Optimal Control Via Occupation Measures and Interval Analysis

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Outline

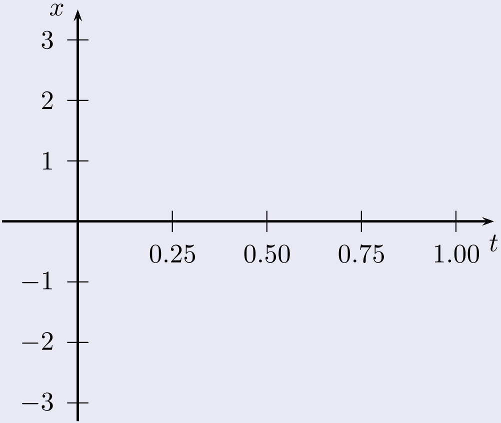
Controlled dynamical system

$$\begin{cases} x(0) & = & x_0 \\ \dot{x}(\tau) & = & f(x(\tau), u(\tau)), \forall \tau \in [0, T], \end{cases}$$

where

- τ is the time,
- x is the state,
- f is a vector field (the dynamics),
- u is the control.

$$\begin{cases} x(0) & = & 2 \\ \dot{x}(\tau) & = & u(\tau), \tau \in [0,1], \\ u(\tau) & \in & [-2,2]. \end{cases}$$



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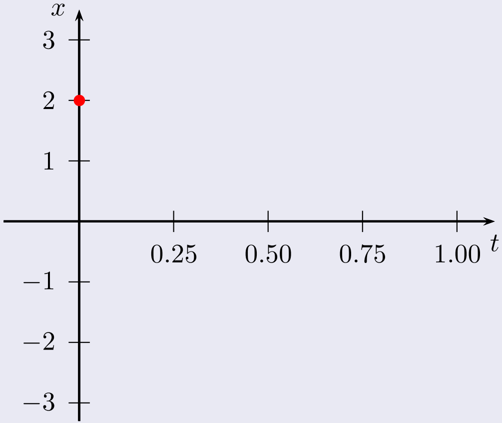

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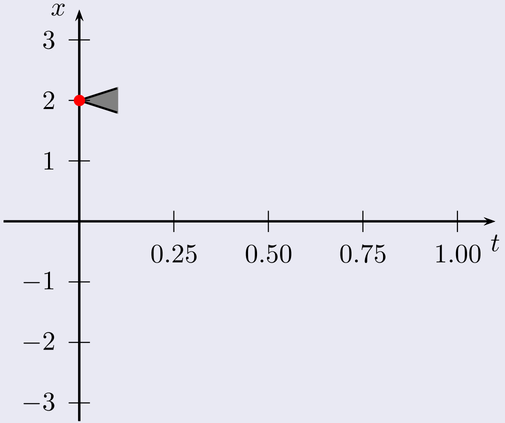

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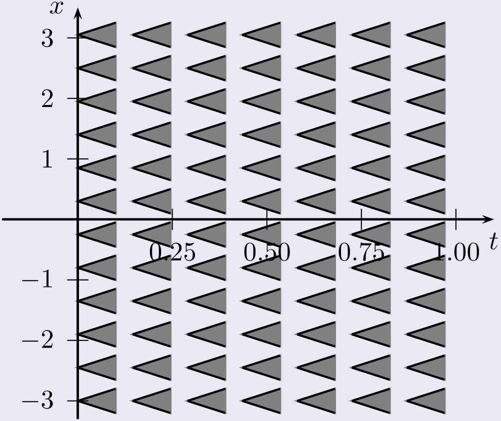

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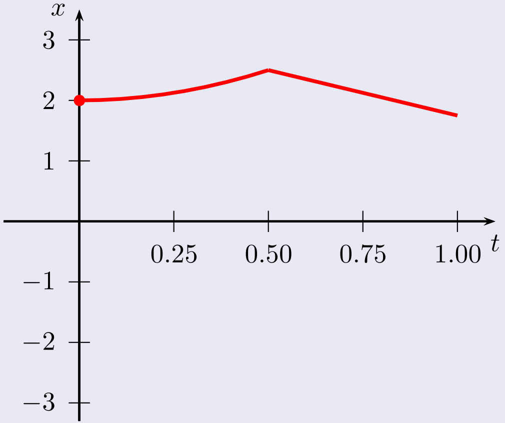

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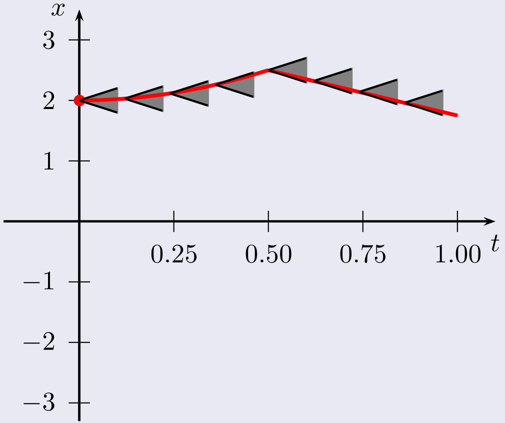

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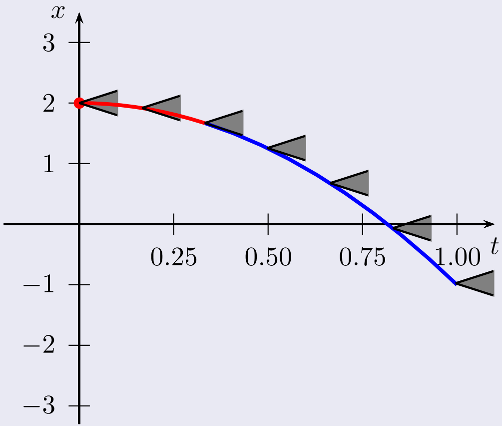

Optimal control problem

$$
\begin{aligned}
J^* = \min_{u:[0,T]\to U} \quad & \int_0^T h(\tau, x(\tau), u(\tau))d\tau + H(x(T)) \\
\text{subject to} \quad & x(0) = x_0 \\
& \dot{x}(\tau) = f(x(\tau), u(\tau)), \forall \tau \in [0, T], \\
& x(\tau) \in X, \forall \tau \in [0, T], \\
& x(T) \in K.
\end{aligned}
$$

where

- U is the set of admissible control,
- h and H are real valued functions.

Definition

Let X be a set and Σ a $\sigma-$algebra over X.
A function $\mu : \Sigma \to \mathbb{R} \cup \{\infty\}$ is called a *measure* if :

- $\forall E \in \Sigma, \mu(E) \geq 0$,
- $\mu(\emptyset) = 0$,
- For all countable collections $\{E_i\}_{i \in \mathbb{N}}$ of pairwise disjoint sets in Σ, one has :

$$\mu\left(\bigcup_{i \in \mathbb{N}} E_i\right) = \sum_{i \in \mathbb{N}} \mu\left(E_i\right).$$

Examples

- The Lebesgue measure λ on $\mathbb{R}$ defined by

$$\lambda([a, b]) = b - a.$$

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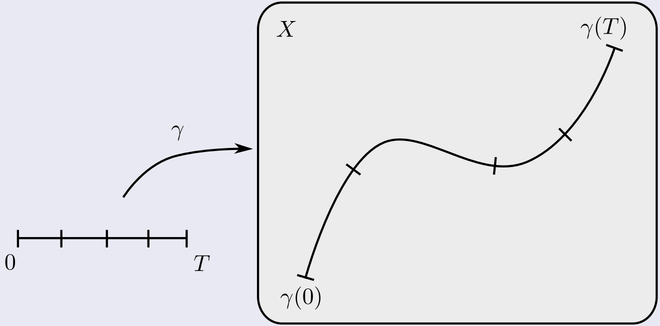
- Let f be a non negative function on $\mathbb{R}$, one can defined a measure λ_f on $\mathbb{R}$ by

$$\lambda_f(E) = \int_E f(x)dx.$$

Occupation measure μ_γ on X

Let $\gamma : [0, T] \mapsto X$ be a parametrized curve, let us define μ_γ by

$$\mu_\gamma(E) = \int_0^T 1_E(\gamma(\tau))d\tau \text{ where } 1_E(x) = \begin{cases} 1 \text{ if } x \in E, \\ 0 \text{ otherwise.} \end{cases}$$



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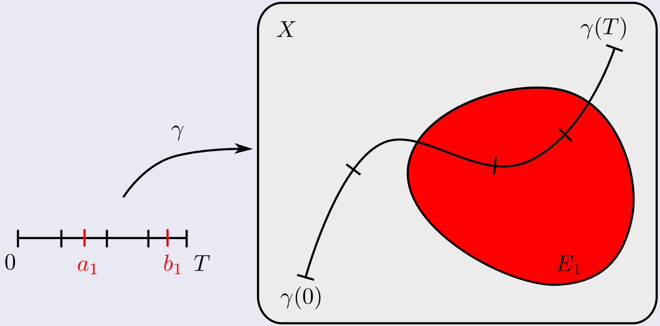

$\mu_\gamma(E_1) = b_1 - a_1$

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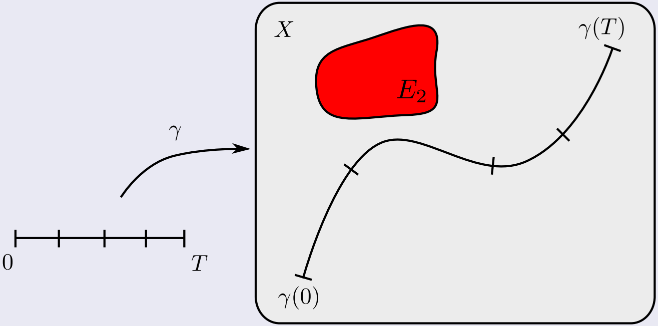


$\mu_\gamma(E_2) = 0$

Proposition

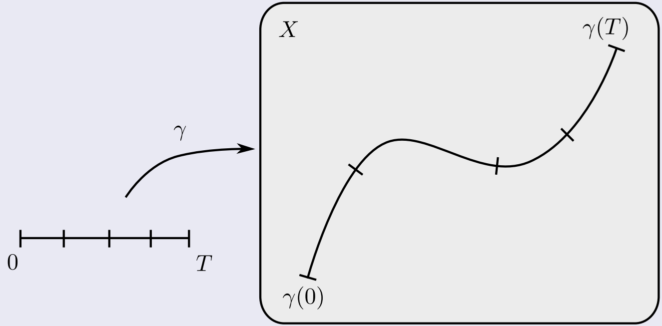
Let $h : X \mapsto \mathbb{R}$ be a function and $\gamma : [0, T] \mapsto X$ be a parametrized curve, then

$$\langle \mu, h \rangle = \int_X h(x) d\mu_\gamma = \int_0^T h(\gamma(\tau)) d\tau$$

Occupation measure ν_γ on K at time T

Let $\gamma : [0, T] \mapsto X$ be a parametrized curve, let us define μ_γ by

$$\nu_\gamma(E) = \delta_{\gamma(T)}(E)$$



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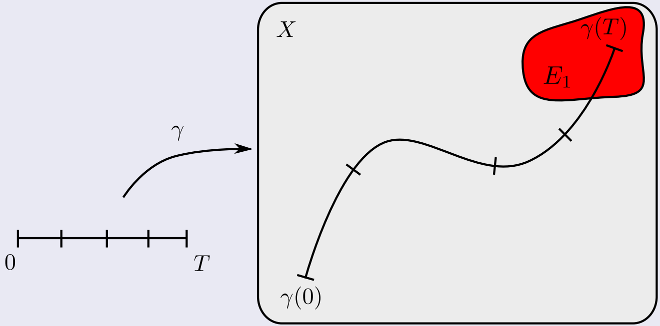

$\nu_\gamma(E_1) = 1$

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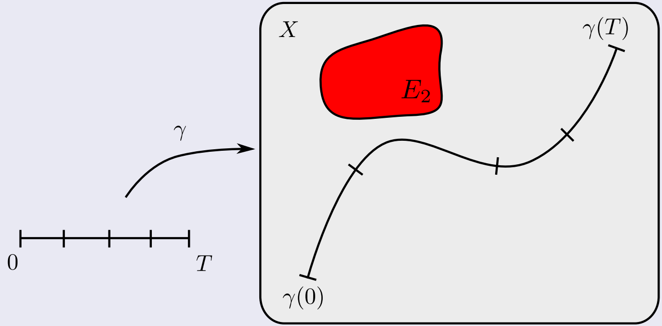

$\nu_\gamma(E_2) = 0$

Theorem

Let

$$J_1^* = \min_{u:[0,T]\to U} \int_0^T h(\tau, x(\tau), u(\tau))d\tau + H(x(T))$$

$$\text{subject to} \quad x(0) = x_0, \dot{x} = f(x,u), \forall \tau \in [0,T],$$
$$x(\tau) \in X, \forall \tau \in [0,T], x(T) \in K.$$

and

$$J_2^* = \min_{\mu,\nu \in \mathcal{M}_+} \langle \mu, h \rangle + \langle \nu, H \rangle$$

$$\text{subject to} \quad \mathcal{L}^*(\mu,\nu) = \delta_{(0,x_0)}$$
$$supp(\mu) \subset [0,T] \times X \times U, supp(\nu) \subset K.$$

One has

$$J_1^* = J_2^*.$$

Remark

$$J_2^* = \min_{\mu,\nu \in \mathcal{M}_+} \quad \langle \mu, h \rangle + \langle \nu, H \rangle$$

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is an infinite dimensional linear programming problem.

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is an infinite dimensional linear programming problem.

R. Vinter. *Convex duality and nonlinear optimal control.* SIAM J. Control Optim. 31, 2 (1993), 518-538

$$x(0) = x_0, \dot{x} = f(x,u) \Leftrightarrow \mathcal{L}^*(\mu,\nu) = \delta_{(0,x_0)}$$

Proof

Let $\varphi : [0,T] \times X \to \mathbb{R}$ be a differentiable function, one has

$$\begin{aligned}\varphi(T,x_T) - \varphi(0,x_0) &= \int_0^T \frac{d\varphi}{dt}(\tau, x(\tau))d\tau \\ &= \int_0^T \frac{\partial\varphi}{\partial t}(\tau,x) + \nabla_x\varphi(\tau,x)\cdot f(x,u)d\tau\end{aligned}$$

$$\int_0^T -\frac{\partial\varphi}{\partial t}(\tau,x) - \nabla_x\varphi(\tau,x)\cdot f(x,u)d\tau + \varphi(T,x_T) = \varphi(0,x_0)$$

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$$\int_{[0,T]\times X\times U} -\frac{\partial \varphi}{\partial t}(\tau, x) - \nabla_x \varphi(\tau, x) \cdot f(x, u) d\mu + \int_K \varphi d\nu = \varphi(0, x_0)$$

$$\langle (\mu, \nu), \mathcal{L}\varphi \rangle = \langle \delta_{(0,x_0)}, \varphi \rangle$$

where

$$\mathcal{L}\varphi = \left(-\frac{\partial \varphi}{\partial t}(t, x) - \nabla_x \varphi(t, x) \cdot f(x, u) \ , \ \varphi(T, x) \right)$$

The operator $\mathcal{L}$ is linear, let us denote by $\mathcal{L}^*$ its transpose , therefore

$$\langle \mathcal{L}^*(\mu, \nu), \varphi \rangle = \langle \delta_{(0,x_0)}, \varphi \rangle$$

i.e.

$$\mathcal{L}^*(\mu, \nu) = \delta_{(0,x_0)}$$

Definition

An interval is a compact subset of $\mathbb{R}$ of the following form :
$[x] = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} \mid \underline{x} \leq x \leq \overline{x}\}$.

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Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a map, one says that $[f] : \mathbb{IR} \to \mathbb{IR}$ is an inclusion map of f if $\forall [x] \in \mathbb{IR}, f([x]) \subset [f]([x])$.

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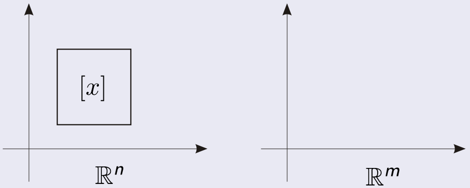

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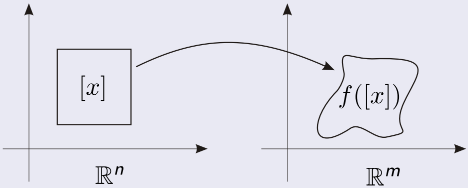

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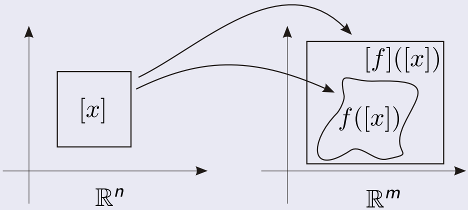

Interval Arithmetic

$$
\begin{aligned}
[x] + [y] &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}] \\
[x] - [y] &= [\underline{x} - \overline{y}, \overline{x} - \underline{y}] \\
[x] \times [y] &= [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}], \\
[x] \div [y] &= [x] \times \left[\frac{1}{\overline{y}}, \frac{1}{\underline{y}}\right], \text{ if } \underline{y}\overline{y} > 0.
\end{aligned}
$$

Interval Arithmetic

$$\begin{array}{rcl} [x]+[y] & = & [\underline{x}+\underline{y}, \overline{x}+\overline{y}] \\ [x]-[y] & = & [\underline{x}-\overline{y}, \overline{x}-\underline{y}] \\ [x]\times[y] & = & [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}], \\ [x]\div[y] & = & [x]\times\left[\frac{1}{\overline{y}}, \frac{1}{\underline{y}}\right], \text{ if } \underline{y}\overline{y}>0. \end{array}$$

Proposition

The four basic interval operations are inclusion maps of $+$, $-$, $\times$ and $\div$ defined on reals.

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If f and g are maps with inclusion maps $[f]$ and $[g]$, then $[f] \circ [g]$ is an inclusion map of $f \circ g$.

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Example

Let $f : \mathbb{R} \to \mathbb{R}$ be

$$f(x) = 1 - 2x + x^2.$$

The map $[f] : \mathbb{IR} \to \mathbb{IR}$

$$[f]([\underline{x}, \overline{x}]) = [1, 1] - [2, 2] \times [\overline{x}, \underline{x}] + [\underline{x}, \overline{x}]^2,$$

is an inclusion map for f.

$$\varphi(x) = 1 - 2x + x^2$$

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$x \in [2, 3]$

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$$x \in [2,3] \quad \Rightarrow \quad 1 \quad \in \quad [\quad 1 \quad , \quad 1 \quad],$$

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$$\begin{array}{llcclcll} x \in [2,3] & \Rightarrow & 1 & \in & [& 1 & , & 1 &], \\ & \Rightarrow & -2x & \in & [& -6 & , & -4 &], \end{array}$$

$$\varphi(x) = 1 - 2x + x^2$$

$$\begin{array}{lcccccccc} x \in [2,3] & \Rightarrow & 1 & \in & [& 1 & , & 1 &], \\ & \Rightarrow & -2x & \in & [& -6 & , & -4 &], \\ & \Rightarrow & x^2 & \in & [& 4 & , & 9 &], \end{array}$$

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\hline
x \in [2,3] & \Rightarrow & \varphi(x) & \in & [& -1 & , & 6 &]. \\
 & & & & [& \underline{\varphi} & , & \overline{\varphi} &].
\end{array}
$$

Main results based on interval analysis

- Interval Arithmetic, Ramon E. Moore, 1966,

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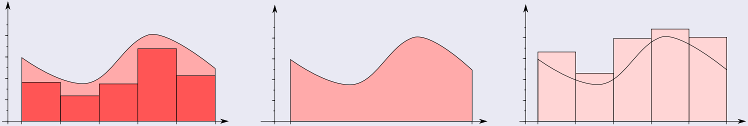
- PDE, algebraic topology, . . .

Lemma - Enclosing

Let X_i be a partition of X, if $\mu \in \mathcal{M}^+$ and $[\varphi]$ an inclusion function for φ then

$$\langle \mu, \varphi \rangle \in \sum_i [\varphi](X_i)\mu(X_i)$$



$$\sum_i \underline{\varphi}(X_i)\mu(X_i) \leq \int_X \varphi(x)\mathrm{d}\mu(x) \leq \sum_i \overline{\varphi}(X_i)\mu(X_i)$$

Relaxation - Main result

$$
\begin{aligned}
J^* = \min_{\mu,\nu \in \mathcal{M}_+} \quad & \langle \mu, h\rangle + \langle \nu, H\rangle \\
\text{subject to} \quad & \mathcal{L}^*(\mu,\nu) = \delta_{(0,x_0)}
\end{aligned} \tag{1}
$$

Let $\{X_i\}$ be a partition of $[0,T] \times X \times U$ and $\{Y_k\}$ be a partition of K. Suppose $\mathcal{P} = \{\varphi\}$ is a finite family of functions with indeterminates t, x.

$$
\begin{aligned}
\underline{J} = \min_{\mu_i,\nu_i \in \mathbb{R}^+} \quad & \sum_{i\in I} \mu_i \underline{h}_i + \sum_{k\in K} \nu_k \underline{H}_k \\
\text{s.t. } \forall \varphi \in \mathcal{P} \quad & \sum_{i\in I} \mu_i \underline{\psi}_i + \sum_{i\in K} \nu_i \underline{\varphi}_i \leq \varphi(0,x_0) \leq \sum_{i\in I} \mu_i \overline{\psi}_i + \sum_{i\in K} \nu_i \overline{\varphi}_i, \\
& \text{where } \psi = -\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial x} f(t,x,u),
\end{aligned} \tag{2}
$$

then

$$
\underline{J} \leq J^*. \tag{3}
$$

Example

$$J^* = \min_{u:[0,T]\to[-1,1]} \int_0^1 x^2(\tau)d\tau + x^2(T)$$

subject to

$$x(0) = 2.82, \dot{x} = u,$$
$$x(\tau) \in [-3,3],$$
$$u(\tau) \in [-1,1],$$
$$x(T) \in [-3,3].$$

becomes

$$J^* = \min_{\mu,\nu\in\mathcal{M}_+} \langle\mu, x^2\rangle + \langle\nu, x^2\rangle$$

subject to

$$\mathcal{L}^*(\mu,\nu) = \delta_{(0,x_0)}$$
$$supp(\mu) \subset [0,1]\times[-3,3]\times[-1,1], supp(\nu) \subset [-3,3].$$

In practice, this infinite LP is relaxed to the following finite dimensional LP :

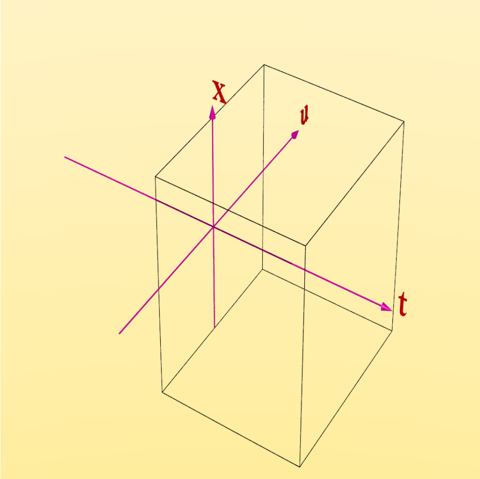
$$\begin{aligned} \underline{J} = \min_{\mu_i,\nu_i \in \mathbb{R}^+} \quad & \sum_{i\in I} \mu_i \underline{h}_i + \sum_{k \in K} \nu_k \underline{H}_k \\ \text{s.t. } \forall \varphi \in \mathcal{P} \quad & \sum_{i\in I} \mu_i \underline{\psi}_i + \sum_{i\in K} \nu_i \underline{\varphi}_i \leq \varphi(0, x_0) \leq \sum_{i\in I} \mu_i \overline{\psi}_i + \sum_{i \in K} \nu_i \overline{\varphi}_i, \\ & \text{where } \psi = -\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial x} f(t,x,u), \end{aligned}$$

where

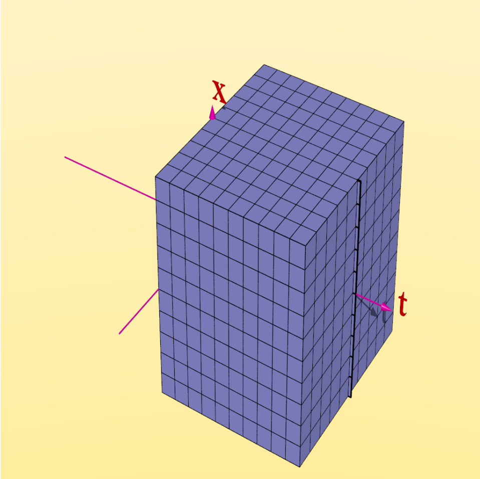
- $\mathcal{P}$ is chosen to be a finite subset of :

$$\mathcal{P} = \{1, t, x, t^2, tx, x^2, t^3, \dots\},$$

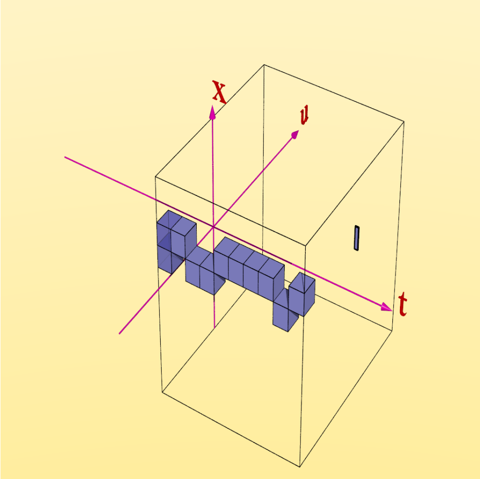
- $\underline{h}_i$, $\underline{H}_i$, $\underline{\varphi}_i$, $\overline{\varphi}_i$, and $\underline{\psi}_i$, $\overline{\psi}_i$ are obtained with interval arithmetic.



$$[0,1] \times [-3,3] \times [-1,1].$$



$n = 10$, $n^3 + n = 1\,010$ variables and $\#(\mathcal{P}) = 10$ constraints.



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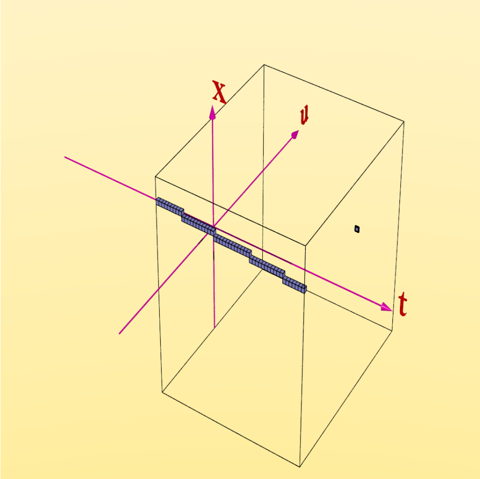
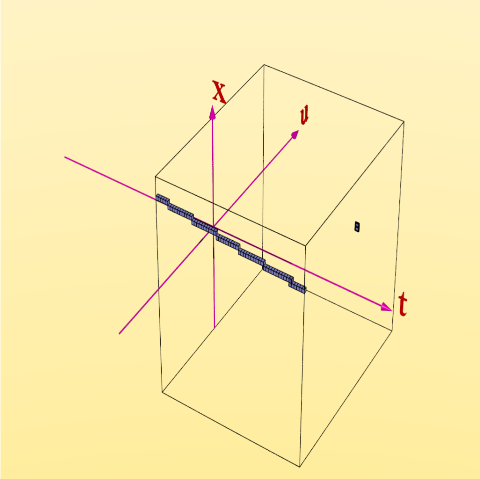
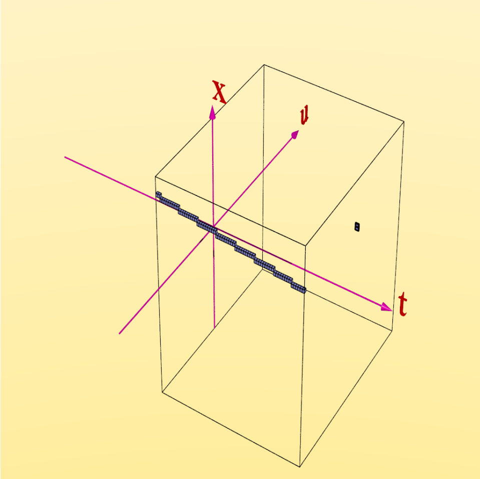


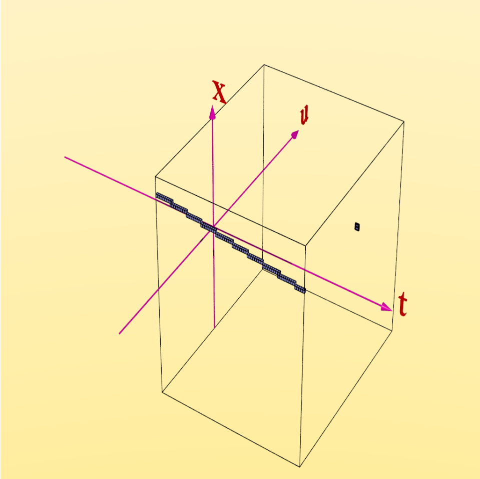
$n = 50,\ n^3 + n = 125\,050$ variables and $\#(\mathcal{P}) = 10$ constraints.



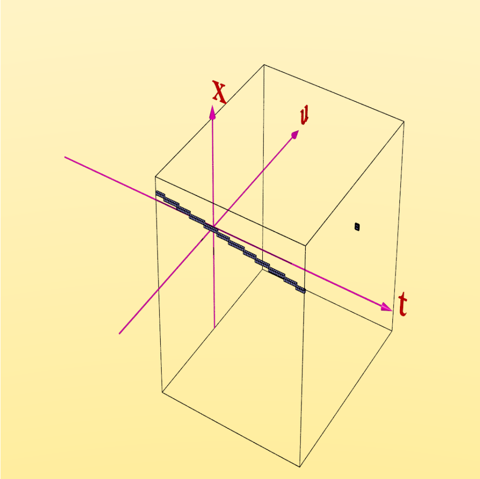
$n = 60,\ n^3 + n = 216\,060$ variables and $\#(\mathcal{P}) = 10$ constraints.



$n = 70,\ n^3 + n = 343\,070$ variables and $\#(\mathcal{P}) = 10$ constraints.



$n = 80,\ n^3 + n = 512\ 080$ variables and $\#(\mathcal{P}) = 10$ constraints.



$n = 90,\ n^3 + n = 729\ 090$ variables and $\#(\mathcal{P}) = 10$ constraints.

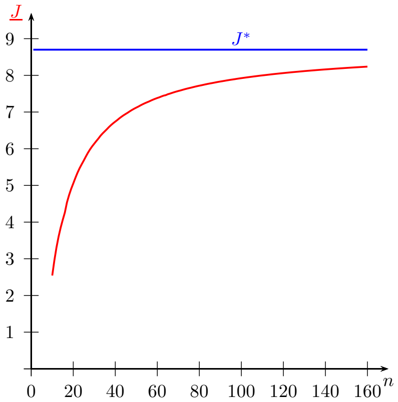


Figure : Guaranteed lower bounds of J^* where $\#(\{X_i\}_i) = n^3 + n$.

Software

- filib - FI_LIB - A fast interval library,

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Future work

- Solve the dual problem to compute a guaranteed upper bound.
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Thank you for your attention.