A new method for integrating ODE based on monotonicity

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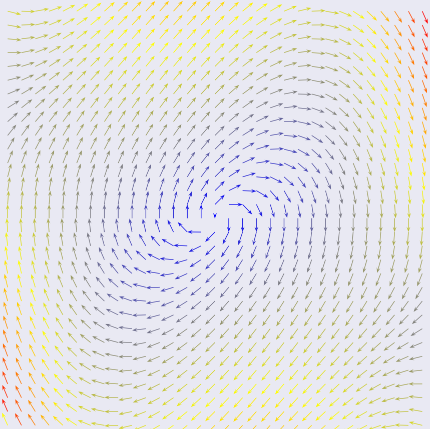
Main goal

Computing the smallest box containing the solution of the initial value problem $\dot{x} = f(x), x(0) \in [x]$



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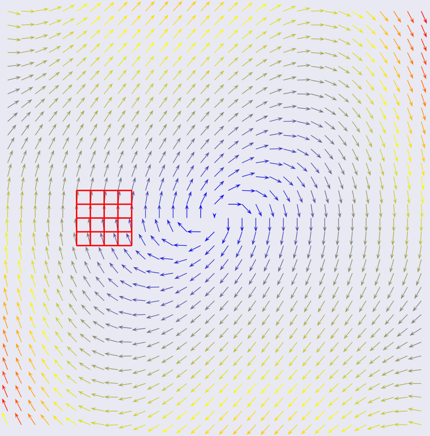

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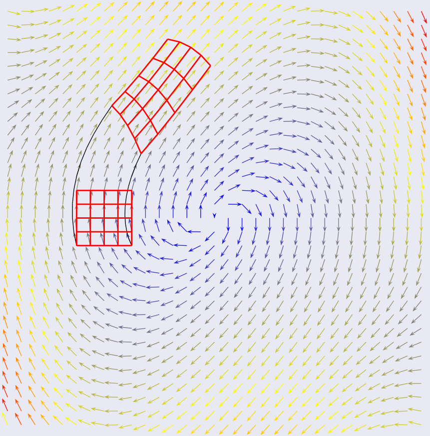

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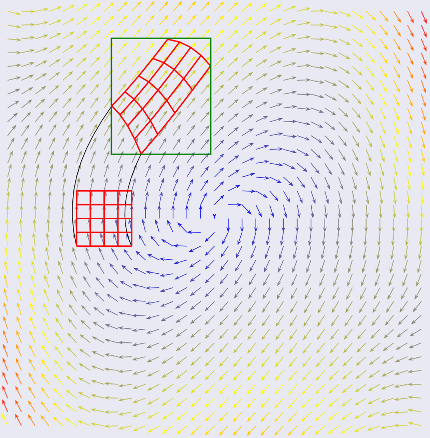

Outline

1. Interval analysis, optimal inclusion function
   - Inclusion function
   - Optimal inclusion function
2. Computing optimal validated solutions for ODE
   - ODE, Dynamical system and flow
   - Derivative of the flow with respect to initial condition
   - Algorithm

Definition

Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. A function $[f] : \mathbb{IR}^n \to \mathbb{IR}^m$ satisfying : $\forall [x] \in \mathbb{IR}^n, f([x]) \subset [f]([x])$ is an inclusion function of f.

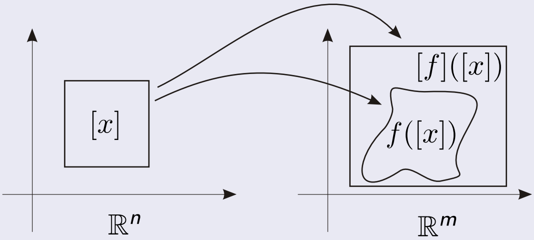

FIG.: Illustration of inclusion function.

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- Interval arithmetic gives a method to compute an inclusion function of a given function defined by an arithmetical expression.

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- In general, the smallest inclusion function is not obtained and one only has : $f([x]) \subsetneqq [f]([x])$.

Theorem

Let $[x]$ be a box of $\mathbb{R}^n$ and f be a differentiable function $\mathbb{R}^n \to \mathbb{R}$. Let us denote by $f_*(x)$ the jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x_1}(x) & \dots & \frac{\partial f}{\partial x_n}(x) \end{pmatrix}$$

Suppose that all components of $f_*([x])$ are non-negative, then $[f(\underline{x}), f(\overline{x})]$ is the smallest interval containing $f([x])$.

Example

Let us consider the function $f : (x_1, x_2) \mapsto 3x_1^2 - 2x_1x_2 + 3x_2^2$.

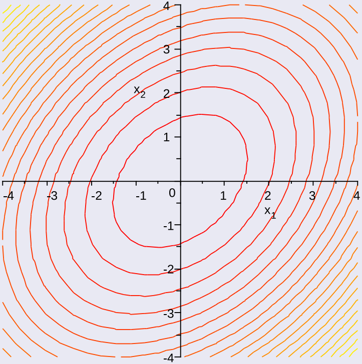

FIG.: Level curves.

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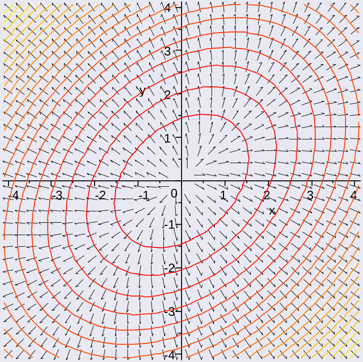
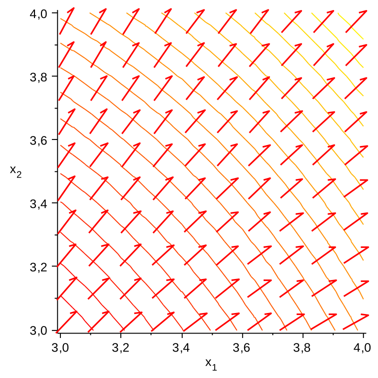


FIG.: Level curves.

Example $f : (x_1, x_2) \mapsto 3x_1^2 - 2x_1x_2 + 3x_2^2$

Since $f_*(x_1, x_2) = \begin{pmatrix} 6x_1 - 2x_2 & -2x_1 + 6x_2 \end{pmatrix}$, one has

$$\{f_*(x_1, x_2) \mid (x_1, x_2) \in [3, 4] \times [3, 4]\} \subset \mathbb{R}^+ \times \mathbb{R}^+.$$



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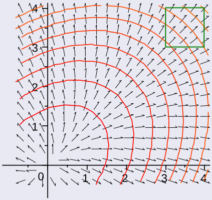

FIG.: Level curves.

- According to the previous theorem, one can conclude that $f([3,4] \times [3,4]) = [f(3,3), f(4,4)] = [36, 52]$.

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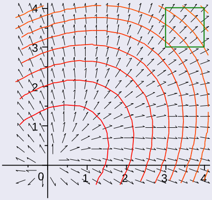


FIG.: Level curves.

- According to the previous theorem, one can conclude that $f([3,4] \times [3,4]) = [f(3,3), f(4,4)] = [36, 52]$.
- This result can be compare to the one obtained applying interval arithmetic : $3 * [3,4]^2 - 2 * [3,4] * [3,4] + 3 * [3,4]^2$, *i.e.* $[22, 78]$.

Corollary

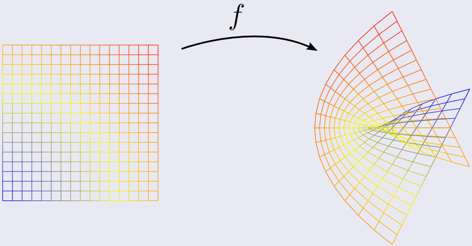
Let $[x]$ be a box of $\mathbb{R}^n$ and f be a differentiable function $\mathbb{R}^n \to \mathbb{R}^m$. Let us denote by $f_*(x)$ the jacobian matrix

$$\left(\frac{\partial f_j}{\partial x_j}(x)\right)_{1\leq i\leq n, 1\leq j\leq n}$$

Suppose that no component of $f_*([x])$ contains 0, then there exists $2m$ corners $\underline{\tilde{x}}_j$ and $\overline{\tilde{x}}_j$ of $[x]$ such that $\prod_{1\leq j\leq n}[f_j(\underline{\tilde{x}}_j), f_j(\overline{\tilde{x}}_j)]$ is the smallest box containing $f([x])$.

Example

$$f : \begin{array}{ccc} \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \mapsto & \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \end{array} \text{ where } \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2 \\ x_1 + x_1 x_2 \end{pmatrix}$$



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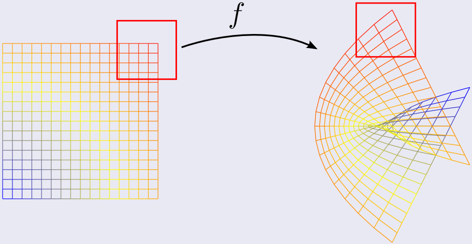

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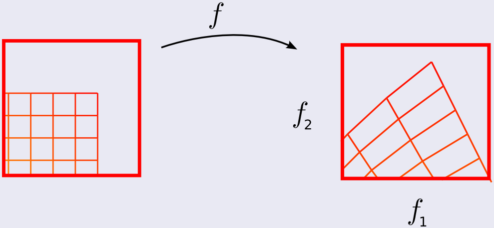

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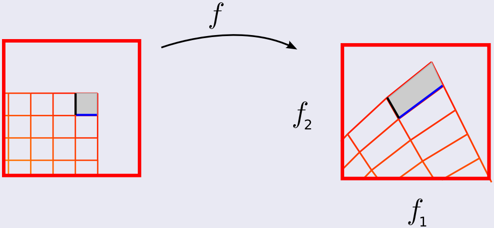
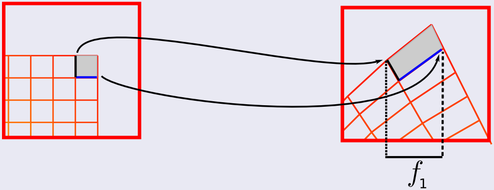


Example

$$f_*(x_1, x_2) = \begin{pmatrix} 2x_1 & -1 \\ 1 + x_2 & x_1 \end{pmatrix} = \begin{pmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{pmatrix}$$



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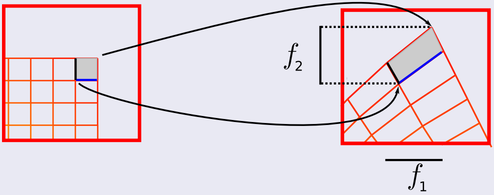

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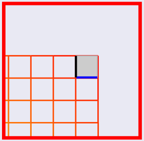

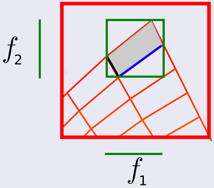

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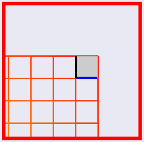
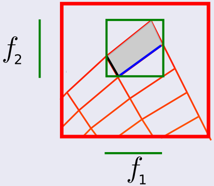
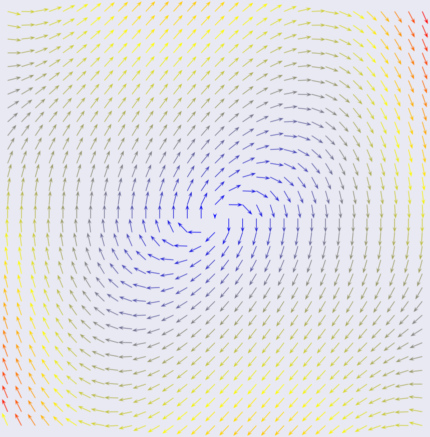




$\begin{pmatrix} [f_1(\underline{x}_1,\overline{x}_2), f_1(\overline{x}_1,\underline{x}_2)] \\ [f_2(\underline{x}_1,\underline{x}_2), f_2(\overline{x}_1,\overline{x}_2)] \end{pmatrix}$ is the smallest box containing $f([\underline{x}_1,\overline{x}_1],[\underline{x}_2,\overline{x}_2])$

$$\begin{cases} \dot{x} = f(x) \\ x \in \mathbb{R}^n \end{cases}, f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n).$$



Let us denote by $\{g^t : \mathbb{R}^n \to \mathbb{R}^n\}_{t\in\mathbb{R}}$ the flow, i.e.

$$\left.\frac{d}{dt}g^t x\right|_{t=0} = f(x) \text{ and } g^0 = Id \tag{1}$$

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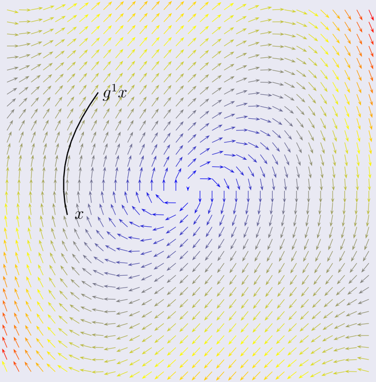
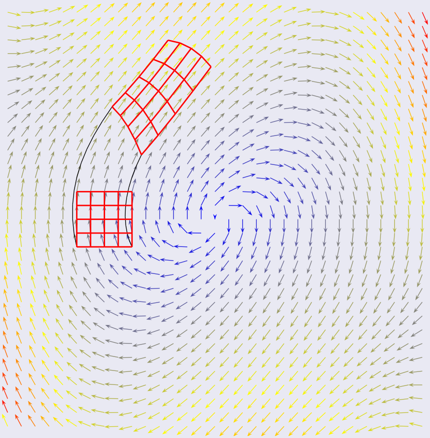
Note that $t \mapsto g^t x$ is the solution of $\dot{x} = f(x)$ satisfying $x(0) = x$.



For a fixed t, g^t is a function from $\mathbb{R}^n \longrightarrow \mathbb{R}^n$

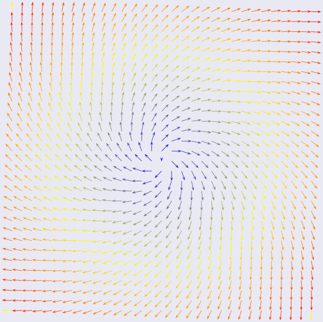


According to the previous theorem, if no component of $g_*^t([x])$ contains 0, then there exists $2n$ corners $\underline{\tilde{x}}_j$ and $\overline{\tilde{x}}_j$ of $[x]$ such that $\prod_{1\leq j\leq n}[g_j^t(\underline{\tilde{x}}_j), g_j^t(\overline{\tilde{x}}_j)]$ is the smallest box containing $g^t([x])$.

Example

Let us consider the following ODE :

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$



Example

One can obtain an explicit solution :

$$\begin{aligned}
\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= \exp(tA) \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \\
&= \begin{pmatrix} e^t \cos(t) & e^t \sin(t) \\ -e^t \sin(t) & e^t \cos(t) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}
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$$\begin{array}{cccc} g^t : & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \mapsto & \begin{pmatrix} e^t \cos(t)\, x_1 + e^t \sin(t)\, x_2 \\ -e^t \sin(t)\, x_1 + e^t \cos(t)\, x_2 \end{pmatrix} \end{array}$$

$$
\begin{array}{cccc}
g^1 : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\
& \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \longmapsto & \begin{pmatrix} e^1 \cos(1)\, x_1 + e^1 \sin(1)\, x_2 \\ -e^1 \sin(1)\, x_1 + e^1 \cos(1)\, x_2 \end{pmatrix}
\end{array}
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$$\begin{aligned} g^1_* &= \begin{pmatrix} \frac{\partial g^1_1}{\partial x_1} & \frac{\partial g^1_1}{\partial x_2} \\ \frac{\partial g^1_2}{\partial x_1} & \frac{\partial g^1_2}{\partial x_2} \end{pmatrix} \\ &= \begin{pmatrix} e^1 \cos(1) & e^1 \sin(1) \\ -e^1 \sin(1) & e^1 \cos(1) \end{pmatrix} \\ &\simeq \begin{pmatrix} 1.468693940 & 2.287355287 \\ -2.287355287 & 1.468693940 \end{pmatrix} \end{aligned}$$

$$g_*^1 \simeq \begin{pmatrix} 1.468693940 & 2.287355287 \\ -2.287355287 & 1.468693940 \end{pmatrix}$$

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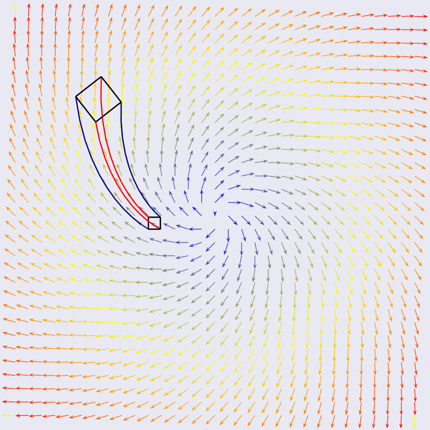
According to the previous theorem :

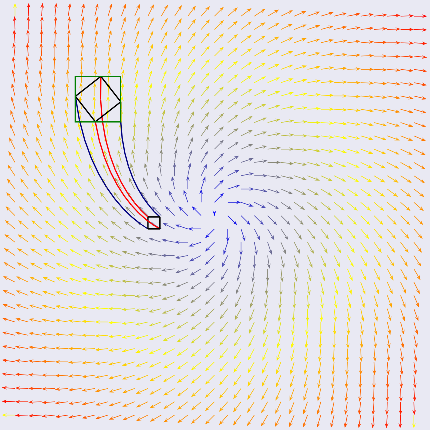
$$\begin{pmatrix} \left[g^1_1 \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} ; g^1_1 \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} \right] \\ \left[g^1_2 \begin{pmatrix} \overline{x}_1 \\ \underline{x}_2 \end{pmatrix} ; g^1_2 \begin{pmatrix} \underline{x}_1 \\ \overline{x}_2 \end{pmatrix} \right] \end{pmatrix}$$

is the smallest box containing $g^1 \begin{pmatrix} [\ \underline{x}_1 ; \overline{x}_1\ ] \\ [\ \underline{x}_2 ; \overline{x}_2\ ] \end{pmatrix}$

$$\left[g_1^1\begin{pmatrix}\underline{x}_1\\ \underline{x}_2\end{pmatrix};g_1^1\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}\right]\times\left[g_2^1\begin{pmatrix}\overline{x}_1\\ \underline{x}_2\end{pmatrix};g_2^1\begin{pmatrix}\underline{x}_1\\ \overline{x}_2\end{pmatrix}\right]$$



$$\left[g_1^1\begin{pmatrix}\underline{x}_1\\ \underline{x}_2\end{pmatrix};g_1^1\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}\right]\times\left[g_2^1\begin{pmatrix}\overline{x}_1\\ \underline{x}_2\end{pmatrix};g_2^1\begin{pmatrix}\underline{x}_1\\ \overline{x}_2\end{pmatrix}\right]$$



Theorem

Suppose that $f:\mathbb{R}^n \to \mathbb{R}^n$ is twice continuously differentiable. Then g_*^t is solution to the initial value problem

$$\begin{aligned} \frac{\partial}{\partial t} g_*^t(x) &= f_*(g^t x) g_*^t(x), \\ g_*^0(x) &= Id \end{aligned}$$

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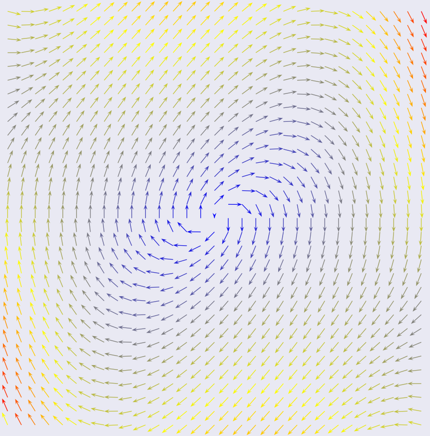
Proof

$$\begin{aligned} \frac{\partial}{\partial t} g_*^t(x) &= \frac{d}{dt} \frac{d}{dx} g^t x \\ &= \frac{d}{dx} \frac{d}{dt} g^t x \\ &= \frac{d}{dx} f(g^t x) \\ &= f_*(g^t x)(g_*^t x) \end{aligned}$$

Example - Van der Pol oscillator

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & (1 - x_1^2)x_2 - x_1 \end{cases}$$



Example - Van der Pol oscillator

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1 - x_1^2)x_2 - x_1 \end{cases}$$

Let us denote by $(a_{i,j})_{1\leq i,j\leq 2}$ the coordinate of g_*^t, *i.e.*

$$g_*^t = \begin{pmatrix} \partial_{x_1} g_1^t & \partial_{x_2} g_1^t \\ \partial_{x_1} g_2^t & \partial_{x_2} g_2^t \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

one has :

$$\frac{\partial}{\partial t}\left(g_*^t\right) = \begin{pmatrix} \dot{a}_{11} & \dot{a}_{12} \\ \dot{a}_{21} & \dot{a}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 - 2x_1x_2 & 1 - x_1^2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Example - Van der Pol oscillator

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1-x_1^2)x_2 - x_1 \\ \dot{a}_{11} &= a_{21} \\ \dot{a}_{12} &= a_{22} \\ \dot{a}_{21} &= -a_{11} - 2x_1x_2a_{11} + a_{21} - x_1^2a_{21} \\ \dot{a}_{22} &= -a_{12} - 2x_1x_2a_{12} + a_{22} - x_1^2a_{22} \end{cases}$$

with the following initial condition

$$\begin{cases} x_1(0) &= x_1^0 \\ x_2(0) &= x_2^0 \\ a_{11}(0) &= 1 \\ a_{12}(0) &= 0 \\ a_{21}(0) &= 0 \\ a_{22}(0) &= 1. \end{cases}$$

Solver

Algorithm

- *Input :*

Algorithm

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  - $\dot{x} = f(x)$

Algorithm

- *Input :*
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  - $[x] \in \mathbb{IR}^n$

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  - t a non-negative real
- Algorithm

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  - Compute rigorously $g_*^t([x])$,

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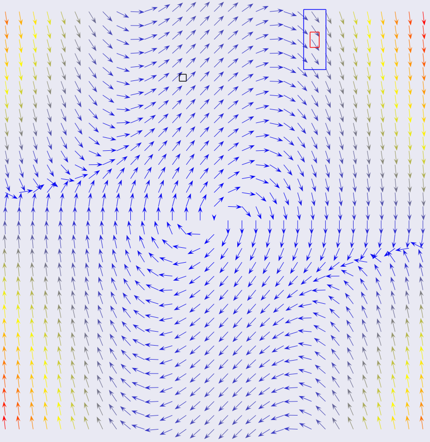
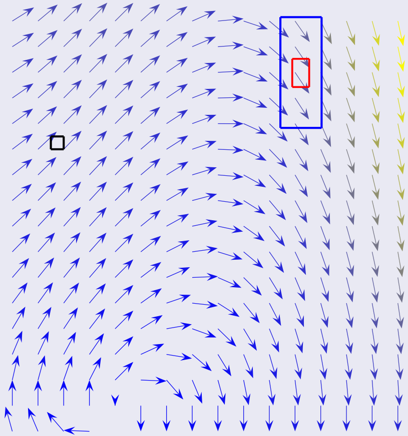
  - Compute rigorously $g_*^t([x])$,
  - if no component of $g_*^t([x])$ contains 0,

Algorithm

- *Input :*
  - $\dot{x} = f(x)$
  - $[x] \in \mathbb{IR}^n$
  - t a non-negative real
- Algorithm
  - Compute rigorously $g_*^t([x])$,
  - if no component of $g_*^t([x])$ contains 0,
  - then compute rigorously the set $\{g_*^t(\tilde{x})\}$ and return the interval hull of this set.



black : initial conditions,
red : our method,
blue : taylor method.



black : initial conditions,
red : our method,
blue : taylor method.

- Thank you for your attention !