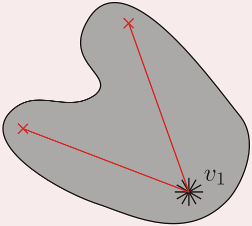
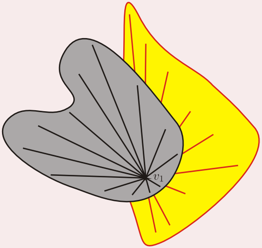


Proposition 2

Let X and Y two v^*-star-shaped set, then $X \cap Y$ and $X \cup Y$ are also v^*-star-shaped.



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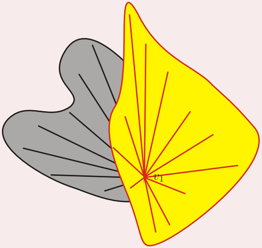

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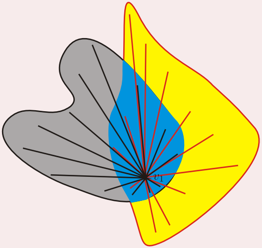
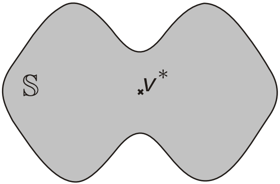


Theorem

If $\mathbb{S} = \{x \in D \subset \mathbb{R}^n \;/\; f(x) \leq 0\}$ *where* f *is a* C^1 *function from* D *to* $\mathbb{R}$, D *a convex set,* v^* *be in* $\mathbb{S}$ *and if*

$$f(x) = 0,\; Df(x).(x - v^*) \leq 0,\; x \in D$$

is inconsistent then v^* *is star a for* $\mathbb{S}$.



▸ Proof idea - Geometric interpretation

Let us prove that $v^* = (0.6, -0.5)$ is a star for the set defined by

$$\mathbb{S} = \{(x,y) \in \mathbb{R}^2, \text{ such that } f(x,y) = x^2 + y^2 + xy - 2 \leq 0\}$$

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$$\Leftrightarrow \left\{ \begin{array}{l} x^2 + y^2 + xy - 2 = 0 \\ \partial_x f(x,y).(x - 0.6) + \partial_y f(x,y).(y + 0.5) \leq 0 \end{array} \right. \quad \text{is inconsistent}$$

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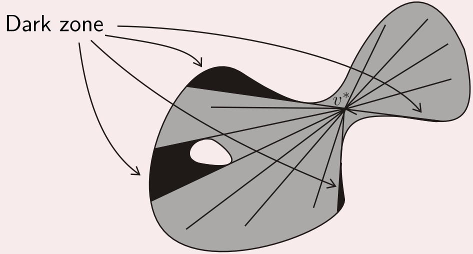
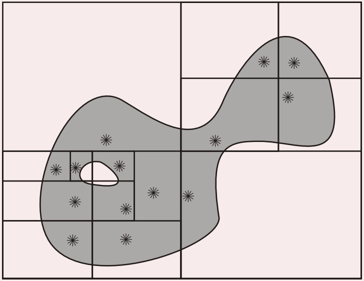
$$\Leftrightarrow \left\{ \begin{array}{l} x^2 + y^2 + xy - 2 = 0 \\ (2x + y)(x - 0.6) + (2y + x)(y + 0.5) \leq 0 \end{array} \right. \quad \text{is inconsistent}$$


Dark zone
v^*

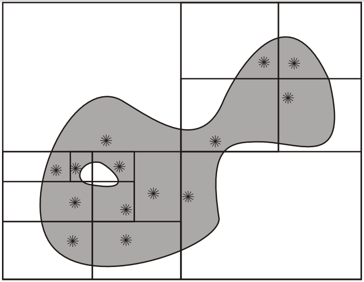
The idea

To divide it with a paving $\mathcal{P}$ such that, on each part p, $\mathbb{S} \cap p$ is star-shaped.



The idea

Let us define the notion of *star-spangled* graph with the relation : $\mathbb{S} \cap p \cap q \neq \emptyset$.



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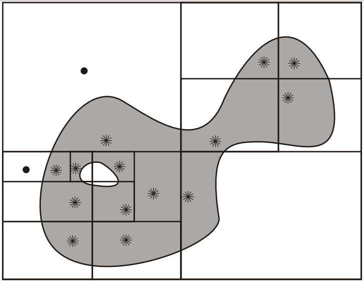

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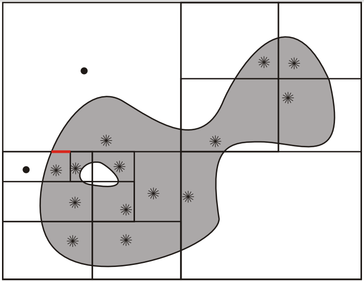

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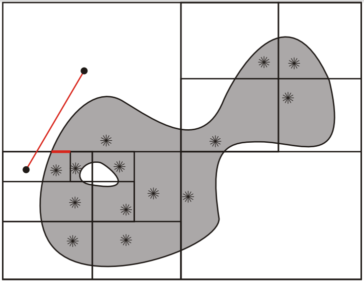

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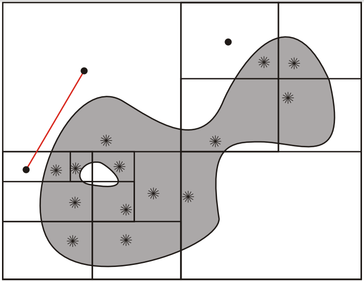

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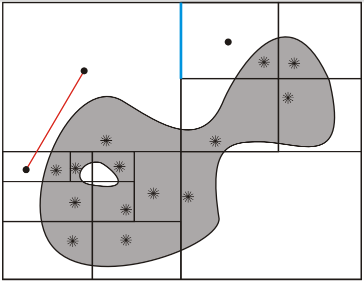

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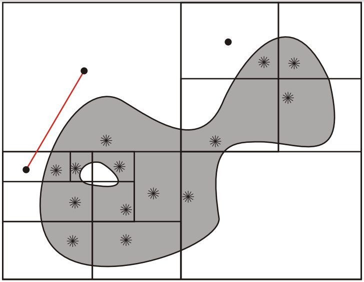

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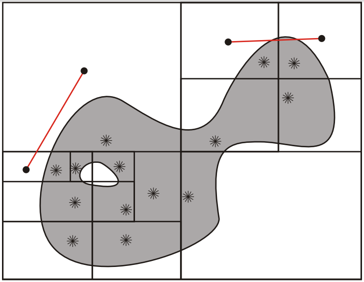

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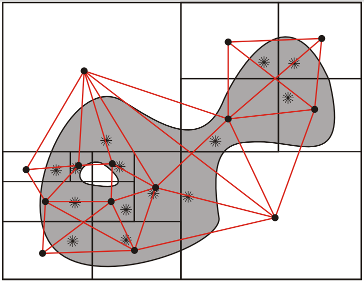
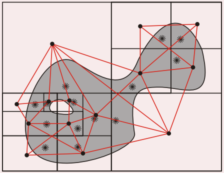


Definition

A *star-spangled graph* of a set $\mathbb{S}$, noted by $\mathcal{G}_{\mathbb{S}}$, is a relation $\mathcal{R}$ on a paving $\mathcal{P}$ where

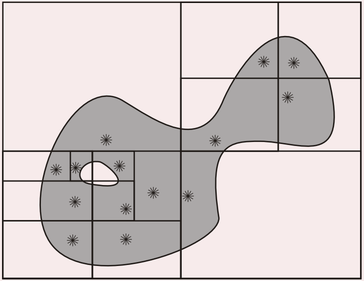
- $\mathcal{P} = (p_i)_{i \in I}$, for all p of $\mathcal{P}$, $\mathbb{S} \cap p$ is star-shaped. And $\mathbb{S} \subset \bigcup_{i \in I} p_i$
- $\mathcal{R}$ is the reflexive and symmetric relation on $\mathcal{P}$ defined by $p \,\mathcal{R}\, q \Leftrightarrow \mathbb{S} \cap p \cap q \neq \emptyset$.

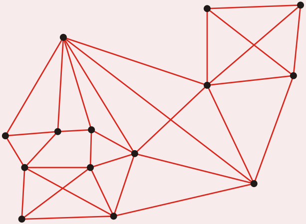


Theorem

Let $\mathcal{G}_{\mathbb{S}}$ be a star-spangled graph of a set $\mathbb{S}$.

$\mathbb{S}$ *is path-connected* $\Leftrightarrow \mathcal{G}_{\mathbb{S}}$ *is connected* .

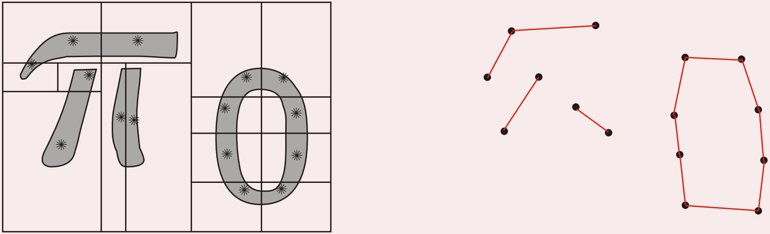




Proof idea

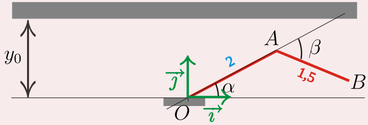
Corollary

Let $\mathcal{G}_{\mathbb{S}}$ be a star-spangled graph of a set $\mathbb{S}$.
$\mathcal{G}_{\mathbb{S}}$ has the same number of connected components than $\mathbb{S}$. i.e. $\pi_0(\mathbb{S}) = \pi_0(\mathcal{G}_{\mathbb{S}})$.



Proof idea

Feasible configuration set, $y_0 = 2.3$



$$\mathbb{S} = \left\{ (\alpha,\beta) \in]-\pi,\pi[^2 / \left\{ \begin{array}{lcl} 2\sin(\alpha) & \in & [0,y_0] \\ 2\sin(\alpha) + 1.5\sin(\alpha+\beta) & \in &]-\infty,y_0] \end{array} \right. \right\}$$

The solver CIA (path-Connected via Interval Analysis)

Feasible configuration set, $y_0 = 2.3$

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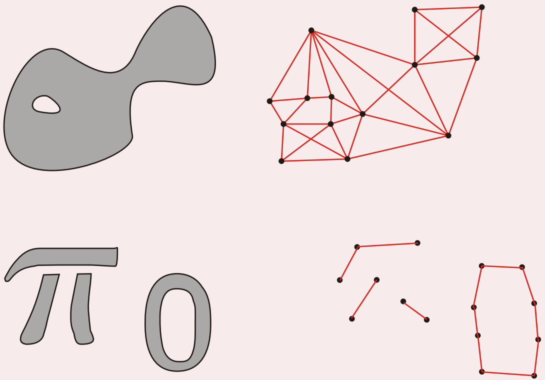
Feasible configuration set, $y_0 = 2.3$

$(\alpha,\beta) \in]-\pi,\pi[^2 \ /$

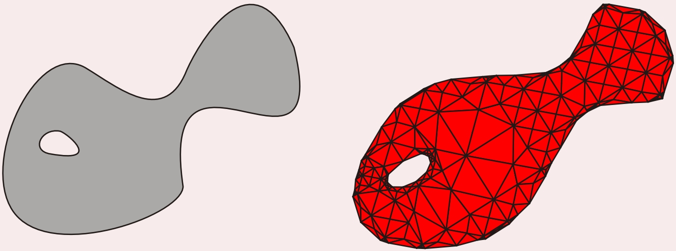
$$\left\{ \begin{array}{lcl} f_1(\alpha,\beta) = 2\sin(\alpha) - y_0 & \leq & 0 \\ f_2(\alpha,\beta) = -2\sin(\alpha) & \leq & 0 \\ f_3(\alpha,\beta) = 2\sin(\alpha) + 1.5\sin(\alpha+\beta) - y_0 & \leq & 0 \end{array} \right.$$

Conclusion



Future work

- Build a triangulation to guarantee more topology properties of a set, e.g. to be able to compute its :
  - homotopy type, Fundamental Group ($\pi_1(\mathbb{S})$)
  - homology groups ($H_1(\mathbb{S})$,$H_2(\mathbb{S})$, ...)
  - Betti number



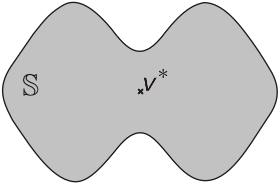
Theorem

If $\mathbb{S} = \{x \in D \subset \mathbb{R}^n \ / \ f(x) \leq 0\}$ *where* f *is a* C^1 *function from* D *to* $\mathbb{R}$, D *a convex set,* v^* *be in* $\mathbb{S}$ *and if*

$$f(x) = 0,\ Df(x).(x - v^*) \leq 0,\ x \in D$$

is inconsistent then v^* *is star a for* $\mathbb{S}$.

(1) is inconsitent $\Leftrightarrow \forall x \in D, f(x) = 0 \Rightarrow Df(x).(x - v^*) > 0$



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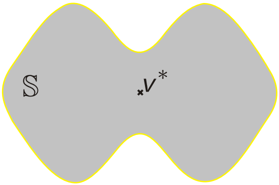

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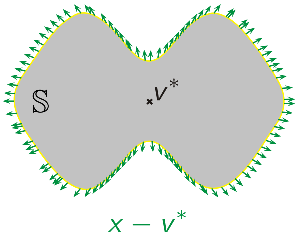

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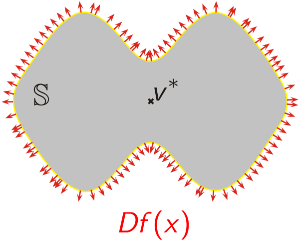


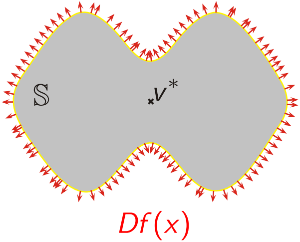
Theorem

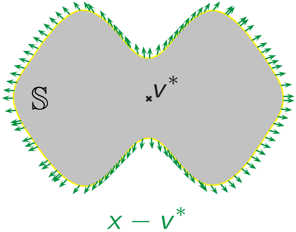
If $\mathbb{S} = \{x \in D \subset \mathbb{R}^n \ / \ f(x) \leq 0\}$ *where* f *is a* C^1 *function from* D *to* $\mathbb{R}$*,* D *a convex set,* v^* *be in* $\mathbb{S}$ *and if*

$$f(x) = 0, \, Df(x).(x - v^*) \leq 0, \, x \in D$$

is inconsistent then v^* *is star a for* $\mathbb{S}$*.*

(1) is inconsitent $\Leftrightarrow \forall x \in D, f(x) = 0 \Rightarrow Df(x).(x - v^*) > 0$





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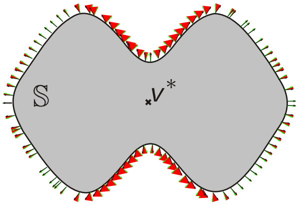

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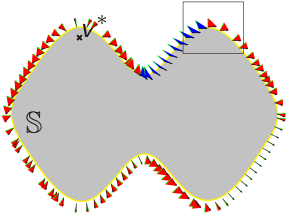
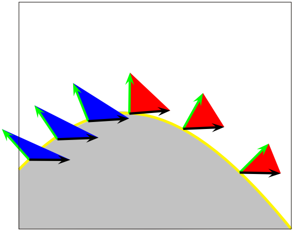
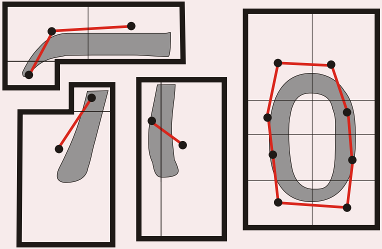




Return

Corollary

Let $\mathcal{G}_{\mathbb{S}}$ be a star-spangled graph of a set $\mathbb{S}$.
$\mathcal{G}_{\mathbb{S}}$ has the same number of connected components than $\mathbb{S}$. i.e.
$\pi_0(\mathbb{S}) = \pi_0(\mathcal{G}_{\mathbb{S}})$.



Return