Holding Time Maximization Preserving Output Performance for Timed Event Graphs

Xavier David-Henriet, Laurent Hardouin, Jörg Raisch, and Bertrand Cottenceau

***Abstract*—The trade-off between energy consumption and execution time (i.e., for a given task, the faster it is achieved, the higher its energy consumption is) is investigated for systems modelled by timed event graphs. In this paper, we aim to increase execution times (and, consequently, lower energy consumption) while preserving input-output and perturbation-output behaviors. Under this condition, the optimal solution is independent of the considered cost functions and is obtained using residuation theory.**

***Index Terms*—Dioid, discrete event systems, manufacturing process, Petri nets.**

I. Introduction

For some systems (e.g., in the field of manufacturing processes or transport networks), the occurrence of an event never disables the occurrence of other events. Such systems are called decision-free: the interesting question is not what the next event is, but when the next events happen. A possible model for such systems is an event graph, a particular Petri net, where each place has exactly one upstream transition and one downstream transition and all arcs have weight 1 [1]. To capture time in event graphs, timed event graphs (TEG) are built by equipping each place with a holding time (i.e., duration a token must spend in a place before enabling the firing of the next transition). It is a well known fact that, over some dioids (or idempotent semirings), the time/event behavior of TEG, under the earliest functioning rule (i.e., transitions with input places fire as soon as they are enabled), is expressed by linear state-space representations [2]. This last property leads to the development, by analogy with classical control theory, of control methods for TEG (e.g., optimal control [3], linear feedback [4], model predictive control [5]).

In the following, the problem of holding time maximization while maintaining acceptable performance is addressed (e.g., for manufacturing process, it could mean reducing the rate of some machines without decreasing the overall production rate). This problem is relevant to reduce energy consumption. Indeed, for a large class of systems, there is a trade-off between energy consumption and execution time: the energy consumption associated with a task (modelled by a place in the TEG) decreases, when its execution time (i.e., the holding time of the place) increases.

Some work has already been done on this problem. In [6], assuming that input dates and deadlines are known, holding times are adjusted at each cycle to minimize a convex energy criterion. The algorithm proposed by the authors is very efficient: the complexity is linear with the number of tasks. In [7], a structural approach is considered. The optimization is done for all inputs, this leads to a modification of the holding times once for all. The performance criterion is on the system throughput (periodicity of the transfer function matrix), which must remain greater than a certain value (specification). After some transformations, the author manages to obtain an integer linear programming problem leading to the minimization of the number of resources and/or the maximization of the holding times. It yields a new system with a throughput above the specification.

In this paper, a structural approach is also considered, and the specification is to preserve the input-output and perturbation-output behavior of the system, not only to maintain the system throughput above a certain value as in [7]. Therefore, in our approach, the transient of the input-output behavior is not modified, which is relevant for systems prone to repetitive start-up procedures. Besides, as the perturbation-output behavior is preserved, the system after holding time maximization and the initial system have the same behavior in case of additive perturbations on the state.

The paper is organized as follows. Necessary algebraic tools are given in section II. In section III, TEG modeling over the dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is recalled. A first contribution, namely the γ-projection onto a series, is introduced in section IV. In section V, a residuation-based approach to solve the problem is presented.

II. Algebraic Preliminaries

The following is a short summary of basic results from dioid theory and residuation theory. The reader is invited to consult [2], [8], [9] for more details.

A. Dioid Theory

A dioid $\mathcal{D}$ is a set endowed with two internal operations denoted $\oplus$ (addition) and $\otimes$ (multiplication, often denoted by juxtaposition), both associative and both having a neutral element denoted ε and e respectively. Moreover, $\oplus$ is commutative and idempotent ($\forall a \in \mathcal{D}, a \oplus a = a$), $\otimes$ is distributive with respect to $\oplus$, and ε is absorbing for $\otimes$ ($\forall a \in \mathcal{D}, \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$).

The operation $\oplus$ induces an order relation $\preceq$ on $\mathcal{D}$, defined by $\forall a, b \in \mathcal{D}, a \succeq b \Leftrightarrow a \oplus b = a$. According to this order relation, $a \oplus b$ is the least upper bound of $\{a, b\}$. A dioid $\mathcal{D}$ is said to be complete if it is closed for infinite sums and if multiplication distributes over infinite sums.

By analogy with linear algebra, $\oplus$ and $\otimes$ are defined for matrices with entries in a dioid. Consider matrices $A, B \in \mathcal{D}^{n\times m}$ and $C \in \mathcal{D}^{m\times p}$, $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$ and $(A \otimes C)_{ij} = \bigoplus_{k=1}^{m} A_{ik} C_{kj}$. Endowed with these operations, the set of square matrices with entries in a complete dioid is also a complete dioid.

Definition 1. *A closure (resp. dual closure) mapping f is an isotone (i.e., order-preserving) projection (i.e., $f \circ f = f$) from a dioid $\mathcal{D}$ into itself, greater than or equal to (resp. less than or equal to) the identity mapping* Id, *i.e., $\forall x \in \mathcal{D}$, $f(x) \succeq x$ (resp. $f(x) \preceq x$).*

The following theorem gives the least solution to some implicit equations in complete dioids. It plays a fundamental role for the study of TEG behavior under the earliest functioning rule.

Theorem 1 (Kleene star theorem)**.** *The implicit equation $x = ax \oplus b$ defined over a complete dioid admits $x = a^* b$ as the least solution with $a^* = \bigoplus_{i\geqslant 0} a^i$ (Kleene star), where $a^0 = e$ and $a^i = a \otimes a^{i-1}$ for $i \geqslant 1$.*

Remark 1 ([4])**.** *$x \mapsto x^*$, defined over a complete dioid, is a closure.*

Proposition 1 ([4])**.** *In a complete dioid $\mathcal{D}$, the greatest solution to inequality $x^* \preceq a^*$ is $x = a^*$.*

Proposition 2 ([10])**.** *Let a, b be two elements in a complete dioid $\mathcal{D}$. $a^* \succeq b^*$ is equivalent to $a^* b^* = b^* a^* = a^*$.*

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B. Residuation Theory

In ordered sets, like dioids, equations $f(x) = b$ may have either no solution, one solution, or multiple solutions. In order to give always a unique answer to the problem of mapping inversion, residuation theory [11], [12] provides, under some assumptions, the greatest solution (in accordance with the considered order) to the inequality $f(x) \preceq b$.

Definition 2 (Residuation)**.** *Let $f : \mathcal{E} \to \mathcal{F}$, with $(\mathcal{E}, \preceq)$ and $(\mathcal{F}, \preceq)$ ordered sets. An isotone mapping f is said to be residuated if for all $y \in \mathcal{F}$, the least upper bound of the subset $\{x \in \mathcal{E} | f(x) \preceq y\}$ exists and lies in this subset. It is denoted $f^\sharp(y)$, and mapping $f^\sharp$ is called the residual of f.*

If the considered ordered sets are complete dioids, $L_a : x \mapsto a \otimes x$ (left-product by a), respectively $R_a : x \mapsto x \otimes a$ (right-product by a), is residuated. Its residual is denoted by $L_a^\sharp(x) = a \backslash\!\!\circ x$ (left-division by a), resp. $R_a^\sharp(x) = x /\!\!\circ a$ (right-division by a). Therefore, $a \backslash\!\!\circ x$ (resp. $x /\!\!\circ a$) denotes the greatest solution y of the inequality $a \otimes y \preceq x$ (resp. $y \otimes a \preceq x$). As left- and right-products are extended to matrices with entries in a complete dioid, left- and right-divisions are also extended to matrices with entries in a complete dioid.

The following theorem gives a fundamental link between Kleene star and left-division (or right-division).

Theorem 2 ([4])**.** *Let $\mathcal{D}$ be a complete dioid and $A \in \mathcal{D}^{p \times n}$. Then, $A \backslash\!\!\circ A \in \mathcal{D}^{n \times n}$ and $A /\!\!\circ A \in \mathcal{D}^{p \times p}$. Moreover, $A /\!\!\circ A = (A /\!\!\circ A)^*$ and $A \backslash\!\!\circ A = (A \backslash\!\!\circ A)^*$.*

III. TEG Description

The behavior of a TEG may be represented by transfer relations in some particular dioids. Hereafter, such a dioid is briefly presented, namely $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ (see [2], [9] for more details), and TEG description in this dioid is recalled.

A. Dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$

Dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is formally the quotient dioid of $\mathbb{B}[\![\gamma, \delta]\!]$ (the set of formal power series in two commutative variables γ and δ, with Boolean coefficients and with exponents in $\mathbb{Z}$), by the equivalence relation $x \mathcal{R} y \Leftrightarrow \gamma^*(\delta^{-1})^* x = \gamma^*(\delta^{-1})^* y$. Dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is complete.

As $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is a quotient dioid, an element of $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ may admit several representatives in $\mathbb{B}[\![\gamma, \delta]\!]$. The representative which is minimal with respect to the number of terms is called the minimum representative.

A simple geometrical interpretation of the previous equivalence relation is available in the (γ, δ)-plane. Consider a monomial $\gamma^k \delta^t \in \mathbb{B}[\![\gamma, \delta]\!]$, its south-east cone is defined as $\{(k', t') | k' \geqslant k \text{ and } t' \leqslant t\}$. The south-east cone of a series in $\mathbb{B}[\![\gamma, \delta]\!]$ is defined as the union of the south-east cones associated with the monomials composing the considered series. For two elements s_1 and s_2 in $\mathbb{B}[\![\gamma, \delta]\!]$, $s_1 \mathcal{R} s_2$ (i.e., s_1 and s_2 are equal in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$) is equivalent to the equality of their south-east cones. Direct consequences of the previous geometrical interpretation are:

- simplification rules in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$

$$\gamma^k \oplus \gamma^l = \gamma^{\min(k,l)} \text{ and } \delta^k \oplus \delta^l = \delta^{\max(k,l)} \qquad (1)$$

- a simple formulation of the order relation for monomials

$$\gamma^n \delta^t \preceq \gamma^{n'} \delta^{t'} \Leftrightarrow n \geqslant n' \text{ and } t \leqslant t'$$

The dater canonically associated with the series s in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is the unique non-decreasing function $d_s : \mathbb{Z} \to \mathbb{Z} \cup \{-\infty, +\infty\}$ such that $s = \bigoplus_{k \in \mathbb{Z}} \gamma^k \delta^{d_s(k)}$. A simple interpretation of the variables γ and δ for daters is available:

- multiplying a series s by γ is equivalent to shifting the argument of the associated dater function by -1
- multiplying a series s by δ is equivalent to shifting the values of the associated dater function by 1

Example 1. *Consider the series $s = \gamma\delta \oplus \gamma^3\delta^4 \oplus \gamma^4\delta^3$ represented by dots in Fig. 1. The south-east cone of s is colored in grey in Fig. 1. The minimum representative of s in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is $\gamma\delta \oplus \gamma^3\delta^4$. This result could be obtained using the simplification rules of* (1)*.*

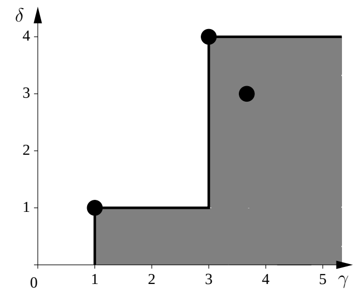

Fig. 1. s and its south-east cone (grey)

Besides,

$$s = \bigoplus_{k \leqslant 0} \gamma^k \delta^{-\infty} \oplus \bigoplus_{k=1,2} \gamma^k \delta \oplus \bigoplus_{k \geqslant 3} \gamma^k \delta^4$$

Therefore, the dater d_s associated with s is given by

$$d_s(k) = \begin{cases} -\infty \text{ if } k \leqslant 0 \\ 1 \text{ if } k = 1, 2 \\ 4 \text{ if } k \geqslant 3 \end{cases}$$

B. Linear state-space representation of TEG in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$

From now on, we only consider TEG with at most one place from a transition to another transition. This assumption is not restrictive, as it is always possible to transform any TEG in an equivalent TEG with at most one place from a transition to another transition.

The dynamics of a TEG may be captured by associating each transition with a series $s \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$, where $d_s(k)$ is defined as the time of firing k of the transition. Therefore, for TEG, γ is a shift operator in the event domain, where an event is interpreted as the firing of the transition, and δ is a shift operator in the time domain.

The transitions of a TEG are divided into three categories:

- state transitions ($x_1, \ldots, x_n$): transitions with at least one input place and one output place
- input transitions ($u_1, \ldots, u_p$): transitions with at least one output place, but no input places
- output transitions ($y_1, \ldots, y_m$): transitions with at least one input place, but no output places

Under the earliest functioning rule (i.e., state and output transitions fire as soon as they are enabled), with respect to a place with initially m tokens and holding time t, the influence of its upstream transition on its downstream transition is a positive shift in the time domain of t time units and a negative shift in the event domain of m events. The complete shift operator is coded by the monomial $\gamma^m \delta^t$ in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. Therefore, consider the place upstream from transition t_i and downstream from transition t_j, the influence of transition t_j on transition t_i is coded by the monomial f_{ij} in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ defined by $f_{ij} = \gamma^{m_{ij}} \delta^{\tau_{ij}}$ where m_{ij} is the initial number of tokens in the place and τ_{ij} is the holding time of the place.

Consequently, a TEG admits a linear state-space representation in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$

$$\begin{cases} x = Ax \oplus Bu \oplus q \\ y = Cx \end{cases} \quad (2)$$

where $x \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^n$ is the state, $u \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^p$ the input, $y \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^m$ the output, and $q \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^n$ the additive perturbation on the state. The perturbation q models, for example, unexpected failure, delays or uncertain parameters such as task duration (see [13]). $A \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times n}$, $B \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times p}$, and $C \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{m\times n}$ are matrices with monomial entries describing the influence of transitions on each other.

According to Th. 1, under the earliest functioning rule, the input-output (resp. perturbation-output) transfer function matrix H (resp. G) of the system is equal to CA^*B (resp. CA^*).

$$y = CA^*Bu \oplus CA^*q = Hu \oplus Gq \quad (3)$$

Therefore, the condition for holding time maximization (preserving the input-output and perturbation-output behaviors) is rephrased in terms of transfer function matrices. The condition is now to preserve the input-output and perturbation-output transfer function matrices.

When an element s of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is used to code information concerning a transition of a TEG, then a monomial $\gamma^k\delta^t$ with $k, t \geqslant 0$ may be interpreted as *at most k events occur strictly before time t* (i.e., $d_s(k) \geqslant t$). An element s of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, used to code a transfer relation between two transitions of a TEG (e.g., an entry of H), is causal (i.e., no anticipation in the time/event domain: all exponents are non-negative) and periodic (i.e., $s = p \oplus qr^*$ with polynomials p, q and a monomial $r \neq e$). For a periodic series s with $r = \gamma^\nu\delta^\tau$, its asymptotic slope $\sigma(s)$ is defined as $\frac{\nu}{\tau}$.

Example 2. *A manufacturing system,composed of three machines M_1 (with transitions x_2 and x_3), M_2 (with transitions x_4 and x_5) and M_3 (with transitions x_6 and x_7), is considered. The system is modelled by the TEG represented in Fig. 2.*

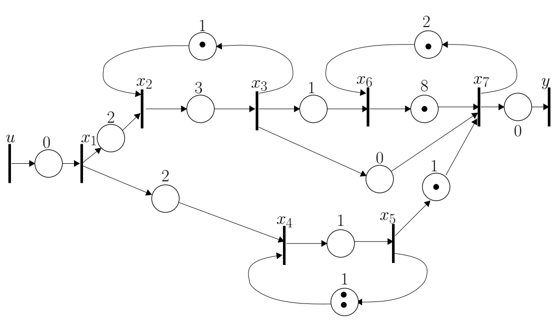

Fig. 2. Manufacturing system

The matrices of the state-space representation are

$$A = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \delta^2 & \varepsilon & \gamma\delta & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \delta^3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \delta^2 & \varepsilon & \varepsilon & \varepsilon & \gamma^2\delta & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \delta & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \delta & \varepsilon & \varepsilon & \varepsilon & \gamma\delta^2 \\ \varepsilon & \varepsilon & e & \varepsilon & \gamma\delta & \gamma\delta^8 & \varepsilon \end{pmatrix} \quad B = \begin{pmatrix} e \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix}$$

$$C = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix}$$

The input-output transfer function matrix, which is computed with the C++ library described in [14], is equal to

$$H = CA^*B = \delta^5 \oplus \left(\gamma\delta^{14} \oplus \gamma^2\delta^{18}\right)\left(\gamma^2\delta^{10}\right)^*$$

The asymptotic slope of H is 0.2, i.e., the average production rate of the system is at most 1 piece every 5 time units.

IV. γ-Projection Onto a Series

In this section, a new mapping from $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ to $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is introduced to combine the event behavior of a series with the time behavior of another series. Its aim is to formalize the following condition: only holding times should differ between the initial system and the system after holding time maximization. First, a preliminary notion is defined: γ-discontinuity.

Definition 3. *Given a series $s \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ and d_s its canonically associated dater. A γ-discontinuity of s is an event k such that $d_s(k) \neq d_s(k-1)$. The set of γ-discontinuities associated with s (i.e., $\{k \in \mathbb{Z} | d_s(k) \neq d_s(k-1)\}$) is denoted $\mathcal{K}_s$.*

As d_s is a non-decreasing function, $d_s(k) > d_s(k-1)$ for $k \in \mathcal{K}_s$. Thus, in TEG, a γ-discontinuity is an event which occurs strictly later than the previous event. It represents a piece of information not given by the previous values of the dater (the south-east cone of $\gamma^k\delta^{d_s(k)}$ is not included in the south-east cone generated by the previous events). Therefore, considering the γ-discontinuities is necessary and sufficient to completely define the series. This assertion is formally shown in the following proposition.

Proposition 3. *Given a series $s \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ and d_s its associated dater, $\bigoplus_{k\in\mathcal{K}_s} \gamma^k\delta^{d_s(k)}$ is the minimum representative of s.*

Proof: This result is a direct consequence of Th. 5.20 in [2, §5.4.2.4]. ■

Second, the new mapping itself is defined: the γ-projection onto a series.

Definition 4. *Given a series $s_0 \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. The γ-projection onto series s_0, denoted $\mathsf{Pr}_{s_0}^{\gamma}$, of a series $s \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is defined as*

$$\mathsf{Pr}_{s_0}^{\gamma}(s) = \bigoplus_{k\in\mathcal{K}_{s_0}} \gamma^k\delta^{d_s(k)}$$

with $\mathcal{K}_{s_0}$ the set of γ-discontinuities of series s_0 and d_s the dater canonically associated with series s.

The γ-projection of series s onto series s_0 is a series combining the event behavior of s_0 represented by its set of γ-discontinuities with the time behavior of s represented by its associated dater d_s.

A direct extension of the γ-projection to the matrix case is possible: consider $S_0 \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times m}$, $\mathsf{Pr}_{S_0}^{\gamma}$ is defined as, for all $S \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times m}$, $\left(\mathsf{Pr}_{S_0}^{\gamma}(S)\right)_{ij} = \mathsf{Pr}_{(S_0)_{ij}}^{\gamma}(S_{ij})$.

Proposition 4. *Consider s and s_0 in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$*

$$\forall k \in \mathbb{Z}, \quad d_{\mathsf{Pr}_{s_0}^{\gamma}(s)}(k) = d_s\left(\max_{l\in\mathcal{K}_{s_0}} \{l | l \leqslant k\}\right)$$

Proof: Let us denote $\tilde{d}$ the non-decreasing mapping, defined by $\tilde{d}(k) = d_s\left(\max_{l\in\mathcal{K}_{s_0}} \{l | l \leqslant k\}\right)$, and k_i, k_{i+1} two successive elements of $\mathcal{K}_{s_0}$ with $k_i < k_{i+1}$. Then, as $\tilde{d}(k) = d_s(k_i)$ for $k_i \leqslant k < k_{i+1}$,

$$\bigoplus_{k\in\mathbb{Z}} \gamma^k\delta^{\tilde{d}(k)} = \mathsf{Pr}_{s_0}^{\gamma}(s)$$

As the dater canonically associated with a series s is the unique non-decreasing function such that $s = \bigoplus_{k\in\mathbb{Z}} \gamma^k\delta^{d_s(k)}$, the dater canonically associated with $\mathsf{Pr}_{s_0}^{\gamma}(s)$ is $\tilde{d}$. ■

In the following proposition, the behavior of the γ-projection with respect to causality and periodicity is investigated.

Proposition 5. *Consider s and s_0 two causal and periodic series with*

- $s = p \oplus qr^*$ *and* $r = \gamma^{\nu}\delta^{\tau}$
- $s_0 = p_0 \oplus q_0 r_0^*$ *and* $r_0 = \gamma^{\nu_0}\delta^{\tau_0}$

$\mathsf{Pr}^{\gamma}_{s_0}(s)$ *is causal and periodic and, if* s_0 *is not a polynomial, then* $\sigma\left(\mathsf{Pr}^{\gamma}_{s_0}(s)\right) = \sigma(s)$.

Proof: For causality, the previous result is obvious. Therefore, only periodicity is considered in the following. If s or s_0 are polynomials (i.e., $\nu = 0$, $\tau = 0$, $\nu_0 = 0$, or $\tau_0 = 0$), $\mathsf{Pr}^{\gamma}_{s_0}(s)$ is also a polynomial and the proposition holds. Thus, only the non-degenerated case is considered: ν, τ, ν_0, and τ_0 are greater than 0.

On the one hand, the periodicity of s_0 determines the structure of $\mathcal{K}_{s_0}$:

$$\mathcal{K}_{s_0} = \mathcal{K}_{p_0} \cup \bigcup_{j \geqslant 0} \mathcal{K}_{q_0,j} \text{ with } \mathcal{K}_{q_0,j} = \{a + j\nu_0 | a \in \mathcal{K}_{q_0}\}$$

$\mathcal{K}_{p_0}$ (resp. $\mathcal{K}_{q_0}$) denotes the set of γ-discontinuities of p_0 (resp. q_0).

On the other hand, the periodicity of s implies the periodicity of d_s:

$$\exists K \geqslant 0, \forall k \geqslant K, d_s(k+\nu) = \tau + d_s(k)$$

Next, consider $j' \in \mathbb{N}$ such that $K' = \min K_{q_0,j'} \geqslant K$ and $k \geqslant K'$, according to Prop. 4,

$$d_{\mathsf{Pr}^{\gamma}_{s_0}(s)}(k + \nu\nu_0) = d_s(k') \text{ with } k' = \max_{l \in \mathcal{K}_{s_0}} \{l | l \leqslant k + \nu\nu_0\} \quad (4)$$

Furthermore, $k \geqslant K' \Rightarrow k + \nu\nu_0 \geqslant K' + \nu\nu_0$ and since $K' \in \mathcal{K}_{s_0}$, $K' + \nu\nu_0 \in \mathcal{K}_{s_0}$ too, i.e.,

$$d_{\mathsf{Pr}^{\gamma}_{s_0}(s)}(K' + \nu\nu_0) = d_s(K' + \nu\nu_0) \quad (5)$$

$k \geqslant K'$ combined with (4) and (5) leads to

$$\begin{aligned} d_{\mathsf{Pr}^{\gamma}_{s_0}(s)}(k + \nu\nu_0) = d_s(k') &\geqslant d_{\mathsf{Pr}^{\gamma}(s)}(K' + \nu\nu_0) \\ &\geqslant d_s(K' + \nu\nu_0) \end{aligned}$$

It implies $k' \geqslant K' + \nu\nu_0 = \min \mathcal{K}_{q_0,j'+\nu}$. Then

$$\begin{aligned} k' &= \max_{l \in \bigcup_{j \geqslant j'+\nu} \mathcal{K}_{q_0,j}} \{l | l \leqslant k + \nu\nu_0\} \\ &= \max_{l \in \bigcup_{j \geqslant j'} \mathcal{K}_{q_0,j}} \{l + \nu\nu_0 | l + \nu\nu_0 \leqslant k + \nu\nu_0\} \\ &= \nu\nu_0 + \tilde{k} \text{ with } \tilde{k} = \max_{l \in \bigcup_{j \geqslant j'} \mathcal{K}_{q_0,j}} \{l | l \leqslant k\} \end{aligned}$$

Hence $\tilde{k} \geqslant K' \geqslant K$ and $\tilde{k} \in \mathcal{K}_{s_0}$ and by using periodicity of s:

$$d_{\mathsf{Pr}^{\gamma}_{s_0}(s)}(k + \nu\nu_0) = d_s(k') = d_s\left(\nu\nu_0 + \tilde{k}\right) = d_s\left(\tilde{k}\right) + \tau\nu_0$$

According to Prop. 4, since $\tilde{k} = \max_{l \in \mathcal{K}_{s_0}} \{l | l \leqslant k\}$, $d_s\left(\tilde{k}\right) = d_{\mathsf{Pr}^{\gamma}_{s_0}(s)}(k)$, then,

$$d_{\mathsf{Pr}^{\gamma}_{s_0}(s)}(k + \nu\nu_0) = d_{\mathsf{Pr}^{\gamma}_{s_0}(s)}(k) + \tau\nu_0$$

Consequently, $\mathsf{Pr}^{\gamma}_{s_0}(s)$ is periodic and

$$\sigma\left(\mathsf{Pr}^{\gamma}_{s_0}(s)\right) = \frac{\nu\nu_0}{\tau\nu_0} = \sigma(s)$$

■

Example 3. *Let* $s_0 = \gamma\delta^{10} \oplus \gamma^3\delta^{13}\left(\gamma^3\delta^3\right)^*$ *and* $s = \delta\left(\gamma^2\delta^5\right)^*$. *Then,* $\mathcal{K}_{s_0}$ *is equal to* $\{1, 3k$ *with* $k \geqslant 0\}$ *and* $\mathsf{Pr}^{\gamma}_{s_0}(s)$ *is equal to* $\gamma\delta \oplus \left(\gamma^3\delta^6 \oplus \gamma^6\delta^{16}\right)\left(\gamma^6\delta^{15}\right)^*$. *In Fig. 3, the series* s, s_0 *and* $\mathsf{Pr}^{\gamma}_{s_0}(s)$ *are represented in the* (γ, δ)*-plane. As expected (see Prop. 5), the throughputs of* s *and* $\mathsf{Pr}^{\gamma}_{s_0}(s)$ *are both equal to* 0.4, *however the periodicities are different:* $\gamma^2\delta^5$ *for* s *but* $\gamma^6\delta^{15}$ *for* $\mathsf{Pr}^{\gamma}_{s_0}(s)$.

The aim of the following propositions is to characterize the series $\mathsf{Pr}^{\gamma}_{s_0}(s)$ as the optimal solution of a problem conserving the set of γ-discontinuities of s_0. First, some important properties of $\mathsf{Pr}^{\gamma}_{s_0}$ are proven.

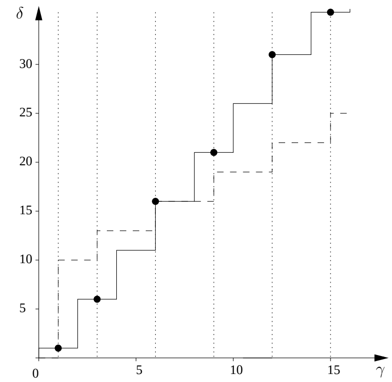

Fig. 3. Representation of s_0 (dashed line), s (continuous line), $\mathsf{Pr}^{\gamma}_{s_0}(s)$ (dots), and $\mathcal{K}_{s_0}$ (dotted line)

Proposition 6. *Given a series* s_0 *in* $\mathcal{M}^{ax}_{in}[\![\gamma, \delta]\!]$, $\mathsf{Pr}^{\gamma}_{s_0}$ *is a dual closure.*

Proof: Obviously, $\mathsf{Pr}^{\gamma}_{s_0}$ is isotone (i.e., $s_1 \succeq s_2 \Rightarrow \mathsf{Pr}^{\gamma}_{s_0}(s_1) \succeq \mathsf{Pr}^{\gamma}_{s_0}(s_2)$). Besides, due to the idempotency of $\oplus$,

$$\begin{aligned} \forall s \in \mathcal{M}^{ax}_{in}[\![\gamma, \delta]\!], s \oplus \mathsf{Pr}^{\gamma}_{s_0}(s) &= \bigoplus_{k \in \mathbb{Z}} \gamma^k \delta^{d_s(k)} \oplus \bigoplus_{k \in \mathcal{K}_{s_0}} \gamma^k \delta^{d_s(k)} \\ &= s \end{aligned}$$

Therefore, $\forall s \in \mathcal{M}^{ax}_{in}[\![\gamma, \delta]\!], \mathsf{Pr}^{\gamma}_{s_0}(s) \preceq s$ or equivalently $\mathsf{Pr}^{\gamma}_{s_0} \preceq \mathsf{Id}$. It remains to show that $\mathsf{Pr}^{\gamma}_{s_0}$ is a projection (i.e., $\mathsf{Pr}^{\gamma}_{s_0} \circ \mathsf{Pr}^{\gamma}_{s_0} = \mathsf{Pr}^{\gamma}_{s_0}$). According to Prop. 4, for $k \in \mathcal{K}_{s_0}$, we have $d_{\mathsf{Pr}^{\gamma}_{s_0}(s)}(k) = d_s(k)$. Thus,

$$\begin{aligned} \mathsf{Pr}^{\gamma}_{s_0}\left(\mathsf{Pr}^{\gamma}_{s_0}(s)\right) &= \bigoplus_{k \in \mathcal{K}_{s_0}} \gamma^k \delta^{d_{\mathsf{Pr}^{\gamma}_{s_0}(s)}(k)} \\ &= \bigoplus_{k \in \mathcal{K}_{s_0}} \gamma^k \delta^{d_s(k)} \\ &= \mathsf{Pr}^{\gamma}_{s_0}(s) \end{aligned}$$

■

Second, an interpretation, in terms of γ-discontinuities, of the image of the γ-projection onto s_0, $\mathsf{Im}\left(\mathsf{Pr}^{\gamma}_{s_0}\right)$, is presented.

Proposition 7. *Given a series* s_0 *in* $\mathcal{M}^{ax}_{in}[\![\gamma, \delta]\!]$, $\mathsf{Im}\left(\mathsf{Pr}^{\gamma}_{s_0}\right)$ *is the set of all series* $s \in \mathcal{M}^{ax}_{in}[\![\gamma, \delta]\!]$ *having a set of* γ*-discontinuities* $\mathcal{K}_s$ *included in* $\mathcal{K}_{s_0}$.

Proof: On the one hand, consider s such that $\mathcal{K}_s \subseteq \mathcal{K}_{s_0}$. Then, as $\mathcal{K}_s \subseteq \mathcal{K}_{s_0} \subseteq \mathbb{Z}$, $s = \bigoplus_{k \in \mathcal{K}_{s_0}} \gamma^k \delta^{d_s(k)}$ (see Prop. 3). Consequently, $s = \mathsf{Pr}^{\gamma}_{s_0}(s)$ and $s \in \mathsf{Im}\left(\mathsf{Pr}^{\gamma}_{s_0}\right)$.

On the other hand, consider $s \in \mathsf{Im}\left(\mathsf{Pr}^{\gamma}_{s_0}\right)$, then there exists s' such that $s = \mathsf{Pr}^{\gamma}_{s_0}(s')$. Thus, $s = \bigoplus_{k \in \mathcal{K}_{s_0}} \gamma^k \delta^{d_{s'}(k)}$ and, according to Prop. 3, $\mathcal{K}_s \subseteq \mathcal{K}_{s_0}$. ■

The following proposition presents an interesting result for dual closures.

Proposition 8. *Consider a dual closure* ϕ *on a dioid* $\mathcal{D}$ *and an element* $s \in \mathcal{D}$. $\phi(s)$ *is the greatest solution of*

$$\begin{cases} x \preceq s \\ x \in \mathsf{Im}(\phi) \end{cases} \quad (6)$$

Proof: As $\phi \preceq \mathsf{Id}$, $\phi(s)$ is a solution of (6). Consider a solution x of (6), as $x \in \mathsf{Im}(\phi)$, $x = \phi(x) \preceq s$. Besides, as ϕ is isotone, $x = \phi(x) \preceq \phi(s)$. ■

Finally, combining the previous propositions (Prop. 6,7,8) leads to the interpretation of $\mathsf{Pr}^{\gamma}_{s_0}(s)$ as the greatest solution of a problem preserving the set of γ-discontinuities of s_0.

Proposition 9. *Consider two series s and s_0 in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, $\mathsf{Pr}^{\gamma}_{s_0}(s)$ is the greatest series less than or equal to s and having a set of γ-discontinuities included in the set of γ-discontinuities of s_0, $\mathcal{K}_{s_0}$, i.e., $\mathsf{Pr}^{\gamma}_{s_0}(s)$ is the greatest solution of*

$$\begin{cases} x \preceq s \\ \mathcal{K}_x \subseteq \mathcal{K}_{s_0} \end{cases}$$

Proof: Consider a series $x \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. According to Prop. 7, $x \in \mathsf{Im}\left(\mathsf{Pr}^{\gamma}_{s_0}\right)$ is equivalent to $\mathcal{K}_x \subseteq \mathcal{K}_{s_0}$. Therefore,

$$\begin{cases} x \preceq s \\ \mathcal{K}_x \subseteq \mathcal{K}_{s_0} \end{cases} \Leftrightarrow \begin{cases} x \preceq s \\ x \in \mathsf{Im}\left(\mathsf{Pr}^{\gamma}_{s_0}\right) \end{cases} \tag{7}$$

As $\mathsf{Pr}^{\gamma}_{s_0}$ is a dual closure (see Prop. 6), $\mathsf{Pr}^{\gamma}_{s_0}(s)$ is, according to Prop. 8, the greatest solution of (7). ∎

The previous proposition is extended to matrices.

Proposition 10. *Consider S and S_0 in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{m\times n}$, $\mathsf{Pr}^{\gamma}_{S_0}(S)$ is the greatest element in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{m\times n}$ less than or equal to S such that the set of γ-discontinuities of an entry of $\mathsf{Pr}^{\gamma}_{S_0}(S)$ is included in the set of γ-discontinuities of the corresponding entry of S_0.*

Proof: As $\left(\mathsf{Pr}^{\gamma}_{S_0}(S)\right)_{ij} = \mathsf{Pr}^{\gamma}_{(S_0)_{ij}}(S_{ij})$, the result is an obvious consequence of Prop. 9. ∎

V. Holding Time Maximization

In this section, a method to obtain the greatest holding time while preserving the input-output and perturbation-output transfer function matrices is introduced. First, we look for the greatest state-matrix A_M greater than or equal to A and preserving the input-output and perturbation-output transfer function matrices. Formally, this means finding the greatest state-matrix A_M such that $A_M \succeq A$, $CA_M^*B = CA^*B$, and $CA_M^* = CA^*$.

Proposition 11. *Consider a TEG represented by matrices (C, A, B), the greatest state-matrix A_M such that $A_M \succeq A$, $CA_M^*B = CA^*B$, and $CA_M^* = CA^*$ is*

$$A_M = CA^* \backslash\!\!\!\!\circ\, CA^*$$

Proof: Preserving the perturbation-output transfer function matrix implies preserving the input-output transfer function matrix: $CA_M^* = CA^* \Rightarrow CA_M^*B = CA^*B$. Therefore, the problem is to find the greatest state-matrix $A_M \succeq A$ such that $CA_M^* = CA^*$. As $A_M \succeq A$ implies $CA_M^* \succeq CA^*$, it is equivalent to find the greatest state-matrix A_M such that $A_M \succeq A$ and $CA_M^* \preceq CA^*$. If $A_M \succeq A$, then $A_M^* \succeq A^*$. Thus, $A_M^* = A^*A_M^*$ (see Prop. 2) and $CA_M^* \preceq CA^* \Leftrightarrow CA^*A_M^* \preceq CA^*$. Using residuation theory, $CA^*A_M^* \preceq CA^* \Leftrightarrow A_M^* \preceq CA^* \backslash\!\!\!\!\circ\, CA^*$. Finally, according to Th. 2 $CA^* \backslash\!\!\!\!\circ\, CA^*$ is a star, then, according to Prop. 1, $A_M^* \preceq CA^* \backslash\!\!\!\!\circ\, CA^* \Leftrightarrow A_M \preceq CA^* \backslash\!\!\!\!\circ\, CA^*$. Clearly, $CA^* \backslash\!\!\!\!\circ\, CA^*$ is a solution. Therefore, $A_M = CA^* \backslash\!\!\!\!\circ\, CA^*$. ∎

Remark 2. *This problem can be rephrased as an optimal feedback control problem: find the greatest feedback F from state to state preserving the perturbation-output transfer function matrix. Formally, this is equivalent to finding the greatest feedback F such that $C(A \oplus F)^* \preceq CA^*$ (see [4]).*

Prop. 11 is combined with the γ-projection onto matrices to obtain a TEG differing from the original TEG only for the holding times. For the state-matrix A, a coefficient A_{ij} represents the influence of transition x_j on transition x_i. According to section III, consider the place upstream from transition x_i and downstream from transition x_j, A_{ij} is the monomial $\gamma^{m_{ij}}\delta^{\tau_{ij}}$ where m_{ij} is the initial number of tokens in the place and τ_{ij} is the holding time of the place, hence, $\mathcal{K}_{A_{ij}} = \{m_{ij}\}$. Furthermore, conserving the places and their initial markings (i.e., constraint of the considered problem) is equivalent to maintaining the set of γ-discontinuities in comparison to the one of the initial system. Consequently, the problem of holding time maximization is to find the greatest solution $A' \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times n}$ of

$$\begin{cases} A' \preceq A_M \\ \forall i,j \quad \mathcal{K}_{A'_{ij}} = \mathcal{K}_{A_{ij}} \end{cases} \tag{8}$$

Next, we show that, with $A_M = CA^* \backslash\!\!\!\!\circ\, CA^*$, the previous problem comes down to finding the greatest solution $A' \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times n}$ of

$$\begin{cases} A' \preceq A_M \\ \forall i,j \quad \mathcal{K}_{A'_{ij}} \subseteq \mathcal{K}_{A_{ij}} \end{cases} \tag{9}$$

According to Prop. 9 and Prop. 10, (9) admits $A_{opt} = \mathsf{Pr}^{\gamma}_{A}(A_M)$ as the greatest solution. We check that A_{opt} is a solution of (8). Indeed, as the entries of A are monomials, $\mathcal{K}_{A_{ij}}$ is either empty or a singleton. If $\mathcal{K}_{A_{ij}} = \emptyset$, then $\mathcal{K}_{A_{opt,ij}} = \mathcal{K}_{A_{ij}}$. Otherwise, $\mathcal{K}_{A_{ij}}$ is a singleton. As the γ-projection onto a series is a dual closure, $A_M \succeq A$ leads to $A_{opt} \succeq \mathsf{Pr}^{\gamma}_{A}(A) = A$. Therefore, $A_{ij} \neq \varepsilon$ implies $A_{opt,ij} \neq \varepsilon$. In terms of γ-discontinuities, $\mathcal{K}_{A_{ij}} \neq \emptyset$ implies $\mathcal{K}_{A_{opt,ij}} \neq \emptyset$. Then,

$$\emptyset \subset \mathcal{K}_{A_{opt,ij}} \subseteq \mathcal{K}_{A_{ij}}$$

Furthermore, as $\mathcal{K}_{A_{ij}}$ is a singleton, $\mathcal{K}_{A_{opt,ij}} = \mathcal{K}_{A_{ij}}$. Consequently, as a solution of (8) is a solution of (9), A_{opt} is the greatest solution of (8).

Therefore, the greatest state-matrix modifying only the holding times and preserving the perturbation-output transfer function matrix is A_{opt}. Besides, the initial system and the system after holding time maximization have the same input-output behavior and the same response in case of perturbations.

Remark 3. *The complexity of the calculation of A_{opt} is in $O(n^3)$ elementary operations on periodic series with n the number of state transitions in the considered TEG. In [7], holding time optimization is solved with an integer linear programming problem with n^2 constraints. However, the problems are not comparable since the maintained characteristics of the system are different between the two approaches. Therefore, the most suitable method might depend on the considered application.*

Example 4. *Consider the system presented in Ex. 2, the calculation is done with the C++ library described in [14], and the source code is available in [15]. The γ-projection onto A is applied to $CA^* \backslash\!\!\!\!\circ\, CA^*$. For example, $\left(CA^* \backslash\!\!\!\!\circ\, CA^*\right)_{45} = \delta^{-1}\left(\gamma^2\delta^{10}\right)^*$ and $A_{45} = \gamma^2\delta$ leads to*

$$\left(\mathsf{Pr}^{\gamma}_{A}\left(CA^* \backslash\!\!\!\!\circ\, CA^*\right)\right)_{45} = \mathsf{Pr}^{\gamma}_{A_{45}}\left(\left(CA^* \backslash\!\!\!\!\circ\, CA^*\right)_{45}\right) = \gamma^2\delta^9$$

The complete solution is

$$A_{opt} = \mathsf{Pr}^{\gamma}_{A}\left(CA^* \backslash\!\!\!\!\circ\, CA^*\right) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \delta^2 & \varepsilon & \gamma\delta & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \delta^3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \delta^{12} & \varepsilon & \varepsilon & \varepsilon & \gamma^2\delta^9 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \delta & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \delta & \varepsilon & \varepsilon & \varepsilon & \gamma\delta^2 \\ \varepsilon & \varepsilon & e & \varepsilon & \gamma\delta & \gamma\delta^8 & \varepsilon \end{pmatrix}$$

In this very simple example, the maximal admissible holding time modifications are

- *increase of the holding time of the place from transition x_1 to transition x_4 by 10 time units*
- *increase of the holding time of the place from transition x_5 to transition x_4 by 8 time units*

The input-output transfer function matrix is indeed preserved

$$CA_{opt}^{*}B = CA^{*}B = \delta^5 \oplus \left(\gamma\delta^{14} \oplus \gamma^2\delta^{18}\right)\left(\gamma^2\delta^{10}\right)^{*}$$

VI. Conclusion

In this paper, we have addressed the problem of holding time maximization, while preserving input-output and perturbation-output behaviors. Using residuation theory, it turns out that a single, optimal solution exists with such a strict constraint. Furthermore, this solution does not depend on the considered cost functions. Therefore, under the considered constraint, holding time maximization is solved by maximizing independently the holding times. To take into account the physical structure of the system, a new projection in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is introduced: the γ-projection onto a series.

Time/event duality in TEG leads to an approach to solve the problem of resource optimization (i.e., deleting useless resources) under the same constraint, using δ-projection onto series. An interesting starting point for an extension of this work is to find an optimal state-matrix conserving a reference model $G \succeq H$, i.e., to find the maximal holding times when a predefined relaxed performance is required.

References

[1] C. G. Cassandras and S. Lafortune, *Introduction to Discrete Event Systems*. Secaucus, NJ, USA: Springer-Verlag New York, Inc., 2006.

[2] F. Baccelli, G. Cohen, G. J. Olsder, and J.-P. Quadrat, *Synchronization and Linearity, An Algebra for Discrete Event Systems*. New York, USA: John Wiley and Sons, 1992, available at www-rocq.inria.fr/metalau/cohen/SED/SED1-book.html.

[3] G. Cohen, S. Gaubert, and J. Quadrat, “From first to second-order theory of linear discrete event systems,” in *12th IFAC World Congress*, Sydney, Jul. 1993.

[4] B. Cottenceau, L. Hardouin, J.-L. Boimond, and J.-L. Ferrier, “Model Reference Control for Timed Event Graphs in Dioids,” *Automatica*, vol. 37, no. 9, pp. 1451–1458, Sep. 2001.

[5] B. De Schutter and T. van den Boom, “Model predictive control for max-plus-linear systems,” in *Proceedings of the 2000 American Control Conference*, Chicago, Illinois, Jun. 2000, pp. 4046–4050.

[6] J. Mao and C. Cassandras, “Optimal control of multi-stage discrete event systems with real-time constraints,” *IEEE Transactions on Automatic Control*, vol. 54, no. 1, pp. 108 –123, jan. 2009.

[7] S. Gaubert, “Resource optimization and (min,+) spectral theory,” *IEEE Trans. on Automat. Contr*, vol. 40, pp. 1931–1934, 1995.

[8] B. Heidergott, G. J. Olsder, and J. W. v. d. Woude, *Max Plus at work : modeling and analysis of synchronized systems : a course on Max-Plus algebra and its applications*, ser. Princeton series in applied mathematics. Princeton (N.J.): Princeton University Press, 2006.

[9] G. Cohen, P. Moller, J.-P. Quadrat, and M. Viot, “Algebraic Tools for the Performance Evaluation of Discrete Event Systems,” *Proceedings of the IEEE*, vol. 77, no. 1, pp. 39–58, Jan. 1989, special issue on Discrete Event Systems.

[10] I. Ouerghi, “Etude et commande de systèmes (max,+)-linéaires soumis à des contraintes,” Ph.D. dissertation, ISTIA, Université d’Angers, Angers, France, 2006, available at www.istia.univ-angers.fr/~hardouin/encadrement.html.

[11] T. S. Blyth and M. F. Janowitz, *Residuation Theory*. Oxford, United Kingdom: Pergamon press, 1972.

[12] G. Cohen, “Residuation and Applications,” in *Algèbres Max-Plus et applications en informatique et automatique*, ser. École de printemps d’informatique théorique, no. 26. Île de Noirmoutier, France: INRIA, mai 1998.

[13] L. Hardouin, C. Maia, B. Cottenceau, and M. Lhommeau, “Max-plus linear observer: Application to manufacturing systems,” in *Proceedings of the 10th International Workshop on Discrete Event Systems, WODES 2010*, September 2010, pp. 171–176.

[14] B. Cottenceau, L. Hardouin, M. Lhommeau, and J.-L. Boimond, “Data processing tool for calculation in dioid,” in *Proceedings of the 5th International Workshop on Discrete Event Systems, WODES 2000*, Ghent, Belgium, 2000, available at www.istia.univ-angers.fr/~hardouin/outils.html.

[15] www.istia.univ-angers.fr/~hardouin/HoldingTimeMaximisation.html.