Interval analysis and Optimal Transport

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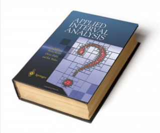
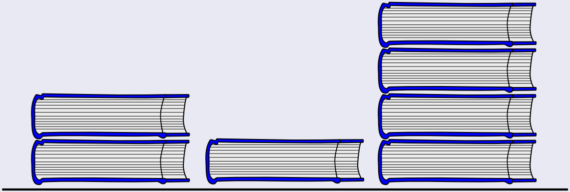
Outline

Example with books



Example in the discrete case



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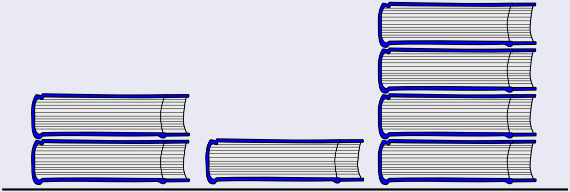

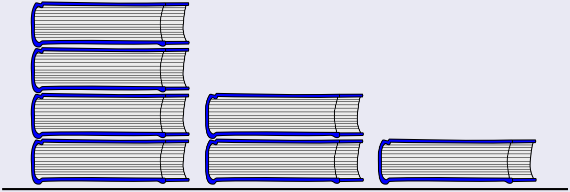

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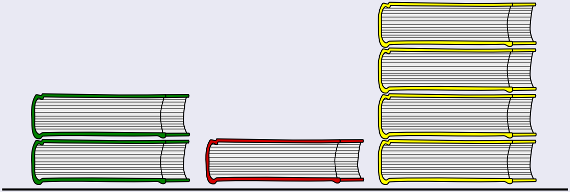

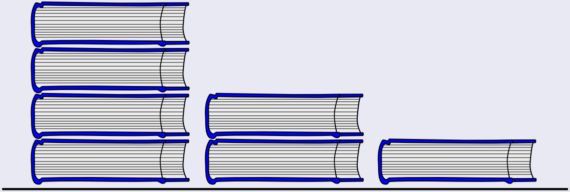

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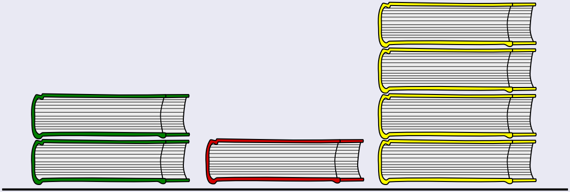

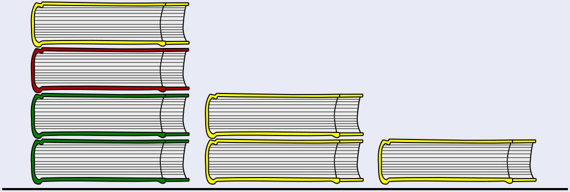

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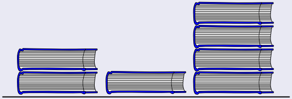
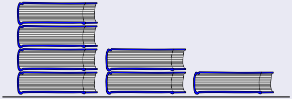


$$\mu = (2, 1, 4)$$

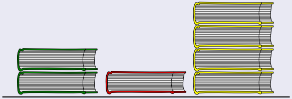


$$\nu = (4, 2, 1)$$

Transportation

	4	2	1
2			
1			
4			

Example in the discrete case



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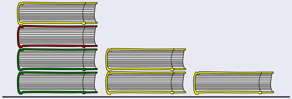

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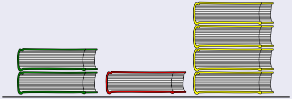

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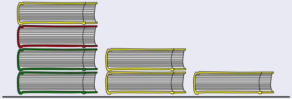


$$\nu = (4, 2, 1)$$

A plan transference π

$$\begin{array}{c|ccc} & 4 & 2 & 1 \\ \hline 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 1 \end{array}, \qquad \pi = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

Plan transference problem

$$\begin{array}{c|ccc} & 4 & 2 & 1 \\ \hline 2 & . & . & . \\ 1 & . & . & . \\ 4 & . & . & . \end{array}$$

Solutions

$$\pi = \begin{array}{c|ccc} & 4 & 2 & 1 \\ \hline 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 1 \end{array}, \quad \tilde{\pi} = \begin{array}{c|ccc} & 4 & 2 & 1 \\ \hline 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 4 & 2 & 2 & 0 \end{array}.$$

Definition - Transference plan

A transference plan (or a transportation) π is a measure on the product space $X \times Y$ such that

$$\begin{cases} \pi(A \times Y) = \mu(A), \\ \pi(X \times B) = \nu(B). \end{cases}$$

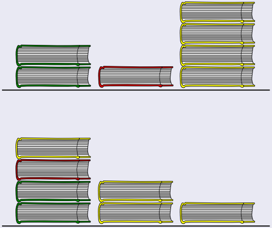
all measurable subsets A of X and B of Y.

In the discrete case

$$\begin{cases} \forall i, \ \sum_j \pi_{ij} = \mu_i, \\ \forall j, \ \sum_i \pi_{ij} = \nu_j. \end{cases}$$

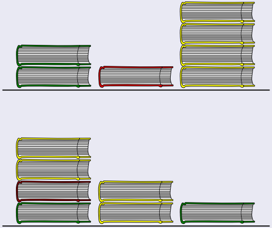
Comparing two plan transferences

Transportation π



$$I(\pi) = 0 + 0 + 1 + 2 + 1 + 1 + 0 = 5$$

Transportation $\tilde{\pi}$



$$I(\tilde{\pi}) = 0 + 2 + 1 + 2 + 2 + 1 + 1 = 9$$

In the discrete case

$$
\begin{aligned}
\min_{\pi \in \mathbb{R}^n \otimes \mathbb{R}^m} \quad & \sum_{i,j} c_{ij}\pi_{ij} \\
\text{subject to} \quad & \forall i, \ \sum_{j} \pi_{ij} = \mu_i, \\
& \forall j, \ \sum_{j} \pi_{ij} = \nu_j.
\end{aligned} \tag{1}
$$

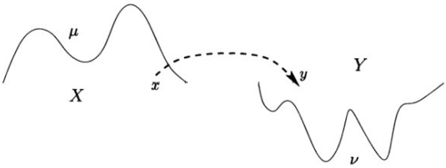
where c_{ij} are non negative real numbers which tells how much it costs to transport one unit of mass from location i to location j.

Kantorovich formulation

The *optimal transportation cost* between μ and ν is the value :

$$\begin{aligned} \mathcal{T}_c(\mu,\nu) = \inf_{\pi\in\mathcal{B}(X\times Y)} \quad & \int_{X\times Y} c(x,y)\mathrm{d}\pi(x,y) \\ \text{subject to} \quad & \pi_X = \mu, \\ & \pi_Y = \nu \end{aligned} \tag{2}$$

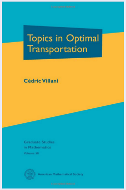
The optimal π's, i.e. those such that $I(\pi) = \mathcal{T}_c(\mu,\nu)$, if they exist, will be called *optimal transference plans*.



Remark

The *optimal transportation problem* is an infinite dimensional linear programming problem.
i.e. I is a linear cost function, and constraints are linear.

1. $c = \|x - y\|^p$, $p > 1$, the strict convexity of c guarantees that, if μ, ν are absolutely continuous with respect to Lebesgue measure, then there is a unique solution to the Kantorovich problem.
2. $c = \|x - y\|^2$, optimal transference plans are the (restrictions of) gradients of convex functions.
3. many others in



Topics in Optimal Transportation, Cédric Villani, AMS (2003)

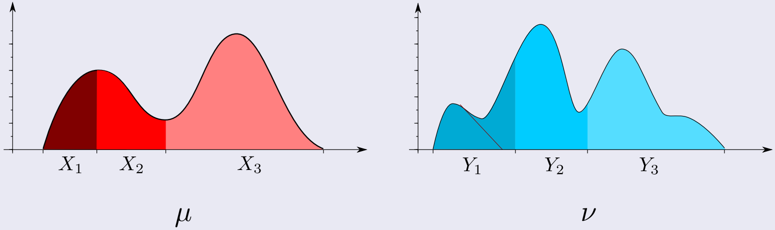
Proposition - Relaxation

Let μ and ν (with support X and Y) be absolutely continuous measures with respect to Lebesgue measure. If $\{X_i\}_i$ and $\{Y_i\}_i$ be finite pavings of X and Y. Suppose that $\mu(X_i) \in [\underline{\mu}_i, \overline{\mu}_i]$, $\nu(Y_j) \in [\underline{\nu}_j, \overline{\nu}_j]$, and $\forall x, y \in X_i \times Y_j, \underline{c}_{ij} \leq c(x,y)$,

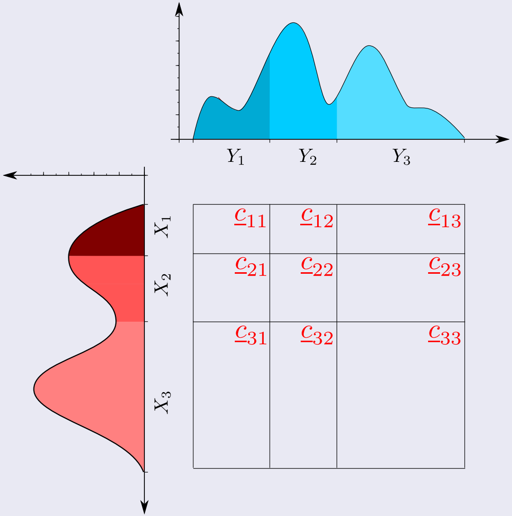
$$\begin{aligned} \underline{\mathcal{T}} = \min_{\pi_{ij} \in \mathbb{R}^n \otimes \mathbb{R}^m} \quad & \sum_{i,j} \underline{c}_{ij} \pi_{ij} \\ \text{subject to} \quad & \forall i,\ \underline{\mu}_i \leq \sum_j \pi_{ij} \leq \overline{\mu}_i, \\ & \forall j,\ \underline{\nu}_j \leq \sum_i \pi_{ij} \leq \overline{\nu}_j, \\ & \forall i, \forall j,\ \pi_{ij} \geq 0. \end{aligned}$$

$$\text{then} \qquad \underline{\mathcal{T}} \leq \mathcal{T}_c(\mu, \nu).$$

Spatial discretization



Spatial discretization



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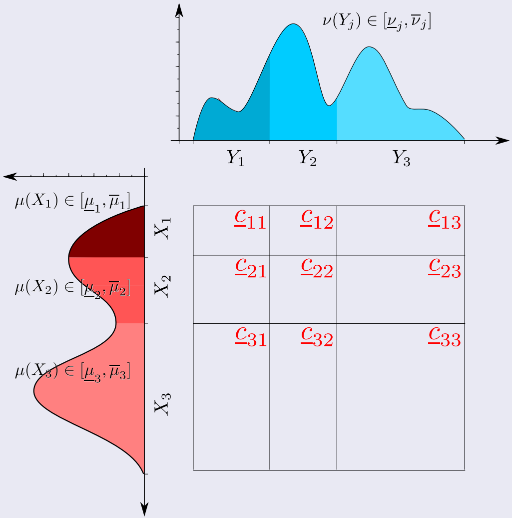
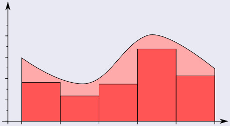


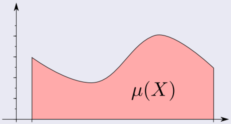
Enclosing

If $\mu = f(x)\mathrm{d}x$, and $[f]$ an inclusion function for f then

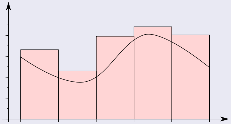
$$\int_X f(x)\mathrm{d}x \in \sum_i [f](X_i)\lambda(X_i)$$



$\underline{\mu}_i$



$\mu(X)$



$\overline{\mu}_i$

$$\sum_i \underline{f}(X_i)\lambda(X_i) \leq \int_X f(x)\mathrm{d}x \leq \sum_i \overline{f}(X_i)\lambda(X_i)$$

Proof

Let $\{X_i\}$, $\{Y_j\}$ be a pavings, let $\pi_{ij} = \pi(X_i \times Y_j)$ then $\forall \pi, \exists \xi_{ij} \in X_i \times Y_j$,

$$\sum_{i,j} c(\xi_{ij})\pi_{ij} = \int_{X \times Y} c(x,y)\mathrm{d}\pi(x,y) \tag{3}$$

Since $\underline{c}_{ij} \leq c(\xi_{ij})$ and $\pi_{ij} \geq 0$, then $\forall \pi$,

$$\sum_{i,j} \underline{c}_{ij}\pi_{ij} \leq \int_{X \times Y} c(x,y)\mathrm{d}\pi(x,y) \tag{4}$$

Proof

Let μ and ν (with support X and Y) be absolutely continuous measures with respect to Lebesgue measure. If $\{X_i\}_i$ and $\{Y_i\}_i$ be finite pavings of X and Y. Suppose that $\mu(X_i) \in [\underline{\mu}_i, \overline{\mu}_i]$, $\nu(Y_j) \in [\underline{\nu}_j, \overline{\nu}_j]$, and $\forall x, y \in X_i \times Y_j, \underline{c}_{ij} \leq c(x, y)$,

$$\begin{aligned} \mathcal{K} = \min_{\pi_{ij} \in \mathbb{R}^n \otimes \mathbb{R}^m} \quad & \sum_{i,j} \underline{c}_{ij} \pi_{ij} \\ \text{subject to} \quad & \forall i,\ \mu_i = \sum_j \pi_{ij} = \mu_i, \\ & \forall j,\ \nu_j = \sum_i \pi_{ij} = \nu_j, \\ & \forall i, \forall j,\ \pi_{ij} \geq 0. \end{aligned}$$

$$\text{then} \qquad \mathcal{K} \leq \mathcal{T}_c(\mu, \nu).$$

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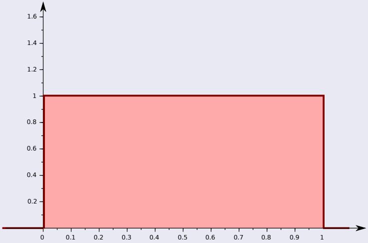
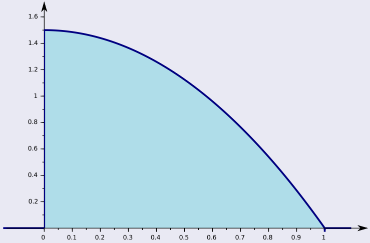
\underline{\mathcal{T}} = \min_{\pi_{ij} \in \mathbb{R}^n \otimes \mathbb{R}^m} \quad & \sum_{i,j} \underline{c}_{ij} \pi_{ij} \\
\text{subject to} \quad & \forall i,\ \underline{\mu}_i \leq \sum_j \pi_{ij} \leq \overline{\mu}_i, \\
& \forall j,\ \underline{\nu}_j \leq \sum_i \pi_{ij} \leq \overline{\nu}_j, \\
& \forall i, \forall j,\ \pi_{ij} \geq 0.
\end{aligned}$$

$$\text{then} \qquad \underline{\mathcal{T}} \leq \mathcal{T}_c(\mu, \nu).$$

Example

$X = Y = [0,1]$





$$\mu = 1\mathrm{d}x \qquad \nu = 1.5(1-y^2)\mathrm{d}y$$

$$c(x,y) = \|x-y\|^2$$

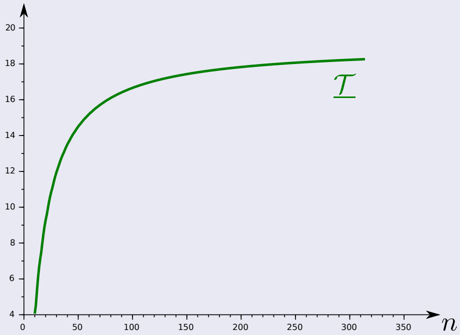


Figure : Guaranteed lower bounds of $\mathcal{T}_c(\mu, \nu)$ where $n = \text{Card}\{X_i\}_i$

Outline

Linear programming - Duality

Primal problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{subject to} \quad & Ax = b, \\ & x \geq 0. \end{aligned}$$

Dual problem

$$\begin{aligned} \max_{y \in \mathbb{R}^m} \quad & b^T y \\ \text{subject to} \quad & y_i \in \mathbb{R}, \\ & A^T y \leq c. \end{aligned} \tag{5}$$

Duality

$$\begin{array}{ll} \inf\limits_{\pi \in \mathcal{B}(X\times Y)} & \displaystyle\int_{X\times Y} c(x,y)\mathrm{d}\pi(x,y) \\ \text{subject to} & \pi_X = \mu, \\ & \pi_Y = \nu \end{array}$$

$$\begin{array}{ll} \sup\limits_{\phi,\psi \in \mathcal{C}_b(X,Y)} & \displaystyle\int_X \varphi(x)\,\mathrm{d}\mu(x) + \int_Y \psi(y)\,\mathrm{d}\nu(y) \\ \text{subject to} & \varphi(x)+\psi(y) \le c(x,y). \end{array} \tag{6}$$

where $\mathcal{C}_b(X,Y)$ denotes the set of all pairs of bounded and continuous functions $\phi : X \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$.

If X is compact and Haussdorff, $\mathcal{C}_b(X)^* = \{\text{Radon measure}\}$

Kantorovich Duality

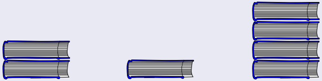
The minimum of the Kantorovich problem is equal to

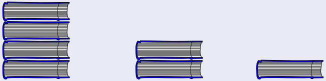
$$\mathcal{T}_c(\mu,\nu) = \sup_{\phi,\psi\in\mathcal{C}_b(X,Y)} \int_X \varphi(x)\,\mathrm{d}\mu(x) + \int_Y \psi(y)\,\mathrm{d}\nu(y) \quad \text{subject to} \quad \varphi(x)+\psi(y)\leq c(x,y). \tag{7}$$

Interpretation in the discrete case

$$\sup_{(\phi_i)\in\mathbb{R}^n,(\psi_j)\in\mathbb{R}^m} \quad \sum_i \phi_i\mu_i + \sum_j \psi_j\nu_i \qquad (8)$$

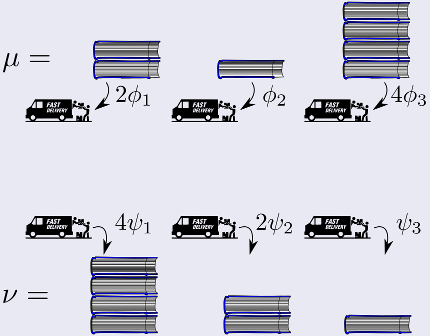
$$\text{subject to} \qquad \phi_i + \psi_j \leq c_{ij}$$

$\mu =$ 

$\nu =$ 

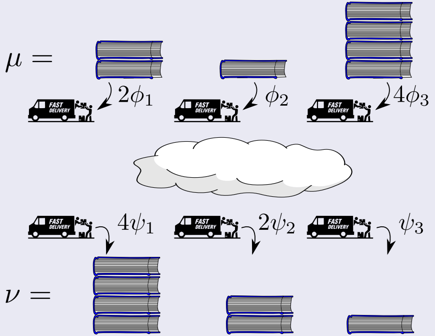
Interpretation in the discrete case

$$\sup_{(\phi_i)\in\mathbb{R}^n,(\psi_j)\in\mathbb{R}^m} \quad \sum_i \phi_i\mu_i + \sum_j \psi_j\nu_i \qquad \text{subject to} \quad \phi_i + \psi_j \leq c_{ij} \tag{9}$$



Interpretation in the discrete case

$$\sup_{(\phi_i)\in\mathbb{R}^n,(\psi_j)\in\mathbb{R}^m} \quad \sum_i \phi_i\mu_i + \sum_j \psi_j\nu_i \qquad (10)$$
$$\text{subject to} \quad \phi_i + \psi_j \leq c_{ij}$$



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$$\overline{\mathcal{T}} = \sup_{(\phi_i)\in\mathbb{R}^n, (\psi_j)\in\mathbb{R}^m} \quad \sum_i \phi_i \overline{\mu}_i + \sum_j \psi_j \overline{\nu}_i \qquad \text{subject to} \quad \phi_i + \psi_j \leq \overline{c}_{ij} \tag{11}$$

$$\text{then} \qquad \mathcal{T}_c(\mu, \nu) \leq \overline{\mathcal{T}}.$$

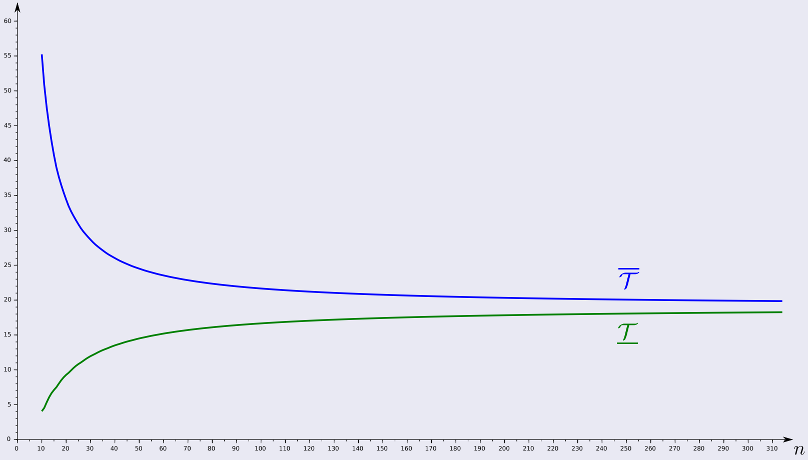

Figure : Guaranteed upper bounds of $\mathcal{T}_c(\mu, \nu)$ where $n = \mathrm{Card}\{X_i\}_i$

Software

- filib - FI_LIB - A fast interval library,

  http://www2.math.uni-wuppertal.de/~xsc/software/filib.html

- GLPK - GNU Linear Programming Kit (GLPK),

  http://www.gnu.org/software/glpk/

- GMP - GNU Multiple Precision Arithmetic Library,

  https://gmplib.org/

- Source code is available on my webpage.

Future work

- Compute guaranteed enclosures of the solution combining linear programming and constraint propagation.
- Generalize this methodology to other problems (D. Henrion & J.B. Lasserre):
  - Probability and Markov Chains
  - Optimal Control with occupation measures (ODE),
  - Others as in *Moments, Positive Polynomials and Their Applications*, J.B Lasserre, Imperial College Press Optimization Series (2009)

Tack för din uppmärksamhet !