Set convexity analysis with Interval Analysis.

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Aims

$$\text{Prove } S = \bigcap_{i=1}^{r} \{x \in D \subset \mathbb{R}^n ; f_i(x) \leq 0\} \text{ is convex}$$

where

- D is a compact subset of $\mathbb{R}^n$,
- $f_i \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}), i \in \{1, \ldots, r\}$.

Outline

Interval analysis is often able to prove that a set defined by inequalities is empty.

Example

Let

$$\begin{array}{rccl} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & (\sin x - x^2 + 1)\cos x \end{array}$$

Let us prove that $S = \{x \in [0; \frac{1}{2}], f(x) = 0\} = \emptyset$

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$$= \quad (\sin[0; \tfrac{1}{2}] + [-\tfrac{1}{4}; 0] + 1) \cos[0; \tfrac{1}{2}]$$

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Conclusion

$0 \notin [f]([0; \frac{1}{2}])$, therefore $S = \emptyset$

Definition (Convex)

A subset X of $\mathbb{R}^n$ is *convex* if

$$\forall x, y \in X, \forall t \in [0,1], tx + (1-t)y \in X$$

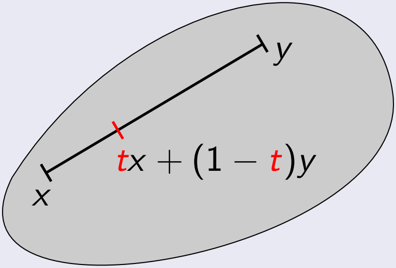

Fig.: Convex set

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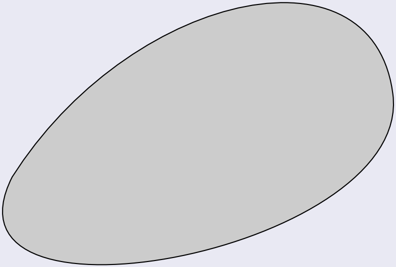

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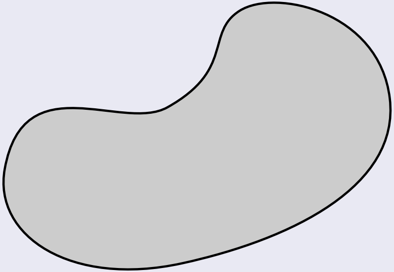

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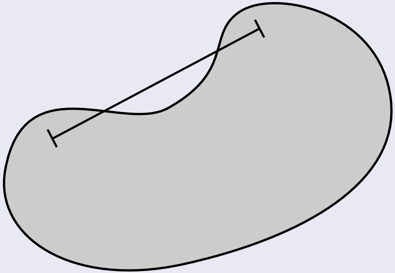

FIG.: Non convex set

Proposition

A and B convex $\Rightarrow A \cap B$ convex.

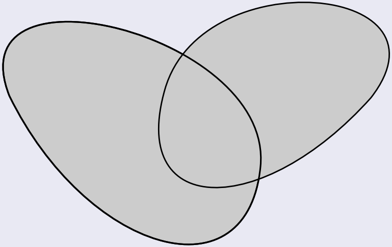

FIG.: Intersection

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A and B convex $\Rightarrow A + B$ convex.

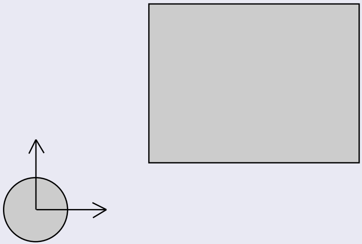

FIG.: Addition

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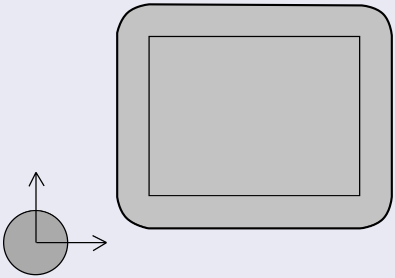
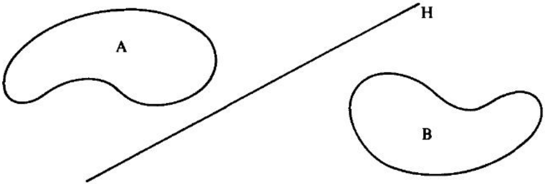


FIG.: Addition

Hahn-Banach Theorem

Let A, B, two convex, nonempty and disjoint sets.
A is closed and B is compact.
Then there is a hyperplane H which strictly separates A and B.

eométricamente la separación significa que A y B se sitúan **«de un lado y de otro de H»**.



Recordemos finalmente que un conjunto A ⊂ E es convexo si

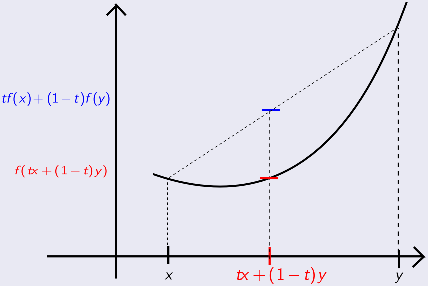
$$tx + (1-t)y \in A \qquad \forall x, y \in A, \qquad \forall t \in [0, 1].$$

FIG.: Haïm Brézis, Analyse fonctionnelle - Théorie et applications

Definition (Convex function)

A real-valued function $f : X \to \mathbb{R}$ is called convex if

$$\forall x \in X, \forall y \in X, \forall t \in [0,1], f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$



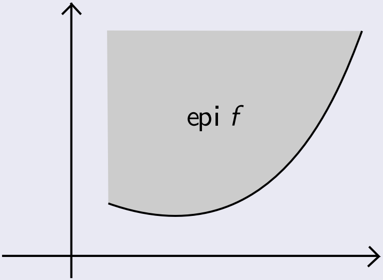
Proposition

1. If f and g are convex functions, then so are $x \mapsto \max(f(x), g(x))$ and $x \mapsto f(x) + g(x)$.
2. If f and g are convex functions and if g is increasing, then $h(x) = g \circ f(x)$ is convex.
3. Convexity is invariant under affine maps : if $x \mapsto f(x)$ is convex with $x \in \mathbb{R}^n$, and $g : y \mapsto Ay + b$ then so is $y \mapsto f \circ g(y) = f(Ay + b)$, where $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^m$.

Definition (epigraph)

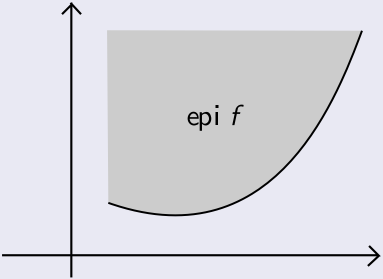
The epigraph of a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is the set of points lying on or above its graph :

$$\text{epi } f = \{(x, y) \,:\, x \in \mathbb{R}^n, \mu \in \mathbb{R}, f(x) \leq y\} \subseteq \mathbb{R}^{n+1}.$$



Proposition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its epigraph is convex.



Proposition

A function $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ is convex if and only if $\forall x, \nabla^2 f(x)$ is semi definite positive (*i.e.* $\forall x, \nabla^2 f(x) \succeq 0$).

Example

- $f : \mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$ is convex since $f''(x) = 2 \geq 0$.
- $g : \mathbb{R}^2 \ni (x, y) \mapsto x^2 + y^2 \in \mathbb{R}$ is convex since

$$\nabla^2 f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

is positive definite.

$$\min_{x \in S \subset \mathbb{R}^n} f(x)$$

theorem - uniqueness

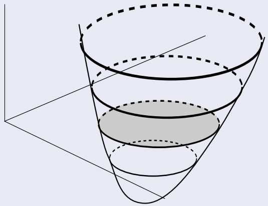
If S is a convex subset of $\mathbb{R}^n$ and $f \in \mathcal{C}^2(D, \mathbb{R})$ a convex function, the following conditions are equivalent

- $\underline{x}$ is a local minimum.
- $\underline{x}$ is a global minimum.

Corollary

Let $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$.
If $\forall x \in D, \nabla^2 f(x) \succeq 0$ then

$$\{x \in D \subset \mathbb{R}^n \mid f(x) \leq 0\} \text{ is a convex set}$$



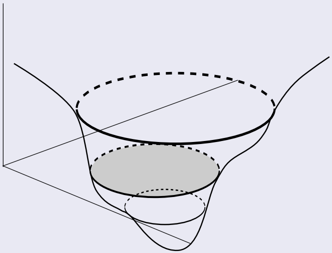
Example

The set $\{x \in \mathbb{R} \mid f(x) = x^2 + y^2 - 1 \leq 0\}$ is convex since $\forall x, \nabla^2 f(x) \succeq 0$.

Not necessary

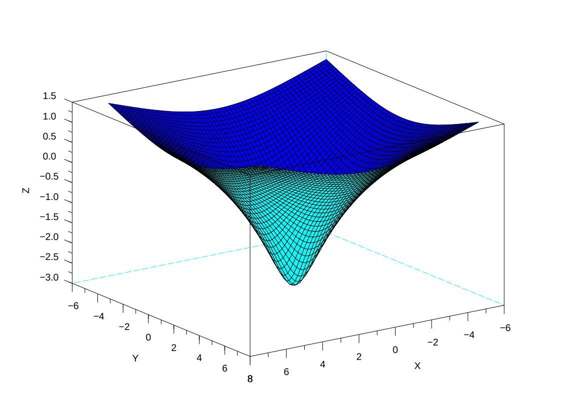
There exists function such that

$$\begin{cases} \{x \in D \subset \mathbb{R}^n \mid f(x) \leq 0\} \text{ is a convex set} \\ \exists x \in D, \nabla^2 f(x) \not\succeq 0 \end{cases}$$



Example

$$f(x, y) = \log(x^2 + y^2 + 1) - 3$$



Let use denote by
S the set $\{x \in D \subset \mathbb{R}^n \mid f(x) \leq 0\}$ and
∂S the set $\{x \in D \subset \mathbb{R}^n \mid f(x) = 0\}$

Theorem

Suppose that

- D is convex,
- S is path-connected,
- $\forall x \in \partial S, \ker \nabla f(x)$ has codimension 1,

$$\text{If } \forall x \in \partial S, \nabla^2 f(x)_{|Ker \nabla f(x)} \succ 0 \text{ then } S \text{ is convex.}$$

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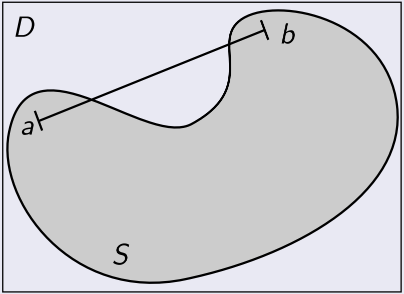
Since D is convex,



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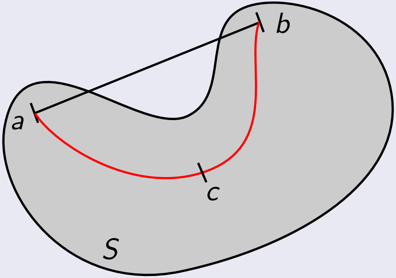

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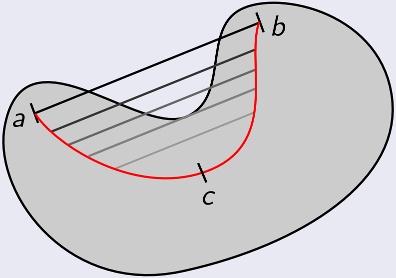

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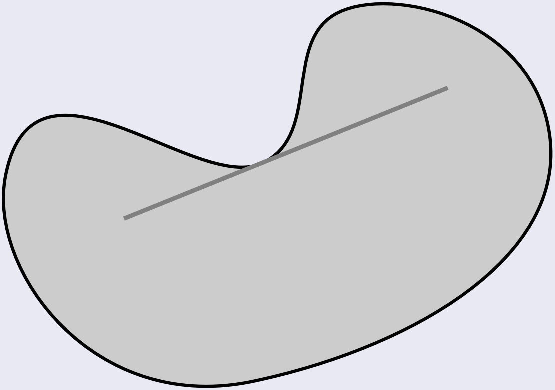
$$\exists x \in \partial S, \exists h \in \ker \nabla f(x), h^T \nabla^2 f(x) h \leq 0$$



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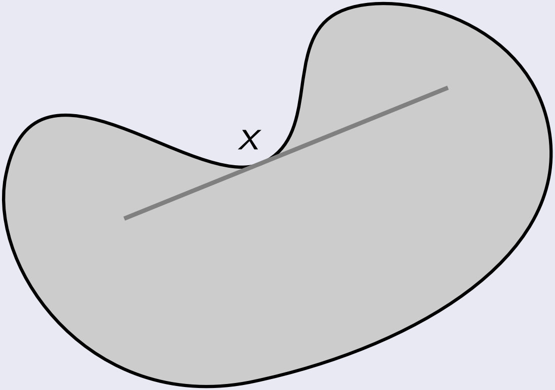

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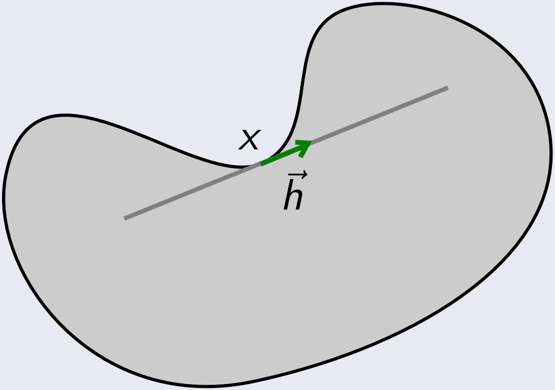

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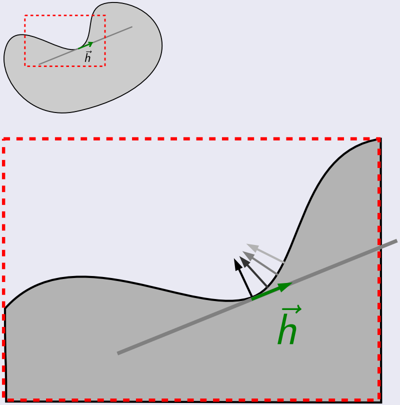

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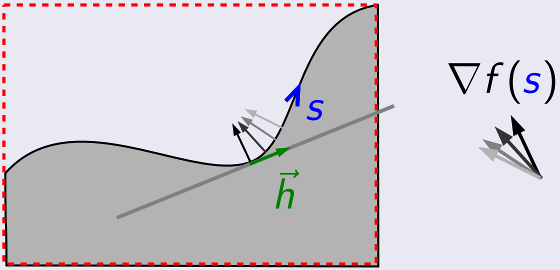

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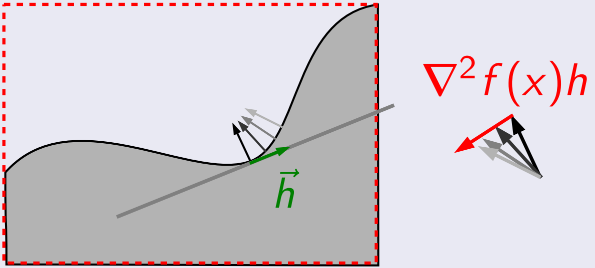
$$\frac{d}{dt}_{|t=0} \nabla f(s(t)) = \frac{d\nabla f(x)}{dx}(s(0)) \frac{d}{dt}_{t=0} s(t) = \nabla^2 f(x) h$$



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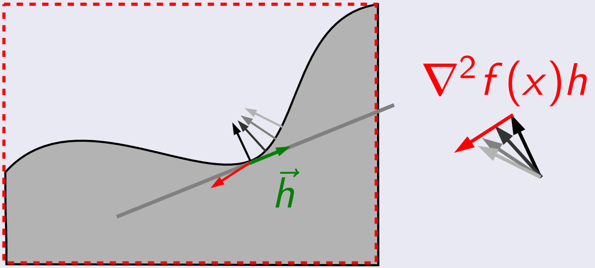

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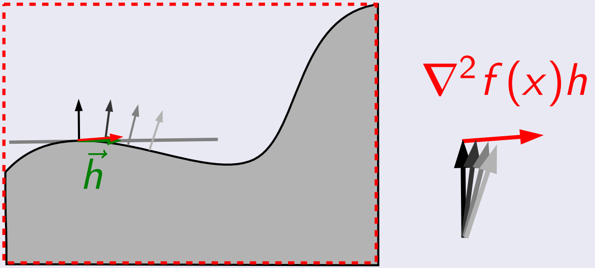

Example

Let f be the function

$$\begin{array}{rccc} f: & [-10,10] & \rightarrow & \mathbb{R} \\ & (x,y) & \mapsto & \log(x^2+y^2+1)-3 \end{array}$$

To prove that $S = \{(x,y) \in [-10,10]^2 \mid f(x,y) \leq 0\}$ is convex, one only has to check that :

$$\begin{cases} S \text{ is path-connected} \\ f(x) = 0 \Rightarrow \nabla f(x) \neq 0 \\ f(x) = 0 \Rightarrow \nabla^2 f(x) \succ 0 \end{cases}$$

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$$\begin{cases} f(x) = 0 \wedge \nabla f(x) = 0 \text{ has no solution} \\ f(x) = 0 \wedge \nabla^2 f(x)_{|\ker \nabla f(x)} \not\succ 0 \text{ has no solution} \end{cases}$$

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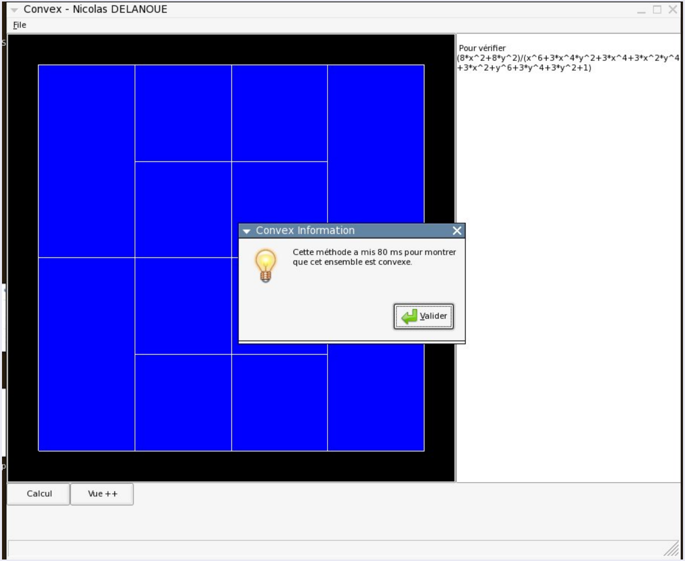
$$\begin{cases} \log(x^2 + y^2 + 1) - 3 = 0 \\ \frac{2x}{y^2+x^2+1} = 0 \\ \frac{2y}{y^2+x^2+1} = 0 \end{cases} \quad \text{has no solution}$$

and

$$\begin{cases} \log(x^2 + y^2 + 1) - 3 = 0 \\ \frac{8x^2+8y^2}{x^6+3x^4y^2+3x^4+3x^2y^4+6x^2y^2+3x^2+y^6+3y^4+3y^2+1} \leq 0 \end{cases} \quad \text{has no solution}$$


Convex - Nicolas DELANOUE
File
Pour vérifier
(8*x^2+8*y^2)/(x^6+3*x^4*y^2+3*x^4+3*x^2*y^4
+3*x^2+y^6+3*y^4+3*y^2+1)
Convex Information
Cette méthode a mis 80 ms pour montrer
que cet ensemble est convexe.
Valider
Calcul
Vue ++

Theorem

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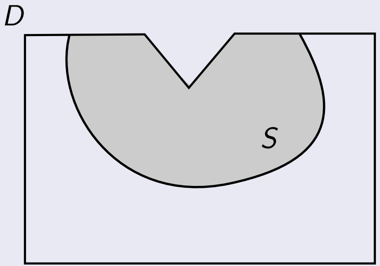
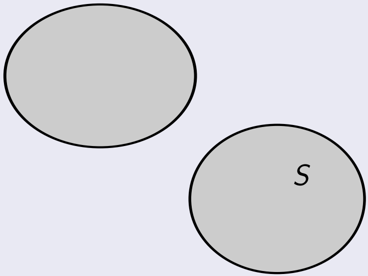
If D is non convex,

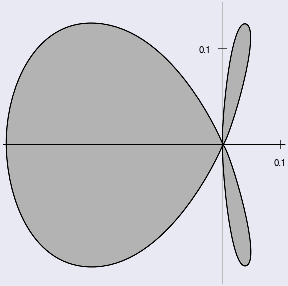


If S is not path-connected,



If $\exists x \in \partial S, \ker \nabla f(x)$ has not codimension 1,

$$\{(x_1, x_2) \mid (x_2^2 + x_1^2)^2 + \frac{x_1 x_2^2}{2} + \frac{x_1^3}{2} \leq 0\}$$



1. CVX : Matlab Software for Disciplined Convex Programming
2. Cylindrical algebraic decomposition

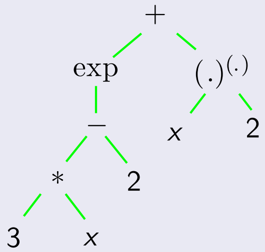
CVX : Matlab Software for Disciplined Convex Programming

- constraint propagation
- symbolic differentiation

- C. Crusius.
  A parser/solver for convex optimization problems.
  PhD thesis, Stanford Univ., USA, 2002.
- M. C. Grant.
  Disciplined convex programming.
  PhD thesis, Stanford Univ., USA, 2004.

CVX : Matlab Software for Disciplined Convex Programming

$x \mapsto f(x) = \exp(3x - 2) + x^2$ is convex since



Cylindrical algebraic decomposition

Tarski's theorem

The solution set of a quantified system of real algebraic equations and inequations is a semialgebraic set (Tarski 1951, Strzebonski 2000).

- Tarski's theorem gives rise to an impractical method (Davenport and Heintz 1988)
- A much more efficient procedure for implementing quantifier elimination is called *cylindrical algebraic decomposition*. It was developed by Collins (1975).

Proposition

If P and Q have a common root then $Res(P, Q) = 0$.

Proof

Let P and Q have a common root a with $\deg P = p$ and $\deg Q = q$.
One has $P(X) = (X - a)\tilde{P}(X)$ and $Q(X) = (X - a)\tilde{Q}(X)$, then :

$$P\tilde{Q} - Q\tilde{P} = 0$$

where $\deg \tilde{P} = p - 1$ and $\deg \tilde{Q} = q - 1$.

Proof

Note that

$$(b_2X^2+b_1X+b_0)(a_2X^2+a_1X+a_0) = c_4X^4+c_3X^3+c_2X^2+c_1X+c_0$$

is equivalent to

$$\begin{pmatrix} b_2 & b_1 & b_0 \end{pmatrix} \begin{pmatrix} a_2 & a_1 & a_1 & 0 & 0 \\ 0 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_2 & a_1 & a_0 \end{pmatrix} = \begin{pmatrix} c_4 & c_3 & c_2 & c_1 & c_0 \end{pmatrix}$$

Proof

$$\exists \tilde{P} \neq 0 \exists \tilde{Q} \neq 0, P\tilde{Q} - Q\tilde{P} = 0$$

is equilavent to det $Syl = 0$ with :

$$Syl = \begin{pmatrix} a_p & \dots & \dots & \dots & \dots & a_0 & 0 & \dots & 0 \\ 0 & \ddots & & & & & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \dots & 0 & a_p & \dots & \dots & \dots & \dots & a_0 \\ b_q & \dots & \dots & \dots & \dots & b_0 & 0 & \dots & 0 \\ 0 & \ddots & & & & & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \dots & 0 & b_q & \dots & \dots & \dots & \dots & b_0 \end{pmatrix}$$

Example

$P(X) = (X-1)(X-2) = X^2 - 3X + 2$
$Q(X) = (X-1)(X-3) = X^2 - 4X + 3$
have common root,

$$\det \begin{pmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 1 & -4 & 3 & 0 \\ 0 & 1 & -4 & 3 \end{pmatrix} = 0$$

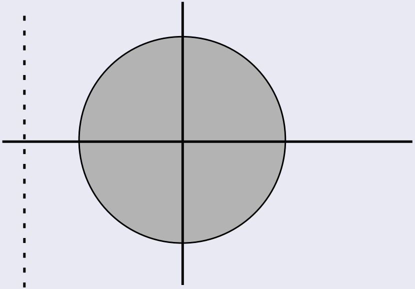
Example

$P(X) = (X-1)(X-2) = X^2 - 3X + 2$
$Q(X) = (X-1)(X-3) = X^2 - 4X + 3$
have common root since

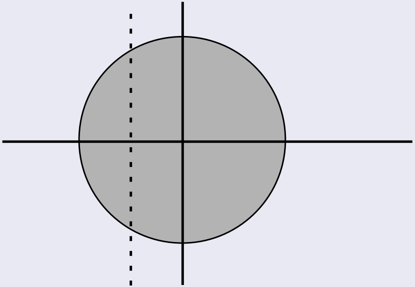
$$\begin{pmatrix} 1 & -3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 1 & -4 & 3 & 0 \\ 0 & 1 & -4 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(X-3)P(X) - (X-2)Q(X) = 0$$

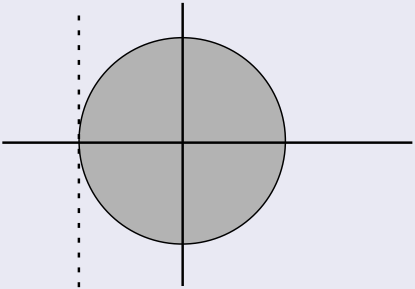
$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \leq 0$$



$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \leq 0$$

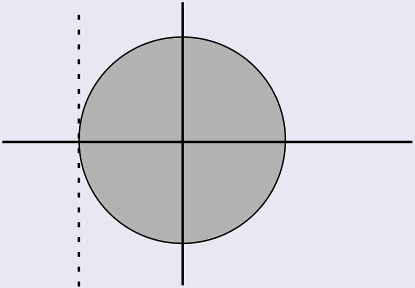


$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \leq 0$$



$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \leq 0$$

$y \mapsto P(x, y)$ and $y \mapsto \frac{d}{dy} P(x, y)$ has a common root.
i.e. $1 * Y^2 + 0 * Y^1 + (X^2 - 1) * Y^0$ and $2 * Y^1$ has a common root.



$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \leq 0$$

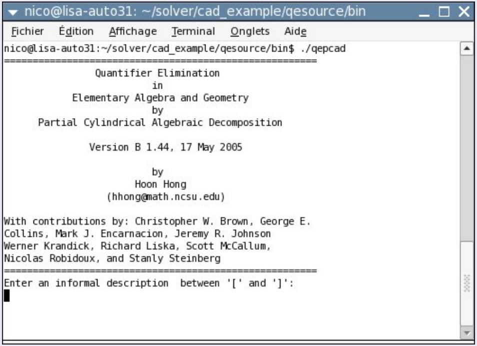
$1 * Y^2 + 0 * Y^1 + (X^2 - 1) * Y^0$ and $2 * Y^1$ has a common root.

$$\det \begin{pmatrix} 1 & 0 & X^2 - 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = 0$$

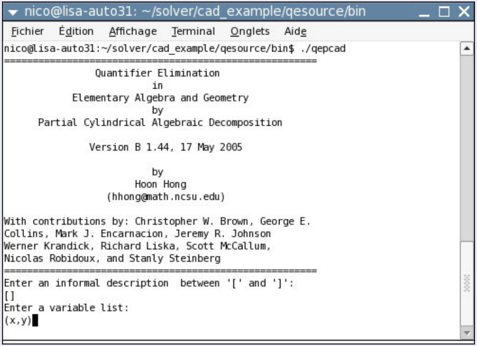
$$\Leftrightarrow (-2)(-2)(X^2 - 1) = 0$$

$$\Leftrightarrow (X^2 - 1) = 0$$

$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \leq 0$$



$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \leq 0$$



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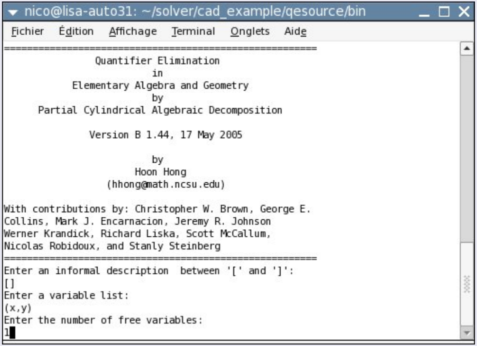

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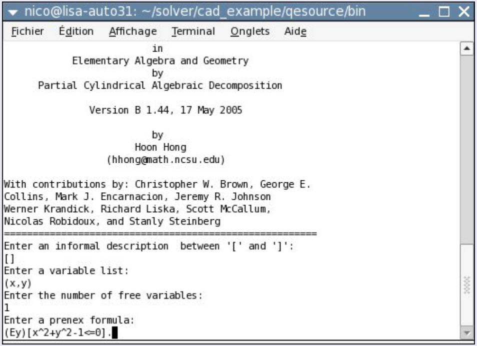

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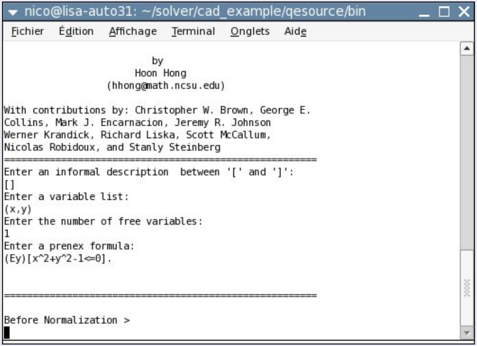

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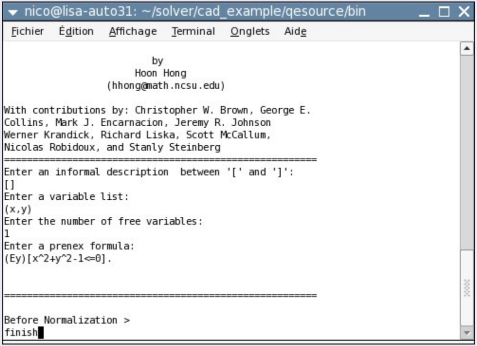
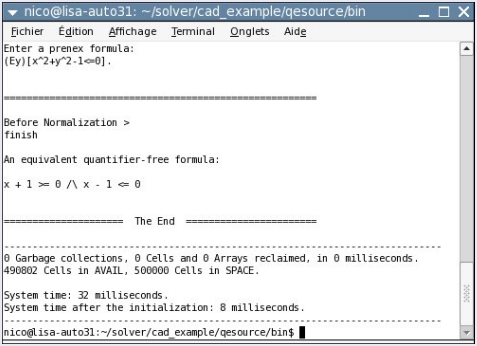


$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \leq 0$$



$$\exists y \in \mathbb{R}, x^2 + y^2 - 1 \leq 0$$

$$\Leftrightarrow (x + 1 \geq 0) \wedge (x - 1 \leq 0)$$

Proposition

Convexity is a semi-algebraic property.

Let S be a semi-algebraic set, S is convex if

$$\forall x \forall y \forall t, (t \in [0,1], x \in S, y \in S \Rightarrow tx + (1-t)y \in S)$$

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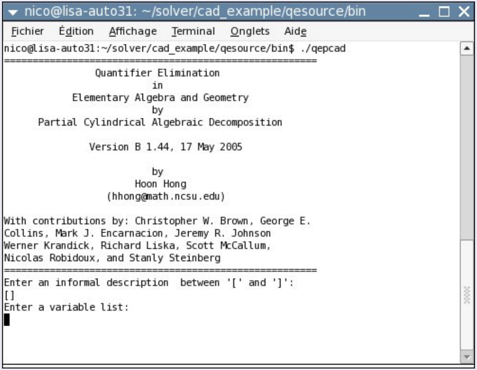
$$\forall x \forall y \forall t, (t \in [0,1], x \in S, y \in S \Rightarrow tx + (1-t)y \in S)$$

Example

Proving that $\{x \in \mathbb{R}, x^2 - 1 \leq 0\} = [-1, 1]$ is convex remains to prove :

$$\forall x \forall y \forall t, 0 \leq t \leq 1 \wedge x^2 - 1 \leq 0 \wedge y^2 - 1 \leq 0 \Rightarrow (tx + (1-t)y)^2 - 1 \leq 0$$

$\{x \in \mathbb{R}, x^2 - 1 \leq 0\} = [-1, 1]$ is convex.



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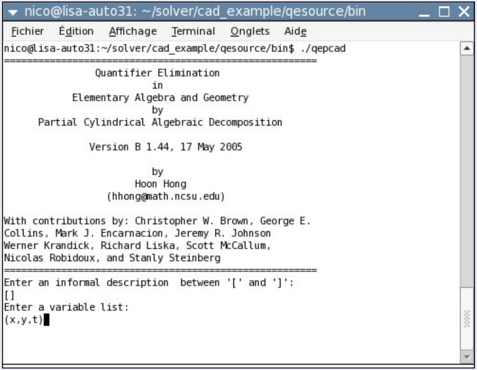


```
nico@lisa-auto31:~/solver/cad_example/qesource/bin$ ./qepcad
=======================================================
                Quantifier Elimination
                          in
            Elementary Algebra and Geometry
                          by
      Partial Cylindrical Algebraic Decomposition

               Version B 1.44, 17 May 2005

                          by
                       Hoon Hong
                  (hhong@math.ncsu.edu)

With contributions by: Christopher W. Brown, George E.
Collins, Mark J. Encarnacion, Jeremy R. Johnson
Werner Krandick, Richard Liska, Scott McCallum,
Nicolas Robidoux, and Stanly Steinberg
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Enter an informal description  between '[' and ']':
[]
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(x,y,t)
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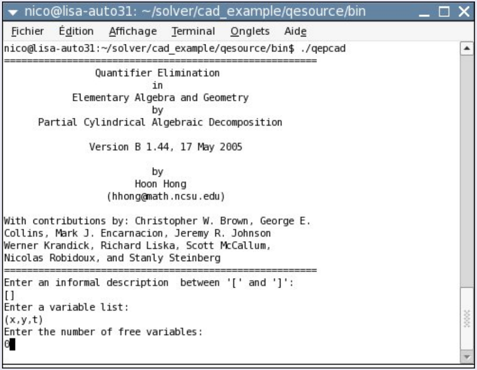
$\{x \in \mathbb{R}, x^2 - 1 \leq 0\} = [-1, 1]$ is convex.



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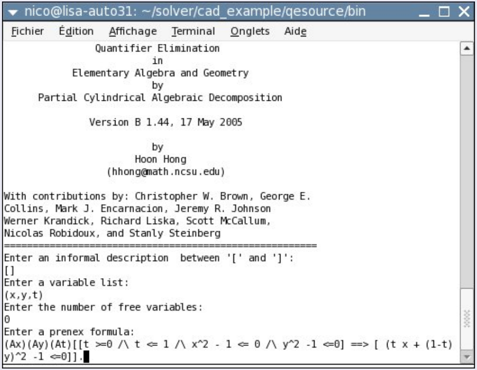
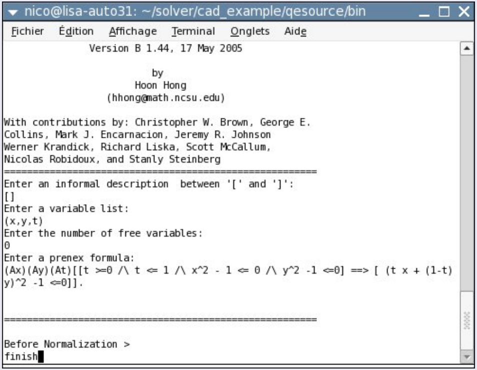
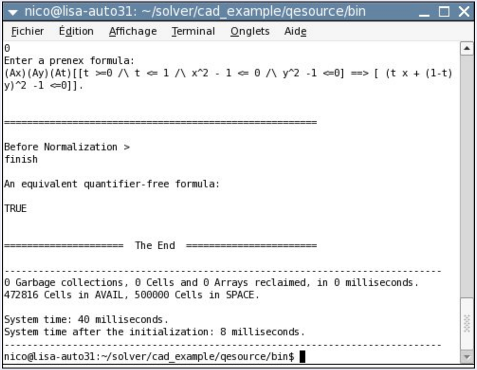


$\{x \in \mathbb{R}, x^2 - 1 \leq 0\} = [-1, 1]$ is convex.



$\{x \in \mathbb{R}, x^2 - 1 \leq 0\} = [-1, 1]$ is convex.



Conclusion

Let use denote by
S the set $\{x \in D \subset \mathbb{R}^n \mid f(x) \leq 0\}$ and
∂S the set $\{x \in D \subset \mathbb{R}^n \mid f(x) = 0\}$

Theorem

Suppose that

- D is convex,
- S is path-connected,
- $\forall x \in \partial S, \ker \nabla f(x)$ has codimension 1,

If $\forall x \in \partial S, \nabla^2 f(x)_{|Ker \nabla f(x)} \succ 0$ then S is convex.

Future works

- Prove that the proposed algorithm terminates in the generic case.
- Expand this method to cases where f_i depends of unknown parameters.