Control and State Estimation for max-plus Linear Systems

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Contents

Control and State Estimation for max-plus Linear Systems

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ABSTRACT

Max-plus linear systems theory was inspired by and originated from classical linear systems theory more than three decades ago, with the purpose of dealing with nonlinear synchronization and delay phenomena in timed discrete event systems in a linear manner. Timed discrete event systems are driven by discrete events, are equipped with a notion of time, and their temporal evolution is entirely characterized by the occurrence of events over time. If their behavior is completely governed by synchronization and delay phenomena, timed discrete event systems can be modeled as max-plus linear systems. On appropriate levels of abstraction, such systems adequately describe many problems in diverse areas such as manufacturing, communication, or transportation networks. The aim of this paper is to provide a thorough survey of current research work in max-plus linear systems. It summarizes the main mathematical concepts required for a theory of max-plus linear systems, including idempotent semirings, residuation theory, fixed point equations in the max-plus algebra, formal power series, and timed-event

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graphs. The paper reviews some recent major achievements in control and state estimation of max-plus linear systems. These include max-plus observer design, max-plus model matching by output or state feedback and observer-based control synthesis. Control is required to be optimal with respect to the so-called just-in-time criterion, which is a common standard in industrial engineering. It implies that the time for all input events is delayed as much as possible while guaranteeing that all output events occur, at the latest, at pre-specified reference times.

1

Introduction

Discrete event systems (DESs) are typically understood as event-driven systems whose state evolutions are completely characterized by the occurrence of discrete events over time. They often provide an adequate level of abstraction when modeling manufacturing systems (e.g., Cohen *et al.*, 1983; Cohen *et al.*, 1985), computer networks (e.g., Cruz, 1991; Boudec and Thiran, 2002) or transportation systems (e.g., Braker, 1993; Farhi *et al.*, 2005; Heidergott *et al.*, 2006; Lotito *et al.*, 2001; Olsder *et al.*, 1998). The diversity of phenomena observed in this class of systems led to the emergence of different modeling frameworks such as finite automata (e.g., Hopcroft *et al.*, 2006), Markov chains (e.g., Norris, 1997), and Petri nets (e.g., Reisig, 1985; Murata, 1989). In the context of control of DESs, Cassandras and Lafortune, 2006 and Seatzu *et al.*, 2012 provide extensive surveys on different modeling paradigms. Timed Event Graphs (TEGs) are a subclass of timed Petri nets where the occurrence of events only depends on delay and synchronization phenomena. The latter, when described in standard algebra, are highly nonlinear. Motivated by this, a special algebra, called max-plus algebra, has been suggested, in which these phenomena are linear. For more than three decades, researchers (e.g., Baccelli *et al.*, 1992; Cohen *et al.*, 1998) have been working to establish a linear systems and control theory in this algebra. Probably the first work on manufacturing systems described in this algebraic framework is due to R.A. Cuninghame-Green (Cuninghame-Green, 1962). In 1981 (see Cohen *et al.*, 1999 for a historical review), the Max-Plus working group of the INRIA started to develop a control theory for dynamical systems that are linear in the max-plus algebra. The underlying idea behind these developments is that, by changing the algebra, the behavior of certain discrete event systems can be described by linear equations. This, in turn, can then be exploited for analysis and control synthesis purposes. Hence, metaphorically speaking, by changing one's glasses, it is possible to reexamine a nonlinear world in a linear way. However, there is a price to be paid. Classical control theory is built on powerful mathematical concepts such as linear algebra and vector spaces. In contrast, the max-plus algebra is a weaker structure, namely an idempotent semiring, or dioid. This implies that addition in this algebra (which corresponds to the standard maximum operation) is not invertible. Despite this detriment, it has been possible to develop a rather elegant control theory for dynamical systems that are linear in the max-plus algebra, and several control strategies have been proposed for this class of systems. Examples are optimal open loop control (Cohen *et al.*, 1999; Lhommeau *et al.*, 2005; Menguy *et al.*, 2000) and optimal state and output feedback control in order to solve the model matching problem (Cottenceau *et al.*, 2001b; Lhommeau *et al.*, 2003a; Maia *et al.*, 2003; Maia *et al.*, 2005; Maia *et al.*, 2011), as well as control strategies forcing the state to stay in a specified set (Amari *et al.*, 2012; Katz, 2007; Maia *et al.*, 2005; Necoara *et al.*, 2009).

This paper provides an overview of the max-plus linear systems theory elaborated in the past three decades, especially with respect to the just-in-time criterion, a common standard in industrial engineering. Optimality, in this criterion, means that all input events are delayed as much as possible while ensuring that the output events occur at or before pre-specified reference times.

The paper is organized as follows: in Section 2, a motivational example is introduced. It represents a simple manufacturing system, and it will be used throughout this paper to illustrate the main concepts developed in subsequent sections. Section 3 briefly summarizes timed event graphs, the class of discrete event systems that is investigated in this paper. In this class, the occurrence of discrete events is only governed by delay and synchronization phenomena. Using the example introduced in Section 2, it is shown how to derive equations that describe the temporal evolution of timed event graphs.

In the following sections, the main mathematical foundations for developing a systems and control theory for max-plus linear systems are summarized. Section 4 provides the necessary algebraic background.

Section 5 investigates maps between idempotent semirings and their properties. Section 6 presents useful mathematical results dealing with fixed point equations in the max-plus algebra. Section 7 reviews residuation theory, which plays an essential role in the process of establishing a max-plus linear systems and control theory. Section 8 presents idempotent semirings of formal power series in the event domain. They prove particularly useful for deriving compact models for TEGs. This is discussed in some detail in Section 9.

The main part of this paper reviews some recent major achievements in control and estimation of max-plus linear systems. Section 10 is dedicated to the state estimation problem in max-plus linear systems, and an observer design inspired by Luenberger's approach (Luenberger, 1971) is presented. Section 11 discusses how to synthesize open-loop and closed-loop (both output and state feedback) control by solving an optimization problem with constraints. Optimality is in the sense of the well-known just-in-time criterion while the constraints reflect requirements imposed by a model matching, or model reference, problem (Hardouin *et al.*, 2011; Maia *et al.*, 2003; Maia *et al.*, 2005). Section 12 introduces an observer-based controller for the case when the state of the plant is not completely measurable or when it is too expensive to measure all the states. The resulting observer-based controller is compared with the output feedback and state-feedback controllers described in Section 11. It turns out that the proposed observer-based controller in general indeed provides better performance than an output feedback controller. Section 13 discusses how various control problems can be posed as specific model matching problems by setting up appropriate reference models. Finally, Section 14 illustrates the main results of this paper for the running manufacturing system example.

2

Motivational example

To illustrate the main topics, an elementary example borrowed from a manufacturing setting is used throughout the paper. Consider an assembly line composed of three machines labelled M_1, M_2 and M_3. Machine M_1 processes parts; for each part the processing time is 2 time units, and the capacity of the machine is 1, *i.e.*, it can process one part at a time. Transportation of a raw part to machine M_1 requires 1 time unit, and transporting a processed part from machine M_1 to machine M_3 will also take 1 time unit. Machine M_2 operates in parallel to M_1. It processes raw parts with a fixed processing time of 5, transportation of raw parts to machine M_2 requires 2 time units, and transporting a processed part from machine M_2 to machine M_3 will take 3 time units. The capacity of machine M_2 is also 1. Machine M_3 produces final parts by assembling pre-processed parts from M_1 and M_2. The processing time is equal to 2, and the machine capacity is 3, *i.e.*, it is able to assemble 3 final parts simultaneously. Naturally, processing on machine M_3 can only start when at least one part from machine M_1 and one part from machine M_2 are available. The corresponding basic structure of our small manufacturing system is shown in Fig. 2.1.

This assembly line is a discrete event dynamic system because

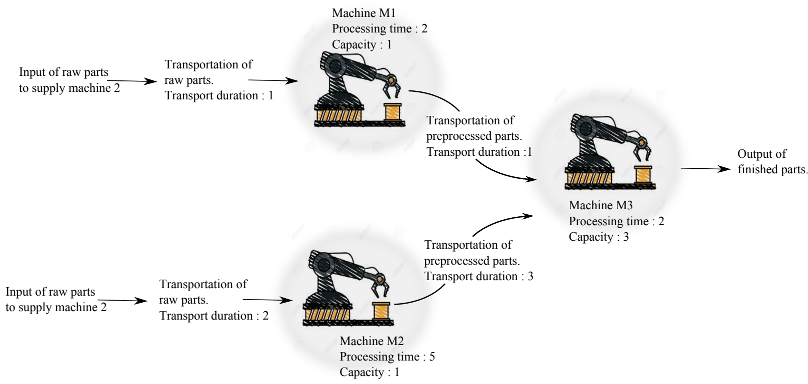


Figure 2.1: Structure of the elementary manufacturing system considered.

its temporal evolution is completely governed by the occurrence of discrete events (e.g., the arrival of raw parts, the start and finish of processing parts on machines, etc.). Note that this example includes synchronization (the processing of parts on machine M_3 cannot begin before parts from the other machines have become available) and delay phenomena, but it does not include logical choice: when parts enter the system, their paths through the system are indeed pre-determined. This precisely characterizes the class of discrete event systems that is considered in this paper – timed discrete event systems whose temporal evolution is exclusively determined by synchronization and delay.

3

Timed event graphs

Petri nets represent a popular framework for modeling discrete event systems. For details and a pedagogical introduction, see, e.g., Cabasino *et al.*, 2013. We will consider a subclass of timed Petri nets exhibiting the properties discussed above, called timed event graphs (TEGs). In this section, we recall the basic properties of TEGs and derive equations that govern their evolution over time.

A TEG is a directed bi-partite graph. It has two types of vertices, namely places and transitions, and directed arcs from places to transitions or from transitions to places. Unlike the general Petri net case, each place has exactly one incoming arc (and therefore one upstream transition) and one outgoing arc (and therefore one downstream transition), and the weight of all arcs is 1. In contrast, transitions may have several (or no) upstream and downstream places. A marking is associated to each place; it represents the number of tokens which are assigned to the place. Graphically, places are represented by circles and transitions by bars. Tokens associated to places are indicated by black bullets inside the circles representing the respective places. The "rules of operation" are as follows: a transition can fire if all its upstream places contain at least one token. If a transition fires, it removes one token

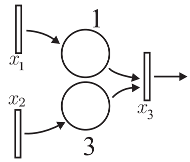


Figure 3.1: Timed event graph with holding times.

from each upstream place and deposits one token in each downstream place. Timing information can be added in different ways: time can either be associated with transitions (representing transition delays) or with places (representing holding times). In the first case, a transition can only fire, if its logical firing condition is satisfied, *i.e.*, all upstream places contain at least one token, and the associated delay has passed. In the second case, a token in a place only contributes to satisfying firing conditions, if it has resided there for at least the required holding time. It can easily be shown that TEGs with transition delays can always be rewritten as equivalent TEGs with holding times, but not the other way round. In the sequel, we consider the more general case, *i.e.*, timed event graphs with holding times.

In Fig. 3.1, an elementary TEG with two places and three transitions is shown. Transitions are labeled x_1, x_2 and x_3. As transitions x_1 and x_2 do not have upstream places, they can fire autonomously. When transition x_1 fires, a token is put in the place between x_1 and x_3, and it has to spend at least 1 time unit there before being able to contribute to the firing of transition x_3. When transition x_2 fires, a token is deposited in the place between x_2 and x_3, where it has to spend at least 3 time units before being able to contribute to the firing of transition x_3. Denoting $x_i(k)$ as the time of the k^{th} firing of transition x_i, it follows that the time for the k^{th} firing of transition x_3 satisfies the following inequality:

$$x_3(k) \geq \max(x_1(k) + 1, x_2(k) + 3). \tag{3.1}$$

Remark 1 (Earliest firing rule). In the following, we will assume that TEGs operate under the so-called earliest firing-rule, i.e, each transition fires as soon as it is enabled. If the TEG in Fig. 3.1 operates under the earliest firing rule, inequality (3.1) becomes an equality.

Example 1 (Manufacturing system). Reconsidering the small manufacturing process introduced in Section 2, we will show how this system can be modeled as a TEG. To do this, the user has to decide which events are important or essential to model the system. Based on Fig. 2.1, the TEG given in Fig. 3.2 can be obtained. The transitions labeled u_1 and u_2 represent the arrival of raw parts. The holding times of their downstream places represent the respective transportation times to machines M_1 and M_2. Transition x_1 represents the start of a processing step on machine M_1, while x_2 represents the end of this processing step; the holding time of the place between x_1 and x_2 corresponds to the processing time of this machine (2 time units). The capacity of machine M_1 is modeled by the number of tokens residing initially in the place with upstream transition x_2 and downstream transition x_1. The fact that this place initially contains one token reflects that the machine has capacity one: if x_1 has fired (and the token in the place has been removed), *i.e.*, if machine M_1 has started processing a workpiece, it cannot accept the next workpiece before x_2 has fired (and a token has again been deposited in this place), *i.e.*, before the machine has finished processing its current workpiece. Machines M_2 and M_3 are modeled in the same way. M_2 has processing time 5 and capacity 1, while M_3 requires a processing time of 2 and has capacity 3. The transportation time from machine M_1 to machine M_3 is modeled by a holding time of 1 associated with the place between transition x_2 and transition x_5. Similarly, the transportation time from machine M_1 to machine M_2 is modeled by a holding time of 3 associated with the place between transition x_4 and transition x_5. Transition y represents the output of the system, *i.e.*, $y(k)$ denotes the time when the k^{th} finished part becomes available.

Assuming the earliest firing rule, we can immeditaley deduce recursive equations for the firing instants of transitions x_i, $i = 1, \ldots 6$, from

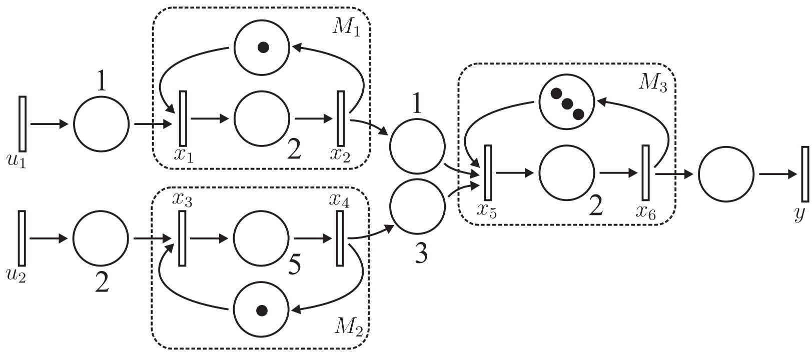


Figure 3.2: Timed event graph of a simple manufacturing system.

Fig. 3.2.

$$x_1(k) = \max(1 + u_1(k), x_2(k-1)) \tag{3.2}$$

$$x_2(k) = x_1(k) + 2 \tag{3.3}$$

$$x_3(k) = \max(u_2(k) + 2, x_4(k-1)) \tag{3.4}$$

$$x_4(k) = x_3(k) + 5 \tag{3.5}$$

$$x_5(k) = \max(x_2(k) + 1, x_4(k) + 3, x_6(k-3)) \tag{3.6}$$

$$x_6(k) = x_5(k) + 2. \tag{3.7}$$

In the same manner, the firing time instants of the output transition y can be determined as

$$y(k) = x_6(k).$$

Events are understood as the firing of transitions, hence the variable k "counts" events. Note that, by convention, the counting of events is started at 0. Note furthermore that the shifts in the event domain in the above equations are being caused by the initial marking of the TEG, *i.e.*, by the number of tokens residing initially in places. For example, in Eq. (3.2), the event shift $x_2(k-1)$ is due to the initial marking of the place between transition x_2 and x_1. This marking implies that the k^{th} firing of transition x_1 depends on the $(k-1)^{st}$ firing of transition x_2, *i.e.*, the k^{th} firing of transition x_1 can only occur when u_1 has fired k times and 1 time unit has subsequently passed and when x_2 has fired

$k-1$ times. Similarly, the event shift $x_6(k-3)$ in Eq. (3.6) is being caused by the fact that the place between x_6 and x_5 initially contains three tokens.

Given a sequence of firing time vectors $u(k) = [u_1(k)\, u_2(k)]^T$, $k = 0, 1, \ldots$, assuming initial conditions $\forall k < 0, x_i(k) = -\infty$, and using Equations (3.2)–(3.7), the resulting firing time vectors $x(k) = [x_1(k)\, x_2(k)\, x_3(k)\, x_4(k)\, x_5(k)\, x_6(k)]^T$ can be determined for $k = 0, 1, \ldots$. If the input is chosen such that it does not slow down the system, e.g., by making an unlimited number of raw parts available at time 0 (which is equivalent to $\forall k \geq 0, u(k) = [0\, 0]^T$), the resulting series of firing time vectors $x(k)$ becomes

$$\underbrace{\begin{bmatrix} 1 \\ 3 \\ 2 \\ 7 \\ 10 \\ 12 \end{bmatrix}}_{x(0)}, \underbrace{\begin{bmatrix} 3 \\ 5 \\ 7 \\ 12 \\ 17 \\ 19 \end{bmatrix}}_{x(1)}, \underbrace{\begin{bmatrix} 5 \\ 7 \\ 12 \\ 17 \\ 20 \\ 22 \end{bmatrix}}_{x(2)}, \underbrace{\begin{bmatrix} 7 \\ 9 \\ 17 \\ 21 \\ 24 \\ 26 \end{bmatrix}}_{x(3)}, \underbrace{\begin{bmatrix} 9 \\ 11 \\ 21 \\ 26 \\ 29 \\ 31 \end{bmatrix}}_{x(4)}, \ldots \tag{3.8}$$

Note that each entry x_i of the vector x is a non-decreasing series, *i.e.*, $x_i(k+1) \geq x_i(k), k = 0, 1, \ldots$. This simply reflects the fact that the $k+1$st firing of a transition cannot occur before its k^{th} firing.

Clearly, determining the firing instants using the recursive Equations (3.2)–(3.7) involves addition and the maximum operation. These equations are non-linear in conventional algebra. However, there is a mathematical structure called idempotent semirings (or dioids) in which the recurrence relations of the firing instants have a linear representation. It is shown in the sequel that these algebraic structures are extremely useful to analyze the performance of the system and to establish a control theory that, to a certain extent, resembles the one developed for systems that are linear in the standard algebra. Many results would not appear so evidently, or even not at all, by not considering these algebraic structures. Switching to such structures is therefore the equivalent of acquiring a new pair of glasses through which the (same) world reveals its properties in a much clearer way.

4

Algebraic setting

This section summarizes the algebraic concepts that are required to obtain a linear representation of the transition firing instants in a timed event graph. This section is necessarily rather technical and does not claim to be exhaustive. For a more exhaustive description, the interested reader is referred to Baccelli *et al.*, 1992.

4.1 Ordered sets

Definition 1 (Order relation)**.** A binary relation $\preceq$ on a set $\mathcal{C}$ is an order relation if the following properties hold for all $a, b, c \in \mathcal{C}$:

- reflexivity: $a \preceq a$,
- anti-symmetry: $(a \preceq b \text{ and } b \preceq a) \Rightarrow a = b$,
- transitivity: $(a \preceq b \text{ and } b \preceq c) \Rightarrow a \preceq c$.

Definition 2 (Ordered set)**.** A set $\mathcal{C}$ endowed with an order relation $\preceq$ is said to be an ordered set and is denoted $(\mathcal{C}, \preceq)$. It is said to be a totally ordered set if any pair of elements in $\mathcal{C}$ can be compared with respect to $\preceq$, *i.e.*, $\forall a, b \in \mathcal{C}$ one can either write $a \preceq b$ or $b \preceq a$. Otherwise $(\mathcal{C}, \preceq)$ is said to be partially ordered.

Example 2 (Ordered sets). A classical example of an ordered set is $(\mathbb{Z}, \leq)$, *i.e.*, the set of (scalar) integers endowed with the classical "less or equal" order relation. Clearly, $(\mathbb{Z}, \leq)$ is totally ordered. In contrast, the ordered set $(\mathbb{Z}^2, \preceq)$, where vectors $\mathbf{x} = [x_1\, x_2]^T$ and $\mathbf{y} = [y_1\, y_2]^T$ are ordered, *i.e.*, $\mathbf{x} \preceq \mathbf{y}$, if $x_1 \leq y_1$ and $x_2 \leq y_2$, is only partially ordered, as it is not possible to compare all pairs of vectors with integer entries. For example, the vectors $[a, b]^T$ and $[b, a]^T$ are not related when $a \neq b$.

Definition 3 (Bounds on ordered sets). Given a non-empty subset $\mathcal{B} \subseteq \mathcal{C}$ of an ordered set $(\mathcal{C}, \preceq)$, element $a \in \mathcal{C}$ is called a lower bound of $\mathcal{B}$ if $\forall b \in \mathcal{B} : a \preceq b$. If $\mathcal{B}$ has a lower bound, its greatest lower bound (glb) is denoted $\bigwedge \mathcal{B}$. Similarly, an element $c \in \mathcal{C}$ is called an upper bound of $\mathcal{B}$ if $\forall b \in \mathcal{B} : b \preceq c$. If $\mathcal{B}$ has an upper bound, its least upper bound (lub) is denoted $\bigvee \mathcal{B}$.

Definition 4 (Lattices). An ordered set $(\mathcal{C}, \preceq)$ is called sup-semi-lattice, if $\forall a, b \in \mathcal{C}$ there exists $a \vee b = \bigvee\{a, b\}$. It is said to be a *complete* sup-semi-lattice, if for every subset $\mathcal{B} \subseteq \mathcal{C}$ there exists a least upper bound, *i.e.*, $\bigvee \mathcal{B}$ exists $\forall \mathcal{B} \subseteq \mathcal{C}$. Analogously, an ordered set $(\mathcal{C}, \preceq)$ is called an inf-semi-lattice, if $\forall a, b, \in \mathcal{C}$ there exists $a \wedge b = \bigwedge\{a, b\}$, and it is said to be a *complete* inf-semi-lattice, if $\forall \mathcal{B} \subseteq \mathcal{C}$, there exists a greatest lower bound $\bigwedge \mathcal{B}$. If an ordered set $(\mathcal{C}, \preceq)$ forms a sup-semi-lattice as well as an inf-semi-lattice, it is called a lattice and denoted $(\mathcal{C}, \vee, \wedge)$. In lattices, the following properties hold $\forall a, b \in \mathcal{C}$:

$$a \preceq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a.$$

A lattice is called a *complete* (or bounded) lattice if it is a complete sup-semi-lattice as well as a complete inf-semi-lattice. The lub of a complete lattice is denoted $\top$ (top element), the glb is denoted $\perp$ (bottom element).

Remark 2. The operations $\vee$ and $\wedge$ of a lattice $(\mathcal{C}, \vee, \wedge)$ are associative, commutative, idempotent, *i.e.*, $\forall a \in \mathcal{C}, a \vee a = a$ and $a \wedge a = a$, and the absorption property holds, *i.e.*, $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$, $\forall a, b \in \mathcal{C}$.

4.2 Idempotent semirings

An idempotent semiring, also called a dioid (see Heidergott *et al.*, 2006, Gondran and Minoux, 2008, Baccelli *et al.*, 1992), is a specific algebraic structure. Below, the definitions and properties of this structure are summarized.

Definition 5 (Monoid)**.** A set $\mathcal{M}$ equipped with a binary operation $\oplus : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a monoid, denoted $(\mathcal{M}, \oplus)$, if $\oplus$ is associative, *i.e.*, $\forall a, b, c \in \mathcal{M}, a \oplus (b \oplus c) = (a \oplus b) \oplus c$, and has a neutral element, denoted ε, *i.e.*, $\forall m \in \mathcal{M}, m \oplus \varepsilon = \varepsilon \oplus m = m$. If $\oplus$ is commutative, *i.e.*, $\forall a, b \in \mathcal{M}, a \oplus b = b \oplus a$, the monoid is said to be commutative.

Definition 6 (Idempotent semiring, dioid)**.** A set $\mathcal{D}$ equipped with two binary operations $\oplus$ and $\otimes$ is an idempotent semiring, or dioid, denoted $(\mathcal{D}, \oplus, \otimes)$, if the following axioms hold (Heidergott *et al.*, 2006,Gondran and Minoux, 1984,Baccelli *et al.*, 1992):

- $(\mathcal{D}, \oplus)$ is a commutative monoid with neutral element ε (also called “zero element”), and $\oplus$ is idempotent, i.e, $\forall a \in \mathcal{D}, a \oplus a = a$,
- $(\mathcal{D}, \otimes)$ is a monoid with neutral element e (also called “one element”),
- $\otimes$ distributes over $\oplus$, *i.e.*, $\forall a, b, c \in \mathcal{D}, (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$ and $c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$,
- ε is absorbing for $\otimes$, *i.e.*, $\forall a \in \mathcal{D}, a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$.

Furthermore, if $(\mathcal{D}, \otimes)$ is a commutative monoid, the idempotent semiring $(\mathcal{D}, \oplus, \otimes)$ is said to be commutative.

Remark 3. If all elements of a dioid (except ε) have a multiplicative inverse, it forms an idempotent semifield.

Remark 4. As in classical algebra, the multiplication sign $\otimes$ is often omitted when unambiguous.

Definition 7 (Canonical order relation)**.** On an idempotent semiring $(\mathcal{D}, \oplus, \otimes)$, a canonical order can be defined by $a \preceq b \iff a \oplus b = b$. Then, $(\mathcal{D}, \preceq)$ becomes a sup-semi-lattice, and $a \vee b = a \oplus b$.

Definition 8 (Complete idempotent semiring). An idempotent semiring is complete if it is closed for infinite sums and if $\otimes$ distributes over infinite sums, *i.e.*, if $\forall c \in \mathcal{D}$ and $\forall \mathcal{X} \subseteq \mathcal{D}$

$$c \otimes (\bigoplus_{x \in \mathcal{X}} x) = \bigoplus_{x \in \mathcal{X}} c \otimes x.$$

Remark 5. An idempotent semiring has the structure of a sup-semi-lattice. Hence, if it is complete, it forms a complete sup-semi-lattice and therefore has a top (or greatest) element $\top$, which corresponds to the sum of all elements in the dioid $\mathcal{D}$, *i.e.* : $\top = \bigoplus_{x \in \mathcal{D}} x$. Furthermore, an idempotent semiring always admits ε as bottom (or minimal) element. A complete idempotent semiring is then a complete sup-semi-lattice with a minimal element. According to Definition 4, a complete idempotent semiring has therefore the structure of a complete lattice for the order $\preceq$.

Definition 9. If $\mathcal{D}$ is a complete idempotent semiring, then the greatest lower bound of $a, b \in \mathcal{D}$ is defined as

$$a \wedge b = \bigoplus_{x \preceq a, x \preceq b} x.$$

$\wedge$ is associative, commutative and idempotent, and the following equivalences hold

$$a = a \oplus b \Leftrightarrow a \succeq b \Leftrightarrow b = a \wedge b. \tag{4.1}$$

Remark 6. From (4.1), it follows that $\oplus$, $\otimes$, $\wedge$ are order preserving, *i.e.*, $\forall a, b, c \in \mathcal{D}$ the following implications hold:

$$\begin{aligned} a \preceq b &\quad \Rightarrow \quad a \otimes c \preceq b \otimes c, \\ a \preceq b &\quad \Rightarrow \quad a \oplus c \preceq b \oplus c, \\ a \preceq b &\quad \Rightarrow \quad a \wedge c \preceq b \wedge c. \end{aligned}$$

This can be easily seen from the following arguments:

$$\begin{aligned} a \preceq b &\Leftrightarrow a \oplus b = b \\ &\Rightarrow (a \oplus b) \otimes c = a \otimes c \oplus b \otimes c = b \otimes c \\ &\Leftrightarrow a \otimes c \preceq b \otimes c. \end{aligned}$$

$$\begin{aligned} a \preceq b &\Leftrightarrow a \oplus b = b \\ &\Rightarrow a \oplus b \oplus c = a \oplus b \oplus c \oplus c = (a \oplus c) \oplus (b \oplus c) = b \oplus c \\ &\Leftrightarrow a \oplus c \preceq b \oplus c. \end{aligned}$$

$$\begin{aligned} a \preceq b &\Leftrightarrow a = a \wedge b \\ &\Rightarrow a \wedge c = (a \wedge b) \wedge c = (a \wedge c) \wedge (b \wedge c) \\ &\Leftrightarrow a \wedge c \preceq b \wedge c. \end{aligned}$$

Remark 7. Relation (4.1) seems to indicate that the operators $\oplus$ and $\wedge$ play a symmetric role. This is indeed true from the lattice point of view since operator $\oplus$ corresponds to $\vee$. This is false, however, if we consider the second operator of the semiring, namely $\otimes$, since $\otimes$ distributes over $\oplus$, but not over $\wedge$. Nevertheless, the following property, often referred to as subdistributivity, holds :

$$\forall a, b, c \in \mathcal{D}, \quad c \otimes (a \wedge b) \preceq (c \otimes a) \wedge (c \otimes b).$$

This follows immediately from $a \wedge b \preceq a$ and $a \wedge b \preceq b$. As the product is order preserving (see Remark 6), we have $c \otimes (a \wedge b) \preceq c \otimes a$ and $c \otimes (a \wedge b) \preceq c \otimes b$ and therefore $c \otimes (a \wedge b) \preceq (c \otimes a) \wedge (c \otimes b)$.

Example 3 (max-plus algebra). $\overline{\mathbb{Z}}_{\max} = (\mathbb{Z} \cup \{-\infty, +\infty\}, \max, +)$ is a complete idempotent semiring. By definition, $a \oplus b = \max(a, b)$, $a \otimes b = a + b$, $a \wedge b = \min(a, b)$, with $\varepsilon = -\infty$, $e = 0$, and $\top = +\infty$. The order $\preceq$ is total and corresponds to the natural order $\leq$.

Example 4 (min-plus algebra). $\overline{\mathbb{Z}}_{\min} = (\mathbb{Z} \cup \{-\infty, +\infty\}, \min, +)$ is a complete idempotent semiring. By definition, $a \oplus b = \min(a, b)$, $a \otimes b = a + b$, $a \wedge b = \max(a, b)$, with $\varepsilon = +\infty$, $e = 0$, and $\top = -\infty$. The order $\preceq$ is total and corresponds to the inverse of the natural order (*i.e.*, $a \preceq b$ iff $a \geq b$).

Definition 10 (Subsemiring)**.** Let $(\mathcal{D}, \oplus, \otimes)$ be a semiring and $\mathcal{C} \subseteq \mathcal{D}$. $(\mathcal{C}, \oplus, \otimes)$ is a subsemiring of $\mathcal{D}$ if $\varepsilon, e \in \mathcal{C}$ and if $\mathcal{C}$ is closed under the operations $\oplus$ and $\otimes$. A subsemiring is complete if it is closed for infinite sums too.

Note that, just as in standard algebra, addition and multiplication can be readily extended to matrices of appropriate dimensions with elements in a dioid $\mathcal{D}$. Namely, matrix addition is done elementwise, and matrix multiplication is performed by multiplying the appropriate rows and columns:

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}, \quad A, B \in \mathcal{D}^{m \times n} \tag{4.2}$$

$$(C \otimes D)_{ik} = \bigoplus_{j=1}^{n} (c_{ij} \otimes d_{jk}), \quad C \in \mathcal{D}^{m \times n}, D \in \mathcal{D}^{n \times p}. \tag{4.3}$$

Furthermore, the order $\preceq$ on $\mathcal{D}$ induces a (partial) order on the set of matrices with entries in $\mathcal{D}$, *i.e.*, for matrices $A, B \in \mathcal{D}^{m \times n}$ the following equivalence holds:

$$A \preceq B \Leftrightarrow a_{ij} \preceq b_{ij} \quad \forall i \in [1, m], \forall j \in [1, n]. \tag{4.4}$$

Remark 8. The set of square $(n \times n)$ matrices with elements in a complete dioid $\mathcal{D}$, together with the operations $\oplus$ and $\otimes$ defined above, is a complete idempotent semiring. The one element (or identity matrix) is the $n \times n$-matrix with entries equal to e on the diagonal and ε elsewhere, it is denoted I_n in the sequel. The zero element in this semiring is the $n \times n$-matrix with all entries equal to ε, it is denoted ε, and the top element is the $n \times n$-matrix with all entries equal to $\top$, denoted $\top$. Note that the order defined in (4.4) is consistent with the definition of the canonical order on this semiring, *i.e.*, $A \preceq B \Leftrightarrow A \oplus B = B$.

Example 5. To get a first idea of the usefulness of investigating the behaviour of TEGs in an appropriate dioid setting, consider the simple manufacturing example from Section 2. If we interprete equations (3.2)

– (3.7) in the dioid $\overline{\mathbb{Z}}_{\max}$, we get the following set of linear equations

$$x_1(k) = x_2(k-1) \oplus 1u_1(k) \tag{4.5}$$
$$x_2(k) = 2x_1(k) \tag{4.6}$$
$$x_3(k) = x_4(k-1) \oplus 2u_2(k) \tag{4.7}$$
$$x_4(k) = 5x_3(k) \tag{4.8}$$
$$x_5(k) = 1x_2(k) \oplus 3x_4(k) \oplus x_6(k-3) \tag{4.9}$$
$$x_6(k) = 2x_5(k). \tag{4.10}$$

For initial conditions $\forall k < 0,\ x_i(k) = \varepsilon$ and an input sequence that corresponds to providing an infinite number of raw parts at time 0, *i.e.*, $\forall k \geq 0, u(k) = [e\ e]^T$, we again obtain the series of firing time vectors (3.8) from (4.5)–(4.10). As the order $\preceq$ in the max-plus algebra coincides with the natural order $\leq$, the series of firing times are clearly non-decreasing in the dioid $\overline{\mathbb{Z}}_{\max}$, *i.e.*, $x_i(k) \preceq x_i(k+1), k = 0, 1, \ldots$

5

Mappings defined over idempotent semirings

In this section, $\mathcal{C}$ and $\mathcal{D}$ refer to complete idempotent semirings. To keep notation reasonably simple, addition in both semirings will be denoted by the same symbol, $\oplus$, and multiplication in both semirings will be denoted by $\otimes$.

Definition 11 (Continuity). A mapping Π from a complete idempotent semiring $\mathcal{D}$ to a complete idempotent semiring $\mathcal{C}$ is lower semi-continuous (denoted $l.s.c.$) if for all finite or infinite sets $\mathcal{X} \subseteq \mathcal{D}$,

$$\Pi(\bigoplus_{x \in \mathcal{X}} x) = \bigoplus_{x \in \mathcal{X}} \Pi(x),$$

and it is upper semi-continuous (denoted $u.s.c$) if for all finite or infinite sets $\mathcal{X} \subseteq \mathcal{D}$

$$\Pi(\bigwedge_{x \in \mathcal{X}} x) = \bigwedge_{x \in \mathcal{X}} \Pi(x).$$

It is continuous if it is both $l.s.c.$ and $u.s.c.$.

Definition 12 (Isotone, antitone, monotone). Let $\Pi : \mathcal{D} \to \mathcal{C}$ be a mapping, with $\mathcal{D}$ and $\mathcal{C}$ two idempotent semirings. Mapping Π is

- isotone if it is order preserving, $i.e.$, $\forall x, x' \in \mathcal{D}$ the following implication holds: $x \preceq x' \Rightarrow \Pi(x) \preceq \Pi(x')$,

- antitone if it inverts the order, *i.e.*, $\forall x, x' \in \mathcal{D}$ the following implication holds: $x \preceq x' \Rightarrow \Pi(x) \succeq \Pi(x')$,

- monotone if is isotone or antitone.

Remark 9. The composition of two monotone mappings is a monotone mapping. In particular, it can be easily checked that the composition of

- two isotone mappings is isotone,

- two antitone mappings is isotone,

- an isotone mapping with an antitone mapping is antitone.

Theorem 1. Let $\Pi : \mathcal{D} \to \mathcal{C}$ be a mapping, with $\mathcal{D}$ and $\mathcal{C}$ two idempotent semirings:

1. if Π is $l.s.c$ then it is isotone,

2. if Π is $u.s.c$ then it is isotone.

Proof.
1. Let $x, x' \in \mathcal{D}$. If $x \preceq x'$, then $x \oplus x' = x'$, then $\Pi(x \oplus x') = \Pi(x) \oplus \Pi(x') = \Pi(x')$ since Π is $l.s.c.$, hence $\Pi(x) \preceq \Pi(x')$.
2. Let $x, x' \in \mathcal{D}$. If $x \preceq x'$, then $x \wedge x' = x$, and $\Pi(x \wedge x') = \Pi(x) \wedge \Pi(x') = \Pi(x)$ since Π is $u.s.c.$, hence $\Pi(x) \preceq \Pi(x')$. $\square$

Remark 10. If $\Pi : \mathcal{D} \to \mathcal{C}$ is an isotone mapping, the following inequality holds

$$\Pi(x \oplus x') \succeq \Pi(x) \oplus \Pi(x') \qquad \forall x, x' \in \mathcal{D},$$

since

$$\left.\begin{array}{lcl} x \oplus x' \succeq x & \Rightarrow & \Pi(x \oplus x') \succeq \Pi(x) \\ x \oplus x' \succeq x' & \Rightarrow & \Pi(x \oplus x') \succeq \Pi(x') \end{array}\right\} \Rightarrow \Pi(x \oplus x') \succeq \Pi(x) \oplus \Pi(x').$$

Moreover, the following inequality holds

$$\Pi(x \wedge x') \preceq \Pi(x) \wedge \Pi(x') \qquad \forall x, x' \in \mathcal{D},$$

as

$$\left.\begin{array}{lcl} x \wedge x' \preceq x & \Rightarrow & \Pi(x \wedge x') \preceq \Pi(x) \\ x \wedge x' \preceq x' & \Rightarrow & \Pi(x \wedge x') \preceq \Pi(x') \end{array}\right\} \Rightarrow \Pi(x \wedge x') \preceq \Pi(x) \wedge \Pi(x').$$

Definition 13 (Homomorphism)**.** A mapping $\Pi : \mathcal{D} \to \mathcal{C}$ is a homomorphism if

$$\forall a, b \in \mathcal{D} \qquad \Pi(a \oplus b) = \Pi(a) \oplus \Pi(b) \text{ and } \Pi(\varepsilon) = \varepsilon, \tag{5.1}$$
$$\Pi(a \otimes b) = \Pi(a) \otimes \Pi(b) \text{ and } \Pi(e) = e. \tag{5.2}$$

A mapping satisfying only (5.1) is said to be a $\oplus$-morphism, *i.e.*, the image of the sum of elements in $\mathcal{D}$ is the sum, in $\mathcal{C}$, of their images. A mapping satisfying only (5.2) is said to be a $\otimes$-morphism, *i.e.*, the image of the product of two elements of $\mathcal{D}$ is the product, in $\mathcal{C}$, of their images.

Definition 14 (Isomorphism)**.** A mapping $\Pi : \mathcal{D} \to \mathcal{C}$ is an isomorphism if the inverse of Π is defined and if Π and its inverse mapping are homomorphisms.

Definition 15 (Equivalence relation)**.** An equivalence relation $\mathcal{R}$ on a set E is a binary relation which is:

- reflexive: $\forall\, x \in E,\ x\mathcal{R}x$,
- symmetric: $\forall\, x, y \in E,\ x\mathcal{R}y \Rightarrow y\mathcal{R}x$,
- transitive: $\forall\, x, y, z \in E,\ (x\mathcal{R}y \text{ and } y\mathcal{R}z) \Rightarrow x\mathcal{R}z$.

Definition 16 (Congruence)**.** A congruence on an idempotent semiring $\mathcal{D}$ is an equivalence relation (denoted $\mathcal{R}$) compatible with the semiring laws, *i.e.*, $\forall a, b, c \in \mathcal{D}$,

$$a\,\mathcal{R}\,b \Rightarrow (a \oplus c)\,\mathcal{R}\,(b \oplus c), \quad (c \otimes a)\,\mathcal{R}\,(c \otimes b), \quad (a \otimes c)\,\mathcal{R}\,(b \otimes c).$$

Theorem 2 (Quotient semiring, Baccelli *et al.*, 1992)**.** Let $\mathcal{D}$ be an idempotent semiring and $\mathcal{R}$ a congruence over $\mathcal{D}$, and let $[a] = \{x \in \mathcal{D} \,|\, x\,\mathcal{R}\,a\}$ denote the equivalence class of $a \in \mathcal{D}$. Then, the quotient set of $\mathcal{D}$ by this congruence, denoted $\mathcal{D}_{/\mathcal{R}}$, is a semiring with the following sum and product:

$$\begin{array}{rcl} [a] \oplus [b] & \triangleq & [a \oplus b], \\ [a] \otimes [b] & \triangleq & [a \otimes b]. \end{array} \tag{5.3}$$

Proof. As $\mathcal{R}$ is a congruence on $\mathcal{D}$, for all $a, a', b, b' \in \mathcal{D}$ such that $[a] = [a']$ and $[b] = [b']$, the following holds:

$$[a \oplus b] = [a' \oplus b] = [a' \oplus b'] \text{ and } [a \otimes b] = [a' \otimes b] = [a' \otimes b'],$$

i.e., the equivalence classes $[a \oplus b]$ and $[a \otimes b]$ are exclusively defined by the equivalence classes $[a]$ and $[b]$, and not by specific elements. The operations on the quotient set given by (5.3) are then perfectly defined, hence the quotient $\mathcal{D}_{/\mathcal{R}}$ inherits the structure of idempotent semiring from $\mathcal{D}$. □

Theorem 3 (Baccelli *et al.*, 1992)**.** Let $\Pi : \mathcal{D} \to \mathcal{C}$ be a homomorphism. Relation $\mathcal{R}_\Pi$ defined by

$$a \, \mathcal{R}_\Pi \, b \iff \Pi(a) = \Pi(b), \quad \forall a, b \in \mathcal{D},$$

is a congruence.

Proof. First, it is clear that $\mathcal{R}_\Pi$ is an equivalence relation. We need to show that it respects the operations $\oplus$ and $\otimes$ in $\mathcal{D}$. This is straightforward, as, due to Π being a homomorphism, $\Pi(a) = \Pi(b)$ implies for all $c \in \mathcal{D}$

- $\Pi(a \oplus c) = \Pi(a) \oplus \Pi(c) = \Pi(b) \oplus \Pi(c) = \Pi(b \oplus c)$, *i.e.*, $(a \oplus c)\mathcal{R}_\Pi(b \oplus c)$,
- $\Pi(c \otimes a) = \Pi(c) \otimes \Pi(a) = \Pi(c) \otimes \Pi(b) = \Pi(c \otimes b)$, *i.e.*, $(c \otimes a)\mathcal{R}_\Pi(c \otimes b)$,
- $\Pi(a \otimes c) = \Pi(a) \otimes \Pi(c) = \Pi(b) \otimes \Pi(c) = \Pi(b \otimes c)$, *i.e.*, $(a \otimes c)\mathcal{R}_\Pi(b \otimes c)$.

□

Definition 17 (Coimage)**.** The quotient of $\mathcal{D}$ by the congruence $\mathcal{R}_\Pi$, denoted $\mathcal{D}_{/\mathcal{R}_\Pi}$, is called the coimage of Π.

Definition 18 (Image)**.** The image of mapping $\Pi : \mathcal{D} \to \mathcal{C}$ is denoted $\mathsf{Im}\Pi$ and is defined as follows:

$$\mathsf{Im}\Pi = \{y \in \mathcal{C} | y = \Pi(x) \text{ for some } x \in \mathcal{D}\}.$$

Remark 11. The mapping $g : \mathcal{D}_{/\mathcal{R}_\Pi} \to \mathsf{Im}\Pi$, with $g([x]_\Pi) = \Pi(x)$, is an isomorphism.

Definition 19. Let $\mathcal{P}$ be the set of mappings from $\mathcal{D}$ to $\mathcal{C}$. The order $\preceq$ defined on $\mathcal{C}$ induces an order on $\mathcal{P}$, for simplicity also denoted $\preceq$, by

$$(\forall \Pi_1, \Pi_2 \in \mathcal{P}) \;\; \Pi_1 \preceq \Pi_2 \Leftrightarrow (\forall x \in \mathcal{D}) \;\; \Pi_1(x) \preceq \Pi_2(x).$$

6

Fixed points of monotone mappings

This section collects some useful results in order to deal with fixed point equations. It will be recalled that iterative algorithms can be used to compute fixed points of equations involving monotone mappings. The results are based on the Knaster-Tarski theorem, which states that the set of fixed points of an isotone mapping Π, defined over a complete lattice, is also a complete lattice. This theorem guarantees the existence of at least one fixed point of Π, and it also guarantees the existence of a least and a greatest fixed point.

In the following, these results are adapted to the setting of semirings by recalling that a complete idempotent semiring is a complete lattice (see Remark 5).

Theorem 4. Let $\Pi : \mathcal{D} \to \mathcal{D}$ be an isotone mapping with $\mathcal{D}$ a complete idempotent semiring. Let $\mathcal{Y} = \{x \in \mathcal{D} | \Pi(x) = x\}$ be the set of fixed points of Π.

1. $\bigwedge_{y \in \mathcal{Y}} \ y$ is the least fixed point of Π, *i.e.*, $\bigwedge_{y \in \mathcal{Y}} \ y \in \mathcal{Y}$, and it satisfies $\bigwedge_{y \in \mathcal{Y}} \ y = \bigwedge \{x \in \mathcal{D} | \Pi(x) \preceq x\}$.

2. $\bigvee_{y \in \mathcal{Y}} \ y$ is the greatest fixed point of Π, *i.e.*, $\bigvee_{y \in \mathcal{Y}} \ y \in \mathcal{Y}$, and it satisfies $\bigvee_{y \in \mathcal{Y}} \ y = \bigvee \{x \in \mathcal{D} | x \preceq \Pi(x)\}$.

Proof. The proof is given for item 1, it is analogous for the second. Hence, by considering $\mathcal{Z} = \{x \in \mathcal{D} \mid \Pi(x) \preceq x\}$, it is sufficient to show that $\bigwedge_{z\in\mathcal{Z}} z \in \mathcal{Y}$ and that $\bigwedge_{y\in\mathcal{Y}} y = \bigwedge_{z\in\mathcal{Z}} z$.

First, we prove that the greatest lower bound of $\mathcal{Z}$ is a fixed point, *i.e.*, $\bigwedge_{z\in\mathcal{Z}} z \in \mathcal{Y}$. Due to isotony of Π (see Remark 10), the following inequality holds: $\Pi(\bigwedge_{z\in\mathcal{Z}} z) \preceq \bigwedge_{z\in\mathcal{Z}} \Pi(z)$. From the definition of $\mathcal{Z}$, it follows that $\forall z \in \mathcal{Z}$, $\Pi(z) \preceq z$, and therefore $\bigwedge_{z\in\mathcal{Z}} \Pi(z) \preceq \bigwedge_{z\in\mathcal{Z}} z$. This implies

$$\Pi(\bigwedge_{z\in\mathcal{Z}} z) \preceq \bigwedge_{z\in\mathcal{Z}} z. \tag{6.1}$$

From (6.1) and isotony of Π, it follows that $\Pi(\Pi(\bigwedge_{z\in\mathcal{Z}} z)) \preceq \Pi(\bigwedge_{z\in\mathcal{Z}} z)$. Hence, from the definition of $\mathcal{Z}$, it follows that $\Pi(\bigwedge_{z\in\mathcal{Z}} z) \in \mathcal{Z}$ and is therefore greater than or equal to the greatest lower bound of $\mathcal{Z}$, *i.e.*,

$$\bigwedge_{z\in\mathcal{Z}} z \preceq \Pi(\bigwedge_{z\in\mathcal{Z}} z). \tag{6.2}$$

(6.1) and (6.2) imply that $\bigwedge_{z\in\mathcal{Z}} z = \Pi(\bigwedge_{z\in\mathcal{Z}} z)$, *i.e.*, $\bigwedge_{z\in\mathcal{Z}} z \in \mathcal{Y}$.

Second, we show that $\bigwedge_{z\in\mathcal{Z}} z = \bigwedge_{y\in\mathcal{Y}} y$. According to the definition of $\mathcal{Y}$ and $\mathcal{Z}$, it holds that $\mathcal{Y} \subseteq \mathcal{Z}$. Therefore, $\bigwedge_{y\in\mathcal{Y}} y \succeq \bigwedge_{z\in\mathcal{Z}} z$. On the other hand, we have shown above that $\bigwedge_{z\in\mathcal{Z}} z \in \mathcal{Y}$ implying $\bigwedge_{y\in\mathcal{Y}} y \preceq \bigwedge_{z\in\mathcal{Z}} z$. Combining both inequalities, we have $\bigwedge_{z\in\mathcal{Z}} z = \bigwedge_{y\in\mathcal{Y}} y$. $\square$

Theorem 4 ensures the existence of both a least and a greatest fixed point of monotone mappings defined over a complete idempotent semiring. Below, a constructive algorithm providing the greatest fixed point is given.

Theorem 5. Let $\Pi : \mathcal{D} \to \mathcal{D}$ be an isotone mapping and $\mathcal{D}$ be a complete idempotent semiring. The greatest fixed point of Π can be obtained by considering the following algorithm

Algorithm 1 Yields the greatest $x_m \in \mathcal{D}$ such that $x_m = \Pi(x_m)$.

Require: $m = 0$, $x_0 = \bigvee \mathcal{D} = \top_{\mathcal{D}}$ and Π an isotone mapping
  $x_{m+1} = \Pi(x_m)$
  while $x_{m+1} \neq x_m$ **do**
    $m = m + 1$
    $x_{m+1} = \Pi(x_m)$
  end while

Proof. Theorem 4 ensures the existence of a greatest fixed point of Π in $\mathcal{D}$. The algorithm terminates when $x_m = \Pi(x_m)$. Hence x_m is a fixed point. Furthermore, $\forall z \in \mathcal{D}$ such that $z = \Pi(z)$ we have $z \preceq x_0 = \bigvee \mathcal{D}$. Then Π being isotone implies $z = \Pi(z) \preceq \Pi(x_0)$ and $z = \Pi^m(z) \preceq \Pi^m(x_0) = x_m$, hence x_m is the greatest fixed point. □

Remark 12. If the greatest fixed point is finite, convergence occurs in a finite numbers of steps. The use of the algorithm is illustrated in Example 12.

Remark 13. A dual algorithm can be used to find the least fixed point of an isotone mapping Π. For this, it is sufficient to start Algorithm 1 with $x_0 = \bigwedge \mathcal{D} = \varepsilon_{\mathcal{D}}$.

For semi-continuous mappings Π, the following holds.

Theorem 6 (Baccelli *et al.*, 1992)**.** Let $\mathcal{D}$ be a complete idempotent semiring and $\Pi : \mathcal{D} \to \mathcal{D}$ be a mapping and $\mathcal{Y} = \{x \in \mathcal{D} | \Pi(x) = x\}$ be the set of fixed points of Π. Then,

1. if Π is lower semi-continuous ($l.s.c.$) then $\bigwedge_{y\in\mathcal{Y}} \ y = \Pi^*(\bigwedge_{x\in\mathcal{D}} \ x)$,

2. if Π is upper semi-continuous ($u.s.c$) then $\bigvee_{y\in\mathcal{Y}} \ y = \Pi_*(\bigvee_{x\in\mathcal{D}} \ x)$,

where

$$\begin{aligned} \Pi^*(x) &= \bigoplus_{i\geq 0} \Pi^i(x), \\ \Pi_*(x) &= \bigwedge_{i\geq 0} \Pi^i(x), \end{aligned}$$

$\Pi^0 = \mathsf{Id}_{\mathcal{D}}$ is the identity mapping, and for all $i \geq 0$, $\Pi^{i+1} = \Pi \circ \Pi^i$.

We now turn to specific implicit inequalities and equations over a complete dioid $\mathcal{D}$. These will play a central role in the modeling of TEGs, and finding a least solution will turn out to be essential for establishing the fastest temporal evolution of TEGs. Note that in the following theorem, the unknown, x, may both be a scalar or a vector in $\mathcal{D}$.

Theorem 7 (see Baccelli *et al.*, 1992, Th. 4.75)**.** The implicit inequality $x \succeq ax \oplus b$ and the equality $x = ax \oplus b$ defined over a complete idempotent semiring $\mathcal{D}$ admit $x = a^*b$ as the least solution, where $a^* = \bigoplus_{i \geq 0} a^i$ (Kleene star operator).

Below, we summarize some properties of the Kleene star operator (*) and the operator $^+$ defined by $a^+ = a(a)^* = a \oplus a^2 \oplus a^3 \ldots$. Note that these properties hold for any complete dioid, and therefore also for square matrices with entries in such a dioid (see Remark 8).

$$(a \oplus b)^* = (a^*b)^* \, a^* = (b^*a)^*b^* \tag{6.3}$$

$$a^*a^* = a^* \text{ and } a^+a^+ = aa^+ \tag{6.4}$$

$$(a^*)^* = a^* \text{ and } (a^+)^+ = a^+ \tag{6.5}$$

$$a\,(ba)^* = (ab)^* \, a \tag{6.6}$$

$$(a^*)^+ = (a^+)^* = a^* \tag{6.7}$$

$$a^+ \preceq a^* \tag{6.8}$$

$$(ab^*)^* = e \oplus a\,(a \oplus b)^* \,. \tag{6.9}$$

$$a^* \preceq b^* \Leftrightarrow a^*b^* = b^*. \tag{6.10}$$

$$ax \preceq x \Leftrightarrow a^*x = x. \tag{6.11}$$

These properties can be shown as follows:

Eq. (6.3): consider the implicit equation $x = (a \oplus b)x \oplus e = ax \oplus bx \oplus e$. According to Theorem 7, its least solution is $(a \oplus b)^*e = (a \oplus b)^*$. According to Theorem 7, its least solution also satisfies $x = a^*(bx \oplus e) = a^*bx \oplus a^*$, which yields the least solution $(a^*b)^*a^*$. The least solution of the implicit equation $x = ax \oplus bx \oplus e$ also satisfies $x = b^*(ax \oplus e) = b^*ax \oplus b^*$, which yields the least solution $(b^*a)^*b^*$.

Eq. (6.4): $a^*a^* = (e \oplus a \oplus a^2 \oplus ...)(e \oplus a \oplus a^2 \oplus ...) = (e \oplus a \oplus a^2 \oplus ...) \oplus (a \oplus a^2 \oplus a^3 ...) \oplus (a^2 \oplus a^3 \oplus a^4 ...) = e \oplus a \oplus a^2 \oplus a^3 \oplus a^4 ...) = a^*$. Similarly, $a^+a^+ = (a \oplus a^2 \oplus ...)(a \oplus a^2 \oplus ...) = (a^2 \oplus a^3 \ldots) \oplus (a^3 \oplus a^4 \ldots) \oplus \ldots = a^2 \oplus a^3 \oplus \ldots = aa^+$.

Eq. (6.5): $(a^*)^* = e \oplus a^* \oplus a^*a^* \oplus ...$ and $a^*a^* = a^*$, hence $(a^*)^* = e \oplus a^* = a^*$. Similarly, $(a^+)^+ = a^+ \oplus a^+a^+ \oplus ...$ and $a^+a^+ = aa^+$ hence $(a^+)^+ = a^+ \oplus aa^+ \oplus a^2a^+ ... = a^+$. The last equality holds since $a^+ = a \oplus a^2 \oplus ... \succeq a^ia^+ = a^{i+1} \oplus a^{i+2} \oplus ..,\ i = 0, 1, \ldots$.

Eq. (6.6): $a(ba)^* = a(e \oplus ba \oplus baba \oplus ...) = a \oplus aba \oplus ababa \oplus ... = (e \oplus ab \oplus abab \oplus ...)a = (ab)^*a$.

Eq. (6.7): $(a^*)^+ = a^* \oplus a^*a^* \oplus ...$ and $a^*a^* = a^*$, hence $(a^*)^+ = a^* \oplus a^* \oplus ... = a^*$. Similarly, $(a^+)^* = e \oplus a^+ \oplus a^+a^+ \oplus \ldots$ and $a^+a^+ = aa^+$. Moreover, since $a^+ \succeq a^i a^+$, $i = 0, 1, \ldots$, $(a^+)^* = e \oplus a^+ = a^*$.

Inequality (6.8): the proof follows directly from the definition of a^+.

Eq. (6.9): from Eqs. (6.3) and (6.6), $a(a \oplus b)^* = a(b^*a)^*b^* = ab^*(ab^*)^* = (ab^*)^+$, hence $e \oplus a(a \oplus b)^* = e \oplus (ab^*)^+ = (ab^*)^*$.

Equiv. (6.10): recall that $e \preceq a^*$, hence due to isotony of the product law (see Remark 6) $b^* \preceq a^*b^*$. Furthermore, $a^* \preceq b^*$ implies $a^*b^* \preceq b^*b^* = b^*$. Hence, the following implication holds $a^* \preceq b^* \Rightarrow a^*b^* = b^*$. On the other hand, $b^* = a^*b^* = a^* \oplus a^*b \oplus a^*b^2 \oplus \ldots \Rightarrow a^* \preceq b^*$. This establishes (6.10).

Equiv. (6.11): first, the definition $a^* = \bigoplus_{i \geq 0} a^i$ implies $a^*x \succeq x$. Furthermore, if $ax \preceq x$ then $a^n x \preceq ... \preceq a^2 x \preceq ax \preceq x$, then $a^*x = \bigoplus_{i \geq 0} a^i x \preceq x$. Hence, $ax \preceq x \Rightarrow a^*x = x$. On the other hand, $a^*x = x \Rightarrow x \succeq ax$, hence equivalence (6.11) holds.

Properties (6.3) – (6.11) hold for any complete dioid. If $\otimes$ is commutative, as, for example, for scalars in the max-plus algebra, (6.3) can be simplified as follows:

$$(a \oplus b)^* = a^*b^*. \tag{6.12}$$

To show this, observe that from Eq. (6.3), the following equality holds: $(a \oplus b)^* = (a^*b)^*a^* = a^* \oplus a^*ba^* \oplus a^*ba^*ba^* \oplus ... = a^* \oplus a^*a^*b \oplus a^*a^*a^*b^2 \oplus ....$ The last equality holds because of the commutativity assumption. Then, since $a^*a^* = a^*$, the following equality is obtained: $(a \oplus b)^* = a^* \oplus a^*b \oplus a^*b^2 \oplus ... = a^*(e \oplus b \oplus b^2 \oplus ...) = a^*b^*$.

Let $A \in D^{n \times n}$ be a matrix, then $A^* = \bigoplus\limits_{i \geq 0} A^i$ with $A^0 = I_n$ the identity matrix, can be computed by an iterative strategy inspired by Gauss' elimination for classical linear systems; below, an algorithm of complexity $\mathcal{O}(n^3)$ is given. Note that the scalar $\left(a_{kk}^{(k-1)}\right)^*$ in the algorithm is equal to e if $\mathrm{a}_{kk}^{(k-1)} \preceq e$. An example for the use of this algorithm can be found at the end of Section 7.

Algorithm 2 Yields $A^* \in \mathcal{D}^{n\times n}$.

Require: $A^* \in \mathcal{D}^{n\times n}$

```
A^(0) = A;
for k=1 to n do
    // Computation of A^(k)
    for i=1 to n do
        for j=1 to n do
            a_ij^(k) = a_ij^(k-1) ⊕ a_ik^(k-1) (a_kk^(k-1))* a_kj^(k-1)
        end for
    end for
end for
A* = I_n ⊕ A^(n)
```

$A^{(0)} = A$;

for k=1 to n **do**

// Computation of $A^{(k)}$

for i=1 to n **do**

for j=1 to n **do**

$a_{ij}^{(k)} = a_{ij}^{(k-1)} \oplus a_{ik}^{(k-1)}(a_{kk}^{(k-1)})^* a_{kj}^{(k-1)}$

end for

end for

end for

$A^* = I_n \oplus A^{(n)}$

7

Residuation theory

In general, mappings defined on ordered sets do not have an inverse. Nevertheless, under some assumptions regarding continuity, residuation theory provides an answer to problems such as: what is the greatest solution of inequality $f(x) \preceq b$? Or, dually, what is the least solution of inequality $f(x) \succeq b$? For historical references about this theory, the reader may consult Blyth, 2005, Blyth and Janowitz, 1972, Croisot, 1956, Cuninghame-Green, 1979. In this section, we consider residuation theory in a semiring framework, as in Chapter 4 of Baccelli *et al.*, 1992 and in Blyth, 2005, Cohen, 1998, Cuninghame-Green, 1979. Note that this theory, regarding the inversion of the order relation, is very close to Galois theory (see Dubreil-Jacotin *et al.*, 1953). For details, the interested reader is invited to consult Birkhoff, 1940, Davey and Priestley, 1990.

Definition 20 (Residual and residuated mapping)**.** Let $\mathcal{D}$, $\mathcal{C}$ be two complete idempotent semirings and $f : \mathcal{D} \to \mathcal{C}$ be an isotone mapping. f is a *residuated mapping* if for all $y \in \mathcal{C}$ there exists a greatest solution to the inequality $f(x) \preceq y$ (hereafter denoted $f^\sharp(y)$). The mapping $f^\sharp : \mathcal{C} \to \mathcal{D}, y \mapsto \bigoplus\{x \in \mathcal{D} | f(x) \preceq y\}$ is called the *residual* of f.

Note that, if equality $f(x) = y$ is solvable, $f^\sharp(y)$ yields its greatest solution.

Theorem 8 (see Baccelli *et al.*, 1992 Th. 4.50,Blyth, 2005)**.** Let $\mathcal{D}$, $\mathcal{C}$ be two complete idempotent semirings and $f : \mathcal{D} \to \mathcal{C}$ be an isotone mapping. The following statements are equivalent:

(i) f is residuated.

(ii) there exists a unique mapping $f^\sharp : \mathcal{C} \to \mathcal{D}$ which is isotone and $u.s.c.$ such that $f \circ f^\sharp \preceq \mathsf{Id}_{\mathcal{C}}$ and $f^\sharp \circ f \succeq \mathsf{Id}_{\mathcal{D}}$, where $\mathsf{Id}_{\mathcal{D}}$ and $\mathsf{Id}_{\mathcal{C}}$ are the identity mappings on $\mathcal{D}$ and $\mathcal{C}$, respectively.

(iii) $f(\varepsilon_{\mathcal{D}}) = \varepsilon_{\mathcal{C}}$ and f is $l.s.c.$.

Theorem 9 (Baccelli *et al.*, 1992, Th. 4.56)**.** Let $\mathcal{D}$, $\mathcal{C}$ be two complete idempotent semirings and $f : \mathcal{D} \to \mathcal{C}$ be a residuated mapping. Then,

$$f \circ f^\sharp \circ f = f \text{ and } f^\sharp \circ f \circ f^\sharp = f^\sharp. \tag{7.1}$$

Proposition 1. Let $\mathcal{D}$, $\mathcal{C}$ be two complete idempotent semirings and $f : \mathcal{D} \to \mathcal{C}$ and $g : \mathcal{D} \to \mathcal{C}$ be two residuated mappings. The greatest solution of equality $f(x) = g(x)$ is equal to the greatest fixed point of the isotone mapping $\Pi : \mathcal{D} \to \mathcal{D}, \Pi(x) = x \wedge g^\sharp(f(x)) \wedge f^\sharp(g(x))$.

Proof. The following equivalences hold:

$$\begin{aligned} f(x) = g(x) \quad &\Leftrightarrow \quad \begin{cases} f(x) \preceq g(x) \\ g(x) \preceq f(x) \end{cases} \\ &\Leftrightarrow \quad \begin{cases} x \preceq f^\sharp(g(x)) \\ x \preceq g^\sharp(f(x)) \end{cases} \\ &\Leftrightarrow \quad x \preceq f^\sharp(g(x)) \wedge g^\sharp(f(x)) \\ &\Leftrightarrow \quad x = x \wedge f^\sharp(g(x)) \wedge g^\sharp(f(x)). \end{aligned}$$

Hence the greatest fixed point of $\Pi(x) = x \wedge f^\sharp(g(x)) \wedge g^\sharp(f(x))$ is the greatest solution of equation $f(x) = g(x)$. Furthermore, the law $\wedge$ and the mappings f, $f^\sharp$, g and $g^\sharp$ being isotone, the mapping Π is isotone. $\square$

As an immediate consequence of Proposition 1, the greatest solution of equation $f(x) = g(x)$ can be obtained using Algorithm 1.

Example 6. Mappings $L_a : x \mapsto a \otimes x$ and $R_a : x \mapsto x \otimes a$ defined over $\mathcal{D}$ are both residuated (see Baccelli *et al.*, 1992, Section 4.4.4). Their residuals are isotone mappings, denoted respectively by $L_a^\sharp(x) = a \mathbin{\circ\!\!\!\!\backslash} x$ ("left division by a") and $R_a^\sharp(x) = x \mathbin{\circ\!\!\!\!/} a$ ("right division by a"). This means that $a \mathbin{\circ\!\!\!\!\backslash} b$ is the greatest solution of the inequality $a \otimes x \preceq b$, while $b \mathbin{\circ\!\!\!\!/} a$ is the greatest solution of $x \otimes a \preceq b$. Note that $\varepsilon \mathbin{\circ\!\!\!\!\backslash} \top = \top \mathbin{\circ\!\!\!\!\backslash} \top = \top$ and that $\forall b \neq \top$ the following equalities hold: $\varepsilon \mathbin{\circ\!\!\!\!\backslash} b = \top$, $\top \mathbin{\circ\!\!\!\!\backslash} b = \varepsilon$. Similarly, $\top \mathbin{\circ\!\!\!\!/} \varepsilon = \top \mathbin{\circ\!\!\!\!/} \top = \top$ and $\forall b \neq \top$, $b \mathbin{\circ\!\!\!\!/} \varepsilon = \top$, $b \mathbin{\circ\!\!\!\!/} \top = \varepsilon$.

In the following, we collect some useful properties of left and right multiplication and their residuals.

$$a(a \mathbin{\circ\!\!\!\!\backslash} x) \preceq x \qquad (x \mathbin{\circ\!\!\!\!/} a)a \preceq x \tag{7.2}$$

$$a \mathbin{\circ\!\!\!\!\backslash} (ax) \succeq x \qquad (xa) \mathbin{\circ\!\!\!\!/} a \succeq x \tag{7.3}$$

$$a(a \mathbin{\circ\!\!\!\!\backslash} (ax)) = ax \qquad ((xa) \mathbin{\circ\!\!\!\!/} a)a = xa \tag{7.4}$$

$$a \mathbin{\circ\!\!\!\!\backslash} (a(a \mathbin{\circ\!\!\!\!\backslash} x)) = a \mathbin{\circ\!\!\!\!\backslash} x \qquad ((x \mathbin{\circ\!\!\!\!/} a)a) \mathbin{\circ\!\!\!\!/} a = x \mathbin{\circ\!\!\!\!/} a \tag{7.5}$$

$$a \mathbin{\circ\!\!\!\!\backslash} (x \wedge y) = a \mathbin{\circ\!\!\!\!\backslash} x \wedge a \mathbin{\circ\!\!\!\!\backslash} y \qquad (x \wedge y) \mathbin{\circ\!\!\!\!/} a = x \mathbin{\circ\!\!\!\!/} a \wedge y \mathbin{\circ\!\!\!\!/} a \tag{7.6}$$

$$(a \oplus b) \mathbin{\circ\!\!\!\!\backslash} x = a \mathbin{\circ\!\!\!\!\backslash} x \wedge b \mathbin{\circ\!\!\!\!\backslash} x \qquad x \mathbin{\circ\!\!\!\!/} (a \oplus b) = x \mathbin{\circ\!\!\!\!/} a \wedge x \mathbin{\circ\!\!\!\!/} b \tag{7.7}$$

$$(ab) \mathbin{\circ\!\!\!\!\backslash} x = b \mathbin{\circ\!\!\!\!\backslash} (a \mathbin{\circ\!\!\!\!\backslash} x) \qquad x \mathbin{\circ\!\!\!\!/} (ba) = (x \mathbin{\circ\!\!\!\!/} a) \mathbin{\circ\!\!\!\!/} b \tag{7.8}$$

$$(a \mathbin{\circ\!\!\!\!\backslash} x)b \preceq a \mathbin{\circ\!\!\!\!\backslash} (xb) \qquad b(x \mathbin{\circ\!\!\!\!/} a) \preceq (bx) \mathbin{\circ\!\!\!\!/} a \tag{7.9}$$

Proof. The proofs are given for the mapping L_a (left multiplication by a) defined over $\mathcal{D}$ and its residual ("left division by a"). The proofs for R_a are similar.
Inequalities (7.2) and (7.3) follow directly from Theorem 8, since $L_a \circ L_a^\sharp \preceq \mathsf{Id}_{\mathcal{D}}$ and $L_a^\sharp \circ L_a \succeq \mathsf{Id}_{\mathcal{D}}$.
Eq. (7.4) and Eq. (7.5) follow directly from Theorem 9, since, $L_a \circ L_a^\sharp \circ L_a = L_a$ and $L_a^\sharp \circ L_a \circ L_a^\sharp = L_a^\sharp$.
Eq. (7.6) holds as, according to Theorem 8, $L_a^\sharp$ is *u.s.c.*.
To prove Eq. (7.7), note that $(a \oplus b)y = ay \oplus by \preceq x \Leftrightarrow (ay \preceq x$ and $by \preceq x)$. Therefore, the following equivalence holds: $y \preceq (a \oplus b) \mathbin{\circ\!\!\!\!\backslash} x \Leftrightarrow (y \preceq a \mathbin{\circ\!\!\!\!\backslash} x$ and $y \preceq b \mathbin{\circ\!\!\!\!\backslash} x) \Leftrightarrow y \preceq (a \mathbin{\circ\!\!\!\!\backslash} x) \wedge (b \mathbin{\circ\!\!\!\!\backslash} x)$. Hence, Eq. (7.7) holds.
To prove Eq. (7.8), observe that $y \preceq (ab) \mathbin{\circ\!\!\!\!\backslash} x \Leftrightarrow aby \preceq x \Leftrightarrow by \preceq a \mathbin{\circ\!\!\!\!\backslash} x \Leftrightarrow$

$y \preceq b \backslash\!\!\!\circ (a \backslash\!\!\!\circ x)$. Hence, Eq. (7.8) holds.
To prove Ineq. (7.9), observe that Eq. (7.2) implies $a(a \backslash\!\!\!\circ x)b \preceq xb \Leftrightarrow (a \backslash\!\!\!\circ x)b \preceq a \backslash\!\!\!\circ (xb)$.

□

Recall from Remark 8 that square matrices with entries from a complete dioid $\mathcal{D}$ form a complete dioid, $\mathcal{D}^{n\times n}$, in their own right. Left and right multiplication in this dioid by a matrix A with entries a_{ij} are then residuated mappings. Hence, for given matrices B and C in $\mathcal{D}^{n\times n}$, the inequalities

$$L_A(X) = AX \preceq B, \qquad L_A : \mathcal{D}^{n\times n} \mapsto \mathcal{D}^{n\times n}, \tag{7.10}$$

$$R_A(X) = XA \preceq C, \qquad R_A : \mathcal{D}^{n\times n} \mapsto \mathcal{D}^{n\times n} \tag{7.11}$$

have maximal solutions $L_A^\sharp(B) = A \backslash\!\!\!\circ B$ ("left division by matrix A"), respectively $R_A^\sharp(C) = C /\!\!\!\circ A$ ("right division by matrix A"). This implies $\forall k, j$

$$\bigoplus_{i=1}^{n} a_{ki}(A \backslash\!\!\!\circ B)_{ij} \preceq b_{kj}$$

and therefore

$$a_{ki}(A \backslash\!\!\!\circ B)_{ij} \preceq b_{kj} \text{ or, equivalently, } (A \backslash\!\!\!\circ B)_{ij} \preceq a_{ki} \backslash\!\!\!\circ b_{kj}, \quad i = 1, \ldots n.$$

As $A \backslash\!\!\!\circ B$ is the greatest solution of (7.10), we can deduce that, for $\forall i, j \in [1, n]$,

$$(A \backslash\!\!\!\circ B)_{ij} = \bigwedge_{k=1}^{n} (a_{ki} \backslash\!\!\!\circ b_{kj}). \tag{7.12}$$

Similarly, it can be shown that, $\forall i, j \in [1, n]$,

$$(C /\!\!\!\circ A)_{ij} = \bigwedge_{k=1}^{n} (c_{ik} /\!\!\!\circ a_{jk}). \tag{7.13}$$

Multiplication of non-square matrices can be handled by suitably padding the involved matrices with ε-rows or columns. Then the greatest solutions of inequalities

$$L_A(X) = AX \preceq B, \qquad L_A : \mathcal{D}^{n\times m} \mapsto \mathcal{D}^{p\times m}, \tag{7.14}$$

$$R_A(X) = XA \preceq C, \qquad R_A : \mathcal{D}^{m\times p} \mapsto \mathcal{D}^{m\times n} \tag{7.15}$$

are obtained as $A \diamond\!\!\backslash B$, with $(A \diamond\!\!\backslash B)_{ij}$ given by (7.12), $\forall i \in [1, n]$, and $\forall j \in [1, m]$, respectively as $C \not\diamond A$, with $(C \not\diamond A)_{ij}$ given by (7.13), $\forall i \in [1, m]$, and $\forall j \in [1, p]$.

Example 7. (Gonçalves *et al.*, 2017) Consider two $p \times n$-matrices A, B with entries from a complete idempotent semiring. Then, Proposition 1 implies that the greatest solution of equation $AX = BX$ is obtained by computing the greatest fixed point of the mapping $\Pi(X) = X \wedge B \diamond\!\!\backslash (AX) \wedge A \diamond\!\!\backslash (BX)$.

Definition 21 (Restricted mappings)**.** Let $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be complete idempotent semirings. Let $f : \mathcal{D} \to \mathcal{C}$ be a mapping and $\mathcal{B} \subseteq \mathcal{D}$. The restricted mapping $f_{|\mathcal{B}} : \mathcal{B} \to \mathcal{C}$ is defined by $f_{|\mathcal{B}} = f \circ \mathsf{Id}_{|\mathcal{B}}$, where $\mathsf{Id}_{|\mathcal{B}} : \mathcal{B} \to \mathcal{D}$ is the canonical injection, *i.e.*, $\forall x \in \mathcal{B},\ \mathsf{Id}_{|\mathcal{B}}(x) = x$. Similarly, let $\mathcal{E} \subseteq \mathcal{C}$ be a set such that $\mathsf{Im} f \subseteq \mathcal{E}$. The restricted mapping ${}_{\mathcal{E}|}f : \mathcal{D} \to \mathcal{E}$ is defined by $f = \mathsf{Id}_{|\mathcal{E}} \circ {}_{\mathcal{E}|}f$, where $\mathsf{Id}_{|\mathcal{E}} : \mathcal{E} \to \mathcal{C}$ is the canonical injection, *i.e.*, $\forall x \in \mathcal{E},\ \mathsf{Id}_{|\mathcal{E}}(x) = x$.

Theorem 10 (Projection into a subsemiring Blyth and Janowitz, 1972)**.** Let $\mathcal{D}$ be a complete semiring and $\mathcal{D}_{sub}$ a complete subsemiring of $\mathcal{D}$. The canonical injection $I_{\mathcal{D}_{sub}} : \mathcal{D}_{sub} \to \mathcal{D}$ is residuated. The residual $I^{\sharp}_{\mathcal{D}_{sub}} = \mathsf{Pr}_{\mathcal{D}_{sub}}$ satisfies:

(i) $\mathsf{Pr}_{\mathcal{D}_{sub}} \circ \mathsf{Pr}_{\mathcal{D}_{sub}} = \mathsf{Pr}_{\mathcal{D}_{sub}}$,

(ii) $\mathsf{Pr}_{\mathcal{D}_{sub}} \preceq \mathsf{Id}_{\mathcal{D}}$, where $\mathsf{Id}_{\mathcal{D}}$ is the identity mapping over $\mathcal{D}$,

(iii) $x \in \mathcal{D}_{sub} \Leftrightarrow \mathsf{Pr}_{\mathcal{D}_{sub}}(x) = x$.

Definition 22 (Closure mapping)**.** A closure mapping is an order preserving mapping $f : \mathcal{D} \to \mathcal{D}$ such that $f \succeq \mathsf{Id}_{\mathcal{D}}$ and $f \circ f = f$.

Theorem 11 (see Cottenceau *et al.*, 2001b)**.** Let $f : \mathcal{D} \to \mathcal{D}$ be a closure mapping. Then, ${}_{\mathsf{Im} f|}f$ is a residuated mapping whose residual is the canonical injection $\mathsf{Id}_{|\mathsf{Im} f}$, *i.e.*,

$$({}_{\mathsf{Im} f|}f)^{\sharp} = \mathsf{Id}_{|\mathsf{Im} f}.$$

Example 8. Mapping $K : \mathcal{D} \to \mathcal{D}$ with $K(x) = x^*$ is a closure mapping. This is easily seen since $\forall x \in \mathcal{D},\ K(x) = x^* \succeq x = \mathsf{Id}_{\mathcal{D}}(x)$ and

$(K \circ K)(x) = (x^*)^* = x^* = K(x)$ (see Property (6.5)). Then, according to Theorem 11, ${}_{\mathsf{Im}K|}K$ is residuated and its residual is $({}_{\mathsf{Im}K|}K)^\sharp = \mathsf{Id}_{|\mathsf{Im}K}$. Hence, if $a \in \mathsf{Im}K$, *i.e.*, $\exists y \in \mathcal{D}$ such that $a = K(y) = y^*$ and therefore $a^* = (y^*)^* = y^* = a$, the greatest solution of inequality $x^* \preceq a$ is $x = a = a^*$. This implies $x \preceq a^* \Leftrightarrow x^* \preceq a^*$.

Example 9. Mapping $P : \mathcal{D} \to \mathcal{D}$ with $P(x) = x^+ = xx^* = x^*x$ is a closure mapping. This is easily seen since $\forall x \in \mathcal{D},\ P(x) = x^+ \succeq x = \mathsf{Id}_{\mathcal{D}}(x)$ and $(P \circ P)(x) = (x^+)^+ = x^+ = P(x)$ (see Property (6.5)). Then, according to Theorem 11, ${}_{\mathsf{Im}P|}P$ is residuated and its residual is $({}_{\mathsf{Im}P|}P)^\sharp = \mathsf{Id}_{|\mathsf{Im}P}$. Hence, if $a \in \mathsf{Im}P$, *i.e.*, $\exists y \in \mathcal{D}$ such that $a = P(y) = y^+$ and therefore $a^+ = (y^+)^+ = y^+ = a$, the greatest solution of inequality $x^+ \preceq a$ is $x = a = a^+$. This implies $x \preceq a^+ \Leftrightarrow x^+ \preceq a^+$.

Remark 14. According to Eq. (6.7), $(a^*)^+ = a^*$, therefore $\mathsf{Im}K \subset \mathsf{Im}P$.

Proposition 2. (Hardouin, 2004; MaxPlus, 1991) Below we summarize some properties involving the left and right residuals and the Kleene star operator.

$$a \circ\!\!\!\!\backslash\, a = (a \circ\!\!\!\!\backslash\, a)^* \qquad a \circ\!\!\!\!/\, a = (a \circ\!\!\!\!/\, a)^* \tag{7.16}$$

$$a^* \circ\!\!\!\!\backslash\, (a^*x) = a^*x \qquad (a^*x) \circ\!\!\!\!/\, a^* = a^*x \tag{7.17}$$

Proof. The proofs are given for the left residual defined over $\mathcal{D}$. The proofs for the right are similar.

Equalities (7.16) : First, by considering Eq. (7.9) (with $x = a$ and $b = x$) the following inequality holds $(a \circ\!\!\!\!\backslash\, a)x \preceq a \circ\!\!\!\!\backslash\, (ax)$. By choosing $x = a \circ\!\!\!\!\backslash\, a$, this yields $(a \circ\!\!\!\!\backslash\, a)(a \circ\!\!\!\!\backslash\, a) \preceq a \circ\!\!\!\!\backslash\, (a(a \circ\!\!\!\!\backslash\, a)) = a \circ\!\!\!\!\backslash\, a$, where the equality follows from Theorem 9. Hence $(a \circ\!\!\!\!\backslash\, a)^2 \preceq a \circ\!\!\!\!\backslash\, a$ and consequently $(a \circ\!\!\!\!\backslash\, a)^k \preceq a \circ\!\!\!\!\backslash\, a$ for all $k \geq 2$, hence $(a \circ\!\!\!\!\backslash\, a)^* = e \oplus \bigoplus_{k=1}^{n} (a \circ\!\!\!\!\backslash\, a)^k = e \oplus a \circ\!\!\!\!\backslash\, a$. Furthermore, Eq. (7.3) implies that $e \preceq a \circ\!\!\!\!\backslash\, a$ (choose $x = e$), then $(a \circ\!\!\!\!\backslash\, a)^* = e \oplus a \circ\!\!\!\!\backslash\, a = a \circ\!\!\!\!\backslash\, a$.
Equalities (7.17) : From Eq. (6.4) we can write , $a^*a^*x = a^*x$, which is equivalent to $a^*x \preceq a^* \circ\!\!\!\!\backslash\, (a^*x)$ according to the definition of residuation. From Eq. (7.7), $a^* \circ\!\!\!\!\backslash\, (a^*x) = (e \oplus a \oplus a^2 \oplus ...) \circ\!\!\!\!\backslash\, (a^*x) = e \circ\!\!\!\!\backslash\, (a^*x) \wedge a \circ\!\!\!\!\backslash\, (a^*x) \wedge a^2 \circ\!\!\!\!\backslash\, (a^*x) \wedge ...$, hence $a^* \circ\!\!\!\!\backslash\, (a^*x) \preceq e \circ\!\!\!\!\backslash\, (a^*x) = (a^*x)$, therefore Eq. (7.17) holds.

$\square$

Example 10 (Matrix operations in $\overline{\mathbb{Z}}_{\max}$)**.** Given the following matrices with entries in $\overline{\mathbb{Z}}_{\max}$,

$$A = \begin{bmatrix} 1 & 4 \\ 5 & 3 \\ \varepsilon & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 3 \\ 2 & 4 \\ 7 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} e & 4 \\ 1 & 3 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & \varepsilon \end{bmatrix}, \text{ and } \quad F = \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix},$$

we get

$$A \oplus B = \begin{bmatrix} a_{11} \oplus b_{11} & a_{12} \oplus b_{12} \\ a_{21} \oplus b_{21} & a_{22} \oplus b_{22} \\ a_{31} \oplus b_{31} & a_{32} \oplus b_{32} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 4 \\ 7 & 2 \end{bmatrix}$$

$$A \otimes C = \begin{bmatrix} \bigoplus_{j=1}^{2}(a_{1j} \oplus c_{j1}) & \bigoplus_{j=1}^{2}(a_{1j} \oplus c_{j2}) \\ \bigoplus_{j=1}^{2}(a_{2j} \oplus c_{j1}) & \bigoplus_{j=1}^{2}(a_{2j} \oplus c_{j2}) \\ \bigoplus_{j=1}^{2}(a_{3j} \oplus c_{j1}) & \bigoplus_{j=1}^{2}(a_{3j} \oplus c_{j2}) \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 5 & 9 \\ 3 & 5 \end{bmatrix}.$$

Now consider the inequality $G \otimes X \preceq F$. As multiplication in the max-plus algebra corresponds to addition in the standard algebra, its (scalar) residual corresponds to standard subtraction, *i.e.*, $1 \otimes x \preceq 4$ admits the solution set $\mathcal{X} = \{x | x \preceq 1 \circ\!\!\!\backslash 4\}$ with $1 \circ\!\!\!\backslash 4 = 4 - 1 = 3$ being the greatest solution. Applying the residuation rule (7.12) in the max-plus algebra to the inequality $G \otimes X \preceq F$ results in:

$$G \circ\!\!\!\backslash F = \begin{bmatrix} \bigwedge_{k=1}^{3} g_{k1} \circ\!\!\!\backslash f_{k1} \\ \bigwedge_{k=1}^{3} g_{k2} \circ\!\!\!\backslash f_{k1} \end{bmatrix} = \begin{bmatrix} 1 \circ\!\!\!\backslash 6 \wedge 3 \circ\!\!\!\backslash 7 \wedge 5 \circ\!\!\!\backslash 8 \\ 2 \circ\!\!\!\backslash 6 \wedge 4 \circ\!\!\!\backslash 7 \wedge \varepsilon \circ\!\!\!\backslash 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

The reader is invited to note that the rules given in Example 6 yield $\varepsilon \circ\!\!\!\backslash 8 = \top$.

Matrix $G \circ\!\!\!\backslash F = [3 \;\; 3]^T$ is the greatest solution for X which ensures $E \otimes X \preceq F$. Indeed,

$$G \otimes (G \circ\!\!\!\backslash F) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} \preceq \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix} = F.$$

Remark 15. Note that residuation achieves equality in the case of scalar multiplication in the max-plus algebra, while this is in general not true for the matrix case (see the above example).

Example 11 (Equation $x = Ax \oplus b$ in $\overline{\mathbb{Z}}_{\max}$ and $\overline{\mathbb{Z}}_{\min}$). Let

$$A = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & 3 \\ 4 & \varepsilon & \varepsilon \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ \varepsilon \\ 8 \end{bmatrix}$$

be matrices with entries in $\overline{\mathbb{Z}}_{\max}$. The least solution of the implicit equation $x = Ax \oplus b$ is equal to A^*b, and A^* can be computed using Algorithm 2. Observe that in $\overline{\mathbb{Z}}_{\max}$, if $a_{ij} \preceq e$ then $a^*_{ij} = e$, else $a^*_{ij} = \top$. Applying the steps of Algorithm 2, we obtain:

$$\begin{aligned} A^{(0)} = A^{(1)} = A^{(2)} &= \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & 3 \\ 4 & \varepsilon & \varepsilon \end{bmatrix} \\ \text{and } A^{(3)} &= \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon \\ 7 & \varepsilon & 3 \\ 4 & \varepsilon & \varepsilon \end{bmatrix}, \end{aligned}$$

therefore

$$A^* = I_3 \oplus A^{(3)} = \begin{bmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \end{bmatrix} \oplus \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon \\ 7 & \varepsilon & 3 \\ 4 & \varepsilon & \varepsilon \end{bmatrix} = \begin{bmatrix} e & \varepsilon & \varepsilon \\ 7 & e & 3 \\ 4 & \varepsilon & e \end{bmatrix}.$$

Consequently, the least solution of the implicit equation $x = Ax \oplus b$ is

$$x = A^*b = \begin{bmatrix} 2 \\ 11 \\ 8 \end{bmatrix}.$$

We now look for the least solution of $x = Ax \oplus b$ in $\overline{\mathbb{Z}}_{\min}$. Recall that the order $\preceq$ defined in $\overline{\mathbb{Z}}_{\min}$ is the opposite of the natural order $\geq$ in $\mathbb{Z}$, that $\varepsilon = +\infty$ and $\top = -\infty$. Applying Algorithm 2, we obtain

$$A^{(0)} = A^{(1)} = A^{(2)} = A^{(3)} = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & 3 \\ 4 & \varepsilon & \varepsilon \end{bmatrix},$$

therefore

$$A^* = I_3 \oplus A^{(3)} = \begin{bmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \end{bmatrix} \oplus \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & 3 \\ 4 & \varepsilon & \varepsilon \end{bmatrix} = \begin{bmatrix} e & \varepsilon & \varepsilon \\ 2 & e & 3 \\ 4 & \varepsilon & e \end{bmatrix}.$$

Consequently, the least solution of the implicit equation $x = Ax \oplus b$ in $\overline{\mathbb{Z}}_{\min}$ is

$$x = A^* b = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$$

Example 12 (Equation $Cx = Dx$ in $\overline{\mathbb{Z}}_{\max}$)**.** In this example, we illustrate how Proposition 1 can be used to find the greatest solution, in $\overline{\mathbb{Z}}_{\max}$, of equation $Cx = Dx$ such that $x \preceq x_0$, where

$$C = \begin{bmatrix} 2 & 5 & \varepsilon \\ 2 & 7 & 7 \\ 7 & 9 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 4 & 4 \\ 6 & 3 & 1 \\ 2 & 5 & 8 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix}$$

are matrices, respectively a vector, with entries in $\overline{\mathbb{Z}}_{\max}$. First, the proposition states

$$Cx = Dx \Leftrightarrow x = (C \backslash\!\!\circ (Dx)) \wedge (D \backslash\!\!\circ (Cx)) \wedge x \triangleq \Pi(x).$$

Then, Algorithm 1 is applied to compute the greatest fixed point of Π. Below, the iteration steps of this algorithm are provided in detail.

$$\begin{aligned}
x_1 &= \Pi(x_0) \\
&= \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} \wedge \left(\begin{bmatrix} 1 & 4 & 4 \\ 6 & 3 & 1 \\ 2 & 5 & 8 \end{bmatrix} \backslash\!\!\circ \left(\begin{bmatrix} 2 & 5 & \varepsilon \\ 2 & 7 & 7 \\ 7 & 9 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} \right)\right) \\
&\qquad \wedge \left(\begin{bmatrix} 2 & 5 & \varepsilon \\ 2 & 7 & 7 \\ 7 & 9 & 1 \end{bmatrix} \backslash\!\!\circ \left(\begin{bmatrix} 1 & 4 & 4 \\ 6 & 3 & 1 \\ 2 & 5 & 8 \end{bmatrix} \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} \right)\right) \\
&= \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} \wedge \left(\begin{bmatrix} 1 & 4 & 4 \\ 6 & 3 & 1 \\ 2 & 5 & 8 \end{bmatrix} \backslash\!\!\circ \begin{bmatrix} 20 \\ 22 \\ 24 \end{bmatrix} \right) \wedge \left(\begin{bmatrix} 2 & 5 & \varepsilon \\ 2 & 7 & 7 \\ 7 & 9 & 1 \end{bmatrix} \backslash\!\!\circ \begin{bmatrix} 19 \\ 21 \\ 23 \end{bmatrix} \right) \\
&= \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} \wedge \begin{bmatrix} 16 \\ 16 \\ 16 \end{bmatrix} \wedge \begin{bmatrix} 16 \\ 14 \\ 14 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 14 \end{bmatrix},
\end{aligned}$$

$$
\begin{aligned}
x_2 &= \Pi(x_1) \\
&= \begin{bmatrix} 15 \\ 14 \\ 14 \end{bmatrix} \wedge (\begin{bmatrix} 1 & 4 & 4 \\ 6 & 3 & 1 \\ 2 & 5 & 8 \end{bmatrix} \backslash\!\!\!\circ (\begin{bmatrix} 2 & 5 & \varepsilon \\ 2 & 7 & 7 \\ 7 & 9 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 14 \\ 14 \end{bmatrix})) \\
&\qquad \wedge (\begin{bmatrix} 2 & 5 & \varepsilon \\ 2 & 7 & 7 \\ 7 & 9 & 1 \end{bmatrix} \backslash\!\!\!\circ (\begin{bmatrix} 1 & 4 & 4 \\ 6 & 3 & 1 \\ 2 & 5 & 8 \end{bmatrix} \begin{bmatrix} 15 \\ 14 \\ 14 \end{bmatrix})) \\
&= \begin{bmatrix} 15 \\ 14 \\ 14 \end{bmatrix} \wedge (\begin{bmatrix} 1 & 4 & 4 \\ 6 & 3 & 1 \\ 2 & 5 & 8 \end{bmatrix} \backslash\!\!\!\circ \begin{bmatrix} 19 \\ 21 \\ 23 \end{bmatrix}) \wedge (\begin{bmatrix} 2 & 5 & \varepsilon \\ 2 & 7 & 7 \\ 7 & 9 & 1 \end{bmatrix} \backslash\!\!\!\circ \begin{bmatrix} 18 \\ 21 \\ 22 \end{bmatrix}) \\
&= \begin{bmatrix} 15 \\ 14 \\ 14 \end{bmatrix} \wedge \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} \wedge \begin{bmatrix} 15 \\ 13 \\ 14 \end{bmatrix} = \begin{bmatrix} 15 \\ 13 \\ 14 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= \Pi(x_2) \\
&= \begin{bmatrix} 15 \\ 13 \\ 14 \end{bmatrix} \wedge (\begin{bmatrix} 1 & 4 & 4 \\ 6 & 3 & 1 \\ 2 & 5 & 8 \end{bmatrix} \backslash\!\!\!\circ (\begin{bmatrix} 2 & 5 & \varepsilon \\ 2 & 7 & 7 \\ 7 & 9 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 13 \\ 14 \end{bmatrix})) \\
&\qquad \wedge (\begin{bmatrix} 2 & 5 & \varepsilon \\ 2 & 7 & 7 \\ 7 & 9 & 1 \end{bmatrix} \backslash\!\!\!\circ (\begin{bmatrix} 1 & 4 & 4 \\ 6 & 3 & 1 \\ 2 & 5 & 8 \end{bmatrix} \begin{bmatrix} 15 \\ 13 \\ 14 \end{bmatrix})) \\
&= \begin{bmatrix} 15 \\ 13 \\ 14 \end{bmatrix} \wedge (\begin{bmatrix} 1 & 4 & 4 \\ 6 & 3 & 1 \\ 2 & 5 & 8 \end{bmatrix} \backslash\!\!\!\circ \begin{bmatrix} 18 \\ 21 \\ 22 \end{bmatrix}) \wedge (\begin{bmatrix} 2 & 5 & \varepsilon \\ 2 & 7 & 7 \\ 7 & 9 & 1 \end{bmatrix} \backslash\!\!\!\circ \begin{bmatrix} 18 \\ 21 \\ 22 \end{bmatrix}) \\
&= \begin{bmatrix} 15 \\ 13 \\ 14 \end{bmatrix} \wedge \begin{bmatrix} 15 \\ 14 \\ 14 \end{bmatrix} \wedge \begin{bmatrix} 15 \\ 13 \\ 14 \end{bmatrix} = \begin{bmatrix} 15 \\ 13 \\ 14 \end{bmatrix}.
\end{aligned}
$$

Hence, the algorithm converges to $\begin{bmatrix} 15 & 13 & 14 \end{bmatrix}^T$, which therefore is the greatest solution of $Cx = Dx$ such that $x \preceq x_0$.

Remark 16. If the greatest fixed point is finite, convergence of Algorithm 1 occurs in a finite number of steps. Moreover, in Butkovič, 2010, it is shown that if $Cx = Dx$ possesses a finite solution, the above algorithm is pseudopolynomial, that is the convergence speed is polynomial according to the finite distance between the value of x_0 and the value of the greatest fixed point x_m.

8

Idempotent semirings of formal power series

As indicated in Example 5, the temporal evolution of a TEG is characterised by sequences $x_i(0), x_i(1), \ldots \in \overline{\mathbb{Z}}_{\max}$. We sometimes refer to such sequences as discrete-event signals, as $x_i(k)$ denotes the time of the k^{th} firing, or occurrence, of transition x_i. Recall that in "standard" systems and control theory, sequences corresponding to the values of a discrete-time signal are often represented by their z-transforms. Formally, the z-transformation "translates" the sequence of signal values into a formal power series in the indeterminant z^{-1}. This can also be done for sequences in $\overline{\mathbb{Z}}_{\max}$. In particular, a sequence in $\overline{\mathbb{Z}}_{\max}$ can be written as a formal power series in the indeterminant γ with coefficients in $\overline{\mathbb{Z}}_{\max}$. This is referred to as the γ-transform of the sequence. It will turn out that the set of such series forms a complete idempotent semiring in its own right, denoted by $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$. As noted in Section 3, the sequences representing the firing times of transitions in timed event graphs are non-decreasing in the order $\preceq$ defined in $\overline{\mathbb{Z}}_{\max}$. We will therefore be particularly interested in non-decreasing power series. They can be identified with another idempotent semiring, denoted by $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$. In subsequent sections, we will make intensive use of the dioid $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ to model timed event graphs and, on the basis of these models, perform controller and observer synthesis for TEGs.

In this section, we will first formally define the dioids $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ and $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$. We will then go on to discuss causal and ultimately periodic non-decreasing series, as these play an important role when modelling TEGs.

Definition 23 (γ-transform)**.** The γ-transform of a sequence s with $s(k) \in \overline{\mathbb{Z}}_{\max}$ is a formal power series in γ with coefficients in $\overline{\mathbb{Z}}_{\max}$ and exponents in $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ defined by

$$s = \bigoplus_{k \in \overline{\mathbb{Z}}} s(k)\gamma^k.$$

The valuation of the series s, denoted $val_\gamma(s)$, is the smallest k such that $s(k) \neq \varepsilon$, and the degree of s, denoted by $deg_\gamma(s)$, is the greatest k such that $s(k) \neq \varepsilon$.

Remark 17. We denote both the sequence and its γ-transform by the same symbol, as no ambiguity will occur.

Remark 18. Since $s\gamma = \bigoplus_{k \in \overline{\mathbb{Z}}} s(k) \otimes \gamma^{k+1} = \bigoplus_{k \in \overline{\mathbb{Z}}} s(k-1) \otimes \gamma^k$, multiplication by γ can be interpreted as a backward shift operation.

Definition 24 (Idempotent semiring $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$)**.** The set of formal power series in γ with exponents in $\overline{\mathbb{Z}}$ and coefficients in $\overline{\mathbb{Z}}_{\max}$, with addition and multiplication (Cauchy product) defined by

$$s \oplus s' = \bigoplus_{k \in \overline{\mathbb{Z}}} \left(s(k) \oplus s'(k)\right) \gamma^k, \tag{8.1}$$

$$s \otimes s' = \bigoplus_{k \in \overline{\mathbb{Z}}} \left(\bigoplus_{k_1 \in \overline{\mathbb{Z}}} \left(s(k_1) \otimes s'(k - k_1)\right) \right) \gamma^k, \tag{8.2}$$

is a complete idempotent semiring, denoted $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$. The zero element in this semiring is the series $\varepsilon(\gamma) = \bigoplus_{k \in \overline{\mathbb{Z}}} -\infty\gamma^k$. The unit element is the formal power series $e(\gamma) = \bigoplus_{k<0} -\infty\gamma^k \oplus 0\gamma^0 \oplus \bigoplus_{k>0} -\infty\gamma^k$. The top element is $\top(\gamma) = \bigoplus_{k \in \overline{\mathbb{Z}}} +\infty\gamma^k$. When there is no ambiguity, these series will be denoted ε, e and $\top$, for short.

As $s \preceq s' \Leftrightarrow s \oplus s' = s'$, the order in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ is defined coefficientwise, *i.e.*, $s \preceq s' \Leftrightarrow \forall k \in \overline{\mathbb{Z}},\ s(k) \preceq s'(k)$. Hence the greatest lower bound of two formal power series s and s' is provided by

$$s \wedge s' = \bigoplus_{k \in \overline{\mathbb{Z}}} \left(s(k) \wedge s'(k)\right) \gamma^k, \tag{8.3}$$

and the greatest solution of inequality $s \otimes x \preceq s'$ is given by

$$\begin{aligned} s \mathbin{\circ\!\!\!\!\setminus} s' &= \bigoplus_{x \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]} \{x | s \otimes x \preceq s'\} \\ &= \bigoplus_{k \in \overline{\mathbb{Z}}} \left(\bigwedge_{k_1 \in \overline{\mathbb{Z}}} \left(s(k_1 - k) \mathbin{\circ\!\!\!\!\setminus} s'(k_1)\right) \right) \gamma^k. \end{aligned} \tag{8.4}$$

Remark 19. It is customary to list only coefficients that are not equal to $-\infty$. Hence, a monomial, *i.e.*, a series with only one non-$(-\infty)$ coefficient, can be written as $t\gamma^k$ instead of $\bigoplus_{j<k}(-\infty)\gamma^j \oplus t\gamma^k \oplus \bigoplus_{j>k}(-\infty)\gamma^j$. Similarly for polynomials, *i.e.*, series with only a finite number of non-$(-\infty)$ coefficients.

8.1 Non-decreasing series

In the sequel we will focus on non-decreasing sequences s, *i.e.*, we require $\forall k,\ s(k-1) \preceq s(k)$. The reason for doing this is that we want sequences to model the firing times of corresponding transitions in TEGs. As the time of the k^{th} firing of a transition, $s(k)$, can clearly not be less than the time of its $(k-1)^{st}$ firing, $s(k-1)$, this establishes non-decreasingness of sequences. For the γ-transform of the sequence s, this requirement translates into (see Remark 18)

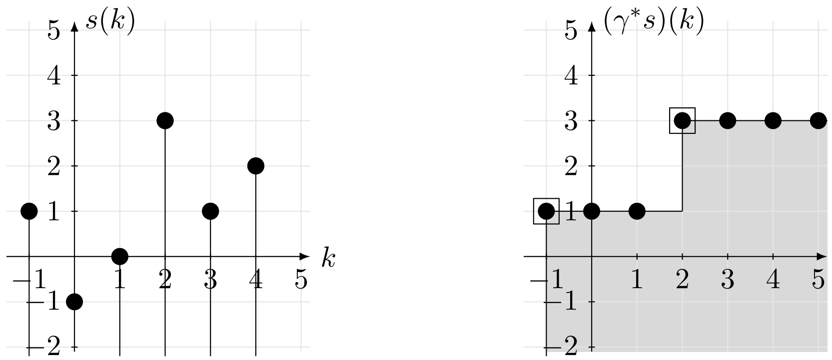
$$\forall k,\ s(k-1) \preceq s(k) \Leftrightarrow \gamma s \preceq s \Leftrightarrow \gamma^* s = s,$$

where $\gamma^* s = \bigoplus_{k \geq 0} \gamma^k s$. We now identify all series s with $\gamma^* s$, *i.e.*, we introduce an equivalence relation $\mathcal{R}_{\gamma^*}$ by

$$s \mathcal{R}_{\gamma^*} s' \iff \gamma^* s = \gamma^* s'.$$

$\mathcal{R}_{\gamma^*}$ respects the laws of the semiring $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ and is therefore a congruence (see Definition 16). The quotient semiring $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ is a semiring where each element is an equivalence class $[s]_{/\mathcal{R}_{\gamma^*}}$, and since $\gamma^*\gamma^* = \gamma^*$, each class $[s]_{/\mathcal{R}_{\gamma^*}}$ admits a greatest element (in the order defined in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$), namely $\gamma^* s$. We can identify each equivalence class $[s]_{/\mathcal{R}_{\gamma^*}}$ with its greatest element $\gamma^* s$. The latter is a non-decreasing series, hence the quotient semiring $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ can be interpreted as the semiring of non-decreasing power series in γ.

The relation between a (not necessarily non-decreasing) series s, the equivalence class $[s]_{/\mathcal{R}_{\gamma^*}}$ and its greatest element $\gamma^* s$ is illustrated in Fig. 8.1. Clearly, $[s]_{/\mathcal{R}_{\gamma^*}}$ is a set of series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$, and it can be visualized as the union of "south-east" cones with apexes $(k, s(k))$. Moreover, Fig. 8.1 illustrates that an equivalence class $[s]_{/\mathcal{R}_{\gamma^*}}$ can also be represented by a minimal series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ (where "minimal" is, again, in the sense of the order defined in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$). In the example shown in Fig. 8.1, this is the polynomial $1\gamma^{-1} \oplus 3\gamma^2$, which can be visualized as the set of "north-west corners" of the gray area.



(a) $s = 1\gamma^{-1} \oplus -1\gamma^0 \oplus 0\gamma \oplus 3\gamma^2 \oplus 1\gamma^3 \oplus 2\gamma^4$

(b) Equivalence class $[s]_{/\mathcal{R}_{\gamma^*}}$ (gray area) with greatest element $(\gamma^* s)$ (black dots) and minimal representative $(1\gamma^{-1} \oplus 3\gamma^2)$ (black rectangles).

Figure 8.1: Relation between series $s \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ and the equivalence class $[s]_{\mathcal{R}_{\gamma^*}} \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ with greatest element $(\gamma^* s)$ and minimal representative.

Note that a minimal representative of an equivalence class is a series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ with the least number of non-$(-\infty)$-coefficients. For algorithmic reasons, we are therefore obviously interested to represent equivalence classes (elements in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$), and hence their greatest elements (which are non-decreasing series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$) by their minimal representatives (which may not be non-decreasing in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$) . Hence, to avoid confusion, we need to distinguish whether we refer to s as a series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ (this is denoted by $s \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$) or as a representation of the equivalence class $[s]_{/\mathcal{R}_{\gamma^*}}$ and therefore the greatest element $\gamma^* s$ in this class. The latter is denoted by $s \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$. The following example illustrates this point.

Example 13. $1\gamma^0 \oplus 3\gamma^2 \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ refers to the power series

$$\bigoplus_{j<0}(-\infty)\gamma^j \oplus 1\gamma^0 \oplus (-\infty)\gamma^1 \oplus 3\gamma^2 \bigoplus_{j>2}(-\infty)\gamma^j,$$

while $1\gamma^0 \oplus 3\gamma^2 \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ refers to the non-decreasing power series

$$\bigoplus_{j<0}(-\infty)\gamma^j \oplus 1\gamma^0 \oplus 1\gamma^1 \bigoplus_{j\geq 2} 3\gamma^j.$$

In the following, we will only work with non-decreasing power series s, hence it is always the latter interpretation, i.e., $s \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$, that will be used.

In particular, the notation $t\gamma^k \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ will refer to the fact that the monomial $t\gamma^k$ is the minimal representative of equivalence class $[t\gamma^k]_{\mathcal{R}_{\gamma^*}} \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ and therefore represents the non-decreasing series $\gamma^* t\gamma^k = \bigoplus_{j\geq k} t\gamma^j$.

As a consequence, the neutral elements of multiplication and addition, and the top element in the dioid $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ will be written, respectively, as

$$\begin{aligned} e &= 0\gamma^0 \\ \varepsilon &= -\infty\gamma^{+\infty} \\ \top &= +\infty\gamma^{-\infty}. \end{aligned}$$

Using this convention, the following computational rules between monomials $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ can be established:

$$\begin{aligned}
t\gamma^k \oplus t\gamma^l &= t\gamma^{\min(k,l)}, && (8.5)\\
t\gamma^k \oplus \tau\gamma^k &= (t \oplus \tau)\gamma^k = \max(t,\tau)\gamma^k, && (8.6)\\
t\gamma^k \oplus \tau\gamma^l &= t\gamma^k \text{ if } t \succeq \tau \text{ and } k \leq l, && (8.7)\\
t\gamma^k \otimes \tau\gamma^l &= (t \otimes \tau)\gamma^{(k+l)} = (t+\tau)\gamma^{(k+l)}, && (8.8)\\
\tau\gamma^l \lozenge\!\!\!\backslash\, t\gamma^k &= (\tau \lozenge\!\!\!\backslash\, t)\gamma^{(k-l)} = (t-\tau)\gamma^{(k-l)}, && (8.9)\\
t\gamma^k \wedge \tau\gamma^l &= (t \wedge \tau)\gamma^{\max(k,l)} = \min(t,\tau)\gamma^{\max(k,l)}. && (8.10)
\end{aligned}$$

Graphically, for monomials in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$,

1. addition $t\gamma^k \oplus \tau\gamma^l$ refers to the union of south-east cones with apexes (k,t) and (l,τ) (see Fig. 8.2(a)),

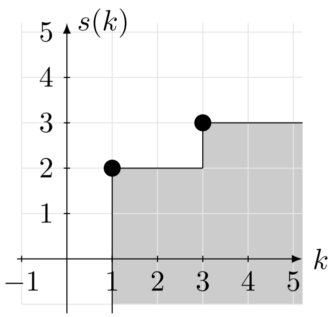
2. multiplication $t\gamma^k \otimes \tau\gamma^l$ refers to the south-east cone of apex $(k+l, t \otimes \tau)$ (see Fig. 8.2(b))

3. greatest lower bound: $t\gamma^k \wedge \tau\gamma^l$ refers to the intersection of the two south-east cones with apexes (k,t) and (l,τ), *i.e.*, the south-east cone with apex $(\max(k,l),(t \wedge \tau))$ (see Fig. 8.2(c)).

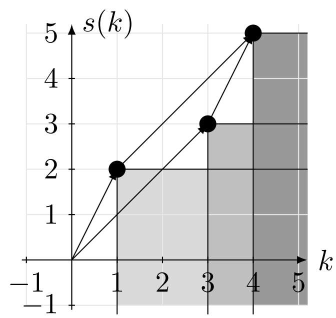
Definition 25 (Polynomial)**.** A polynomial in $p \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ is defined as the sum of a finite number of monomials, *i.e.*, $p = \bigoplus_{i=1}^{m} t_i\gamma^{n_i}$, $0 < m < \infty$. According to simplification rules Eq. (8.5) and Eq. (8.6) a polynomial can always be written in canonical form, with $n_1 < n_2 < \ldots < n_m$ and $t_1 \prec t_2 \prec \ldots \prec t_m$, where "$t_i \prec t_{i+1}$" means "$t_i \preceq t_{i+1}$ but $t_i \neq t_{i+1}$". When a polynomial p is in canonical form, its valuation and degree are given by $val_\gamma(p) = n_1$ and its degree $deg_\gamma(p) = n_m$.

Example 14. The following polynomial $p = 3\gamma^1 \oplus 2\gamma^3 \oplus 4\gamma^2$ admits the following canonical form $p = 3\gamma^1 \oplus 4\gamma^2$ with $val_\gamma(p) = 1$ and $deg_\gamma(p) = 2$.

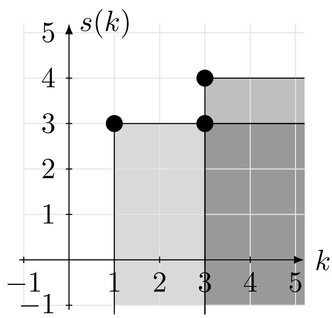
Definition 26 (Operations between polynomials)**.** Operations between polynomials $p = \bigoplus_{i=1}^{m} t_i\gamma^{n_i}$ and $p' = \bigoplus_{j=1}^{m'} t'_j\gamma^{n'_j}$ are a straightforward



(a) $2\gamma^1 \oplus 3\gamma^3$



(b) $2\gamma \otimes 3\gamma^3 = 5\gamma^4$



(c) $3\gamma^1 \wedge 4\gamma^3 = 3\gamma^3$

Figure 8.2: Graphical representation of operations in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$.

extension of the corresponding operations between monomials. They are summarized below, together with the involved computational complexity.

$$\begin{aligned}
p \oplus p' &= \bigoplus_{i=1}^{m} t_i \gamma^{n_i} \oplus \bigoplus_{j=1}^{m'} t_j \gamma^{n'_j}, \\
&\quad \text{with complexity } \mathcal{O}(m+m'), \\
p \otimes p' &= \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m'} (t_i \otimes t'_j) \gamma^{n_i + n'_j}, \\
&\quad \text{with complexity } \mathcal{O}(m.m'), \\
p \wedge p' &= \bigoplus_{i=1}^{m} (t_i \gamma^{n_i} \wedge p') = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m'} (t_i \wedge t'_j) \gamma^{max(n_i, n'_j)}, \\
&\quad \text{with complexity } \mathcal{O}(m.m'), \\
p' \backslash\!\!\!\circ p &= \bigwedge_{i=1}^{m} \bigoplus_{j=1}^{m'} (t'_j \backslash\!\!\!\circ t_i) \gamma^{n_i - n'_j}, \\
&\quad \text{with complexity } \mathcal{O}(m.m').
\end{aligned}$$

Example 15. Let $p = 3\gamma^1 \oplus 7\gamma^3$ and $p' = 5\gamma^3 \oplus 8\gamma^5$ be two polynomials in canonical form. By applying the formulas given above, the following results are obtained :

$$\begin{aligned}
p \oplus p' &= 3\gamma^1 \oplus 7\gamma^3 \oplus 8\gamma^5, \\
p \otimes p' &= 8\gamma^4 \oplus 11\gamma^6 \oplus 12\gamma^6 \oplus 15\gamma^8 = 8\gamma^4 \oplus 12\gamma^6 \oplus 15\gamma^8, \\
p \wedge p' &= 5\gamma^3 \oplus 7\gamma^5 \\
p' \backslash\!\!\!\circ p &= (-2\gamma^{-2} \oplus 2\gamma^0) \wedge (-5\gamma^{-4} \oplus -1\gamma^{-2}) = -2\gamma^{-2} \oplus -1\gamma^0.
\end{aligned}$$

8.2 Causal series

Definition 27 (Causality of a series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$)**.** A series $s \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ is causal if $s = \varepsilon$ or if both $val_\gamma(s) \geq 0$ and $s \succeq 0\gamma^{val_\gamma(s)}$, where $val_\gamma(s)$ refers to the valuation in γ of series s.

Consequently, the coefficients t_i of all monomials composing a causal series s are greater than or equal to 0. If a series encodes the sequence of firing times of a transition, causality means that both the event numbers and the associated firing times are nonnegative. Further justification for relying on causal series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ will be given in Section 9, where they reflect the transfer behaviour of TEGs.

The set of causal elements of $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ has a complete semiring structure and is denoted $\overline{\mathbb{Z}}^{+}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$. Obviously, $\overline{\mathbb{Z}}^{+}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ is a complete sub-dioid of $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$.

Theorem 12. The canonical injection $\mathsf{Id}_{|\overline{\mathbb{Z}}^{+}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}} : \overline{\mathbb{Z}}^{+}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}} \to \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ is residuated and its residual is denoted $\mathsf{Pr}_+ : \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}} \to \overline{\mathbb{Z}}^{+}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$.

Proof. Direct application of Theorem 10. □

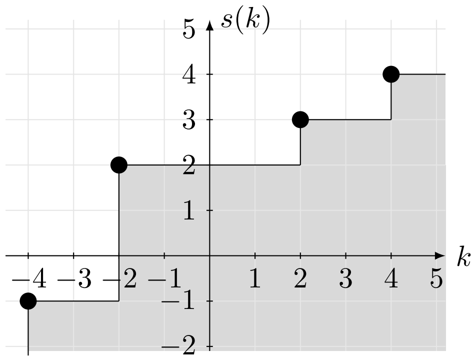
$\mathsf{Pr}_+(s)$ is the greatest causal series less than or equal to series $s \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$, and it is referred to as the causal projection of s. From a practical point of view, for all series $s \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$, the causal projection $\mathsf{Pr}_+(s)$ is obtained by:

$$\mathsf{Pr}_+(s) = \mathsf{Pr}_+\left(\bigoplus_{k\in\overline{\mathbb{Z}}} s(k)\gamma^k\right) = \bigoplus_{k\in\overline{\mathbb{Z}}} s_+(k)\gamma^k \tag{8.11}$$

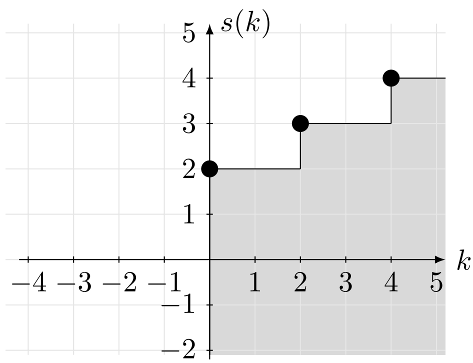
where

$$s_+(k) = \begin{cases} s(k) & \text{if } \; k \geq 0 \text{ and } s(k) \succeq e, \\ \varepsilon & \text{otherwise.} \end{cases}$$

Example 16 (Causal projection of a series)**.** Given a non-causal series $s = -1\gamma^{-4} \oplus 2\gamma^{-2} \oplus 3\gamma^2 \oplus 4\gamma^4 \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$, its causal projection $s_{caus} = \mathsf{Pr}_+(s) = 2\gamma^0 \oplus 3\gamma^2 \oplus 4\gamma^4 \in \overline{\mathbb{Z}}^{+}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$. Graphically, the causal projection of a series s can be represented as the series that covers the same area in the first quadrant as s, but is devoid of any points in the other quadrants. In Fig. 8.3 the series s and its causal projection $\mathsf{Pr}_+(s)$ are shown.



(a) $s = -1\gamma^{-4} \oplus 2\gamma^{-2} \oplus 3\gamma^{2} \oplus 4\gamma^{4}$



(b) $\mathsf{Pr}_{+}(s) = 2\gamma^{0} \oplus 3\gamma^{2} \oplus 4\gamma^{4}$

Figure 8.3: Causal projection of a (non-causal) series $s \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$.

Proposition 3. The following properties hold $\forall s_1, s_2 \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$:

$$s_1 \preceq s_2 \Rightarrow \mathsf{Pr}_{+}(s_1) \preceq \mathsf{Pr}_{+}(s_2), \tag{8.12}$$
$$\mathsf{Pr}_{+}(\mathsf{Pr}_{+}(s_1)) = \mathsf{Pr}_{+}(s_1), \tag{8.13}$$
$$\mathsf{Pr}_{+}(s_1) \preceq s_1, \tag{8.14}$$
$$\mathsf{Pr}_{+}(s_1) \otimes \mathsf{Pr}_{+}(s_2) \preceq \mathsf{Pr}_{+}(s_1 \otimes s_2), \tag{8.15}$$
$$\mathsf{Pr}_{+}(s_1) \wedge \mathsf{Pr}_{+}(s_2) = \mathsf{Pr}_{+}(s_1 \wedge s_2), \tag{8.16}$$
$$\mathsf{Pr}_{+}(s_1) \oplus \mathsf{Pr}_{+}(s_2) = \mathsf{Pr}_{+}(s_1 \oplus s_2), \tag{8.17}$$

and $\forall s_1, s_2 \in \overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$:

$$\mathsf{Pr}_+(s_1) = s_1, \tag{8.18}$$

$$\mathsf{Pr}_+(s_1) \oplus \mathsf{Pr}_+(s_2) = \mathsf{Pr}_+(s_1 \oplus s_2) = s_1 \oplus s_2, \tag{8.19}$$

$$\mathsf{Pr}_+(s_1) \otimes \mathsf{Pr}_+(s_2) = \mathsf{Pr}_+(s_1 \otimes s_2) = s_1 \otimes s_2. \tag{8.20}$$

Proof.
Eq. (8.12) : $\mathsf{Pr}_+ = (\mathsf{Id}_{|\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}})^\sharp$ is an isotone mapping according to Theorem 8.
Eqs. (8.13) and (8.14) : According to Theorem 10, Pr_+ is idempotent and $\mathsf{Pr}_+ \preceq \mathsf{Id}_{\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}}$.
Eq. (8.15) : Eq. (8.14) yields $\mathsf{Pr}_+(s_1) \preceq s_1$ and $\mathsf{Pr}_+(s_2) \preceq s_2$, hence isotony of the product law yields $\mathsf{Pr}_+(s_1) \otimes \mathsf{Pr}_+(s_2) \preceq s_1 \otimes s_2$. By applying causal projection on both sides, Eq. (8.12) yields $\mathsf{Pr}_+(\mathsf{Pr}_+(s_1) \otimes \mathsf{Pr}_+(s_2)) \preceq \mathsf{Pr}_+(s_1 \otimes s_2)$. The semiring $\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ is a subsemiring of $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$, hence it is closed under addition and multiplication (see Definition 10), hence $\mathsf{Pr}_+(\mathsf{Pr}_+(s_1) \otimes \mathsf{Pr}_+(s_2)) = \mathsf{Pr}_+(s_1) \otimes \mathsf{Pr}_+(s_2)$, and therefore $\mathsf{Pr}_+(s_1) \otimes \mathsf{Pr}_+(s_2) \preceq \mathsf{Pr}_+(s_1 \otimes s_2)$.
Eq. (8.16) : $\mathsf{Pr}_+ = (\mathsf{Id}_{|\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}})^\sharp$, hence according to Theorem 8, Pr_+ is *u.s.c.*.
Eq. (8.17) : according to (8.11),

$$\begin{aligned}
\mathsf{Pr}_+(s_1 \oplus s_2) &= \left(\bigoplus_{k \in \overline{\mathbb{Z}}} (s_1 \oplus s_2)_+(k)\gamma^k\right) \\
&= \bigoplus_{k \in \overline{\mathbb{Z}}} ((s_1)_+(k) \oplus (s_2)_+(k))\gamma^k \\
&= \mathsf{Pr}_+(s_1) \oplus \mathsf{Pr}_+(s_2).
\end{aligned}$$

Eqs. (8.18), (8.19) and (8.20) : are implied by the fact that $\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ is a subsemiring of $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ and Eq. (8.13).

□

Remark 20. According to Eqs. (8.17) and (8.16), Pr_+ is both a $\oplus$-morphism and a $\wedge$-morphism.

Remark 21. The causal projection $\mathsf{Pr}_+(H)$ of a matrix $H \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{p\times m}_{/\mathcal{R}_{\gamma^*}}$ is the $p \times m$ matrix of the projections of its entries H_{ij}, *i.e.*,

$$(\mathsf{Pr}_+(H))_{ij} = \mathsf{Pr}_+(H_{ij});\ \ i = 1, \dots, p, j = 1, \dots, m$$

8.3 Ultimately periodic series

Definition 28 (Ultimately periodic series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$)**.** A series $s \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ is said to be ultimately periodic if it can be written as $s = p \oplus q \otimes r^*$, where $p = \bigoplus_{i=1}^{m} t_i\gamma^{n_i}$ is a polynomial referring to a transient phase, e.g., the start-up of the system, $q = \bigoplus_{j=1}^{n} T_j\gamma^{N_j}$ is a polynomial representing the periodical behavior, *i.e.*, the pattern that will be repeated periodically, and $r = \tau\gamma^{\nu}$ is a monomial describing the periodicity. Then the ratio (in standard algebra) $\sigma(s) = \frac{\nu}{\tau}$ is the asymptotic slope (or throughput) of the series, *i.e.*, once the periodic regime is reached, an event occurs ν times every τ time units.

In the following, we will always assume that the polynomials p and q are in canonical form (see Definition 25).

Definition 29 (Proper Representation)**.** An ultimately periodic series $s = \left(\bigoplus_{i=1}^{m} t_i\gamma^{n_i}\right) \oplus \left(\bigoplus_{j=1}^{n} T_j\gamma^{N_j}\right)(\tau\gamma^{\nu})^*$ in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ is in proper form if

$$n_i < n_{i+1},\ N_j < N_{j+1,}\ n_m < N_1,\ t_m \prec T_1$$
$$\text{and } N_n - N_1 < \nu,\ t_1 \circ\!\!\!\backslash T_n \prec \tau.$$

Definition 30. A proper form $s = p \oplus qr^*$ is said to be at least as simple as another proper form $s = p' \oplus q'r'^*$ if

$$n_m \le n'_m,\ t_m \preceq t'_m \ \text{ and }\ \nu \le \nu',\ \tau \preceq \tau'.$$

Theorem 13. An ultimately periodical series s admits a simplest representation. This simplest representation is the canonical form of the series s.

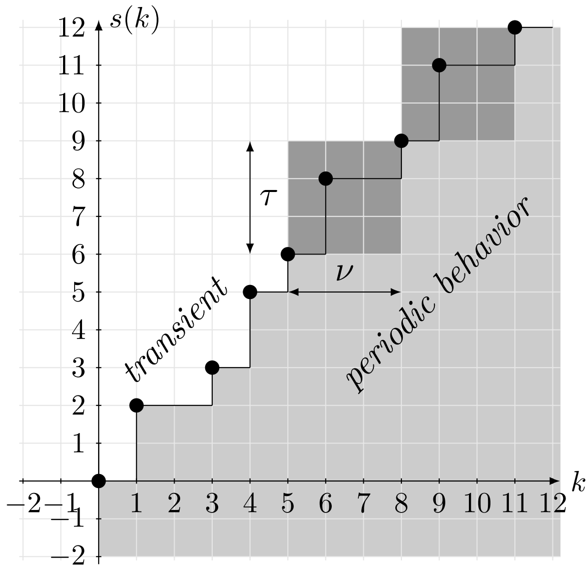


Figure 8.4: Graphical representation of ultimately periodic series $s = e \oplus 2\gamma \oplus 3\gamma^3 \oplus 5\gamma^4 \oplus (6\gamma^5 \oplus 8\gamma^6)(3\gamma^3)^*$.

The canonical form of a series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ is defined analogously to the canonical form of a polynomial (see Definition 25).

Example 17 (Ultimately periodic series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$)**.** Considering the series $s \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$

$$s = 0\gamma^0 \oplus 2\gamma \oplus 3\gamma^3 \oplus 5\gamma^4 \oplus 6\gamma^5 \oplus 8\gamma^6 \\ \oplus 9\gamma^8 \oplus 11\gamma^9 \oplus 12\gamma^{11} \oplus 14\gamma^{12} \oplus \dots$$

This series is ultimately periodic and can be rewritten in simplest form as

$$s = \underbrace{0\gamma^0 \oplus 2\gamma^1 \oplus 3\gamma^3 \oplus 5\gamma^4}_{p} \oplus \underbrace{\left(6\gamma^5 \oplus 8\gamma^6\right)}_{q} \underbrace{\left(3\gamma^3\right)^*}_{r^*}.$$

The graphical representation of this series is given in Fig. 8.4. The reader is invited to check that this series is in proper form according to Definition 30 and that it is the simplest, hence the canonical form. This can be checked by observing that the polynomials p and q are in canonical form and that ν, τ, $val_\gamma(q)$ cannot be reduced to represent this series. Of course this series admits an infinite number of representations, *e.g.*, it can be written as $s = 0\gamma^0 \oplus 2\gamma^1 \oplus 3\gamma^3 \oplus 5\gamma^4 \oplus 6\gamma^5 \oplus \left(8\gamma^6 \oplus 9\gamma^8 \oplus 11\gamma^9 \oplus 12\gamma^11\right)\left(6\gamma^6\right)^*$. The canonical form, however, is essential when comparing series.

The operations between ultimately periodic series can be deduced from the corresponding operations between polynomials, and it can be checked that the results are ultimately periodic. In particular,

$$
\begin{aligned}
s" \triangleq & s \oplus s' = p \oplus p' \oplus qr^* \oplus q'r'^* \\
& \quad \text{has asymptotic slope} \frac{\nu"}{\tau"} = \min(\frac{\nu}{\tau}, \frac{\nu'}{\tau'}), \\
s" \triangleq & s \otimes s' = pp' \oplus p'qr^* \oplus pq'r'^* \oplus qq'r^*r'^* \\
& \quad \text{has asymptotic slope } \frac{\nu"}{\tau"} = \min(\frac{\nu}{\tau}, \frac{\nu'}{\tau'}), \\
s" \triangleq & s \wedge s' = p \wedge p' \oplus (p \wedge q'r'^*) \oplus (qr^* \wedge p') \oplus (qr^* \wedge q'r'^*) \\
& \quad \text{has asymptotic slope } \frac{\nu"}{\tau"} = \max(\frac{\nu}{\tau}, \frac{\nu'}{\tau'}).
\end{aligned}
$$

Furthermore,

$$
\begin{array}{rl}
\text{if } \frac{\nu'}{\tau'} \geq \frac{\nu}{\tau} \text{ then} & s" \triangleq s' \backslash\!\!\!\backslash s = (p' \backslash\!\!\!\backslash (p \oplus qr^*)) \wedge ((q'r'^*) \backslash\!\!\!\backslash (qr^*)) \\
& \text{has asymptotic slope } \frac{\nu"}{\tau"} = \frac{\nu}{\tau}, \\
\text{else} & s" \triangleq s' \backslash\!\!\!\backslash s = \varepsilon.
\end{array}
$$

Note that, by convention, expressions for the asymptotic slope of ultimately periodic series are usually written in the standard algebra.

Example 18 (Operations over ultimately periodic series)**.** Let $s = e \oplus 3\gamma^4(3\gamma^5)^*$ and $s' = 3\gamma \oplus 4\gamma^2(1\gamma^4)^*$ be two ultimately periodic series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$. By considering the simplification rules (8.5)–(8.8), we obtain the following results:

$$
\begin{array}{rcl}
s \oplus s' & = & e \oplus 3\gamma^4(3\gamma^5)^* \oplus 3\gamma^1 \oplus 4\gamma^2(1\gamma^4)^* \\
& = & e \oplus 3\gamma^4 \oplus 6\gamma^9 \oplus 9\gamma^{14} \oplus 12\gamma^{19} ... \\
& & \oplus 3\gamma^1 \oplus 4\gamma^2 \oplus 5\gamma^6 \oplus 6\gamma^{10} \oplus 7\gamma^{14} ... \\
& = & e \oplus 3\gamma^4 \oplus 6\gamma^9 \oplus 9\gamma^{14} \oplus ... \oplus 3\gamma \oplus 4\gamma^2 \oplus 5\gamma^6 \\
& = & e \oplus 3\gamma^1 \oplus 4\gamma^2 \oplus 5\gamma^6 \oplus 6\gamma^9 \oplus 9\gamma^{14} \oplus ... \\
& = & e \oplus 3\gamma^1 \oplus 4\gamma^2 \oplus 5\gamma^6 \oplus 6\gamma^9(3\gamma^5)^*, \\
s \otimes s' & = & (e \oplus 3\gamma^4(3\gamma^5)^*) \otimes (3\gamma^1 \oplus 4\gamma^2(1\gamma^4)^*) \\
& = & (e \oplus 3\gamma^4 \oplus 6\gamma^9 \oplus 9\gamma^{14} \oplus 12\gamma^{19} ...) \\
& & \otimes(3\gamma^1 \oplus 4\gamma^2 \oplus 5\gamma^6 \oplus 6\gamma^{10} \oplus 7\gamma^{14} ...) \\
& = & (3\gamma^1 \oplus 4\gamma^2 \oplus 5\gamma^6 \oplus 6\gamma^{10} \oplus 7\gamma^{14} ...) \\
& & \oplus(6\gamma^5 \oplus 7\gamma^6 \oplus 8\gamma^{10} \oplus 9\gamma^{14} \oplus 10\gamma^{18} ...) \\
& & \oplus(9\gamma^{10} \oplus 10\gamma^{11} \oplus 11\gamma^{15} \oplus 12\gamma^{19} \oplus 13\gamma^{23} ...) \oplus ... \\
& = & (3\gamma^1 \oplus 4\gamma^2) \oplus (6\gamma^5 \oplus 7\gamma^6) \oplus (9\gamma^{10} \oplus 10\gamma^{11}) \oplus ... \\
& = & (3\gamma^1 \oplus 4\gamma^2) \oplus (6\gamma^5 \oplus 7\gamma^6)(3\gamma^5)^*.
\end{array}
$$

9

TEG description in an idempotent semiring

TEGs constitute a subclass of timed Petri nets in which each place has exactly one upstream and one downstream transition and all arcs have weight 1. A TEG description, as the one obtained in Section 3, can be transformed into a max-plus or a min-plus linear model and *vice versa*. To obtain an algebraic model in $\overline{\mathbb{Z}}_{\max}$, a "dater" function is associated to each transition.

In a first step, we partition the set of transitions in the investigated TEG into

- a set of transitions x_i, $1 = 1, \ldots, n$, with both upstream and downstream places; these transitions are referred to as internal transitions,
- a set of transitions y_i, $i = 1, \ldots, m$, with only upstream places; these transitions are referred to as output transitions;
- a set of transitions with only downstream places; these transitions are referred to as input transitions; the set of input transitions is further partitioned into
  - controllable input transitions u_i, $i = 1, \ldots p$, *i.e.*, transitions with freely assignable firing times, and

– uncontrollable input transitions w_i, $i = 1, \dots, l$, *i.e.*, transitions with unknown firing times. These can be interpreted as disturbances (see below).

We assume the following. The firing times of output transitions are immediately observable by an outside agent, e.g., a controller. Furthermore, each output transition y_i has precisely one upstream place, and the upstream transition of this place is an internal one. Finally, the upstream place of each output transition has zero holding time and initially contains zero tokens. This implies that the firing times of each output transition immediately indicate the corresponding firing times of selected internal transitions.

With respect to uncontrollable input transitions, the following assumptions are in place. Each uncontrollable input transition has precisely one downstream place, and the downstream transition of this place is an internal one. Moreover, the downstream place of each uncontrollable input transition has zero holding time and initially contains zero tokens. This implies that each uncontrollable input transition can be interpreted as a disturbance acting on precisely one internal transition. More precisely, the kth firing of the affected internal transition may not occur before the kth firing of the uncontrollable input. Conversely, if the firings of the uncontrollable input transition occur sufficiently early such that the firings of the associated internal transition are not delayed, the disturbance is non-functional.

With each transition, we associate a dater function, which, for simplicity, is denoted by the same symbol as the transition. Hence, $x_i : \overline{\mathbb{Z}} \to \overline{\mathbb{Z}}_{\max}$ is the dater function associated with the internal transition x_i, and $x_i(k)$ represents the time (date) of this transition's k^{th} firing (see Baccelli *et al.*, 1992,Heidergott *et al.*, 2006).

By collecting the internal dater functions x_i, $i = 1, \dots n$, in the vector function $x : \overline{\mathbb{Z}} \to \overline{\mathbb{Z}}^n_{\max}$, the controllable input dater functions: u_i, $i = 1, \dots p$, in $u : \overline{\mathbb{Z}} \to \overline{\mathbb{Z}}^p_{\max}$, the uncontrollable input dater functions w_i, $i = 1, \dots l$, in $w : \overline{\mathbb{Z}} \to \overline{\mathbb{Z}}^l_{\max}$ and the output dater functions y_i, $i = 1, \dots m$, in $y : \overline{\mathbb{Z}} \to \overline{\mathbb{Z}}^m_{\max}$, a TEG satisfying the above assumptions and operating under the earliest firing rule can be described by the following max-plus linear system:

$$x(k) = \bigoplus_{j=0}^{N_a} A_j x(k-j) \oplus \bigoplus_{j=0}^{N_b} B_j u(k-j) \oplus R_0 w(k), \qquad (9.1)$$

$$y(k) = C_0 x(k). \qquad (9.2)$$

The nonnegative integer N_a is equal to the maximal number of tokens initially contained in places between internal transitions, the nonnegative integer N_b is equal to the maximal number of tokens initially contained in places between controllable input transitions and internal transitions, and the entries of matrices $A_j \in \overline{\mathbb{Z}}_{\max}^{n\times n}$, $B_j \in \overline{\mathbb{Z}}_{\max}^{n\times p}$, $R_0 \in \overline{\mathbb{Z}}_{\max}^{n\times l}$, and $C_0 \in \overline{\mathbb{Z}}_{\max}^{m\times n}$ represent the structure of the TEG. More specifically, if there is a place with upstream transition x_q and downstream transition x_i, and if this place initially contains j tokens, then the entry $(A_j)_{iq}$ is the holding time of this place, otherwise $(A_j)_{iq} = \varepsilon$. Similarly, if there is a place with upstream transition u_q and downstream transition x_i, and if this place initially contains j tokens, then the entry $(B_j)_{iq}$ is the holding time of this place, otherwise $(B_j)_{iq} = \varepsilon$. Moreover, $(R_0)_{iq} = e$, if there is a place with upstream transition w_q and downstream transition x_i, otherwise $(R_0)_{iq} = \varepsilon$, and $(C_0)_{iq} = e$, if there is a place with upstream transition x_q and downstream transition y_i, otherwise $(C_0)_{iq} = \varepsilon$. The above assumptions regarding uncontrollable input transitions imply that each column of matrix R_0 has precisely one entry equal to e, each row has at most one entry equal to e, and all other entries are equal to ε. Similarly, the assumptions regarding the output transitions imply that each row of matrix C_0 has precisely one entry equal to e, each column has at most one entry equal to e, and all other entries are equal to ε.

The following example illustrates how a TEG is modeled as a max-plus linear system.

Example 19. In Fig. 9.1, a TEG with $p = 1$ controllable input transition, $l = 2$ uncontrollable input transitions, and $m = 1$ measurable output transition is depicted. Clearly, $N_a = 2$ and $N_b = 0$. Hence, the TEG is represented by the max-plus linear system in Eqns. (9.1),(9.2), where

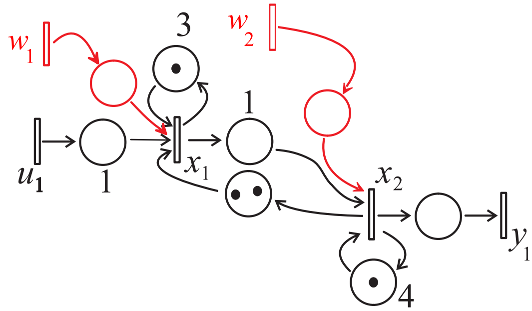


Figure 9.1: TEG with 1 controllable transition u_1, 2 uncontrollable transitions (w_1, w_2), and 1 measurable output transition y_1.

the system matrices are

$$A_0 = \begin{bmatrix} \varepsilon & \varepsilon \\ 1 & \varepsilon \end{bmatrix}, \ A_1 = \begin{bmatrix} 3 & \varepsilon \\ \varepsilon & 4 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} \varepsilon & e \\ \varepsilon & \varepsilon \end{bmatrix}, \ B_0 = \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix},$$
$$C_0 = \begin{bmatrix} \varepsilon & e \end{bmatrix}, \ R_0 = \begin{bmatrix} e & \varepsilon \\ \varepsilon & e \end{bmatrix}.$$

Note that matrix $R_0 = I_2$, where I_2 is the identity matrix in $\overline{\mathbb{Z}}^2_{\max}$. This implies that both internal transitions are directly affected by independent disturbances. The disturbance w incorporates lower bounds for $x(0)$. In particular, $w(0) \preceq x(0)$ (see Baccelli *et al.*, 1992, p. 245, for a discussion about compatible initial conditions).

The sequence of firing times of each transition in a TEG is clearly non-decreasing and can therefore be written as a formal power series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ (see Definition 24 and Section 8.1).

The TEG model (9.1), (9.2) can therefore be rephrased compactly as:

$$x = Ax \oplus Bu \oplus Rw, \tag{9.3}$$
$$y = Cx, \tag{9.4}$$

where the entries of $u \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^p_{/\mathcal{R}_{\gamma^*}}$, $y \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^m_{/\mathcal{R}_{\gamma^*}}$, $x \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^n_{/\mathcal{R}_{\gamma^*}}$, and $w \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^l_{/\mathcal{R}_{\gamma^*}}$ represent formal power series associated with the controllable input transitions, the output transitions, the internal transitions and the uncontrollable input transitions, respectively. Matrices $A \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{n\times n}$, $B \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{n\times p}$, $R \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{n\times l}$, and $C \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{m\times n}$ represent the links between transitions, including the holding time and the number of initial tokens contained in the place between a pair of transitions, and they are defined as follows:

$$
\begin{aligned}
A &= \bigoplus_{j=0}^{N_a} \gamma^j A_j,\\
B &= \bigoplus_{j=0}^{N_b} \gamma^j B_l,\\
R &= \gamma^0 R_0 = R_0,\\
C &= \gamma^0 C_0 = C_0.
\end{aligned}
$$

Clearly, this γ-domain representation carries the same information with respect to a TEG as the event domain equations (9.1),(9.2), but in a more compact form.

Remark 22. It has become customary in the max-plus literature to refer to x as the state of the model (9.3), (9.4) , e.g., Baccelli *et al.*, 1992. Note that this does not coincide with the standard notion of state. The latter would correspond to the distribution of tokens over the places in the considered TEG. In the sequel, we will also refer to x as state, and the terms "state estimation" and "state feedback" are to be understood in this sense.

By considering Theorem 7, the least solutions of (9.1),(9.2) are given by:

$$
x = A^*Bu \oplus A^*Rw \tag{9.5}
$$

$$
y = CA^*Bu \oplus CA^*Rw, \tag{9.6}
$$

where $CA^*B \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{m\times p}$ and $CA^*R \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{m\times l}$ are the control/output and the disturbance/output transfer matrices. Matrices CA^*B and A^*B represent the earliest behavior of the system, and the uncontrollable input vector w is only able to delay the firing times of internal and output transitions, *i.e.*, to delay the occurrence of the corresponding events. In a manufacturing scenario, this could, for example, describe disturbances due to machine breakdown or delays due to an unexpected failure in component supply.

Example 20. The TEG depicted in Fig. 9.1 can be described in the form (9.3),(9.4) with system matrices

$$A = \begin{bmatrix} 3\gamma^1 & e\gamma^2 \\ 1\gamma^0 & 4\gamma^1 \end{bmatrix},\ B = \begin{bmatrix} 1\gamma^0 \\ \varepsilon \end{bmatrix},$$
$$C = \begin{bmatrix} \varepsilon & e \end{bmatrix},\ R = \begin{bmatrix} e & \varepsilon \\ \varepsilon & e \end{bmatrix}.$$

The entry $A_{22} = 4\gamma^1$ represents the place linking transition x_2 to itself and denotes that this place has a holding time of four time units and initially contains one token. Similarly, the entry $A_{12} = e\gamma^2$ represents the place linking transition x_2 to transition x_1 and denotes that this place has zero holding time and initially contains two tokens, etc.

To compute the transfer matrices in Eqns. (9.5) and (9.6), we need to compute A^*. This can be done by applying Algorithm 2.

$$A^{(0)} = A = \begin{bmatrix} 3\gamma^1 & e\gamma^2 \\ 1\gamma^0 & 4\gamma \end{bmatrix},$$
$$A^{(1)} = \begin{bmatrix} 3\gamma^1(3\gamma^1)^* & e\gamma^2(3\gamma^1)^* \\ 1\gamma^0(3\gamma^1)^* & 4\gamma^1 \oplus 7\gamma^4(3\gamma^1)^* \end{bmatrix},$$
$$A^{(2)} =$$
$$\begin{bmatrix} 3\gamma^1 \oplus 6\gamma^2 \oplus 9\gamma^3 \oplus 12\gamma^4 \oplus 15\gamma^5 \oplus 18\gamma^6 \oplus 21\gamma^7(4\gamma^4)^* & e\gamma^2(4\gamma^1)^* \\ 1\gamma^0(4\gamma^1)^* & (4\gamma^1)^* \end{bmatrix}.$$

The reader is invited to compute $a_{11}^{(2)}$ by using the computation rules introduced in Section 8 to provide

$$\begin{aligned} a_{11}^{(2)} &= a_{11}^{(1)} \oplus a_{12}^{(1)}(a_{22}^{(1)})^* a_{21}^{(1)} \\ &= 3\gamma^1(3\gamma^1)^* \oplus \gamma^2(3\gamma^1)^*(4\gamma^1 \oplus 7\gamma^4(3\gamma^1)^*)^* 1(3\gamma^1)^* \\ &= 3\gamma^1 \oplus 6\gamma^2 \oplus 9\gamma^3 \oplus 12\gamma^4 \oplus 15\gamma^5 \oplus 18\gamma^6 \oplus 21\gamma^7(4\gamma^4)^*. \end{aligned}$$

Therefore,

$$\begin{aligned}A^* &= I_2 \oplus A^{(2)} \\ &= \begin{bmatrix} e \oplus 3\gamma^1 \oplus 6\gamma^2 \oplus 9\gamma^3 \oplus 12\gamma^4 \oplus 15\gamma^5 \oplus 18\gamma^6 \oplus 21\gamma^7(4\gamma^4)^* & \gamma^2(4\gamma^1)^* \\ 1\gamma^0(4\gamma^1)^* & (4\gamma^1)^* \end{bmatrix}\end{aligned}$$

This computation can be easily carried out by using the toolbox `MinMaxGD`, a C++ library developed in order to handle ultimately periodic series (see Cottenceau *et al.*, 2000, Hardouin *et al.*, 2016), which is based on Algorithm 2. From matrix A^*, it is easy to obtain the control/output and disturbance/output transfer matrices:

$$\begin{aligned}CA^*B &= \begin{bmatrix} \varepsilon & e \end{bmatrix} A^* \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix} \\ &= 2\gamma^0(4\gamma)^*, && (9.7) \\ CA^*R &= \begin{bmatrix} \varepsilon & e \end{bmatrix} A^* \begin{bmatrix} e & \varepsilon \\ \varepsilon & e \end{bmatrix} \\ &= \begin{bmatrix} 1\gamma^0(4\gamma^1)^* & (4\gamma^1)^* \end{bmatrix}. && (9.8)\end{aligned}$$

Each entry of these matrices is a causal ultimately periodic series representing a transfer relation. For example, $CA^*B = 2\gamma^0(4\gamma)^*$ represents the transfer relation between the controllable input u and the output y. Given a sequence of firing times of the controllable input transition, encoded in a (non-decreasing) series u, the series $y = CA^*Bu \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ encodes the sequence of firing times of the output transition, if the TEG operates under the earliest firing rule and if the uncontrollable input does not slow down the system, *i.e.*, $CA^*Bu \succeq CA^*Rw$. Suppose this is the case and the controllable input transition fires infinitely often at time 0 (this is referred to as applying an impulse at the input). Hence, the series encoding the sequence of firing times of the controllable input transition is $0\gamma^0 \oplus 0\gamma^1 \oplus 0\gamma^2 \oplus \dots$ or, in canonical form, $u = 0\gamma^0$, *i.e.*, the neutral element of multiplication in the dioid $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$. Therefore, $y = CA^*B$. Hence, as in standard systems and control theory, transfer relations represent the impulse response of a system. Clearly, as an impulse applied to the input does not restrain the evolution of the system, the impulse response represents the fastest system behaviour. In our example, $y = CA^*B = 2\gamma^0(4\gamma)^*$. As the polynomial representing the transient part in this series vanishes, the series is immediately periodic. This is, of course, reflected in the sequence of firing times of the output transition, namely $2, 6, 10, 14, \ldots$.

Let us now apply a different control input by letting the control input transition fire at times $5, 7, 7, 12, 12$ (and then no more). The corresponding series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ is (in canonical form) $u = 5\gamma^0 \oplus 7\gamma^1 \oplus 12\gamma^3 \oplus +\infty\gamma^5$. The term $+\infty\gamma^5$ denotes that after the fifth firing of the transition at time 12, it will not fire another time. We still assume that the uncontrollable input does not slow down the system, *i.e.*, $CA^*Bu \succeq CA^*Rw$. Then,

$$
\begin{aligned}
y &= CA^*Bu = (2\gamma^0)(4\gamma)^*(5\gamma^0 \oplus 7\gamma^1 \oplus 12\gamma^3 \oplus +\infty\gamma^5) \\
&= 7\gamma^0 \oplus 11\gamma^1 \oplus 15\gamma^2 \oplus 19\gamma^3 \oplus 23\gamma^4 \oplus +\infty\gamma^5,
\end{aligned}
$$

where the second line of the above expression is obtained by applying the simplification rules (8.5)–(8.8). Consequently, in this case the firing times of the output transition will be $7, 11, 15, 19, 23$.

In this paper, we assume that the investigated TEGs are structurally controllable and structurally observable. This is a standard assumption for max-plus linear systems, which is not restrictive in practice. We first explain, what these properties formally mean.

Definition 31. (Structural Controllability Baccelli *et al.*, 1992) A TEG is said to be *structurally controllable* if every internal transition can be reached by a path from at least one controllable input transition.

Definition 32. (Structural Observability Baccelli *et al.*, 1992) A TEG is said to be *structurally observable* if, from every internal transition, there exists a path to at least one output transition.

Not surprisingly, structural controllability (respectively structural observability) of TEGs can be evaluated from the corresponding transfer matrices:

Theorem 14. (Hardouin, 2004; Spacek *et al.*, 1995) A TEG is structurally controllable if and only if the transfer matrix A^*B is such that at least one entry in each row is different from ε.

Theorem 15. (Hardouin, 2004; Spacek *et al.*, 1995) A TEG is structurally observable if and only if the transfer matrix CA^* is such that at least one entry in each column is different from ε.

Let us briefly revisit Example 20. There, all entries of the matrices A, B, C, R are causal polynomials in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$, and all entries in the resulting transfer matrix CA^*B, respectively CA^*R, are causal ultimately periodic series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$. This raises the question whether system models (9.3),(9.4) with only causal polynomial entries in the matrices A, B, C, R will always give rise to transfer matrices with exclusively causal ultimately periodic entries and, reversely, whether, any such transfer matrix can always be represented by (9.3),(9.4) with only causal polynomial entries.

Definition 33 (Realizability)**.** A series $s \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ is said to be realizable if there exist three matrices A, B and C of appropriate dimension and with entries that are are causal polynomials, such that $s = CA^*B$. A matrix is said to be realizable if its entries are realizable.

In other words, a series s is realizable if it corresponds to the transfer function of a timed event graph.

Theorem 16 (Cohen *et al.*, 1989, Baccelli *et al.*, 1992, Theorem 5.39)**.** Let H be a matrix in $(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{q\times p}$. The matrix H is realizable if and only if all entries of H are ultimately periodic and causal.

The following remarks show how to obtain a TEG realizing a causal ultimately periodic series.

Remark 23. Let $s = p \oplus qr^*$ be an ultimately periodic and causal series in canonical form, with $p = \bigoplus_{i=1}^{n} t_i\gamma^{n_i}$, $q = \bigoplus_{j=1}^{m} T_j\gamma^{N_j}$ and $r = \tau\gamma^{\nu}$, with $\nu,\tau \in \mathbb{N}$. This series can be realized as:

$$\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} = \begin{pmatrix} r & \varepsilon \\ e & \varepsilon\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} \oplus \begin{pmatrix} q\\ p\end{pmatrix} u \tag{9.9}$$

$$y = \begin{pmatrix}\varepsilon & e\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}$$

where $\zeta_1, \zeta_2, y, u \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$.

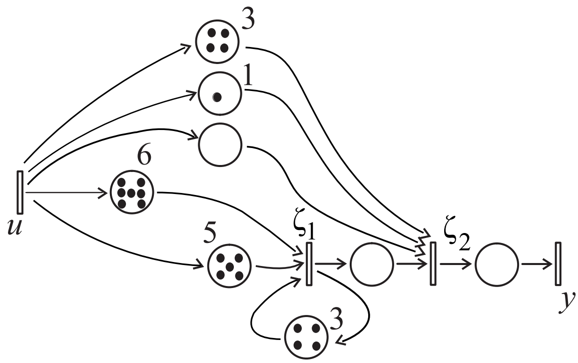


Figure 9.2: Timed Event Graph realization of series $s = p \oplus qr^* = e \oplus 1\gamma \oplus 3\gamma^4 \oplus (5\gamma^5 \oplus 6\gamma^7)(3\gamma^4)^*$.

Remark 24. From the realization (9.9) of the ultimately periodic and causal series s, it is straightforward to obtain the corresponding timed event graph. To do this, we introduce transitions ζ_1, ζ_2, u and y. Furthermore, we introduce a place linking the transition ζ_1 to itself, reflecting the monomial r, *i.e.*, equipped with a holding time of τ and initially containing ν tokens. Then we introduce m places corresponding to the monomials of polynomial q. They link the transition u to the transition ζ_1, have holding times T_j and initially hold N_j tokens, $j = 1, \ldots, m$. We additionally introduce n places corresponding to the monomials of polynomial p. They link the transition u to the transition ζ_2, have holding times t_i and initially hold n_i tokens, $i = 1, \ldots, n$. Finally, we introduce two places, one linking ζ_1 to ζ_2 and the other linking ζ_2 to the output transition y, both with holding time 0 and initially without tokens.

Example 21. Let $s = p \oplus qr^* = e \oplus 1\gamma \oplus 3\gamma^4 \oplus (5\gamma^5 \oplus 6\gamma^7)(3\gamma^4)^*$ be a series. The realization (9.9) then becomes

$$\begin{aligned} \zeta_1 &= (3\gamma^4)\zeta_1 \oplus (5\gamma^5 \oplus 6\gamma^7)u \\ \zeta_2 &= \zeta_1 \oplus (e \oplus 1\gamma \oplus 3\gamma^4)u \\ y &= \zeta_2. \end{aligned}$$

The corresponding TEG is shown in Fig. 9.2. When operated under an earliest firing policy, the firing times of transitions ζ_1, ζ_2, u and y are related by the following set of difference equations.

$$\begin{aligned}
\zeta_1(k) &= 3\zeta_1(k-4) \oplus 5u(k-5) \oplus 6u(k-7) \\
\zeta_2(k) &= \zeta_1(k) \oplus u(k) \oplus 1u(k-1) \oplus 3u(k-4) \\
y(k) &= \zeta_2(k).
\end{aligned}$$

10

Max-plus observer

This section deals with the estimation of the firing times of internal transitions of a TEG, if they are not directly observable. They are estimated on the basis of the TEG model, the measured firing times of the output transitions and the known controllable input transitions. According to Remark 22, this is usually referred to as the state estimation problem for TEGs. This problem has been addressed in different ways (see Spacek *et al.*, 1995; Gazarik and Kamen, 1997; Markele Ferreira Candido *et al.*, 2013; Lotito *et al.*, 2001). In the following, an observer structure directly inspired by the Luenberger observer in classical linear systems theory (e.g., Luenberger, 1971) is considered (Hardouin *et al.*, 2010b; Hardouin *et al.*, 2010a). This observer structure, depicted in Fig. 10.1, is composed of two parts:
(i) the simulator is, except for the disturbance term Rw, a copy of the system model (9.3), (9.4), and is therefore characterized by the matrices $A \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{n\times n}$, $B \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{n\times p}$, $C \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{m\times n}$. As the simulator is initialized by $\hat{x}_i(k) = \varepsilon, \forall k \leq 0$, and as the disturbance term can only slow down the system behavior, the simulator represents the fastest possible behavior of the system (9.3), (9.4).
(ii) The observer matrix $L \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{n\times m}$ is used to feed back

information from the measurable system output into the simulator in order to take the effect of the disturbance w into account. L is chosen to

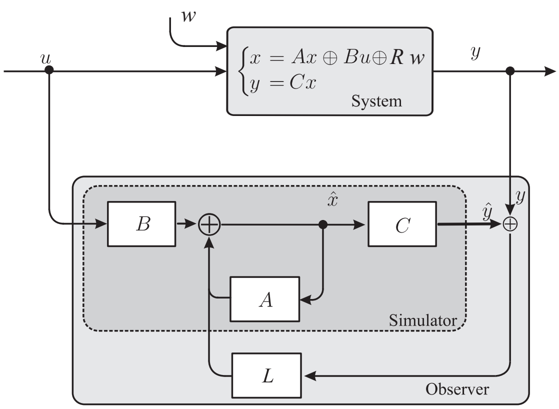


Figure 10.1: The observer structure of max-plus linear systems.

be the greatest matrix (in the order defined in the dioid $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$) such that $\hat{x} \preceq x$. This will make the firing times of the internal transitions $\hat{x}_i$ in the observer, which can be interpreted as estimates of the corresponding firing times of the internal system transitions x_i, $i = 1, \ldots, n$, as large as possible, but without ever being later than the latter.

With the matrices A, B, C and R characterizing the system model assumed to be known, x and y are provided by Eqns. (9.5),(9.6). According to Fig. 10.1, the observer equations, in analogy to the Luenberger observer, are given by:

$$\begin{aligned} \hat{x} &= A\hat{x} \oplus Bu \oplus L(\hat{y} \oplus y) = A\hat{x} \oplus Bu \oplus LC\hat{x} \oplus LCx \\ &= (A \oplus LC)\hat{x} \oplus Bu \oplus LCx. \end{aligned}$$

Because of Theorem 7,

$$\begin{aligned} \hat{x} &= (A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCx \\ &= (A \oplus LC)^*Bu \oplus (A \oplus LC)^*LC(A^*Bu \oplus A^*Rw), \end{aligned} \tag{10.1}$$

where the latter follows from Eq. (9.5).

By using Eqns. (6.3),(6.6), the following equality is obtained:

$$(A \oplus LC)^* = A^*(LCA^*)^*. \tag{10.2}$$

Inserting (10.2) into (10.1), we get

$$\hat{x} \;=\; A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^*LCA^*Bu \oplus A^*(LCA^*)^*LCA^*Rw.$$

Recalling that $(LCA^*)^*LCA^* = (LCA^*)^+$ (see Example 9), this equation may be written as

$$\hat{x} \;=\; A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^+Bu \oplus A^*(LCA^*)^+Rw.$$

Since $(LCA^*)^* \succeq (LCA^*)^+ = (LCA^*)^*LCA^*$ and multiplication is isotone, the observer equation can be simplified to

$$\begin{aligned} \hat{x} &= A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^+Rw \\ &= (A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCA^*Rw, \end{aligned} \tag{10.3}$$

where the latter equality follows from Eq. (10.2).

As indicated above, the objective is to compute the greatest observer matrix L, denoted by L_{opt}, such that $\hat{x} \preceq x$. Hence, we require the greatest L satisfying the following inequality $\forall u, w$:

$$(A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCA^*Rw \preceq A^*Bu \oplus A^*Rw. \tag{10.4}$$

Equivalently, we require

$$(A \oplus LC)^*B \;\preceq\; A^*B \quad \text{and} \tag{10.5}$$

$$(A \oplus LC)^*LCA^*R \;\preceq\; A^*R. \tag{10.6}$$

The following Lemmas yield conditions for the observer matrix L such that (10.5) and (10.6) hold. They also provide the greatest observer matrices satisfying (10.5) and (10.6).

Lemma 1 (Hardouin *et al.*, 2010a)**.** The following equivalence holds:

$$(A \oplus LC)^*B = A^*B \Leftrightarrow L \preceq L_1 = (A^*B) \phi\!\!/ (CA^*B)$$

Proof. Note that if all entries of L are ε, inequality $(A \oplus LC)^*B \preceq A^*B$ is trivially satisfied with equality. As matrix multiplication and the Kleene star operation are isotone, any other solution of $(A \oplus LC)^*B \preceq A^*B$ will also achieve equality. Furthermore,

$$
\begin{aligned}
& (A \oplus LC)^*B \preceq A^*B \\
\Leftrightarrow \quad & (A^*LC)^*A^*B \preceq A^*B && \text{because of (6.3)} \\
\Leftrightarrow \quad & (A^*LC)^* \preceq (A^*B)\phi(A^*B) && \text{as multiplication is residuated} \\
\Leftrightarrow \quad & (A^*LC)^* \preceq ((A^*B)\phi(A^*B))^* && \text{because of (7.16)} \\
\Leftrightarrow \quad & A^*LC \preceq (A^*B)\phi(A^*B) && \text{see Example 8} \\
\Leftrightarrow \quad & L \preceq A^*\circ\!\!\!\!\backslash(A^*B)\phi(A^*B)\phi C && \text{as multiplication is residuated} \\
\Leftrightarrow \quad & L \preceq A^*\circ\!\!\!\!\backslash(A^*B)\phi(CA^*B) && \text{because of (7.8)} \\
\Leftrightarrow \quad & L \preceq (A^*B)\phi(CA^*B) && \text{because of (7.17)}
\end{aligned}
$$

□

Lemma 2 (Hardouin *et al.*, 2010a)**.** The following equivalence holds:

$$(A \oplus LC)^*LCA^*R \preceq A^*R \Leftrightarrow L \preceq L_2 = (A^*R)\phi(CA^*R).$$

Proof.

$$
\begin{aligned}
& (A \oplus LC)^*LCA^*R \preceq A^*R \\
\Leftrightarrow \quad & A^*(LCA^*)^*LCA^*R \preceq A^*R && \text{because of (10.2)} \\
\Leftrightarrow \quad & (LCA^*)^*LCA^*R \preceq A^*\circ\!\!\!\!\backslash(A^*R) && \text{as multiplication is residuated} \\
\Leftrightarrow \quad & (LCA^*)^*LCA^*R \preceq A^*R && \text{because of (7.17)} \\
\Leftrightarrow \quad & (LCA^*)^*LCA^*A^*R \\
& \quad = (LCA^*)^+A^*R \preceq A^*R && \text{because of (6.4) and the def. of } a^+ \\
\Leftrightarrow \quad & (LCA^*)^+ \preceq (A^*R)\phi(A^*R) && \text{as multiplication is residuated} \\
& \quad\quad = ((A^*R)\phi(A^*R))^* && \text{because of (7.16).}
\end{aligned}
$$

According to Remark 14, $((A^*R)\phi(A^*R))^*$ is in $\mathsf{Im}P$, where $P : x \to x^+$. Therefore, by applying the result presented in Example 9, this inequality may be written as follows :

$$
\begin{aligned}
& LCA^* \preceq (A^*R)\phi(A^*R) \\
\Leftrightarrow \quad & L \preceq ((A^*R)\phi(A^*R))\phi(CA^*) && \text{as multiplication is residuated} \\
\Leftrightarrow \quad & L \preceq (A^*R)\phi(CA^*A^*R) && \text{because of (7.8)} \\
\Leftrightarrow \quad & L \preceq (A^*R)\phi(CA^*R) = L_2 && \text{because of (6.5).}
\end{aligned}
$$

□

An immediate consequence of Lemma 1 and Lemma 2 is that the greatest L that satisfies requirements (10.5) and (10.6) is $L_1 \wedge L_2$. Therefore, the following proposition holds.

Proposition 4 (Hardouin *et al.*, 2010a)**.** $L_{opt} = L_1 \wedge L_2$ is the greatest observer matrix L such that $\forall(u, w)$:

$$\hat{x} = A\hat{x} \oplus Bu \oplus Ly \quad \preceq \quad x = Ax \oplus Bu \oplus Rw.$$

Note that Lemma 1, Lemma 2 and Proposition 4 can be restricted to the respective causal projections. Hence, $L_{1_+} = \mathsf{Pr}_+(L_1)$ is the greatest causal solution of requirement (10.5), $L_{2_+} = \mathsf{Pr}_+(L_2)$ is the greatest causal solution of requirement (10.6), and $L_{opt_+} = \mathsf{Pr}_+(L_{opt})$ represents the greatest causal solution that guarantees $\hat{x} \preceq x$ for all u, w.

Below we show that the greatest causal solution L_{opt_+} achieves equality between estimated output $\hat{y}$ and measured output y (see Proposition 5). As a preliminary step, we show in the following lemma that the causal observer matrix $\tilde{L} = C^T$ also achieves equality between $\hat{y}$ and y. However, by construction, $\tilde{L} \preceq L_{opt_+}$, therefore the state estimate $\tilde{x}$ generated by an observer with matrix $\tilde{L}$ will in general not be as good as the estimate $\hat{x}$ generated by an observer with matrix L_{opt_+}, *i.e.*, $\tilde{x} \preceq \hat{x} \preceq x$.

Lemma 3. The observer matrix $\tilde{L} = C^T$, where $(\cdot)^T$ denotes the matrix transpose, ensures equality between estimated output $\hat{y} = C\hat{x}$ and measured output y.

Proof. From Eq. (10.3), equality of $\hat{y}$ and y for all u, w is equivalent to

$$C(A \oplus \tilde{L}C)^* B = CA^* B, \tag{10.7}$$

$$\text{and} \quad C(A \oplus \tilde{L}C)^* \tilde{L}CA^* R = CA^* R. \tag{10.8}$$

By assumption, the entries of matrix $C \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{m\times n}_{/\mathcal{R}_{\gamma^*}}$ are in $\{\varepsilon, e\}$ and precisely one is equal to e in each row (see Section 9). Hence, $\tilde{L}C = C^T C \preceq I_n$, where I_n is the $n \times n$ identity matrix in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$, and $C\tilde{L} = CC^T = I_m$. This implies that $A \oplus \tilde{L}C \preceq A \oplus I_n$ and, because the Kleene star operation is isotone,

$$\begin{aligned}(A \oplus \tilde{L}C)^* &\preceq (A \oplus I_n)^* \\ &= (A^* I_n)^* A^* \;\; \text{because of (6.3)} \\ &= A^* \text{ because of (6.4) and (6.5)} \\ &\preceq (A \oplus \tilde{L}C)^*,\end{aligned}$$

where the latter inequality follows from $A \preceq A \oplus \tilde{L}C$ and the fact that the Kleene star operation is isotone. Hence, equality $(A \oplus \tilde{L}C)^* = A^*$ holds, and Eq. (10.7) is proven. To prove Eq. (10.8), we insert the equality $(A \oplus \tilde{L}C)^* = A^*$ into the left hand side of (10.8) to get $CA^*\tilde{L}CA^*R$. Because of $\tilde{L}C \preceq I_n$, order preservingness of matrix multiplication and (6.4), we get

$$CA^*\tilde{L}CA^*R \preceq CA^*I_nA^*R = CA^*R.$$

Since $A^* \succeq I_n$ and because of order preservingness of matrix multiplication, we also get

$$CA^*\tilde{L}CA^*R \succeq C\tilde{L}CA^*R = CA^*R,$$

where the latter equality follows from $C\tilde{L} = I_m$. Consequently $CA^*\tilde{L}CA^*R = CA^*R$, and Eq. (10.8) has also been proven. □

Proposition 5. Matrix $L_{opt_+} = \mathsf{Pr}_+(L_{opt})$ ensures equality between the estimated output $\hat{y}$ and the measured output y.

Proof. In order to prove equality between the estimated output $\hat{y}$ and the measured output y, we recall that

$$\begin{aligned} y =& \;\; CA^*Bu \oplus CA^*Rw, \\ \hat{y} =& \;\; C(A \oplus L_{opt_+}C)^*Bu \oplus C(A \oplus L_{opt_+}C)^*L_{opt_+}CA^*Rw.\end{aligned}$$

Hence, we need to show

$$CA^*B = C(A \oplus L_{opt_+}C)^*B \text{ and} \tag{10.9}$$

$$CA^*R = C(A \oplus L_{opt_+}C)^*L_{opt_+}CA^*R, \tag{10.10}$$

respectively. Note first that $\tilde{L} = C^T$ is causal, since C is causal, and recall from the proof of Lemma 3 that $(A \oplus \tilde{L}C)^* = A^*$, hence

$(A \oplus \tilde{L}C)^*B = A^*B$ and $C(A \oplus \tilde{L}C)^*B = CA^*B$. Second, $(A \oplus \tilde{L}C)^*\tilde{L}CA^*R = A^*\tilde{L}CA^*R \preceq A^*R$ as $\tilde{L}C \preceq I_n$. According to Theorem 12, L_{opt_+} is the greatest causal matrix such that $L_{opt_+} \preceq L_{opt}$, hence, according to Lemma 1 and Lemma 2, matrix L_{opt_+} is the greatest causal matrix such that $(A \oplus LC)^*B = A^*B$ and $(A \oplus LC)^*LCA^*R \preceq A^*R$. Therefore, $\tilde{L} \preceq L_{opt_+}$. Since matrix L_{opt_+} is such that $C(A \oplus L_{opt_+}C)^*L_{opt_+}CA^*R \preceq CA^*R$, and since $\tilde{L} \preceq L_{opt_+}$ the following inequality holds:

$$C(A \oplus \tilde{L}C)^*\tilde{L}CA^*R \preceq C(A \oplus L_{opt_+}C)^*L_{opt_+}CA^*R \preceq CA^*R.$$

According to the proof of Lemma 3, the left hand side term is equal to CA^*R. Therefore, we conclude that $C(A \oplus L_{opt_+}C)^*L_{opt_+}CA^*R = CA^*R$, and $y = \hat{y}$ for all u, w when the observer matrix L_{opt_+} is used. $\square$

11

Control of max-plus linear systems

In Section 9, it was shown how a TEG can be modelled as a max-plus linear system. In particular, we introduced the system (9.3), (9.4) as a compact model, where all entities are in the diod $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$. In Section 10, we discussed how, on the basis of this model and measured firing times of all output transitions, an optimal state estimate can be obtained. More precisely, the observer introduced in that section provides the closest estimates of the firing times of all internal transitions while guaranteeing that the estimates are never greater than the actual firing times.

In this Section, we will discuss how to control a TEG on the basis of the model (9.3), (9.4). We will investigate both open-loop and closed-loop control. In either case, the aim is to compute the controllable input of the system in order to achieve a desired performance. In the context of TEGs, this means to determine the firing times of the controllable input transitions. For example, in a transportion network or a computer network, the dispatch times of vehicles or data packets need to be computed. In a manufacturing context, it has to be decided when processing of different raw workpieces should be started. In all cases, one wants to ensure that the firing times of the output transition are "in time", *i.e.*, not later than at pre-specified dates. Under this restriction, we typically aim at maximising the firing times of the controllable input transitions. This is referred to as a "just-in-time" policy, and it has been popular in industrial applications because it avoids unnecessary internal stocks. This, in turn, reduces the use of resources and therefore financial cost. For example, in a manufacturing scenario where products have to be finished and delivered to the customer at pre-specified times, it is advantageous to feed raw material and unprocessed workpieces into the system as late as possible since this minimizes the need for internal storage.

In Section 11.1, we discuss two open-loop control strategies, while output feedback and state feedback control will be investigated in Sections 11.2 and 11.3. Later, in Section 12, we will combine the results on state estimation from Section 10 and on state feedback from Section 11.3 to provide an observer-based optimal feedback control policy.

11.1 Optimal open-loop control

11.1.1 Restricting the plant output

As an open-loop policy only makes sense if there are no disturbances acting on the system, the uncontrollable input w in (9.3), (9.4) is neglected, *i.e.*, $w = \varepsilon$, to give the following plant model (in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$):

$$\begin{aligned} x &= Ax \oplus Bu, \\ y &= Cx. \end{aligned}$$

Hence, $y = H_{yu}u$, where the $m \times p$ plant transfer matrix is $H_{yu} \triangleq CA^*B$. Following results given in Cohen *et al.*, 1989, we assume that a desired sequence of firing times for the output transitions are given. This reference sequence is encoded in the power series $z \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^m_{/\mathcal{R}_{\gamma^*}}$. Adopting a just-in-time policy, we seek the maximal control input $u \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^p_{/\mathcal{R}_{\gamma^*}}$ such that $y \preceq z$. In other words, we aim at firing all input transitions as late as possible while making sure that the output transitions fire no later than specified by z. As multiplication is isotone, this implies that applying the resulting control input will give rise to the largest output y respecting the specification.

Hence we aim at solving the optimization problem

$$u_{opt} = \bigoplus_{u \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}} u$$

$$\text{subject to } y = CA^*Bu \preceq z.$$

As multiplication is residuated (see Example 6), the solution is directly obtained by

$$u_{opt} = (CA^*B) \backslash\!\!\!\circ\, z. \tag{11.1}$$

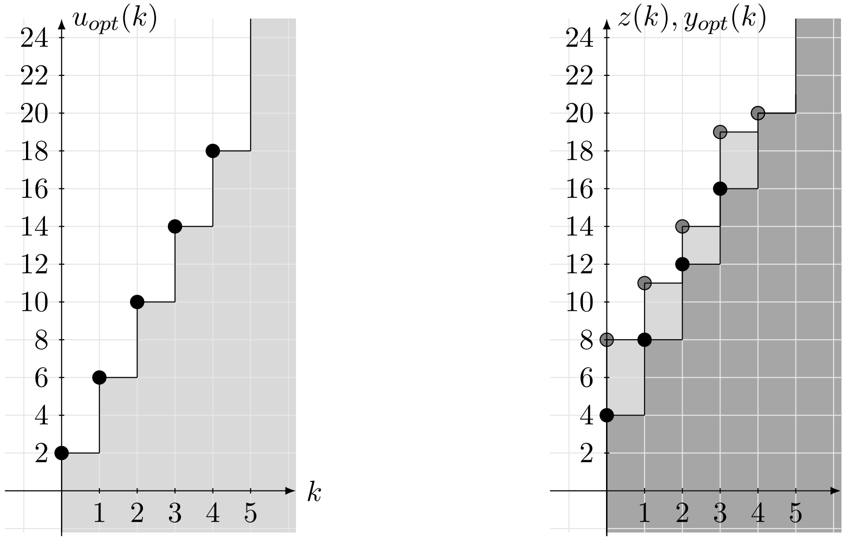
Example 22. Let us consider the TEG given in Fig. 9.1, with transfer relation $H_{yu} = CA^*B = 2\gamma^0(4\gamma)^*$ (see Eq. (9.7)), and the following desired firing times of the output transition: $z(k) = \{8, 11, 14, 19, 20\}$ for $k \in [0, 4]$ and $z(k) = +\infty$ for $k \geq 5$. Recall that It is straightforward to write this sequence in a polynomial form in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$, namely $z = 8\gamma^0 \oplus 11\gamma^1 \oplus 14\gamma^2 \oplus 19\gamma^3 \oplus 20\gamma^4 \oplus +\infty\gamma^5$. Then, by using the algorithm and computational rules introduced in Section 8.1, we obtain

$$\begin{aligned} u_{opt} &= (CA^*B) \backslash\!\!\!\circ\, z \\ &= (2\gamma^0(4\gamma)^*) \backslash\!\!\!\circ\, (8\gamma^0 \oplus 11\gamma^1 \oplus 14\gamma^2 \oplus 19\gamma^3 \oplus 20\gamma^4 \oplus +\infty\gamma^5) \\ &= 2\gamma^0 \oplus 6\gamma^1 \oplus 10\gamma^2 \oplus 14\gamma^3 \oplus 18\gamma^4 \oplus +\infty\gamma^5 \end{aligned}$$

as the optimal (in the sense of the just-in-time criterion) control input . The corresponding optimal output is given by:

$$\begin{aligned} y_{opt} &= H_{yu}u_{opt} \\ &= (2\gamma^0(4\gamma)^*)(2\gamma^0 \oplus 6\gamma^1 \oplus 10\gamma^2 \oplus 14\gamma^3 \oplus 18\gamma^4 \oplus +\infty\gamma^5) \\ &= 4\gamma^0 \oplus 8\gamma^1 \oplus 12\gamma^2 \oplus 16\gamma^3 \oplus 20\gamma^4 \oplus +\infty\gamma^5 \end{aligned}$$

Fig. 11.1 depicts the series u_{opt} and y_{opt}. It illustrates that optimal control implies firing the input transition at times $2, 6, 10, 14, 18$ (and then no more). The resulting output series y_{opt} is the greatest one ensuring that $y_{opt} \preceq z$. Graphically, the reader is invited to note that this optimal control minimizes the area between the graphs representing y and z. Furthermore, there always exists at least one index i such that $y_{opt}(i) = z(i)$, in this example $i = 4$.



(a) Optimal control u_{opt}

(b) Desired output z (light gray) and optimal output y_{opt} (dark gray)

Figure 11.1: Optimal control input and corresponding output for Example 22.

11.1.2 Restricting the transfer matrix

We now take a different view on open-loop control, advocated initially in Hardouin *et al.*, 1997. Instead of assuming a reference series that the plant output needs to match from below as closely as possible, we now investigate the scenario shown in Fig. 11.2. Recall that the

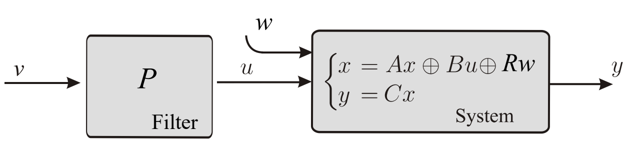


Figure 11.2: System with filter P.

plant transfer matrix from u to y is $H_{yu} \triangleq CA^*B \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{m\times p}_{/\mathcal{R}_{\gamma^*}}$. We aim at computing the maximal causal open-loop controller, or prefilter,

$P \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{p\times p}_{/\mathcal{R}_{\gamma^*}}$ such that the resulting transfer matrix from v to y, $H_{yv} = H_{yu}P$, is bounded from above by a reference transfer matrix G_{ref}, *i.e.*, we require that

$$H_{yv}(P) \triangleq (CA^*B)P \preceq G_{ref}.$$

Hence, as in the previous case, we want all controllable input transitions to fire as late as possible, but now under the restriction of an upper bound for the open-loop transfer matrix, *i.e.*, this transfer matrix may not be slower than G_{ref} . As multiplication is isotone, using the greatest possible prefilter P will lead to the maximal transfer matrix $H_{yv}(P)$ respecting this restriction. As $H_{yv}(P)$ will therefore match the reference transfer matrix as closely as possible, the described approach can be understood as a specific model matching procedure. Formally, this leads to the following optimization problem:

$$P_{opt} = \bigoplus_{P\in\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{p\times p}_{/\mathcal{R}_{\gamma^*}}} P$$
$$\text{subject to } (CA^*B)P \preceq G_{ref}.$$

As multiplication is residuated (see Example 6), the solution is directly obtained by

$$P_{opt} = (CA^*B) \backslash\!\!\circ\, G_{ref}. \tag{11.2}$$

We require a causal prefilter, therefore we subsequently determine the causal projection $P_{opt_+} \triangleq \mathsf{Pr}_+(P_{opt})$, *i.e.*, the largest causal prefilter in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{p\times p}_{/\mathcal{R}_{\gamma^*}}$ that is less than or equal to P_{opt}.

As will be discussed in Section 13, a variety of objectives can be considered and translated into an appropriate reference model G_{ref}. An elementary and popular choice is to consider a strategy called "neutral control". It aims to keep the input/output transfer relation unchanged while delaying the firing of control input transitions as much as possible. Hence, this strategy minimizes the number of tokens in the system while preserving input/output performance. To achieve this, the reference model is chosen as $G_{ref} = CA^*B$, which leads to the prefilter $P_{opt} = (CA^*B) \backslash\!\!\circ\, (CA^*B)$. As we require a causal prefilter, we take the causal projection $P_{opt_+} = \mathsf{Pr}_+(P_{opt})$. The resulting open-loop transfer matrix is then $H_{yv} = CA^*BP_{opt_+}$. Intuitively, causal control can only delay the firing of input transitions, hence $H_{yv} \succeq CA^*B$. On the other hand, solution of the addressed optimisation problem gives $H_{yv} \preceq CA^*B$. Hence, $H_{yv} = CA^*B = G_{ref}$, *i.e.*, the reference model is perfectly matched in this case. This is formally shown in Section 13.1.

Example 23. Consider again the TEG in Fig. 9.1, with the transfer relation from the controllable input u to the output y given by (9.7), *i.e.*, $.CA^*B = 2\gamma^0(4\gamma)^*$. Suppose the reference model is $G_{ref} = CA^*B$, *i.e.*, we aim at solving the neutral control problem described above. According to Eq. (11.2), the optimal prefilter is

$$P_{opt} = (CA^*B)\diamond\!\!\!\backslash G_{ref} = (2\gamma^0(4\gamma)^*)\diamond\!\!\!\backslash(2\gamma^0(4\gamma)^*) = (4\gamma)^*,$$

which is clearly causal and ultimately periodic and therefore, because of Theorem 16, realizable. As expected, in this case, the reference model is perfectly matched. Indeed, $CA^*BP_{opt} = (2\gamma^0(4\gamma)^*)(4\gamma)^* = 2\gamma^0(4\gamma)^* = G_{ref}$. To realize the optimal prefilter $P_{opt} = (4\gamma)^*$, we follow the procedure given in Example 23 (for $p = \varepsilon$, $q = e$, and $r = 4\gamma$). This provides

$$\begin{aligned} \zeta_1 &= 4\gamma\zeta_1 \oplus v \\ \zeta_2 &= \zeta_1 \\ u &= \zeta_2 \end{aligned}$$

or, equivalently,

$$u = 4\gamma u \oplus v.$$

As γ is the backward shift operator, the latter can be written as

$$u(k) = 4u(k-1) \oplus v(k).$$

The corresponding TEG realizing the optimal prefilter is shown in Fig. 11.3. Its effect clearly is that the transition u can fire only once every 4 time units.

11.2 Output feedback control

To deal with unknown disturbances, one needs to incorporate feedback into the control law. In this subsection, we consider linear output

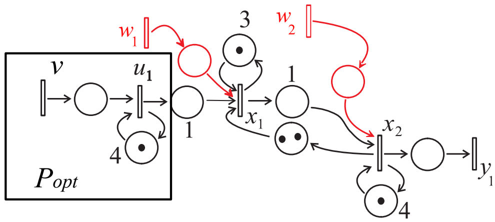


Figure 11.3: TEG with precompensator P_{opt}.

feedback of the form $u = P(v \oplus Fy)$, where $P \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{p\times p}$ and $F \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{p\times m}$. According to Eq. (9.6), the output of the controlled system is

$$y = CA^*Bu \oplus CA^*Rw = CA^*BP(v \oplus Fy) \oplus CA^*Rw.$$

Hence, the control input to the system is

$$u = P(v \oplus Fy) = Pv \oplus PF(CA^*Bu \oplus CA^*Rw),$$

which yields, by applying Theorem 7, the following control input and output:

$$\begin{aligned} u &= (PFCA^*B)^*Pv \oplus (PFCA^*B)^*PFCA^*Rw, \\ &\triangleq H_{uv}(P,F)v \oplus H_{uw}(P,F)w, \qquad (11.3) \\ y &= (CA^*BPF)^*CA^*BPv \oplus (CA^*BPF)^*CA^*Rw, \\ &\triangleq H_{yv}(P,F)v \oplus H_{yw}(P,F)w. \qquad (11.4) \end{aligned}$$

We now pose the following optimal control problem: find controller matrices P and F such that the transfer matrix $H_{uv}(P,F)$ from v to u is maximal (in the sense of the order in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$), while the transfer matrix $H_{yv}(P,F)$ from the external input v to the output y is less than or equal to (in the same order) a reference transfer matrix, denoted G_{ref}. This implies that the closed loop input/output behavior will not be slower than the reference model, but, under this constraint, the effect of v on the firings of all control input transitions will be delayed as much as possible. This, as discussed earlier, is often referred to as a "just-in-time" policy.

Formally, this optimization problem is stated as follows:

$$\begin{array}{rl} \bigoplus\limits_{P,F} & H_{uv}(P,F) \\ \textit{subject to} & H_{yv}(P,F) \preceq G_{ref}. \end{array} \tag{11.5}$$

We first address the bound imposed by the reference model and reformulate the constraint on $H_{yv}(P,F)$ by a constraint on $H_{uv}(P,F)$.

Lemma 4. The following equivalence holds:

$$H_{yv}(P,F) \preceq G_{ref} \Leftrightarrow H_{uv}(P,F) \preceq (CA^*B) \backslash\!\!\!\circ\, G_{ref}. \tag{11.6}$$

Proof. According to Eq. (6.6), $H_{yv}(P,F)$ can be rewritten as

$$\begin{array}{rcl} H_{yv}(P,F) & = & (CA^*BPF)^*CA^*BP \\ & = & CA^*B(PFCA^*B)^*P \\ & = & CA^*BH_{uv}(P,F). \end{array}$$

As matrix right multiplication is a residuated mapping, the constraint $CA^*BH_{uv}(P,F) \preceq G_{ref}$ is equivalent to $H_{uv}(P,F) \preceq (CA^*B) \backslash\!\!\!\circ\, G_{ref}$. □

As multiplication is isotone, an immediate consequence of the above Lemma is that the maximal transfer matrix H_{uv} satisfying the constraints imposed by the reference model will also lead to the maximal H_{yv}. Hence, the transfer matrix H_{yv} will match the reference model G_{ref} as closely as possible from below. In this sense, the optimal control problem addressed in this subsection can also be interpreted as a model matching problem.

We next provide an upper bound for the matrix P.

Lemma 5. Controller P is bounded from above as follows:

$$P \preceq (CA^*B) \backslash\!\!\!\circ\, G_{ref} = P_{opt}. \tag{11.7}$$

Proof. The Kleene star operator definition implies that $I_p \preceq (PFCA^*B)^*$, hence $P \preceq (PFCA^*B)^*P = H_{uv}(P,F) \preceq (CA^*B) \backslash\!\!\!\circ\, G_{ref}$. □

Lemma 6. The output feedback control $u = P_{opt}(v \oplus Fy)$, with $P_{opt} = (CA^*B) \backslash\!\!\!\circ\, G_{ref}$ and $F = \varepsilon$, solves the optimization problem (11.5).

Proof. It is obvious that

$$H_{uv}(P_{opt}, \varepsilon) = (P_{opt}\varepsilon CA^*B)^* P_{opt} = P_{opt} = (CA^*B) \backslash\!\!\!\circ\, G_{ref},$$

which, according to Lemma 4, is the maximal solution to the optimization problem (11.5). □

Proposition 6. The output feedback control $u = P_{opt}(v \oplus Fy)$, with $P_{opt} = (CA^*B) \backslash\!\!\!\circ\, G_{ref}$ and

$$F \preceq F_{opt} = P_{opt} \backslash\!\!\!\circ\, P_{opt} /\!\!\!\circ\, (CA^*BP_{opt}), \tag{11.8}$$

solves the optimization problem (11.5).

Proof. By considering P_{opt} and $H_{uv}(P_{opt}, F) = (P_{opt}FCA^*B)^* P_{opt}$, the following equivalences hold:

$$(P_{opt}FCA^*B)^* P_{opt} \preceq \underbrace{(CA^*B) \backslash\!\!\!\circ\, G_{ref}}_{P_{opt}}$$

$$\begin{aligned}
&\Leftrightarrow \quad P_{opt}(FCA^*BP_{opt})^* \preceq P_{opt} && \text{because of (6.6)}\\
&\Leftrightarrow \quad (FCA^*BP_{opt})^* \preceq P_{opt} \backslash\!\!\!\circ\, P_{opt} && \text{as multipl. is residuated}\\
&\Leftrightarrow \quad (FCA^*BP_{opt})^* \preceq (P_{opt} \backslash\!\!\!\circ\, P_{opt})^* && \text{because of Eq. (7.16)}\\
&\Leftrightarrow \quad FCA^*BP_{opt} \preceq (P_{opt} \backslash\!\!\!\circ\, P_{opt})^* && \text{see Example 8}\\
&\Leftrightarrow \quad FCA^*BP_{opt} \preceq P_{opt} \backslash\!\!\!\circ\, P_{opt} && \text{because of Eq. (7.16)}\\
&\Leftrightarrow \quad F \preceq P_{opt} \backslash\!\!\!\circ\, P_{opt} /\!\!\!\circ\, (CA^*BP_{opt}) && \text{as multipl. is residuated.}
\end{aligned}$$

As $H_{uv}(P_{opt}, \varepsilon) = (CA^*B) \backslash\!\!\!\circ\, G_{ref}$ (Lemma 6) and

$$\begin{aligned}
H_{uv}(P_{opt}, \varepsilon) \quad \preceq \quad & H_{uv}(P_{opt}, F) = (P_{opt}FCA^*B)^* P_{opt}, \preceq (CA^*B) \backslash\!\!\!\circ\, G_{ref},\\
& \forall\, F \text{ satisfying (11.8)},
\end{aligned}$$

we conclude that $H_{uv}(P_{opt}, F) = (CA^*B) \backslash\!\!\!\circ\, G_{ref}$ for all F satifying (11.8). □

Remark 25. The controller matrices P_{opt}, F_{opt} are not necessarily causal, hence they may not be realizable according to Theorem 16. To make them realizable (see Example 23), each entry must be projected into the causal subsemiring $\overline{\mathbb{Z}}^{+}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$ using the projector given in Theorem 12. We will then consider $P_{opt_+} = \mathsf{Pr}_+(P_{opt})$ and $F_{opt_+} = \mathsf{Pr}_+(F_{opt})$. The resulting matrices P_{opt_+} and F_{opt_+} are the greatest causal controller matrices satisfying Lemma 5 and Proposition 6, respectively.

Remark 26. Note that for $P = P_{opt_+}$, any causal controller matrix $F \preceq F_{opt_+}$ solves the optimization problem (11.5). This feedback matrix $F \preceq F_{opt_+}$ injects the output y into the control input u and therefore makes it possible to react to disturbances. Furthermore, the sum and product being order preserving, $F_1 \preceq F_2$ implies that

$$H_{uv}(P, F_1)v \oplus H_{uw}(P, F_1)w \preceq H_{uv}(P, F_2)v \oplus H_{uw}(P, F_2)w \quad \forall v, w, \tag{11.9}$$

hence the pair (P_{opt_+}, F_{opt_+}) will lead to the greatest causal control $u = P(v \oplus Fy)$, *i.e.*, it delays the firing dates of all control input transitions as much as possible. In the sense of this stricter notion of the just-in-time criterion, (P_{opt_+}, F_{opt_+}) is therefore indeed the optimal causal output feedback controller. The resulting closed loop transfer matrix is $H_{uv}(P_{opt_+}, F_{opt_+}) = \mathsf{Pr}_+((CA^*B) \backslash\!\!\circ G_{ref}) = P_{opt_+}$.

11.3 State feedback control

In this subsection, we allow the feedback controller to access more information. Namely, instead of the firing times of the output transitions, it now "sees" the firing times of all internal transitions of the TEG to be controlled, *i.e.*, information that is encoded in the series x. This, in the following, will be referred to as state feedback control. Inserting the feedback control equation

$$u = P(v \oplus Kx), \tag{11.10}$$

where $P \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{p \times p}$, $K \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}})^{p \times n}$, into Eq. (9.5), we get

$$u = P(v \oplus K(A^*Bu \oplus A^*Rw)).$$

By applying Theorem 7, this yields the following transfer relations:

$$\begin{aligned} u &= (PKA^*B)^*Pv \oplus (PKA^*B)^*PKA^*Rw \\ &\triangleq N_{uv}(P, K)v \oplus N_{uw}(P, K)w. \end{aligned}$$

Inserting this into Eqns. (9.5) and (9.6) results in

$$\begin{aligned} x &= A^*B(PKA^*B)^*Pv \oplus A^*B(PKA^*B)^*PKA^*Rw \oplus A^*Rw \\ &= A^*B(PKA^*B)^*Pv \oplus (A^*BPK)^*A^*Rw, \\ y &= CA^*BP(KA^*BP)^*v \oplus C(A^*BPK)^*A^*Rw \\ &\triangleq N_{yv}(P,K)v \oplus N_{yw}(P,K)w. \end{aligned}$$

We now pose a model matching problem in total analogy to subsection 11.2. We aim at maximising the closed loop transfer matrix $N_{uv}(P,K)$ under the constraint that the closed loop transfer matrix $N_{yv}(P,K)$ is bounded from above by a given reference model G_{ref}. This will lead to the maximal $N_{yv}(P,K)$, hence $N_{yv}(P,K)$ will match the reference model G_{ref} as closely as possible from below. Formally, this model matching problem can be written as

$$\begin{aligned} \bigoplus_{P,K} \quad & N_{uv}(P,K) \\ \textit{subject to} \quad & N_{yv}(P,K) \preceq G_{ref}. \end{aligned} \tag{11.11}$$

Proposition 7. The state feedback control $u = P_{opt}(v \oplus Kx)$, with

$$P_{opt} = (CA^*B) \backslash\!\!\!\circ\, G_{ref} \tag{11.12}$$

$$K \preceq P_{opt} \backslash\!\!\!\circ\, P_{opt} /\!\!\!\circ\, (A^*BP_{opt}) = K_{opt}, \tag{11.13}$$

solves the optimization problem (11.11) and achieves $N_{uv}(P_{opt},K) = (CA^*B) \backslash\!\!\!\circ\, G_{ref}$.

Proof. The proof uses the same steps as the one of Proposition 6, but replaces CA^*BP_{opt} by A^*BP_{opt}. □

As in the output feedback case, in order to obtain realizable controllers, we need to consider the causal projection introduced in Theorem 12, *i.e.*, $P_{opt+} = \mathsf{Pr}_+(P_{opt})$ and $K_{opt+} = \mathsf{Pr}_+(K_{opt})$.

As in the output feedback case, for $P = P_{opt+}$ any causal controller matrix $K \preceq K_{opt+}$ will lead to the maximal closed loop transfer matrix from v to u. However, because of the influence of the disturbance w, the actual control input u, *i.e.*, the firing times of the control input transitions, will in general be different for different K. Using the same argument as in Subsection 11.2, it is straightforward to show that $K_1 \preceq K_2$ implies that $\forall v, w$

$$N_{uv}(P_{opt_+}, K_1)v \oplus N_{uw}(P_{opt_+}, K_1)w \preceq N_{uv}(P_{opt_+}, K_2)v \oplus N_{uw}(P_{opt_+}, K_2)w, \qquad (11.14)$$

hence the pair (P_{opt_+}, K_{opt_+}) will lead to the greatest u, *i.e.*, delay the firing times of all control input transitions as much as possible. In the sense of this stricter notion of the just-in-time criterion, (P_{opt_+}, K_{opt_+}) is therefore indeed the optimal causal state feedback controller.

12

Observer-based control

As in standard control theory, internal variables are often not measurable or too expensive to measure. Hence, in this section, we propose an observer-based feedback structure. In this structure, instead of the unknown firing times of the plant's internal transitions, collected in the vector x, their estimates provided by the observer proposed in Section 10, represented by the vector $\hat{x}$, are fed back to the plant control input. This observer-based control structure is depicted in Fig. 12.1. We will also compare the resulting feedback strategy with the output feedback discussed in the previous section .

Formally, the observer-based control scheme is characterized by

$$u_M = P(v \oplus M\hat{x}),$$

where, according to Eq. (10.3),

$$\hat{x} \quad = \quad (A \oplus L_{opt}C)^* B u_M \oplus (A \oplus L_{opt}C)^* L_{opt} C A^* R w.$$

Inserting the observer equation into the feedback equation results in

$$u_M = Pv \oplus PM(A \oplus L_{opt}C)^* B u_M \oplus PM(A \oplus L_{opt}C)^* L_{opt} C A^* R w.$$

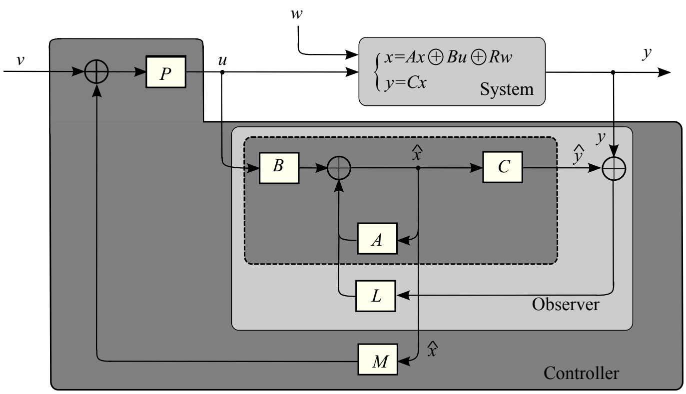


Figure 12.1: Observer-based control scheme for max-plus linear systems.

Using Theorem 7 and Eq. (6.6), we get

$$\begin{aligned} u_M &= (PM(A \oplus L_{opt}C)^*B)^*Pv \\ &\quad \oplus (PM(A \oplus L_{opt}C)^*B)^*PM(A \oplus L_{opt}C)^*L_{opt}CA^*Rw \\ &= P(M(A \oplus L_{opt}C)^*BP)^*v \qquad (12.1) \\ &\quad \oplus PM((A \oplus L_{opt}C)^*BPM)^*(A \oplus L_{opt}C)^*L_{opt}CA^*Rw \\ &\triangleq T_{uv}(P,M)v \oplus T_{uw}(P,M)w. \qquad (12.2) \end{aligned}$$

Inserting (12.1) into the plant equation (9.5) results in

$$\begin{aligned} x &= A^*BP(M(A \oplus L_{opt}C)^*BP)^*v \oplus A^*Rw \oplus \\ &\quad A^*BPM((A \oplus L_{opt}C)^*BPM)^*(A \oplus L_{opt}C)^*L_{opt}CA^*Rw \\ &= A^*BP(M(A \oplus L_{opt}C)^*BP)^*v \oplus A^*Rw \oplus \\ &\quad A^*BPM((A^*L_{opt}C)^*A^*BPM)^*(A^*L_{opt}C)^*A^*L_{opt}CA^*Rw \\ &= A^*BP(M(A \oplus L_{opt}C)^*BP)^*v \oplus A^*Rw \oplus \\ &\quad (A^*BPM(A^*L_{opt}C)^*)^*A^*BPM(A^*L_{opt}C)^*A^*L_{opt}CA^*Rw \\ &= A^*BP(M(A \oplus L_{opt}C)^*BP)^*v \oplus \\ &\quad ((A^*BPM(A^*L_{opt}C)^*)^+A^*L_{opt}CA^*R \oplus A^*R)w, \end{aligned}$$

where the reformulation of the right hand side makes use of Eq. (6.3) for step 1, Eq. (6.6) for step 2, and the definition of the $^+$ operator for step 3. Finally, inserting this into the plant output equation $y = Cx$

gives

$$
\begin{aligned}
y &= CA^*BP(M(A \oplus L_{opt}C)^*BP)^*v && (12.3)\\
&\quad \oplus (C(A^*BPM(A^*L_{opt}C)^*)^+A^*L_{opt}CA^*R \oplus CA^*R)w \\
&\triangleq T_{yv}(P,M)v \oplus T_{yw}(P,M)w. && (12.4)
\end{aligned}
$$

As in Section 11, the aim of model matching is to find controller matrices P and M such that the transfer matrix $T_{uv}(P,M)$ is maximal while respecting the constraint $T_{yv}(P,M) \preceq G_{ref}$. This can be formally stated as:

$$
\begin{aligned}
&\bigoplus_{P,M} && T_{uv}(P,M) && (12.5)\\
&\textit{subject to} && T_{yv}(P,M) \preceq G_{ref}.
\end{aligned}
$$

Lemma 4 and Lemma 5 carry over from the output feedback case in a straightforward way. The resulting Lemma 7 restates the constraints for the transfer matrix $T_{yv}(P,M)$ in terms of the transfer matrix $T_{uv}(P,M)$, while Lemma 8 provides an upper bound for the controller matrix P.

Lemma 7. The following equivalence holds:

$$T_{yv}(P,M) \preceq G_{ref} \Leftrightarrow T_{uv}(P,M) \preceq (CA^*B) \circ\backslash G_{ref}. \quad (12.6)$$

Lemma 8. Controller P is upper-bounded as follows:

$$P \preceq (CA^*B) \circ\backslash G_{ref} = P_{opt}. \quad (12.7)$$

Proposition 8. The control $u_M = P_{opt}(v \oplus M\hat{x})$ with

$$
\begin{aligned}
P_{opt} &= (CA^*B) \circ\backslash G_{ref} && (12.8)\\
M &\preceq P_{opt} \circ\backslash P_{opt} \circ/ (A^*BP_{opt}) = M_{opt} = K_{opt}, && (12.9)
\end{aligned}
$$

solves the optimization problem (12.5) and achieves $T_{uv}(P_{opt},M) = (CA^*B) \circ\backslash G_{ref}$.

Proof. Lemma 1 and Proposition 4 imply that $(A \oplus L_{opt}C)^*B = A^*B$. Therefore the following equivalences hold:

$$
\begin{aligned}
& CA^*BP_{opt}(M(A \oplus L_{opt}C)^*BP_{opt})^* \preceq G_{ref}\\
\Leftrightarrow \quad & CA^*BP_{opt}(MA^*BP_{opt})^* \preceq G_{ref}\\
\Leftrightarrow \quad & M \preceq P_{opt} \circ\backslash P_{opt} \circ/ (A^*BP_{opt}),
\end{aligned}
$$

where the latter follows from Proposition 7. □

As in the output and state feedback cases, in order to obtain realizable controllers we need to consider the causal projections introduced in Theorem 12. The overall observer-based control scheme is therefore given by the causal controller matrices $P_{opt_+} = \mathsf{Pr}_+(P_{opt})$ and $M_{opt_+} = \mathsf{Pr}_+(M_{opt})$, together with the causal observer matrix $L_{opt_+} = \mathsf{Pr}_+(L_{opt})$.

As in the output feedback case, for $P = P_{opt_+}$ any controller matrix $M \preceq M_{opt_+}$ will lead to the maximal closed loop transfer matrix from v to u. However, because of the inflence of the disturbance w, the actual control input u will in general be different for different controller matrices M. Taking into account that $(A \oplus L_{opt_+}C)^*B = A^*B$ and using the same argument as in Subsection 11.2, it is straightforward to show that $M_1 \preceq M_2$ implies that $\forall v, w$

$$T_{uv}(P_{opt_+}, M_1)v \oplus T_{uw}(P_{opt_+}, M_1)w \preceq \\ T_{uv}(P_{opt_+}, M_2)v \oplus T_{uw}(P_{opt_+}, M_2)w, \quad (12.10)$$

hence the pair (P_{opt_+}, M_{opt_+}) will lead to the greatest u, *i.e.*, delay the firing times of all control input transitions as much as possible. In the sense of this stricter notion of the just-in-time criterion, (P_{opt_+}, M_{opt_+}), together with the observer matrix L_{opt_+}, therefore constitutes the optimal causal observer-based feedback scheme.

An obvious implication from the above discussion is that controller synthesis and observer synthesis are independent – this is an analogon to the well-known separation principle in standard linear systems theory. In other words, we can first determine the greatest causal observer matrix L_{opt_+} to ensure that the estimated plant output is the same as the original output. Second, we can find the greatest causal feedback matrices P_{opt_+}, M_{opt_+} to maximize the closed-loop transfer matrix from v to u while guaranteeing that the resulting closed-loop transfer matrix from v to y is less than or equal to a desired transfer matrix G_{ref}.

Proposition 9. Denote the control inputs generated by optimal causal output feedback, optimal causal state feedback, and optimal causal observer-based feedback by $u_{F_{opt_+}}$, $u_{K_{opt_+}}$, and $u_{M_{opt_+}}$, respectively. For all v, w, they are ordered as follows:

$$u_{F_{opt_+}} \preceq u_{M_{opt_+}} \preceq u_{K_{opt_+}}.$$

Proof. We recall that

$$\begin{aligned} u_{F_{opt_+}} &= P_{opt_+}(v \oplus F_{opt_+} y), \\ u_{K_{opt_+}} &= P_{opt_+}(v \oplus K_{opt_+} x), \\ u_{M_{opt_+}} &= P_{opt_+}(v \oplus M_{opt_+} \hat{x}). \end{aligned}$$

Note first that from Eqns. (11.8), (12.9) and (7.8), it follows that $F_{opt} = M_{opt} \phi C$, then by considering Eq. (7.2) the following inequality holds $F_{opt} C \preceq M_{opt}$. Since Pr_+ is an isotone mapping and according to Eq. (8.15), the following inequality holds : $\mathsf{Pr}_+(F_{opt})\mathsf{Pr}_+(C) \preceq \mathsf{Pr}_+(F_{opt}C) \preceq \mathsf{Pr}_+(M_{opt}) = M_{opt_+}$. By assumption, the entries of matrix $C \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{m\times n}_{/\mathcal{R}_{\gamma^*}}$ are in $\{\varepsilon, e\}$ and precisely one is equal to e in each row (see Section 9), so the entries of this matrix are causal and $\mathsf{Pr}_+(C) = C$. Hence $F_{opt_+} C = \mathsf{Pr}_+(F) C \preceq M_{opt_+}$ and $F_{opt_+} C\hat{x} \preceq M_{opt_+}\hat{x}$. According to Corollary 5, the proposed observer ensures $C\hat{x} = \hat{y} = y$, hence $F_{opt_+} y \preceq M_{opt_+}\hat{x}$. By recalling that the addition and product laws are order preserving, we get:

$$u_{F_{opt_+}} = P_{opt_+}(v \oplus F_{opt_+} y) \preceq u_{Mopt_+} = P_{opt_+}(v \oplus M_{opt_+}\hat{x}).$$

Furthermore, according to Proposition 4, the proposed observer guarantees that $\hat{x} \preceq x$, and, according to Proposition 8, $M_{opt_+} = K_{opt_+}$. Hence,

$$u_{M_{opt_+}} = P_{opt_+}(v \oplus M_{opt_+}\hat{x}) \preceq u_{K_{opt_+}} = P_{opt_+}(v \oplus K_{opt_+} x).$$

□

Proposition 9 states that the optimal observer-based controller will delay the firings of the control input transitions at least as much as the optimal output feedback controller. In the sense of this stricter notion of the just-in-time criterion, the former will therefore perform at least as well as the latter. For instance, in a manufacturing setting, the optimal observer-based controller would never start individual processes earlier than the optimal output feedback controller, but nevertheless guarantee that the same temporal deadlines are met.

13

Reference model design

In the previous sections, we discussed control strategies aiming at matching a reference model G_{ref}. In those approaches, the reference model imposes a bound that control needs to guarantee. In this sense, it depicts an essential aspect of the desired behavior of the controlled system. In this section, we will discuss how to choose reference models that reflect common objectives encountered in control theory, namely, neutral control, stabilization and decoupling. As the reference model provides an upper bound for the closed loop transfer matrix, and as control in the investigated framework can only delay the firing of input transitions, the reference model, irrespective of the considered objective, always needs to be greater than or equal to the open loop system transfer matrix.

13.1 Neutral control

In this case, the reference model is chosen as

$$G_{ref} = CA^*B. \tag{13.1}$$

This means that the closed loop behavior of the system, reflected by the transfer matrix from the external input v to the output y, may not be slower than the open loop behavior, given by the transfer matrix CA^*B. Hence the purpose of control is to delay control inputs as much as possible, but without slowing down the system. Note that, as control can only delay the system, choosing the reference model (13.1) and solving the resulting control problem will necessarily lead to exact model matching. Indeed, in the output feedback case, by considering Eqns: (11.7), (11.8) and (13.1), the optimal controller matrices are

$$\begin{aligned} P_{opt} &= (CA^*B) \backslash\!\!\circ (CA^*B), \\ F_{opt} &= P_{opt} \backslash\!\!\circ P_{opt} \circ\!\!/ (CA^*BP_{opt}). \end{aligned}$$

Note that, because of (7.16), $P_{opt} = P_{opt}^*$. Therfore, using (7.17), $P_{opt} \backslash\!\!\circ P_{opt} = P_{opt}$. Furthermore, according to (7.4), we can rewrite $CA^*BP_{opt} = CA^*B((CA^*B) \backslash\!\!\circ (CA^*B)) = CA^*B$. Inserting these expressions into the equation for F_{opt} provides

$$F_{opt} = (CA^*B) \backslash\!\!\circ (CA^*B) \circ\!\!/ (CA^*B).$$

The control law is obtained by taking the causal projections, $u = P_{opt_+}(v \oplus F_{opt_+} y)$. This leads to the following transfer functions $H_{uv} = P_{opt_+} = \mathsf{Pr}_+((CA^*B) \backslash\!\!\circ (CA^*B))$ and $H_{yv} = CA^*BP_{opt_+}$. By recalling that $P_{opt_+} = \mathsf{Pr}_+(P_{opt}) \preceq P_{opt} = (CA^*B) \backslash\!\!\circ (CA^*B)$, the following inequality holds :

$$H_{yv} = CA^*BP_{opt_+} \preceq CA^*BP_{opt},$$

where $CA^*BP_{opt} = CA^*B((CA^*B) \backslash\!\!\circ (CA^*B)) = CA^*B$, and the latter equality comes from (7.4). Hence the following inequality holds

$$H_{yv} \preceq CA^*B. \tag{13.2}$$

Furthermore, note that $P_{opt} = ((CA^*B) \backslash\!\!\circ (CA^*B))^*$ (because of Eq. (7.16)), hence $P_{opt} \succeq I_p$ (see Theorem 7 for the Kleene star definition) Moreover, the identity matrix I_p is causal, therefore Eq. (8.13) yields the following equality $\mathsf{Pr}_+(I_p) = I_p$. The causal projection of the controller is then such that $\mathsf{Pr}_+(P_{opt}) = P_{opt_+} \succeq \mathsf{Pr}_+(I_p) = I_p$ since the mapping Pr_+ is isotone. Hence $I_p \preceq P_{opt_+}$, and therefore: $CA^*B = CA^*BI_p \preceq CA^*BP_{opt_+} = H_{yv}$, which, together with Inequality (13.2), yields $H_{yv} = CA^*B$, that is the input/output behavior is unchanged.

13.2 Stabilization

For a timed event graph, the property of stability means that tokens do not accumulate indefinitely inside the graph or, equivalently, that the marking remains bounded for all inputs. This property is obtained when all transitions of the TEG fire with the same average frequency. It has been shown that a structurally controllable and observable TEG can be made stable by using output feedback, (see Cohen *et al.*, 1984; MaxPlus, 1991; Cottenceau *et al.*, 2001a; Cottenceau *et al.*, 2003). Moreover, stability may be obtained without changing the system throughput, where the throughput of a system with multiple inputs or outputs is defined as the minimum of all the throughputs of the entries in the corresponding transfer matrix. The following theorem formalizes this result.

Theorem 17 (Cohen *et al.*, 1984)**.** Any structurally controllable and observable timed event graph can be made stable by causal output feedback without altering its original throughput.

In order to achieve stability, it is then sufficient to choose (and to achieve) a reference model $G_{ref} \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{m\times p}_{/\mathcal{R}_{\gamma^*}}$ such that all its entries exhibit the same throughput, *i.e.*, $\sigma(G_{ref_{ij}}) = \frac{\nu_{ref}}{\tau_{ref}}$, $i = 1, \dots m, j = 1, \dots p$. From the constraint that $G_{ref} \succeq CA^*B$, it follows immediately that we require

$$\frac{\nu_{ref}}{\tau_{ref}} \leq \min_{i,j}(\sigma((CA^*B)_{ij})) = \sigma(CA^*B). \tag{13.3}$$

A straightforward way of obtaining such a reference model is to choose

$$G_{ref} = (CA^*B) \otimes (\tau_{ref}\gamma^{\nu_{ref}})^*. \tag{13.4}$$

Because of the definition of the Kleene star operator, it is immediately clear that indeed $CA^*B \preceq G_{ref}$. Furthermore, from (13.3) and the operational rules for periodic series in Section 9, we can deduce that the throughput of each entry of (13.4) is the required value $\frac{\nu_{ref}}{\tau_{ref}}$. Possible refinements in the definition of the reference model have been suggested in Cottenceau *et al.*, 2003; Gaubert, 1995.

By considering Eqns. (11.7), (11.8) and (13.4), the optimal controller matrices are:

$$
\begin{aligned}
P_{opt} &= (CA^*B) \diamond\!\!\!\backslash (CA^*B(\tau_{ref}\gamma^{\nu_{ref}})^*),\\
F_{opt} &= P_{opt} \diamond\!\!\!\backslash P_{opt} \diamond\!\!\!/ (CA^*BP_{opt}).
\end{aligned}
$$

By taking their causal projections, we get the control law $u = P_{opt_+}(v \oplus F_{opt_+} y)$. As discussed in Section 11.2, this control leads to the following transfer matrices

$$
\begin{aligned}
H_{uv} &= P_{opt_+} = \mathsf{Pr}_+((CA^*B) \diamond\!\!\!\backslash ((CA^*B)(\tau_{ref}\gamma^{\nu_{ref}})^*))\\
H_{yv} &= CA^*BH_{uv} = CA^*BP_{opt_+} \preceq G_{ref}.
\end{aligned} \tag{13.5}
$$

According to (7.3), the following inequality holds :

$$
(CA^*B) \diamond\!\!\!\backslash (CA^*B(\tau_{ref}\gamma^{\nu_{ref}})^*) \succeq (\tau_{ref}\gamma^{\nu_{ref}})^*,
$$

and, since Pr_+ is isotone, the following holds too

$$
\mathsf{Pr}_+((CA^*B) \diamond\!\!\!\backslash (CA^*B(\tau_{ref}\gamma^{\nu_{ref}})^*)) \succeq \mathsf{Pr}_+((\tau_{ref}\gamma^{\nu_{ref}})^*).
$$

Furthermore, since $(\tau_{ref}\gamma^{\nu_{ref}})^*$ is in $\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$, we have $\mathsf{Pr}_+((\tau_{ref}\gamma^{\nu_{ref}})^*) = (\tau_{ref}\gamma^{\nu_{ref}})^*$, then the latter inequality becomes

$$
\mathsf{Pr}_+((CA^*B) \diamond\!\!\!\backslash (CA^*B(\tau_{ref}\gamma^{\nu_{ref}})^*)) \succeq (\tau_{ref}\gamma^{\nu_{ref}})^*.
$$

Then, as multiplication is isotone,

$$
\begin{aligned}
H_{yv} &= CA^*B\mathsf{Pr}_+((CA^*B) \diamond\!\!\!\backslash (CA^*B(\tau_{ref}\gamma^{\nu_{ref}})^*))\\
&\succeq CA^*B(\tau_{ref}\gamma^{\nu_{ref}})^*\\
&= G_{ref}.
\end{aligned}
$$

This, together with (13.5), yields the equality $H_{yv} = G_{ref}$. Hence, the desired reference model is indeed achieved, and the closed loop system is therefore stable.

13.3 Decoupling problem

As pointed out in Section 9, uncontrollable input transitions can be interpreted as disturbances acting on a TEG. The vector of the corresponding daters, w, influences the output as in Eq. (9.6),*i.e.*, $y = CA^*Bu \oplus CA^*Rw$. From this, it is obvious that a lower bound for the output is given by CA^*Rw. It then makes sense to choose the maximal u, *i.e.*, to fire all control input transitions as late as possible, without further slowing down the system. Using output feedback of the form $u = P(v \oplus Fy)$, we obtain the closed loop equations (see Eq. (11.3) and Eq.((11.4))

$$\begin{aligned}
u &= (PFCA^*B)^*Pv \oplus (PFCA^*B)^*PFCA^*Rw, \\
&\triangleq H_{uv}(P,F)v \oplus H_{uw}(P,F)w, \\
y &= (CA^*BPF)^*CA^*BPv \oplus (CA^*BPF)^*CA^*Rw, \\
&\triangleq H_{yv}(P,F)v \oplus H_{yw}(P,F)w.
\end{aligned}$$

Formally, our control objective is therefore to maximize $H_{uw}(P,F)$ while restricting the closed loop transfer matrix from w to y to be less than or equal to $G_{ref} = CA^*R$, *i.e.*,

$$\begin{aligned}
&\bigoplus_{P,F} && H_{uw}(P,F) \\
&\textit{subject to} && H_{yw}(P,F) \preceq CA^*R.
\end{aligned} \tag{13.6}$$

Lemma 9. The constraint in the optimization problem (13.6) is satisfied if

$$PF \preceq (CA^*B) \backslash\!\!\circ (CA^*R) \phi (CA^*R). \tag{13.7}$$

Proof. The following equivalences hold

$$\begin{aligned}
H_{yw}(P,F) &= (CA^*BPF)^*CA^*R \preceq CA^*R \\
&\Leftrightarrow (CA^*BPF)^* \preceq (CA^*R) \phi (CA^*R) \\
&\qquad \text{as multiplication is residuated} \\
&\Leftrightarrow (CA^*BPF)^* \preceq ((CA^*R) \phi (CA^*R))^* \\
&\qquad \text{because of Eq. (7.16)} \\
&\Leftrightarrow CA^*BPF \preceq (CA^*R) \phi (CA^*R) \\
&\qquad \text{see Example 8} \\
&\Leftrightarrow PF \preceq (CA^*B) \backslash\!\!\circ (CA^*R) \phi (CA^*R) \\
&\qquad \text{as multiplication is residuated.}
\end{aligned}$$

Furthermore, as multiplication and the Kleene star operation are order preserving, we have that

$$P_1F_1 \preceq P_2F_2 \Rightarrow H_{uw}(P_1,F_1) \preceq H_{uw}(P_2,F_2),$$

hence

$$(PF)_{opt} = (CA^*B)\mathbin{\circ\!\!\!\!\backslash}(CA^*R)\mathbin{\circ\!\!\!/}(CA^*R) \tag{13.8}$$

holds for any solution of the optimization problem (13.6). □

As the optimal solution is specified as the product of the two controller matrices, there is a degree of freedom. First we choose $P = I_p$, providing the following bound for the feedback controller.

Proposition 10. If $P = I_p$, the constraint in the optimization problem (13.6) is satisfied for all F such that

$$F \preceq (CA^*B)\mathbin{\circ\!\!\!\!\backslash}(CA^*R)\mathbin{\circ\!\!\!/}(CA^*R) = F_{opt}^{DP}. \tag{13.9}$$

Proof. Direct application of Lemma 9 for $P = I_p$. □

Proposition 11. If $F = F_{opt}^{DP}$, the constraint in the optimization problem (13.6) is satisfied for all P such that

$$\begin{aligned} P &\preceq ((CA^*B)\mathbin{\circ\!\!\!\!\backslash}(CA^*R)\mathbin{\circ\!\!\!/}(CA^*R))\mathbin{\circ\!\!\!/}((CA^*B)\mathbin{\circ\!\!\!\!\backslash}(CA^*R)\mathbin{\circ\!\!\!/}(CA^*R)) \\ &= F_{opt}^{DP}\mathbin{\circ\!\!\!/}F_{opt}^{DP} = P_{opt}^{DP}. \end{aligned} \tag{13.10}$$

Proof. Direct application of Lemma 9 for $F = F_{opt}^{DP}$. □

Because of (13.8) and (7.4),

$$P_{opt}^{DP}F_{opt}^{DP} = (F_{opt}^{DP}\mathbin{\circ\!\!\!/}F_{opt}^{DP})F_{opt}^{DP} = F_{opt}^{DP} = (CA^*B)\mathbin{\circ\!\!\!\!\backslash}(CA^*R)\mathbin{\circ\!\!\!/}(CA^*R). \tag{13.11}$$

Therefore, according to Lemma 9, the controller matrices $(P_{opt}^{DP}, F_{opt}^{DP})$ indeed solve the optimization problem (13.6).

Remark 27. To be realizable these controller matrices need to be projected into $\overline{\mathbb{Z}}_{\max}^{+}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$. The resulting causal controller matrices are then $P_{opt_+}^{DP} = \mathsf{Pr}_+(F_{opt}^{DP}\mathbin{\circ\!\!\!/}F_{opt}^{DP})$ and $F_{opt_+}^{DP} = \mathsf{Pr}_+(F_{opt}^{DP})$. Because multiplication is order preserving, $P_{opt_+}^{DP} \preceq P_{opt}^{DP}$, $F_{opt_+}^{DP} \preceq F_{opt}^{DP}$ and because of (13.11), we get

$$P_{opt_+}^{DP}F_{opt_+}^{DP} \preceq (CA^*B)\mathbin{\circ\!\!\!\!\backslash}(CA^*R)\mathbin{\circ\!\!\!/}(CA^*R),$$

i.e., the causal controller matrices $(P^{DP}_{opt_+}, F^{DP}_{opt_+})$ indeed satisfy the constraint in the optimization problem (13.6). As, furthermore, they are the maximal controller matrices doing so, they constitute a causal solution to the optimization problem (13.6).

Remark 28. Problem (13.6) is sometimes called the Modified Disturbance Decoupling Problem, see Lhommeau *et al.*, 2002; Lhommeau *et al.*, 2003a; Lhommeau *et al.*, 2003b; Shang *et al.*, 2013; Shang *et al.*, 2016; Shang *et al.*, 2014. Note, however, that "disturbance decoupling" in the standard control literature refers to a scenario where the output of the closed loop system is totally unaffected by, *i.e.*, decoupled from, the disturbance input. In our case, we look for the maximal controllers, and therefore the maximal *control* input that does not affect the closed loop output.

14

Application to a manufacturing system

In this section, we will discuss controller synthesis for the simple manufacturing system introduced in Example 1 and depicted in Fig. 3.2. In addition, we now introduce six uncontrollable input transitions $w_1, \dots w_6$ to independently model disturbances acting on the firing of all internal transitions x_i. This is shown in Fig. 14.1. Hence, a delay in (or failure of) the firing of transition w_i will lead to delay in (or failure of) the firing of transtion x_i, $i = 1, \dots 6$.

Denoting the vector of dater functions associated with the uncontrollable input transitions, the control input transitions, the internal transitions, and the output transition by $w \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^6_{/\mathcal{R}_{\gamma^*}}$, $u \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^2_{/\mathcal{R}_{\gamma^*}}$, $x \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^6_{/\mathcal{R}_{\gamma^*}}$, and $y \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]_{/\mathcal{R}_{\gamma^*}}$, respectively, and using the results from Section 9, the TEG model representing our modified manufacturing systems is decribed by

$$\begin{aligned} x &= Ax \oplus Bu \oplus Rw \\ y &= Cx, \end{aligned}$$

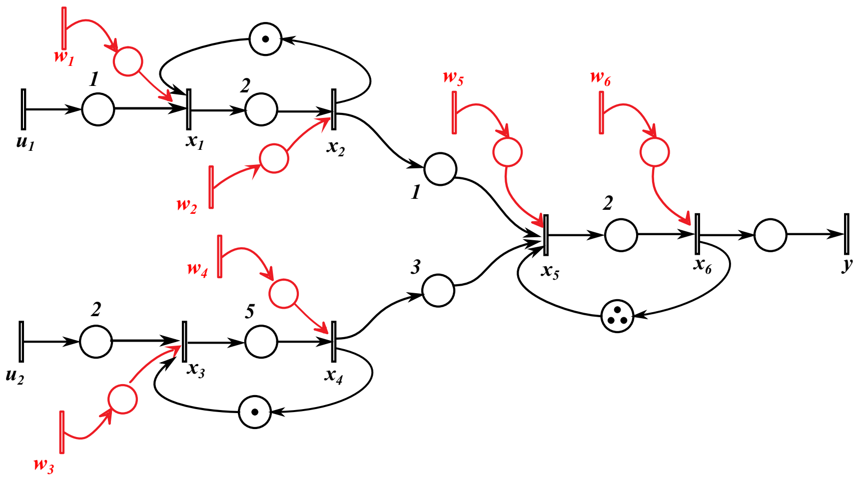


Figure 14.1: TEG model representing simple manufacturing system with disturbances.

where

$$A = \begin{bmatrix} \varepsilon & \gamma^1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 2\gamma^0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \gamma^1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 5\gamma^0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1\gamma^0 & \varepsilon & 3\gamma^0 & \varepsilon & \gamma^3 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2\gamma^0 & \varepsilon \end{bmatrix}, \quad B = \begin{bmatrix} 1\gamma^0 & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & 2\gamma^0 \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix},$$

$$R = \begin{bmatrix} e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix}, \quad C = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix}.$$

Using Theorem 7, we obtain the following transfer matrix between u and y:

$$CA^*B = \begin{bmatrix} 6\gamma^0(2\gamma)^* & 12\gamma^0(5\gamma)^* \end{bmatrix}.$$

The throughput of this transfer matrix is

$$\sigma(CA^*B) = \min(\sigma(CA^*B)_{11}, \sigma(CA^*B)_{12}) = \min(1/2, 1/5) = 1/5,$$

implying that the system, in the disturbance-free case, can finish one workpiece every 5 time units. Following the discussion in Section 13.2, we proceed to stabilize the system by using output feedback to match the reference model

$$G_{ref} = (5\gamma)^* CA^*B = \begin{bmatrix} 6\gamma^0(5\gamma)^* & 12\gamma^0(5\gamma)^* \end{bmatrix}.$$

By Proposition 6, we obtain the optimal output feedback controller as $u = P_{opt}(v \oplus F_{opt}y)$, where

$$P_{opt} = (CA^*B) \circ\!\!\!\!\backslash\, G_{ref} = \begin{bmatrix} (5\gamma)^* & 6\gamma^0(5\gamma)^* \\ -6\gamma^0(5\gamma)^* & (5\gamma)^* \end{bmatrix},$$

$$F_{opt} = P_{opt} \circ\!\!\!\!\backslash\, P_{opt} \circ\!\!\!\!/\, (CA^*BP_{opt}) = \begin{bmatrix} -6\gamma^0(5\gamma)^* \\ -12\gamma^0(5\gamma)^* \end{bmatrix}.$$

According to Definition 27, neither the prefilter matrix P_{opt} nor the output feedback matrix F_{opt} are causal. Consequently, according to Theorem 16, neither of them is realizable. We therefore compute their causal projections $P_{opt+} = \mathsf{Pr}_+(P_{opt})$ and $F_{opt+} = \mathsf{Pr}_+(F_{opt})$, *i.e.*, the greatest causal matrices in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{2\times 2}_{/\mathcal{R}_{\gamma^*}}$, respectively $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{2\times 1}_{/\mathcal{R}_{\gamma^*}}$, such that $P_{opt+} \preceq P_{opt}$, respectively $F_{opt+} \preceq F_{opt}$. Using Theorem 12, we obtain:

$$P_{opt+} = \mathsf{Pr}_+(P_{opt}) = \begin{bmatrix} (5\gamma)^* & 6\gamma^0(5\gamma)^* \\ 4\gamma^2(5\gamma)^* & (5\gamma)^* \end{bmatrix},$$

$$F_{opt+} = \mathsf{Pr}_+(F_{opt}) = \begin{bmatrix} 4\gamma^2(5\gamma)^* \\ 3\gamma^3(5\gamma)^* \end{bmatrix}.$$

To realize the resulting output feedback law $u = P_{opt+}(v \oplus F_{opt+}y)$, we rewrite this as

$$\begin{aligned}
\xi' &= F_{opt+}y \\
&= \begin{bmatrix} (5\gamma)^* & \varepsilon \\ \varepsilon & (5\gamma)^* \end{bmatrix} \begin{bmatrix} 4\gamma^2 \\ 3\gamma^3 \end{bmatrix} y, \\
\xi &= P_{opt+}(v \oplus \xi') \\
&= \begin{bmatrix} (5\gamma)^* & \varepsilon \\ \varepsilon & (5\gamma)^* \end{bmatrix} \begin{bmatrix} e & 6\gamma^0 \\ 4\gamma^2 & e \end{bmatrix} (v \oplus \xi') \quad \text{and} \\
u &= \xi.
\end{aligned}$$

The former two expressions are the solutions of the implicit equations

$$\xi' = \begin{bmatrix} 5\gamma & \varepsilon \\ \varepsilon & 5\gamma \end{bmatrix} \xi' \oplus \begin{bmatrix} 4\gamma^2 \\ 3\gamma^3 \end{bmatrix} y \quad \text{and} \tag{14.1}$$

$$\xi = \begin{bmatrix} 5\gamma & \varepsilon \\ \varepsilon & 5\gamma \end{bmatrix} \xi \oplus \begin{bmatrix} e & 6\gamma^0 \\ 4\gamma^2 & e \end{bmatrix} (v \oplus \xi'). \tag{14.2}$$

With $\xi = (\xi_1, \xi_2)^T$, $\xi' = (\xi_3, \xi_4)^T$, and $v = (v_1, v_2)^T$, and by recalling that γ represents the backward shift operator, the overall control law can be restated as a set of difference equations in the max-plus algebra:

$$\begin{aligned}
\xi_3(k) &= 5\xi_3(k-1) \oplus 4y(k-2), \\
\xi_4(k) &= 5\xi_4(k-1) \oplus 3y(k-3), \\
\xi_1(k) &= 5\xi_1(k-1) \oplus \xi_3(k) \oplus v_1(k) \oplus 6\xi_4(k) \oplus 6v_2(k), \\
\xi_2(k) &= 5\xi_2(k-1) \oplus 4\xi_3(k-2) \oplus 4v_1(k-2) \oplus \xi_4(k) \oplus v_2(k), \\
u_1(k) &= \xi_1(k), \\
u_2(k) &= \xi_2(k).
\end{aligned}$$

The overall control scheme can now be implemented as a "control TEG", as indicated in Fig. 14.2, where the the pre-filter part (consisting of transitions ξ_1 and ξ_2) and the output feedback part (consisting of transitions ξ_3 and ξ_4) appear in the shown gray areas. For example, the first line in the above set of equations means that transition ξ_3 has two upstream places with holding times 5 and 4 and containing initially one, respectively two tokens. The upstream transitions of these places are ξ_3, respectively y. The other equations are similarly mapped into TEG elements to result in Fig. 14.2.

As a side remark, note that the control part shown in Fig. 14.2 can also be written in terms of min-plus equations. To do this, one associates a counter function to each transition. For example, $\xi_3 : \overline{\mathbb{Z}} \to \overline{\mathbb{Z}}_{\min}$ is the counter function associated with transition ξ_3, and $\xi_3(t)$ denotes the number of firings of this transition up to time t. The respective part of the TEG in Fig. 14.2 implies that the number of firings of transition ξ_3, up to time t, is the minimum of the number of firings of this transition up to time $t-5$ plus 1 and the number of firings of transition y up to time $t-4$ plus 2. Interpreting the other control transitions in the same way, we get the following set of equations in the min-plus algebra. Recall that, in the min-plus algebra, addition $\oplus$ corresponds to the standard minimum-operation while multiplication corresponds to standard addition (see Example 4).

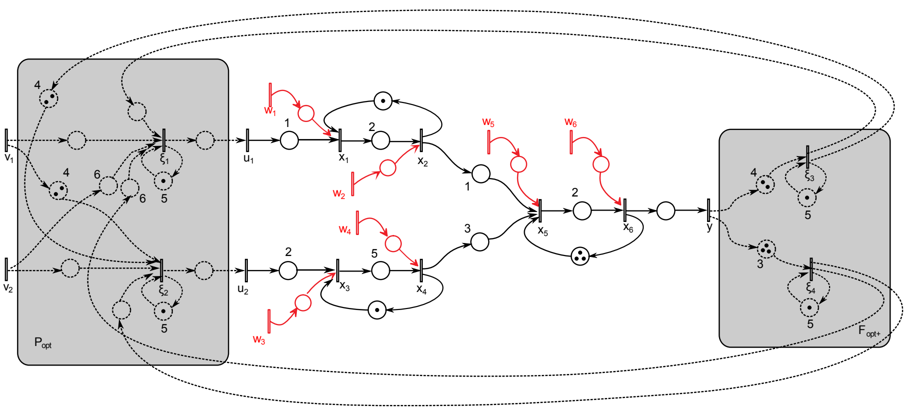


Figure 14.2: TEG implementation of causal output feedback control $u = P_{opt_+}(v \oplus F_{opt_+} y)$.

$$
\begin{aligned}
\xi_3(t) &= 1\xi_3(t-5) \oplus 2y(t-4),\\
\xi_4(t) &= 1\xi_4(t-5) \oplus 3y(t-3),\\
\xi_1(t) &= 1\xi_1(t-5) \oplus \xi_3(t) \oplus \xi_4(t-6) \oplus v_1(t) \oplus v_2(t-6),\\
\xi_2(t) &= 1\xi_2(t-5) \oplus 2\xi_3(t-4) \oplus 2v_1(t-4) \oplus \xi_4(t) \oplus v_2(t),\\
u_1(t) &= \xi_1(t),\\
u_2(t) &= \xi_2(t).
\end{aligned}
$$

Next, we construct observer and observer-based control as discussed in Sections 10 and 12. According to Lemma 1, Lemma 2, and Proposition 4, the optimal observer matrix is obtained as

$$\begin{aligned} L_{opt} &= L_1 \wedge L_2 = (A^*B)\phi(CA^*B) \wedge (A^*R)\phi(CA^*R) \\ &= \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0\gamma^3(2\gamma^3)^* & (2\gamma^3)^* \end{bmatrix}^T, \end{aligned}$$

According to Proposition 8, the matrices P_{opt} and M_{opt} for optimal observer-based control $u = P_{opt}(v \oplus M_{opt}\hat{x})$ are computed as

$$\begin{aligned} P_{opt} &= (CA^*B)\diamond G_{ref} \\ &= \begin{bmatrix} (5\gamma)^* & 6\gamma^0(5\gamma)^* \\ -6\gamma^0(5\gamma)^* & (5\gamma)^* \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} M_{opt} &= P_{opt}\diamond P_{opt}\phi(A^*BP_{opt}) \\ &= \left[\begin{matrix} -1\gamma^0(5\gamma)^* & -3\gamma^0(5\gamma)^* & 4\gamma^0(5\gamma)^* & -1\gamma^0(5\gamma)^* \\ -7\gamma^0(5\gamma)^* & -9\gamma^0(5\gamma)^* & -2\gamma^0(5\gamma)^* & -7\gamma^0(5\gamma)^* \end{matrix} \right. \\ &\qquad\qquad \left. \begin{matrix} -4\gamma^0(5\gamma)^* & -6\gamma^0(5\gamma)^* \\ -10\gamma^0(5\gamma)^* & -12\gamma^0(5\gamma)^* \end{matrix} \right]. \end{aligned}$$

While L_{opt} is causal, *i.e.*, $L_{opt+} = \mathsf{Pr}(L_{opt}) = L_{opt}$, this is clearly not true for the matrices P_{opt} and M_{opt}. For the latter, we therefore need to compute their causal projections $P_{opt+} = \mathsf{Pr}_+(P_{opt})$ and $M_{opt+} = \mathsf{Pr}_+(M_{opt})$. Using Theorem 12, we obtain:

$$\begin{aligned} P_{opt+} &= \begin{bmatrix} (5\gamma)^* & 6\gamma^0(5\gamma)^* \\ 4\gamma^2(5\gamma)^* & (5\gamma)^* \end{bmatrix}, \\ M_{opt+} &= \left[\begin{matrix} 4\gamma^1(5\gamma)^* & 2\gamma^1(5\gamma)^* & 4\gamma^0(5\gamma)^* & 4\gamma^1(5\gamma)^* \\ 3\gamma^2(5\gamma)^* & 1\gamma^2(5\gamma)^* & 3\gamma^1(5\gamma)^* & 3\gamma^2(5\gamma)^* \end{matrix} \right. \\ &\qquad\qquad \left. \begin{matrix} 1\gamma^1(5\gamma)^* & 4\gamma^2(5\gamma)^* \\ 0\gamma^2(5\gamma)^* & 3\gamma^3(5\gamma)^* \end{matrix} \right]. \end{aligned}$$

To realize the observer equations, we rewrite them as

$$\begin{aligned} \hat{x} &= A\hat{x} \oplus Bu \oplus \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \xi_5 & \xi_6 \end{bmatrix}^T, \\ \begin{bmatrix} \xi_5 \\ \xi_6 \end{bmatrix} &= \begin{bmatrix} 0\gamma^3(2\gamma^3)^* \\ (2\gamma^3)^* \end{bmatrix} y. \end{aligned}$$

The latter equation is a solution of

$$\begin{bmatrix} \xi_5 \\ \xi_6 \end{bmatrix} = \begin{bmatrix} 2\gamma^3 & \varepsilon \\ \varepsilon & 2\gamma^3 \end{bmatrix} \begin{bmatrix} \xi_5 \\ \xi_6 \end{bmatrix} \oplus \begin{bmatrix} 0\gamma^3 \\ e \end{bmatrix} y.$$

Taking into account that γ is the backward shift operator, the observer can be written as the following set of difference equations in the max-plus algebra:

$$\begin{aligned}
\xi_5(k) &= 2\xi_5(k-3) \oplus y(k-3), \\
\xi_6(k) &= 2\xi_6(k-3) \oplus y(k), \\
\hat{x}_1(k) &= \hat{x}_2(k-1) \oplus 1u_1(k), \\
\hat{x}_2(k) &= 2\hat{x}_1(k), \\
\hat{x}_3(k) &= \hat{x}_4(k-1) \oplus 2u_2(k), \\
\hat{x}_4(k) &= 5\hat{x}_3(k), \\
\hat{x}_5(k) &= 1\hat{x}_2(k) \oplus 3\hat{x}_4(k) \oplus \hat{x}_6(k-3) \oplus \xi_5(k), \\
\hat{x}_6(k) &= 2\hat{x}_5(k) \oplus \xi_6(k).
\end{aligned}$$

These equations are easily implemented as a TEG, as shown in Fig. 14.3.

Realization and implementation of the prefilter P_{opt+} is identical as in the output feedback case. It can be written as the following set of difference equations in the max-plus algebra and implemented by the TEG shown in the gray box on the top left of Fig. 14.3.

$$\begin{aligned}
\xi_1(k) &= 5\xi_1(k-1) \oplus \xi_7(k) \oplus v_1(k) \oplus 6\xi_8(k) \oplus 6v_2(k), \\
\xi_2(k) &= 5\xi_2(k-1) \oplus 4\xi_7(k-2) \oplus 4v_1(k-2) \oplus \xi_8(k) \oplus v_2(k), \\
u_1(k) &= \xi_1(k), \\
u_2(k) &= \xi_2(k).
\end{aligned}$$

It only remains to show how to realize and implement the feedback matrix M_{opt+}. To do this, recall that

$$\begin{aligned}
\begin{bmatrix} \xi_7 \\ \xi_8 \end{bmatrix} &= M_{opt+}\hat{x} \\
&= \begin{bmatrix} (5\gamma)^* & \varepsilon \\ \varepsilon & (5\gamma)^* \end{bmatrix} \begin{bmatrix} 4\gamma^1 & 2\gamma^1 & 4\gamma^0 & 4\gamma^1 & 1\gamma^1 & 4\gamma^2 \\ 3\gamma^2 & 1\gamma^2 & 3\gamma^1 & 3\gamma^2 & 0\gamma^2 & 3\gamma^3 \end{bmatrix} \hat{x}
\end{aligned}$$

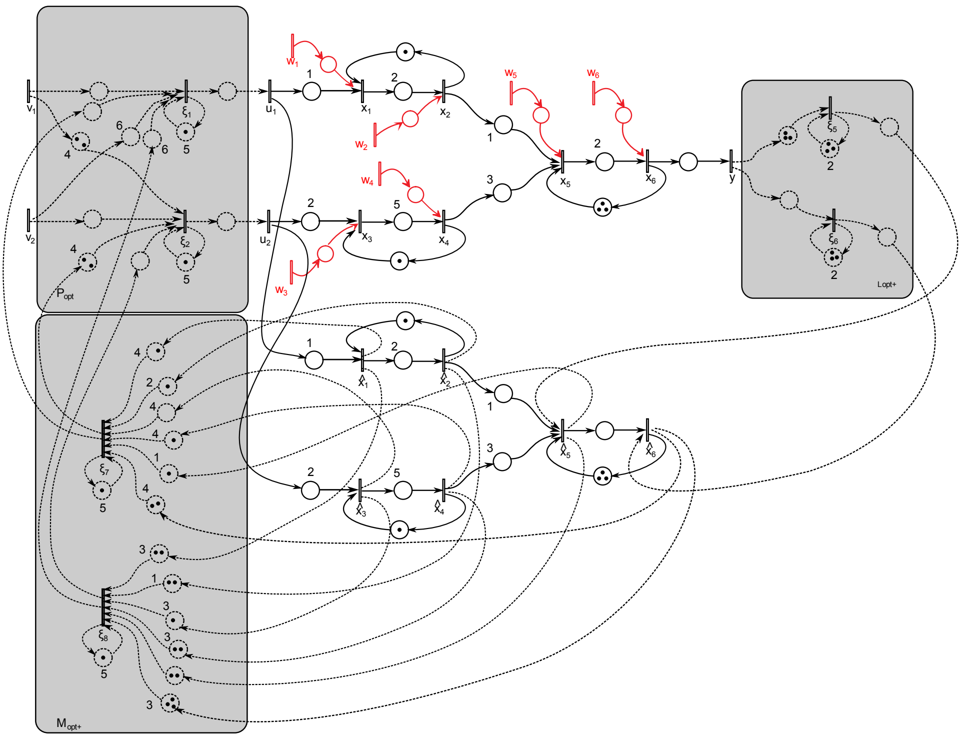


Figure 14.3: TEG implementation of causal observer-based controller $u = P_{opt_+}(v \oplus M_{opt_+}\hat{x})$.

is a solution of

$$\begin{bmatrix} \xi_7 \\ \xi_8 \end{bmatrix} = \begin{bmatrix} 5\gamma & \varepsilon \\ \varepsilon & 5\gamma \end{bmatrix} \begin{bmatrix} \xi_7 \\ \xi_8 \end{bmatrix} \oplus \begin{bmatrix} (5\gamma)^* & \varepsilon \\ \varepsilon & (5\gamma)^* \end{bmatrix} \begin{bmatrix} 4\gamma^1 & 2\gamma^1 & 4\gamma^0 & 4\gamma^1 & 1\gamma^1 & 4\gamma^2 \\ 3\gamma^2 & 1\gamma^2 & 3\gamma^1 & 3\gamma^2 & 0\gamma^2 & 3\gamma^3 \end{bmatrix} \hat{x}.$$

This can be written as a set of two difference equations in the max-plus algebra, which can be readily implemented by the TEG shown in the lower left gray box in Fig. 14.3.

$$\begin{aligned}\xi_7(k) &= 5\xi_7(k-1) \oplus 4\hat{x}_1(k-1) \oplus 2\hat{x}_2(k-1) \oplus 4\hat{x}_3(k) \\ &\quad \oplus 4\hat{x}_4(k-1) \oplus 1\hat{x}_5(k-1) \oplus 4\hat{x}_6(k-2) \\ \xi_8(k) &= 5\xi_8(k-1) \oplus 3\hat{x}_1(k-2) \oplus 1\hat{x}_2(k-2) \oplus 3\hat{x}_3(k-1) \\ &\quad \oplus 3\hat{x}_4(k-2) \oplus \hat{x}_5(k-2) \oplus 3\hat{x}_6(k-3).\end{aligned}$$

Recall that by Proposition 9, the control input generated by the observer-based scheme is, for all v and w, greater than or equal to the control input generated by the output feedback scheme

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Conclusions

This paper provides a survey on recent work on control and state estimation for max-plus linear systems based on the just-in-time criterion. It aims to be self-contained and therefore summarizes the main mathematical concepts that are needed for developing a systems and control theory for max-plus linear systems. It provides an in-depth discussion of observer design and addresses the model matching problem, both in an open-loop and a closed-loop scenario. In the latter case, it distinguishes between output feedback and feedback of the real, respectively, estimated state. It also discusses how different application objectives can be translated into appropriate reference models to be used in the model matching problem. We have tried to make the required theoretical concepts palatable by including a large number of small examples. We have additionally provided a running example from the area of manufacturing systems to explain and visualize max-plus linear modeling and control concepts. We emphasize that this area remains an active field of research. In particular, there are a number of attempts to broaden the class of systems that can be addressed using dioid-based methods. Examples are systems which allow asymmetric interactions between subsystems, with primary subsystems providing time window constraints for secondary ones without being affected by the latter ones, e.g., David-Henriet *et al.*, 2015; David-Henriet *et al.*, 2016. This phenomenon, which is also called partial synchronization, is common in transportation systems, where, e.g., the departures of buses would be synchronized by the arrival of trains, but not the other way round. Another extension addresses a class of timed discrete event systems that operate under resource sharing constraints, e.g., Moradi *et al.*, 2017. Clearly, resource sharing is an important phenomenon in many application areas, but cannot be modeled (or treated) in the standard TEG framework. Another generalization is the control of weight-balanced timed event graphs (e.g., Cottenceau *et al.*, 2014; Trunk *et al.*, 2017b; Trunk *et al.*, 2017a), a class of timed discrete event systems that allows to model practically important phenomena such as splitting and batching of processes. With these and other extensions of the currently available theory, it will be possible to address more "real-life" application problems.

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