A new method for integrating ODE based on monotonicity

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Main goal

Computing an $\epsilon$ approximation of the smallest box containing the solution at $t$ of the initial value problem $\dot{x} = f(x), x(0) \in [x]$
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Known results
Interval analysis, optimal inclusion function
Computing optimal validated solutions for ODE

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Computing an \( \epsilon \) approximation of the smallest box containing the solution at \( t \) of the initial value problem \( \dot{x} = f(x), x(0) \in [x] \)
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   - Wrapping effect

2 Interval analysis, optimal inclusion function
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   - Optimal inclusion function

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   - ODE, Dynamical system and flow
   - Derivative of the flow with respect to initial condition
   - Algorithm
There exists guaranteed *convergent* methods to find $x(t)$ such that

\[
\begin{cases}
\dot{x} = f(x) \\
x(0) = x_0
\end{cases}
\]

- Picard-Lindelöf operator,
- Taylor models,
- Automatic differentiation.
There exists guaranteed *convergent* methods to find $x(t)$ such that

$$
\begin{cases}
\dot{x} &= f(x) \\
x(0) &= x_0
\end{cases}
$$

$\Delta t = 1/6,$
There exists guaranteed *convergent* methods to find $x(t)$ such that

$$\begin{cases}
\dot{x} = f(x) \\
x(0) = x_0
\end{cases}$$

$\Delta t = 1/7$
There exists guaranteed *convergent* methods to find $x(t)$ such that

\[
\begin{align*}
\dot{x} &= f(x) \\
x(0) &= x_0
\end{align*}
\]

$\Delta t = 1/8$
There exists guaranteed *convergent* methods to find $x(t)$ such that

\[
\begin{aligned}
\dot{x} &= f(x) \\
x(0) &= x_0
\end{aligned}
\]

$\Delta t = 1/9$
There exists guaranteed *convergent* methods to find $x(t)$ such that

$$\begin{cases}
\dot{x} &= f(x) \\
x(0) &= x_0
\end{cases}$$

$\Delta t = 1/10$
There exists guaranteed *convergent* methods to find $x(t)$ such that

$$\begin{cases}
\dot{x} &= f(x) \\
x(0) &= x_0
\end{cases}$$

$\Delta t = 1/11$
There exists guaranteed *convergent* methods to find $x(t)$ such that
\[
\begin{aligned}
\dot{x} &= f(x) \\
x(0) &= x_0
\end{aligned}
\]

$\Delta t = 1/12$
There exists guaranteed *convergent* methods to find $x(t)$ such that

\[
\begin{align*}
\dot{x} &= f(x) \\
 x(0) &= x_0
\end{align*}
\]
There exists guaranteed *convergent* methods to find $x(t)$ such that

\[
\begin{cases}
\dot{x} = f(x) \\
x(0) = x_0
\end{cases}
\]
There exists guaranteed *convergent* methods to find \( x(t) \) such that

\[
\begin{align*}
\dot{x} &= f(x) \\
 x(0) &= x_0
\end{align*}
\]

\( \Delta t = 1/15 \)
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**Definition**

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \).

A function \([f] : \mathbb{I}\mathbb{R}^n \rightarrow \mathbb{I}\mathbb{R}^m\) satisfying:

\[ \forall [x] \in \mathbb{I}\mathbb{R}^n, f([x]) \subset [f]([x]) \]

is an inclusion function of \( f \).

**Figure:** Illustration of inclusion function.
remark

- Interval arithmetic gives a method to compute an inclusion function of a given function defined by an arithmetical expression.
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- Interval arithmetic gives a method to compute an inclusion function of a given function defined by an arithmetical expression.

- In general, the smallest inclusion function is not obtained and one only has: $f([x]) \not\subseteq [f([x])]$. 
**Definition**

Let $[x]$ be a box of $\mathbb{R}^n$ and $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$. Let us denote by $f_*(x)$ the jacobian matrix

$$
\begin{pmatrix}
\frac{\partial f}{\partial x_1}(x) & \cdots & \frac{\partial f}{\partial x_n}(x)
\end{pmatrix}
$$

**Theorem**

Suppose that all components of $f_*([x])$ are non-negative, then $[f(x), f(\bar{x})]$ is the range of $[x]$ under $f$. 
Example

Let us consider the function $f : (x_1, x_2) \mapsto 3x_1^2 - 2x_1x_2 + 3x_2^2$.

**Figure**: Level curves.
Example

Let us consider the function $f : (x_1, x_2) \mapsto 3x_1^2 - 2x_1x_2 + 3x_2^2$.

**Figure:** Level curves.
Example $f : (x_1, x_2) \mapsto 3x_1^2 - 2x_1x_2 + 3x_2^2$

Since $f_*(x_1, x_2) = (6x_1 - 2x_2, -2x_1 + 6x_2)$, one has

$$\{f_*(x_1, x_2) \mid (x_1, x_2) \in [3, 4] \times [3, 4]\} \subset \mathbb{R}^+ \times \mathbb{R}^+.$$ 

**Figure:** Level curves.
Example $f : (x_1, x_2) \mapsto 3x_1^2 - 2x_1x_2 + 3x_2^2$

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Example $f : (x_1, x_2) \mapsto 3x_1^2 - 2x_1x_2 + 3x_2^2$

Since $f_\ast(x_1, x_2) = (6x_1 - 2x_2, -2x_1 + 6x_2)$, one has

$$\{f_\ast(x_1, x_2) \mid (x_1, x_2) \in [3, 4] \times [3, 4]\} \subset \mathbb{R}^+ \times \mathbb{R}^+.$$

**Figure**: Level curves.
Example $f : (x_1, x_2) \mapsto 3x_1^2 - 2x_1x_2 + 3x_2^2$

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Since $f_*(x_1, x_2) = (6x_1 - 2x_2, -2x_1 + 6x_2)$, one has

$$\left\{ f_*(x_1, x_2) \mid (x_1, x_2) \in [3, 4] \times [3, 4] \right\} \subset \mathbb{R}^+ \times \mathbb{R}^+.$$
Example $f : (x_1, x_2) \mapsto 3x_1^2 - 2x_1x_2 + 3x_2^2$

\textbf{Figure:} Level curves.

According to the previous theorem, one can conclude that $f([3, 4] \times [3, 4]) = [f(3, 3), f(4, 4)] = [36, 52]$. 
Example $f : (x_1, x_2) \mapsto 3x_1^2 - 2x_1x_2 + 3x_2^2$

Figure: Level curves.

- According to the previous theorem, one can conclude that $f([3, 4] \times [3, 4]) = [f(3, 3), f(4, 4)] = [36, 52]$.
- This result can be compared to the one obtained applying interval arithmetic: $3 \times [3, 4]^2 - 2 \times [3, 4] \times [3, 4] + 3 \times [3, 4]^2$, i.e. $[22, 78]$. 
Corollary

Let $[x]$ be a box of $\mathbb{R}^n$ and $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$. Let us denote by $f_\star(x)$ the jacobian matrix

$$
\left( \frac{\partial f_j}{\partial x_j}(x) \right)_{1 \leq i \leq n, 1 \leq j \leq n}
$$

Suppose that no component of $f_\star([x])$ contains 0, then there exists $2m$ corners $\tilde{x}_j$ and $\bar{x}_j$ of $[x]$ such that $\prod_{1 \leq j \leq n}[f_j(\tilde{x}_j), f_j(\bar{x}_j)]$ is the smallest box containing $f([x])$.

Proof

Apply $m$ times the previous theorem.
Example

\[ f : \mathbb{R}^2 \to \mathbb{R}^2 \]

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\mapsto
\begin{pmatrix}
  f_1 \\
  f_2
\end{pmatrix}
\]

where

\[
\begin{pmatrix}
  f_1 \\
  f_2
\end{pmatrix}
= \begin{pmatrix}
  x_1^2 - x_2 \\
  x_1 + x_1 x_2
\end{pmatrix}
\]
Example

\[ f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]

\[
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\mapsto
\begin{pmatrix}
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\end{pmatrix}
\]

where

\[
\begin{pmatrix}
    f_1 \\
    f_2
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\begin{pmatrix}
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\end{pmatrix}
\]
**Example**

\[ f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]

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  x_2
\end{pmatrix}
\mapsto
\begin{pmatrix}
  f_1 \\
  f_2
\end{pmatrix}
\]

where \[
\begin{pmatrix}
  f_1 \\
  f_2
\end{pmatrix} = \begin{pmatrix}
  x_1^2 - x_2 \\
  x_1 + x_1x_2
\end{pmatrix}
\]
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Example

\[ f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}
\]

where
\[
\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2 \\ x_1 + x_1x_2 \end{pmatrix}
\]
Example

\[ f_\ast(x_1, x_2) = \begin{pmatrix} 2x_1 & -1 \\ 1 + x_2 & x_1 \end{pmatrix} = \begin{pmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{pmatrix} \]
Example

\[ f_*(x_1, x_2) = \begin{pmatrix} 2x_1 & -1 \\ 1 + x_2 & x_1 \end{pmatrix} = \begin{pmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{pmatrix} \]
Example

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**Example**

\[ f_*(x_1, x_2) = \begin{pmatrix} 2x_1 & -1 \\ 1 + x_2 & x_1 \end{pmatrix} = \begin{pmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{pmatrix} \]

\[ \left( \begin{array}{c} f_1(x_1, x_2), f_1(x_1, x_2) \\ f_2(x_1, x_2), f_2(x_1, x_2) \end{array} \right) \]

is the smallest box containing

\[ f([x_1, x_1], [x_2, x_2]) \]
\[ \begin{cases} \dot{x} = f(x) \\ x \in \mathbb{R}^n \end{cases}, f \in C^\infty(\mathbb{R}^n, \mathbb{R}^n). \]
Definition

Let us denote by \( \{g^t : \mathbb{R}^n \rightarrow \mathbb{R}^n \}_{t \in \mathbb{R}} \) the flow associated to the vector field \( f : \)

\[
\frac{d}{dt} g^t \bigg|_{t=0} x = f(x) \text{ and } g^0 = Id
\]  (1)

Note that \( t \mapsto g^t x \) is the solution of \( \dot{x} = f(x) \) satisfying \( x(0) = x \).
Definition

Let us denote by \( \{g^t : \mathbb{R}^n \rightarrow \mathbb{R}^n\}_{t \in \mathbb{R}} \) the flow associated to the vector field \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \):

\[
\frac{d}{dt} g^t x \bigg|_{t=0} = f(x) \text{ and } g^0 = I_d
\]  

(1)

Note that \( t \mapsto g^t x \) is the solution of \( \dot{x} = f(x) \) satisfying \( x(0) = x \).
Remark

For a fixed $t$, $g^t$ is a function from $\mathbb{R}^n \rightarrow \mathbb{R}^n$
According to the previous theorem, if no component of $g^t([x])$ contains 0, then there exists $2n$ corners $\tilde{x}_j$ and $\bar{x}_j$ of $[x]$ such that $\prod_{1 \leq j \leq n}[g^t_j(\tilde{x}_j), g^t_j(\bar{x}_j)]$ is the smallest box containing $g^t([x])$. 
Example

Let us consider the following ODE:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}
\]
Example

One can obtain an explicit solution:

\[
\begin{pmatrix}
  x_1(t) \\
  x_2(t)
\end{pmatrix} = \exp(tA) \begin{pmatrix}
  x_1(0) \\
  x_2(0)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  e^t \cos(t) & e^t \sin(t) \\
  -e^t \sin(t) & e^t \cos(t)
\end{pmatrix} \begin{pmatrix}
  x_1(0) \\
  x_2(0)
\end{pmatrix}
\]
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\begin{pmatrix}
    x_1(t) \\
    x_2(t)
\end{pmatrix}
= \exp(tA)
\begin{pmatrix}
    x_1(0) \\
    x_2(0)
\end{pmatrix}
= 
\begin{pmatrix}
    e^t \cos(t) & e^t \sin(t) \\
    -e^t \sin(t) & e^t \cos(t)
\end{pmatrix}
\begin{pmatrix}
    x_1(0) \\
    x_2(0)
\end{pmatrix}
\]

\[
g^t : \mathbb{R}^2 \rightarrow \mathbb{R}^2
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\mapsto 
\begin{pmatrix}
    e^t \cos(t) x_1 + e^t \sin(t) x_2 \\
    -e^t \sin(t) x_1 + e^t \cos(t) x_2
\end{pmatrix}
\]
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Algorithm $g^1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\mapsto
\begin{pmatrix}
  e^1 \cos(1) x_1 + e^1 \sin(1) x_2 \\
  -e^1 \sin(1) x_1 + e^1 \cos(1) x_2
\end{pmatrix}
\]

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$g^1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\mapsto
\begin{pmatrix}
  e^1 \cos (1) x_1 + e^1 \sin (1) x_2 \\
  -e^1 \sin (1) x_1 + e^1 \cos (1) x_2
\end{pmatrix}
\]

$g^1_* = \left( \begin{array}{cc}
\frac{\partial g^1_1}{\partial x_1} & \frac{\partial g^1_1}{\partial x_2} \\
\frac{\partial g^1_2}{\partial x_1} & \frac{\partial g^1_2}{\partial x_2}
\end{array} \right)$

$\approx \left( \begin{array}{cc}
e^1 \cos (1) & e^1 \sin (1) \\
-e^1 \sin (1) & e^1 \cos (1)
\end{array} \right)$

$\approx \left( \begin{array}{cc}
1.468693940 & 2.287355287 \\
-2.287355287 & 1.468693940
\end{array} \right)$
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\[ g_1^1 \simeq \begin{pmatrix} 1.468693940 & 2.287355287 \\ -2.287355287 & 1.468693940 \end{pmatrix} \]

According to the previous theorem:

\[ \left[ g_1^1 \left( x_1, x_2 \right) ; g_1^2 \left( x_1, x_2 \right) \right] = \text{the smallest box containing} \]

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\[ g^1_\ast \simeq \begin{pmatrix}
1.468693940 & 2.287355287 \\
-2.287355287 & 1.468693940
\end{pmatrix} \]

According to the previous theorem:

\[
\begin{pmatrix}
\begin{bmatrix}
g_1^1(x_1, x_2) & g_1^1(\overline{x}_1, \overline{x}_2)
\end{bmatrix} \\
\begin{bmatrix}
g_2^1(\overline{x}_1, \overline{x}_2) & g_2^1(x_1, \overline{x}_2)
\end{bmatrix}
\end{pmatrix}
\]

is the smallest box containing \(g^1\left(\begin{bmatrix}x_1 & \overline{x}_1 \\ x_2 & \overline{x}_2\end{bmatrix}\right)\)
( \[
\begin{bmatrix}
g_{11}(x_1, x_2) & g_{11}(\bar{x}_1, \bar{x}_2) \\
g_{21}(\bar{x}_1, x_2) & g_{21}(x_1, \bar{x}_2)
\end{bmatrix}
\] )
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\[
\begin{pmatrix}
g_1^1(x_1, x_2) & g_1^1(\bar{x}_1, \bar{x}_2) \\
g_2^1(\bar{x}_1, \bar{x}_2) & g_2^1(x_1, \bar{x}_2)
\end{pmatrix}
\]

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Theorem

Suppose that \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is twice continuously differentiable. Then \( g^t_* \) is solution to the initial value problem

\[
\frac{\partial}{\partial t} g^t_*(x) = f_*(g^t_*)g^t_*(x), \\
g^0_*(x) = Id
\]
Theorem

Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is twice continuously differentiable. Then $g^t_\ast$ is solution to the initial value problem

$$\frac{\partial}{\partial t} g^t_\ast(x) = f_\ast(g^t x) g^t_\ast(x),$$

$$g^0_\ast(x) = Id$$

Proof

$$\frac{\partial}{\partial t} g^t_\ast(x) = \frac{\partial}{\partial t} \frac{d}{dx} g^t x$$

$$= \frac{d}{dx} \frac{\partial}{\partial t} g^t x$$

$$= \frac{d}{dx} f(g^t x)$$

$$= f_\ast(g^t x)(g^t x)$$
Example

\[
\begin{align*}
\dot{x}_1 &= x_2 - x_1 \\
\dot{x}_2 &= -x_1 + x_1x_2
\end{align*}
\]

Let us denote by \((a_{i,j})_{1 \leq i,j \leq 2}\) the coordinate of \(g^t_*, \) i.e.

\[
g^t_* = \begin{pmatrix}
\partial_{x_1} g_1^t & \partial_{x_2} g_1^t \\
\partial_{x_1} g_2^t & \partial_{x_2} g_2^t
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\]

one has:

\[
\begin{pmatrix}
\dot{a}_{11} & \dot{a}_{12} \\
\dot{a}_{21} & \dot{a}_{22}
\end{pmatrix} = \begin{pmatrix}
-1 & 1 \\
-1 + x_2 & x_1
\end{pmatrix} \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\]
Example - $n + n^2$ dimensional initial value problem

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= (1 - x_1^2)x_2 - x_1 \\
\dot{a}_{11} &= -a_{11} + a_{21} \\
\dot{a}_{12} &= -a_{12} + a_{22} \\
\dot{a}_{21} &= (-1 + x_2)a_{11} + x_1a_{21} \\
\dot{a}_{22} &= (-1 + x_2)a_{12} + x_1a_{22}
\end{align*}
\]

with the following initial condition

\[
\begin{align*}
x_1(0) &= x_1^0 \\
x_2(0) &= x_2^0 \\
\begin{pmatrix}
a_{11}(0) & a_{12}(0) \\
a_{21}(0) & a_{22}(0)
\end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{align*}
\]
Example - $n + n^2$ dimensional initial value problem
Algorithm

- **Input:**

$$\dot{x} = f(x)$$

$$x \in \mathbb{R}^n$$

$t$ a real, $\epsilon$ a real.
Known results
Interval analysis, optimal inclusion function
Computing optimal validated solutions for ODE

ODE, Dynamical system and flow
Derivative of the flow with respect to initial condition
Algorithm

Algorithm

- **Input**: 
  - \( \dot{x} = f(x) \)
Algorithm

- **Input:**
  - $\dot{x} = f(x)$
  - $[x] \in \mathbb{IR}^n$
Algorithm

- **Input:**
  - $\dot{x} = f(x)$
  - $[x] \in \mathbb{IR}^n$
  - $t$ a real, $\epsilon$ a real.
**Algorithm**

- **Input:**
  - $\dot{x} = f(x)$
  - $[x] \in \mathbb{IR}^n$
  - $t$ a real, $\epsilon$ a real.

- **Main steps**
Algorithm

Input:
- $\dot{x} = f(x)$
- $[x] \in \mathbb{R}^n$
- $t$ a real, $\epsilon$ a real.

Main steps
- Compute rigorously $g^t([x])$ (with non a convergent method),
Algorithm

- **Input**:
  - \( \dot{x} = f(x) \)
  - \([x] \in \mathbb{IR}^n\)
  - \(t\) a real, \(\epsilon\) a real.

- **Main steps**
  - Compute rigorously \(g^t_*(\{x\})\) (with non a convergent method),
  - if no component of \(g^t_*(\{x\})\) contains 0,
Algorithm

- **Input:**
  - $\dot{x} = f(x)$
  - $[x] \in \mathbb{IR}^n$
  - $t$ a real, $\epsilon$ a real.

- **Main steps**
  - Compute rigorously $g^t_\ast([x])$ (with non a convergent method),
  - if no component of $g^t_\ast([x])$ contains 0,
  - then rigorously integrate the $2n$ initial value problems
    $\dot{x} = f(x), x(0) = \tilde{x},$
Algorithm

- **Input:**
  - $\dot{x} = f(x)$
  - $[x] \in \mathbb{IR}^n$
  - $t$ a real, $\epsilon$ a real.

- **Main steps**
  - Compute rigorously $g^t([x])$ (with non a convergent method),
  - if no component of $g^t([x])$ contains 0,
  - then rigorously integrate the $2n$ initial value problems $\dot{x} = f(x)$, $x(0) = \tilde{x}$,
  - return the interval hull of those results.
Convergence

This method is convergent as soon as no component of \( g_t([x]) \) contains 0.
Convergence

This method is convergent as soon as no component of $g^*(x)$ contains 0.

**Figure:** $a_{11}, a_{12}, a_{21}, a_{22}$ as function of $t$. 
Convergence

This method is convergent as soon as no component of $g^t([x])$ contains 0.

\[ a_{11}, a_{12}, a_{21}, a_{22} \text{ as function of } t. \]
Convergence

This method is convergent as soon as no component of $g^*_t([x])$ contains 0.

Figure: $a_{11}, a_{12}, a_{21}, a_{22}$ as function of $t$. 
Implementation in C++ based on the interval library filib,
- Implementation in C++ based on the interval library filib,
- Adomian decomposition method,
- Implementation in C++ based on the interval library filib,
- Adomian decomposition method,
- Merci pour votre attention!
• Implementation in C++ based on the interval library filib,
• Adomian decomposition method,
• Merci pour votre attention!
• Vladimir Arnold.