Interval analysis and Optimal Transport

Nicolas Delanoue - Mehdi Lhommeau - Philippe Lucidarme
LARIS - Universite d’Angers - France

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Outline

1 Introduction to Optimal Transport
   - Transportation
   - Optimal Transport
   - Some known results

2 A lower bound of the optimal value
   - Finite dimensional relaxation

3 An upper bound of the optimal value
   - Duality
   - Finite dimensional relaxation

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Example with books

Example in the discrete case
Example in the discrete case
Example in the discrete case
Example in the discrete case
Example in the discrete case

\[ \mu = (2, 1, 4) \]

\[ \nu = (4, 2, 1) \]

Transportation

\[
\begin{array}{ccc}
4 & 2 & 1 \\
2 & 2 & 2 \\
1 & 1 & 1 \\
4 & 4 & 4 \\
\end{array}
\]
**Example in the discrete case**

\[ \mu = (2, 1, 4) \]

\[ \nu = (4, 2, 1) \]

**Transportation**

\[
\begin{array}{ccc}
4 & 2 & 1 \\
\hline \\
2 & & \\
1 & & \\
4 & & \\
\end{array}
\]
Example in the discrete case

\[ \mu = (2, 1, 4) \]
\[ \nu = (4, 2, 1) \]

A plan transference \( \pi \)

\[
\begin{array}{c|ccc}
4 & 2 & 1 \\
\hline
2 & 2 & 0 & 0 \\
1 & 1 & 0 & 0 \\
4 & 1 & 2 & 1 \\
\end{array}
\]

\[
\pi = \begin{pmatrix}
2 & 0 & 0 \\
1 & 0 & 0 \\
1 & 2 & 1 \\
\end{pmatrix}
\]
Plan transference problem

\[
\begin{array}{c|ccc}
 & 4 & 2 & 1 \\
\hline
2 & . & . & . \\
1 & . & . & . \\
4 & . & . & . \\
\end{array}
\]

Solutions

\[
\pi = \begin{array}{c|ccc}
 & 4 & 2 & 1 \\
\hline
2 & 2 & 0 & 0 \\
1 & 1 & 0 & 0 \\
4 & 1 & 2 & 1 \\
\end{array}, \quad \tilde{\pi} = \begin{array}{c|ccc}
 & 4 & 2 & 1 \\
\hline
2 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 \\
4 & 2 & 2 & 0 \\
\end{array}.
\]
Definition - Transference plan

A transference plan (or a transportation) $\pi$ is a measure on the product space $X \times Y$ such that

$$\left\{ \begin{array}{l} \pi(A \times Y) = \mu(A), \\ \pi(X \times B) = \nu(B). \end{array} \right.$$ 

all measurable subsets $A$ of $X$ and $B$ of $Y$.

In the discrete case

$$\left\{ \begin{array}{l} \forall i, \sum_j \pi_{ij} = \mu_i, \\ \forall j, \sum_i \pi_{ij} = \nu_j. \end{array} \right.$$
Comparing two plan transferences

Transportation $\pi$

\[ I(\pi) = 0 + 0 + 1 + 2 + 1 + 1 + 0 = 5 \]

Transportation $\tilde{\pi}$

\[ I(\tilde{\pi}) = 0 + 2 + 1 + 2 + 2 + 1 + 1 = 9 \]
In the discrete case

\[
\min_{\pi \in \mathbb{R}^n \times \mathbb{R}^m} \sum_{i,j} c_{ij} \pi_{ij}
\]

subject to \( \forall i, \sum_j \pi_{ij} = \mu_i \), \( \forall j, \sum_i \pi_{ij} = \nu_j \). \( (1) \)

where \( c_{ij} \) are non-negative real numbers which tells how much it costs to transport one unit of mass from location \( i \) to location \( j \).
Kantorovich formulation

The optimal transportation cost between $\mu$ and $\nu$ is the value:

$$T_c(\mu, \nu) = \inf_{\pi \in B(X \times Y)} \int_{X \times Y} c(x, y) d\pi(x, y)$$

subject to $\pi_X = \mu$, $\pi_Y = \nu$ \hspace{1cm} (2)

The optimal $\pi$’s, i.e. those such that $I(\pi) = T_c(\mu, \nu)$, if they exist, will be called optimal transference plans.
Remark

The *optimal transportation problem* is an infinite dimensional linear programming problem.

i.e. $I$ is a linear cost function, and constraints are linear.
1. \( c = \|x - y\|^p, \ p > 1 \), the strict convexity of \( c \) guarantees that, if \( \mu, \nu \) are absolutely continuous with respect to Lebesgue measure, then there is a unique solution to the Kantorovich problem.

2. \( c = \|x - y\|^2 \), optimal transference plans are the (restrictions of) gradients of convex functions.

3. many others in

Topics in Optimal Transportation, Cédric Villani, AMS (2003)
Proposition - Relaxation

Let $\mu$ and $\nu$ (with support $X$ and $Y$) be absolutely continuous measures with respect to Lebesgue measure. If $\{X_i\}_i$ and $\{Y_i\}_i$ be finite pavings of $X$ and $Y$. Suppose that $\mu(X_i) \in [\underline{\mu}_i, \overline{\mu}_i]$, $\nu(Y_j) \in [\underline{\nu}_j, \overline{\nu}_j]$, and $\forall x, y \in X_i \times Y_j, c_{ij} \leq c(x, y)$,

\[
\bar{T} = \min_{\pi_{ij} \in \mathbb{R}^n \otimes \mathbb{R}^m} \sum_{i,j} c_{ij} \pi_{ij}
\]

subject to

$\forall i, \mu_i \leq \sum_j \pi_{ij} \leq \overline{\mu}_i,$

$\forall j, \nu_j \leq \sum_i \pi_{ij} \leq \overline{\nu}_j,$

$\forall i, \forall j, \pi_{ij} \geq 0.$

then $\bar{T} \leq T_{c}(\mu, \nu)$.
Spatial discretization

\[ \mu \]

\[ \nu \]
Spatial discretization

\[
\begin{array}{ccc}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{array}
\]
Spatial discretization

\[ \nu(Y_j) \in [\nu_j, \bar{\nu}_j] \]

\[ \mu(X_1) \in [\mu_1, \bar{\mu}_1] \]

\[ \mu(X_2) \in [\mu_2, \bar{\mu}_2] \]

\[ \mu(X_3) \in [\mu_3, \bar{\mu}_3] \]

\[
\begin{array}{ccc}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{array}
\]
Enclosing

If $\mu = f(x)dx$, and $[f]$ an inclusion function for $f$ then

$$\int_{X} f(x)dx \in \sum_{i} [f](X_{i})\lambda(X_{i})$$

$$\sum_{i} f(X_{i})\lambda(X_{i}) \leq \int_{X} f(x)dx \leq \sum_{i} \bar{f}(X_{i})\lambda(X_{i})$$
Proof

Let \( \{X_i\}, \{Y_j\} \) be a pavings, let \( \pi_{ij} = \pi(X_i \times Y_j) \) then \( \forall \pi, \exists \xi_{ij} \in X_i \times Y_j, \)

\[
\sum_{i,j} c(\xi_{ij})\pi_{ij} = \int_{X \times Y} c(x, y)d\pi(x, y)
\]  \hspace{1cm} (3)

Since \( c_{ij} \leq c(\xi_{ij}) \) and \( \pi_{ij} \geq 0, \)
then \( \forall \pi, \)

\[
\sum_{i,j} c_{ij}\pi_{ij} \leq \int_{X \times Y} c(x, y)d\pi(x, y)
\]  \hspace{1cm} (4)
Proof

Let $\mu$ and $\nu$ (with support $X$ and $Y$) be absolutely continuous measures with respect to Lebesgue measure. If $\{X_i\}_i$ and $\{Y_i\}_i$ be finite pavings of $X$ and $Y$. Suppose that $\mu(X_i) \in [\underline{\mu}_i, \overline{\mu}_i]$, $\nu(Y_j) \in [\underline{\nu}_j, \overline{\nu}_j]$, and $\forall x, y \in X_i \times Y_j, c_{ij} \leq c(x, y),$

$$K = \min_{\pi_{ij} \in \mathbb{R}^n \otimes \mathbb{R}^m} \sum_{i,j} c_{ij} \pi_{ij}$$

subject to $\forall i, \mu_i = \sum_j \pi_{ij} = \mu_i,$

$\forall j, \nu_j = \sum_i \pi_{ij} = \nu_j,$

$\forall i, \forall j, \pi_{ij} \geq 0.$

then $K \leq \mathcal{T}_c(\mu, \nu).$
Proof

Let $\mu$ and $\nu$ (with support $X$ and $Y$) be absolutely continuous measures with respect to Lebesgue measure. If $\{X_i\}_i$ and $\{Y_i\}_i$ be finite pavings of $X$ and $Y$. Suppose that $\mu(X_i) \in [\underline{\mu}_i, \overline{\mu}_i]$, $\nu(Y_j) \in [\underline{\nu}_j, \overline{\nu}_j]$, and $\forall x, y \in X_i \times Y_j, c_{ij} \leq c(x, y)$,

\[
\mathcal{T} = \min_{\pi_{ij} \in \mathbb{R}^n \otimes \mathbb{R}^m} \sum_{i,j} c_{ij} \pi_{ij}
\]

subject to

\[
\forall i, \mu_i \leq \sum_j \pi_{ij} \leq \overline{\mu}_i,
\]

\[
\forall j, \nu_j \leq \sum_i \pi_{ij} \leq \overline{\nu}_j,
\]

$\forall i, \forall j, \pi_{ij} \geq 0$.

then \( \mathcal{T} \leq \mathcal{T}_c(\mu, \nu) \).
Example

$X = Y = [0, 1]$  

$\mu = 1dx$  

$\nu = 1.5(1 - y^2)dy$  

$c(x, y) = \|x - y\|^2$
Figure: Guaranteed lower bounds of $\mathcal{T}_c(\mu, \nu)$ where $n = \text{Card}\{X_i\}_i$.
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4. Conclusion - Future work
Primal problem

\[
\min_{x \in \mathbb{R}^n} \quad c^T x \\
\text{subject to} \quad Ax = b, \quad x \geq 0.
\]

Dual problem

\[
\max_{y \in \mathbb{R}^m} \quad b^T y \\
\text{subject to} \quad y_i \in \mathbb{R}, \quad A^T y \leq c.
\]
Duality

\[ \inf_{\pi \in B(X \times Y)} \int_{X \times Y} c(x, y) d\pi(x, y) \]

subject to
\[ \begin{align*}
\pi_X &= \mu, \\
\pi_Y &= \nu
\end{align*} \]

\[ \sup_{\phi, \psi \in C_b(X, Y)} \int_X \phi(x) d\mu(x) + \int_Y \psi(y) d\nu(y) \]

subject to
\[ \phi(x) + \psi(y) \leq c(x, y). \]  

where \( C_b(X, Y) \) denotes the set of all pairs of bounded and continuous functions \( \phi : X \to \mathbb{R} \) and \( \psi : Y \to \mathbb{R} \).

If \( X \) is compact and Haussdorff, \( C_b(X)^* = \{ \text{Radon measure} \} \).
Kantorovich Duality

The minimum of the Kantorovich problem is equal to

\[ T_c(\mu, \nu) = \sup_{\phi, \psi \in C_b(X, Y)} \int_X \phi(x) \, d\mu(x) + \int_Y \psi(y) \, d\nu(y) \]

subject to \( \phi(x) + \psi(y) \leq c(x, y). \)
**Interpretation in the discrete case**

\[
\sup_{(\phi_i) \in \mathbb{R}^n, (\psi_j) \in \mathbb{R}^m} \left( \sum_i \phi_i \mu_i + \sum_j \psi_j \nu_i \right)
\]

subject to \(\phi_i + \psi_j \leq c_{ij}\)

\[
\mu = \begin{array}{ccc}
\text{book} & \text{book} & \text{book} \\
\text{book} & \text{book} & \text{book} \\
\text{book} & \text{book} & \text{book} \\
\end{array}
\]

\[
\nu = \begin{array}{ccc}
\text{book} & \text{book} & \text{book} \\
\text{book} & \text{book} & \text{book} \\
\text{book} & \text{book} & \text{book} \\
\end{array}
\]
Interpretation in the discrete case

\[ \sup_{(\phi_i)\in \mathbb{R}^n, (\psi_j)\in \mathbb{R}^m} \sum_i \phi_i \mu_i + \sum_j \psi_j \nu_i \]

subject to \[ \phi_i + \psi_j \leq c_{ij} \]

(9)

\[ \mu = \begin{array}{c}
2\phi_1 \\
\phi_2 \\
4\phi_3
\end{array} \]

\[ \nu = \begin{array}{c}
4\psi_1 \\
2\psi_2 \\
\psi_3
\end{array} \]
Interpretation in the discrete case

\[
\sup_{(\phi_i) \in \mathbb{R}^n, (\psi_j) \in \mathbb{R}^m} \sum_i \phi_i \mu_i + \sum_j \psi_j \nu_j
\]

subject to \( \phi_i + \psi_j \leq c_{ij} \)

\[\mu = \begin{array}{c}
\text{2\phi_1} \\
\text{\phi_2} \\
\text{4\phi_3}
\end{array}\]

\[\nu = \begin{array}{c}
\text{4\psi_1} \\
\text{2\psi_2} \\
\text{\psi_3}
\end{array}\]
Proposition - Relaxation

Let $\mu$ and $\nu$ (with support $X$ and $Y$) be absolutely continuous measures with respect to Lebesgue measure. If $\{X_i\}_i$ and $\{Y_i\}_i$ be finite pavings of $X$ and $Y$. Suppose that $\mu(X_i) \in [\underline{\mu}_i, \overline{\mu}_i]$, $\nu(Y_j) \in [\underline{\nu}_j, \overline{\nu}_j]$, and $\forall x, y \in X_i \times Y_j$, $c(x, y) \leq \overline{c}_{ij}$,

\[
\overline{T} = \sup_{(\phi_i) \in \mathbb{R}^n, (\psi_j) \in \mathbb{R}^m} \sum_i \phi_i \underline{\mu}_i + \sum_j \psi_j \overline{\nu}_j
\]

subject to $\phi_i + \psi_j \leq \overline{c}_{ij}$

then $\mathcal{T}_c(\mu, \nu) \leq \overline{T}$. 

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Figure: Guaranteed upper bounds of $\mathcal{T}_c(\mu, \nu)$ where $n = \text{Card}\{X_i\}_i$. 

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Software

- **filib - FI_LIB - A fast interval library,**
  
  [http://www2.math.uni-wuppertal.de/~xsc/software/filib.html](http://www2.math.uni-wuppertal.de/~xsc/software/filib.html)

- **GLPK - GNU Linear Programming Kit (GLPK),**
  

- **GMP - GNU Multiple Precision Arithmetic Library,**
  
  [https://gmplib.org/](https://gmplib.org/)

- Source code is available on my webpage.
Future work

- Compute guaranteed enclosures of the solution combining linear programming and constraint propagation.
- Generalize this methodology to other problems (D. Henrion & J.B. Lasserre):
  - Probability and Markov Chains
  - Optimal Control with occupation measures (ODE),
  - Others as in *Moments, Positive Polynomials and Their Applications*, J.B Lasserre, Imperial College Press Optimization Series (2009)

Tack för din uppmanhet!