Set convexity analysis with Interval Analysis.

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Aims

Prove
$$S = \bigcap_{i=1}^{r} \{x \in D \subset \mathbb{R}^{n}; f_{i}(x) \leq 0\}$$
 is convex

where

- D is a compact subset of \mathbb{R}^n ,
- $f_i \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}), i \in \{1, \ldots, r\}.$

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Outline

Introduction - Motivation

- Convex set
- Convex function
- Application optimization
- 2 Sufficient condition for set convexity
 - A weak sufficient condition
 - Not necessary
 - Sufficient condition
 - Proof idea, by contradiction
 - Example



Conclusion

Introduction - Motivation

Sufficient condition for set convexity theorem discussion and other methods Conclusion

Convex set Hahn-Banach Theorem Convex function Application optimization

Interval analysis is often able to prove that a set defined by inequalities is empty.

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Example

Let

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Example

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$$\begin{array}{rcl} f: & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & (\sin x - x^2 + 1) \cos x \end{array}$$

et us prove that $S = \{x \in [0; \frac{1}{2}], f(x) = 0\} = \emptyset$

One defines
$$\begin{cases} [f]: & \mathbb{IR} \to & \mathbb{IR} \\ & [x] \mapsto & (\sin[x] - [x]^2 + 1) \cos[x] \end{cases}$$

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 $[f]([0; \frac{1}{2}]) = (\sin[0; \frac{1}{2}] - [0; \frac{1}{2}]^2 + 1)\cos[0; \frac{1}{2}]$

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Example

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$$\begin{array}{rcl} f: & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & (\sin x - x^2 + 1) \cos x \end{array}$$

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$$= (\sin[0; \frac{1}{2}] - [0; \frac{1}{4}] + 1) \cos[0; \frac{1}{2}]$$

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$$= (\sin[0; \frac{1}{2}] + [-\frac{1}{4}; 0] + 1) \cos[0; \frac{1}{2}]$$

$$= ([0; \sin \frac{1}{2}] + [\frac{3}{4}; 1])[\cos \frac{1}{2}; 1]$$

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Convex set Hahn-Banach Theorem Convex function Application optimization

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$$= [\frac{3}{4}; 1 + \sin \frac{1}{2}] \times [\cos \frac{1}{2}; 1]$$

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 $[f]([0; \frac{1}{2}]) \subset [0.65818; 1.4795]$

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Convex set Hahn-Banach Theorem Convex function Application optimization

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 $[f]([0; \frac{1}{2}]) \subset [0.65818; 1.4795]$

Conclusion

$$0
ot\in [f]([0;rac{1}{2}]),$$
 therefore $S=\emptyset$

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Convex set Hahn-Banach Theorem Convex function Application optimization

Definition (Convex)

A subset X of \mathbb{R}^n is *convex* if

$$\forall x, y \in X, \forall t \in [0, 1], tx + (1 - t)y \in X$$



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Proposition

A and B convex \Rightarrow A \cap B convex.



FIG.: Intersection

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Convex set Hahn-Banach Theorem Convex function Application optimization

Proposition

A and B convex \Rightarrow A + B convex.



Convex set Hahn-Banach Theorem Convex function Application optimization

Proposition

A and B convex \Rightarrow A + B convex.



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Hahn-Banach Theorem

Let A, B, two convex, nonempty and disjoint sets. A is closed and B is compact. Then there is a hyperplane U which strictly convertes A or

Then there is a hyperplane H which strictly separates A and B.



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Definition (Convex function)

A real-valued function $f: X \to \mathbb{R}$ is called convex if

$$\forall x \in X, \forall y \in X, \forall t \in [0,1], f(tx+(1-t)y) \le tf(x)+(1-t)f(y).$$



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Proposition

- If f and g are convex functions, then so are $x \mapsto \max(f(x), g(x))$ and $x \mapsto f(x) + g(x)$.
- 2 If f and g are convex functions and if g is increasing, then $h(x) = g \circ f(x)$ is convex.
- Convexity is invariant under affine maps : if x → f(x) is convex with x ∈ ℝⁿ, and g : y → Ay + b then so is y → f ∘ g(y) = f(Ay + b), where A ∈ ℝ^{n×m}, b ∈ ℝ^m.

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Definition (epigraph)

The epigraph of a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is the set of points lying on or above its graph :

epi $f = \{(x, y) : x \in \mathbb{R}^n, \mu \in \mathbb{R}, f(x) \le y\} \subseteq \mathbb{R}^{n+1}.$



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Proposition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its epigraph is convex.



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Proposition

A function $f \in C^2(\mathbb{R}^n, \mathbb{R})$ is convex if and only if $\forall x, \nabla^2 f(x)$ is semi definite positive (*i.e.* $\forall x, \nabla^2 f(x) \succeq 0$).

Example

- $f : \mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$ is convex since $f''(x) = 2 \ge 0$.
- $g: \mathbb{R}^2
 i (x, y) \mapsto x^2 + y^2 \in \mathbb{R}$ is convex since

$$\nabla^2 f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

is positive definite.

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$$\min_{x\in S\subset\mathbb{R}^n}f(x)$$

theorem - uniqueness

If S is a convex subset of \mathbb{R}^n and $f \in C^2(D, \mathbb{R})$ a convex function, the following conditions are equivalent

- \underline{x} is a local minimum.
- \underline{x} is a global minimum.

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A weak sufficient condition Not necessary Sufficient condition Proof idea, by contradiction Example

Corollary

Let $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$. If $\forall x \in D, \nabla^2 f(x) \succeq 0$ then

 $\{x \in D \subset \mathbb{R}^n \mid f(x) \leq 0\}$ is a convex set



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Example

The set $\{x \in \mathbb{R} \mid f(x) = x^2 + y^2 - 1 \le 0\}$ is convex since $\forall x, \nabla^2 f(x) \succeq 0$.

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Not necessary

There exists function such that

$$\left\{ \begin{array}{l} \{x \in D \subset \mathbb{R}^n \mid f(x) \leq 0\} \text{ is a convex set} \\ \exists x \in D, \nabla^2 f(x) \not\succeq 0 \end{array} \right.$$



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Example

$$f(x, y) = \log(x^2 + y^2 + 1) - 3$$



A weak sufficient condition Not necessary Sufficient condition Proof idea, by contradiction Example

Let use denote by *S* the set $\{x \in D \subset \mathbb{R}^n \mid f(x) \leq 0\}$ and ∂S the set $\{x \in D \subset \mathbb{R}^n \mid f(x) = 0\}$

Theorem

Suppose that

- D is convex,
- S is path-connected,
- $\forall x \in \partial S$, ker $\nabla f(x)$ has codimension 1,

If $\forall x \in \partial S, \nabla^2 f(x)_{|Ker \nabla f(x)} \succ 0$ then S is convex.

A weak sufficient condition Not necessary Sufficient condition **Proof idea, by contradiction** Example

Let use denote by *S* the set $\{x \in D \subset \mathbb{R}^n \mid f(x) \le 0\}$ and ∂S the set $\{x \in D \subset \mathbb{R}^n \mid f(x) = 0\}$

Theorem

Suppose that

- D is convex,
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If S is non convex then $\exists x \in \partial S, \nabla^2 f(x)|_{Ker \nabla f(x)} \neq 0$

A weak sufficient condition Not necessary Sufficient condition Proof idea, by contradiction Example

If S is non convex then $\exists x \in \partial S, \nabla^2 f(x)_{|Ker\nabla f(x)} \not\succeq 0$

Since D is convex,



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A weak sufficient condition Not necessary Sufficient condition **Proof idea, by contradiction** Example

If S is non convex then $\exists x \in \partial S, \nabla^2 f(x)_{|Ker\nabla f(x)} \not\succ 0$

Since S is path-connected,



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A weak sufficient condition Not necessary Sufficient condition **Proof idea, by contradiction** Example

If S is non convex then $\exists x \in \partial S, \nabla^2 f(x)|_{Ker \nabla f(x)} \neq 0$



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A weak sufficient condition Not necessary Sufficient condition **Proof idea, by contradiction** Example

If S is non convex then
$$\exists x \in \partial S, \nabla^2 f(x)_{|Ker\nabla f(x)} \neq 0$$

i.e.

 $\exists x \in \partial S, \exists h \in \ker \nabla f(x), h^T \nabla^2 f(x) h \leq 0$



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A weak sufficient condition Not necessary Sufficient condition **Proof idea, by contradiction** Example

$$\exists x \in \partial S, \exists h \in \ker \nabla f(x), h^T \nabla^2 f(x) h \leq 0$$



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A weak sufficient condition Not necessary Sufficient condition **Proof idea, by contradiction** Example

$$\exists x \in \partial S, \exists h \in \ker \nabla f(x), h^T \nabla^2 f(x) h \leq 0$$

 $s(0) = x, \dot{s}(0) = h$ $\frac{d}{dt} \int_{t=0}^{t} \nabla f(s(t)) = \frac{d\nabla f(x)}{dx} (s(0)) \frac{d}{dt} \int_{t=0}^{t} s(t) = \nabla^2 f(x) h$ $\nabla^2 f(x)h$

A weak sufficient condition Not necessary Sufficient condition **Proof idea, by contradiction** Example

$\exists x \in \partial S, \exists h \in \ker \nabla f(x), h^T \nabla^2 f(x) h \leq 0$



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A weak sufficient condition Not necessary Sufficient condition **Proof idea, by contradiction** Example

$x \in \partial S, h \in \ker \nabla f(x), h^T \nabla^2 f(x) h \ge 0$



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A weak sufficient condition Not necessary Sufficient condition Proof idea, by contradiction Example

Example

Let f be the function

$$egin{array}{rcl} f:& [-10,10]&
ightarrow&\mathbb{R}\ & (x,y)&\mapsto& \log(x^2+y^2+1)-3 \end{array}$$

To prove that $S = \{(x, y) \in [-10, 10]^2 \mid f(x, y) \le 0\}$ is convex, one only has to check that :

 $\begin{cases} S \text{ is path-connected} \\ f(x) = 0 \Rightarrow \nabla f(x) \neq 0 \\ f(x) = 0 \Rightarrow \nabla^2 f(x) \succ 0 \end{cases}$

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A weak sufficient condition Not necessary Sufficient condition Proof idea, by contradiction Example

$$\begin{cases} f(x) = 0 \Rightarrow \nabla f(x) \neq 0\\ f(x) = 0 \Rightarrow \nabla^2 f(x)_{|\ker \nabla f(x)} \succ 0 \end{cases}$$

 $\begin{cases} f(x) = 0 \land \nabla f(x) = 0 \text{ has no solution} \\ f(x) = 0 \land \nabla^2 f(x)_{|\ker \nabla f(x)} \not\succeq 0 \text{ has no solution} \end{cases}$

$$\begin{cases} f(x) = 0 \land \nabla f(x) = 0 \text{ has no solution} \\ f(x) = 0 \land h \in \ker \nabla f(x) \land h^T \nabla^2 f(x) h \le 0 \text{ has no solution} \end{cases}$$

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A weak sufficient condition Not necessary Sufficient condition Proof idea, by contradiction Example

$$\begin{cases} f(x) = 0 \land \nabla f(x) = 0 \text{ has no solution} \\ f(x) = 0 \land \nabla f(x)h = 0 \land h^T \nabla^2 f(x)h \le 0 \text{ has no solution} \end{cases}$$

$$\begin{cases} \log(x^2 + y^2 + 1) - 3 = 0\\ \frac{2x}{y^2 + x^2 + 1} = 0 & \text{has no solution}\\ \frac{2y}{y^2 + x^2 + 1} = 0 \end{cases}$$

and

$$\begin{cases} \log(x^2 + y^2 + 1) - 3 = 0\\ \frac{8x^2 + 8y^2}{x^6 + 3x^4 y^2 + 3x^4 + 3x^2 y^4 + 6x^2 y^2 + 3x^2 + y^6 + 3y^4 + 3y^2 + 1} \le 0 \end{cases}$$
 has no solution

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A weak sufficient condition Not necessary Sufficient condition Proof idea, by contradiction Example



Conditions Others methods

Theorem

Suppose that

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- S is path-connected,
- $\forall x \in \partial S$, ker $\nabla f(x)$ has codimension 1,

If $\forall x \in \partial S, \nabla^2 f(x)_{|Ker \nabla f(x)} \succ 0$ then S is convex.

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Conditions Others methods

If D is non convex,



Conditions Others methods

If S is not path-connected,



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Conditions Others methods

If $\exists x \in \partial S$, ker $\nabla f(x)$ has not codimension 1,

$$\{(x_1, x_2) \mid (x_2^2 + x_1^2)^2 + \frac{x_1 x_2^2}{2} + \frac{x_1^3}{2} \le 0\}$$



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Conditions Others methods

- CVX : Matlab Software for Disciplined Convex Programming
- Q Cylindrical algebraic decomposition

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Conditions Others methods

CVX : Matlab Software for Disciplined Convex Programming

- constraint propagation
- symbolic differentiation
- C. Crusius.

A parser/solver for convex optimization problems. PhD thesis, Stanford Univ., USA, 2002.

• M. C. Grant.

Disciplined convex programming. PhD thesis, Stanford Univ., USA, 2004.

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Conditions Others methods

CVX : Matlab Software for Disciplined Convex Programming

$$x \mapsto f(x) = \exp(3x - 2) + x^2$$
 is convex since



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Conditions Others methods

Cylindrical algebraic decomposition

Tarski's theorem

The solution set of a quantified system of real algebraic equations and inequations is a semialgebraic set (Tarski 1951, Strzebonski 2000).

- Tarski's theorem gives rise to an impractical method (Davenport and Heintz 1988)
- A much more efficient procedure for implementing quantifier elimination is called *cylindrical algebraic decomposition*. It was developed by Collins (1975).

Conditions Others methods

Proposition

If P and Q have a common root then Res(P, Q) = 0.

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Conditions Others methods

Proof

Let P and Q have a common root a with deg P = p and deg Q = q. One has $P(X) = (X - a)\tilde{P}(X)$ and $Q(X) = (X - a)\tilde{Q}(X)$, then : $P\tilde{Q} - Q\tilde{P} = 0$ where deg $\tilde{P} = p - 1$ and deg $\tilde{Q} = q - 1$.

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Conditions Others methods

Proof

Note that

$$(b_2X^2+b_1X+b_0)(a_2X^2+a_1X+a_0)=c_4X^4+c_3X^3+c_2X^2+c_1X+c_0$$

is equivalent to

$$\begin{pmatrix} b_2 & b_1 & b_0 \end{pmatrix} \begin{pmatrix} a_2 & a_1 & a_1 & 0 & 0 \\ 0 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_2 & a_1 & a_0 \end{pmatrix} = \begin{pmatrix} c_4 & c_3 & c_2 & c_1 & c_0 \end{pmatrix}$$

Conditions Others methods

Proof

$$\exists \tilde{P} \neq 0 \exists \tilde{Q} \neq 0, P \tilde{Q} - Q \tilde{P} = 0$$

is equilavent to det Syl = 0 with :



Conditions Others methods

Example

$$P(X) = (X - 1)(X - 2) = X^{2} - 3X + 2$$

$$Q(X) = (X - 1)(X - 3) = X^{2} - 4X + 3$$

have common root,

$$\det \left(\begin{array}{rrrr} 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 1 & -4 & 3 & 0 \\ 0 & 1 & -4 & 3 \end{array} \right) = 0$$

Conditions Others methods

Example

$$P(X) = (X - 1)(X - 2) = X^{2} - 3X + 2$$

$$Q(X) = (X - 1)(X - 3) = X^{2} - 4X + 3$$

have common root since

$$\begin{pmatrix} 1 & -3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 1 & -4 & 3 & 0 \\ 0 & 1 & -4 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(X-3)P(X) - (X-2)Q(X) = 0$$

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Conditions Others methods

$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \le 0$$



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Conditions Others methods

$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \le 0$$



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Conditions Others methods

$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \le 0$$



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Conditions Others methods

$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \le 0$$

 $y \mapsto P(x, y)$ and $y \mapsto \frac{d}{dy}P(x, y)$ has a common root. *i.e.* $1 * Y^2 + 0 * Y^1 + (X^2 - 1) * Y^0$ and $2 * Y^1$ has a common root.



Conditions Others methods

$$\exists y\in\mathbb{R}, x^2+y^2-1\leq 0$$

$$1*Y^2+0*Y^1+(X^2-1)*Y^0 \text{ and } 2*Y^1 \text{ has a common root}.$$

$$\det \begin{pmatrix} 1 & 0 & X^2 - 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = 0$$
$$\Leftrightarrow (-2)(-2)(X^2 - 1) = 0$$
$$\Leftrightarrow (X^2 - 1) = 0$$

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Conditions Others methods

 $\exists y \in \mathbb{R}, x^2 + y^2 - 1 \le 0$

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Quantifier Elimination	
in	
Elementary Algebra and Geometry	
by	
Partial Cylindrical Algebraic Decomposition	
Version B 1.44, 17 May 2005	
by	
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With contributions by: Christopher W. Brown, George E. Collins, Mark J. Encarnacion, Jeremy R. Johnson	
Werner Krandick, Richard Liska, Scott McCallum,	
Nicolas Robidoux, and Stanly Steinberg	
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Conditions Others methods

 $\exists y \in \mathbb{R}, x^2 + y^2 - 1 \le 0$

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by	
Partial Cylindrical Algebraic Decomposition	
Version B 1.44, 17 May 2005	
by	
Hoon Hong	
(hhong@math.ncsu.edu)	
With contributions by: Christopher W. Brown, George E. Collins, Mark J. Encarnacion, Jeremy R. Johnson	
Werner Krandick, Richard Liska, Scott McCallum, Nicolas Robidoux, and Stanly Steinberg	
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Conditions Others methods

$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \le 0$$

 $\Leftrightarrow (x + 1 \ge 0) \land (x - 1 \le 0)$

Nicolas Delanoue, Didier Henrion Set convexity analysis with Interval Analysis.

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Conditions Others methods

Proposition

Convexity is a semi-algebraic property.

Let S be a semi-algebraic set, S is convex if

 $\forall x \forall y \forall t, (t \in [0, 1], x \in S, y \in S \Rightarrow tx + (1 - t)y \in S)$

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Example

Proving that $\{x \in \mathbb{R}, x^2 - 1 \le 0\} = [-1, 1]$ is convex remains to prove :

$$\forall x \forall y \forall t, 0 \leq t \leq 1 \land x^2 - 1 \leq 0 \land y^2 - 1 \leq 0 \Rightarrow (tx + (1 - t)y)^2 - 1 \leq 0$$

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Conditions Others methods

$\{x \in \mathbb{R}, x^2 - 1 \leq 0\} = [-1, 1]$ is convex.

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Conclusion Future works

Conclusion

Let use denote by *S* the set $\{x \in D \subset \mathbb{R}^n \mid f(x) \leq 0\}$ and ∂S the set $\{x \in D \subset \mathbb{R}^n \mid f(x) = 0\}$

Theorem

Suppose that

- D is convex,
- S is path-connected,
- $\forall x \in \partial S$, ker $\nabla f(x)$ has codimension 1,

If $\forall x \in \partial S, \nabla^2 f(x)|_{Ker\nabla f(x)} \succ 0$ then S is convex.

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Conclusion Future works

Future works

- Prove that the proposed algorithm terminates in the generic case.
- Expand this method to cases where *f_i* depends of unknown parameters.

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