

# Weight-Balanced Timed Event Graphs to Model Periodic Phenomena in Manufacturing Systems

Bertrand Cottenceau, Laurent Hardouin, Johannes Trunk

**Abstract**—Timed Event Graphs (TEGs) are suitable to model manufacturing systems in which synchronization and delay phenomena appear. Since the eighties, TEGs are studied as a class of linear Discrete Event Systems in idempotent semirings such as the  $(\min,+)$  algebra. In this paper, we consider the class of Weighted TEGs (WTEGs) that corresponds to TEGs where the edges have integer weights. By introducing non unitary weights, WTEGs widen the class of manufacturing systems that can be modeled, especially systems with batches and duplications. Moreover, a subclass of WTEGs called Weight-Balanced TEGs (WB-TEGs) can be studied with the algebraic tools that stem from the theory of  $(\min,+)$  linear systems. In this work, the focus lies on some modeling issues for manufacturing systems. Besides cutting and palletization operations, it is shown that WB-TEGs are also well adapted to describe periodic routing policies and, in a symmetrical way, how to merge flows similarly to a multiplexer. In order to simplify the modeling step, a class of Cyclo-Weighted TEGs is introduced. It is an extension of WTEGs where the weights of the edges can change according to a periodic sequence. Finally, we propose some elements of modeling that can be described by Cyclo-Weighted TEGs or equivalently with an input-output transfer relation in an appropriate idempotent semiring of operators.

**Abstract**—[Note to practitioners] The  $(\min,+)$  linear system theory used in this work aims at obtaining linear models for a subclass of man-designed systems, such as automated manufacturing systems or traffic networks. This theory has many analogies with the conventional linear system theory (for continuous systems) and it provides the basis to develop a specific control theory for man made systems. More precisely, the theory of  $(\min,+)$  linear systems is well suited to systems where the prevailing phenomena are synchronizations, delays, duplications and batches. These phenomena arise for example in operations such as assembly/matching, cutting/lot splitting and palletization/lot making. Among the possible representations, we can describe these systems by transfer functions obtained by the combination of a finite number of basic operators. This is analogous to block diagram in the conventional system theory, *i.e.*, a transfer function describes the complete input-output behavior of a system. In the context of manufacturing systems, a transfer function describes the way a system maps an input flow of materials (raw part inputs) into an output flow (finished parts), without the necessity of simulation tools to predict this. Moreover, the transfer function thus obtained can be used to compute controllers in order to regulate the internal flows of a system, for instance to decrease internal stocks. In the case of an automated system, the obtained controllers can be implemented on a Programmable Logic Controller (PLC) as supplementary code. This paper focuses on the use of these algebraic tools in the model process of manufacturing systems and in particular on their ability to describe splitting and merging flows of materials.

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**Index Terms**—Discrete-Event Systems, Petri nets, Weighted Timed Event Graphs, Dioids, Operators, Routing, Multiplexing

## I. INTRODUCTION

Discrete Event Systems (DESs) are popular for the modeling process of manufacturing systems on a high level of abstraction. In DESs, we focus on instantaneous phenomena called *events* that correspond to the state changes. For instance, in a manufacturing system, an event corresponds to the arrival of a raw part, to the departure of a finished part or to the beginning (resp. to the end) of an operation. Besides the

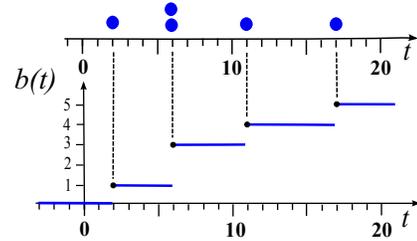


Figure 1. Event list and counter function.

logical order in which events occur, in many applications, the time which elapses between consecutive events is important. For a manufacturing system, we generally want to know the production rate which is the mean value of the number of parts produced in time unit. For a timed DES, we can represent the occurrences of events on a time line, or equivalently by a list of pairs  $\langle \text{event}, \text{date} \rangle$  (sorted by increasing dates). In Fig.1, the occurrences of event  $b$  (depicted by big dots) correspond to the list  $\langle b, 2 \rangle, \langle b, 6 \rangle, \langle b, 6 \rangle, \langle b, 11 \rangle, \langle b, 17 \rangle$ : one event  $b$  occurs at date 2, two events  $b$  occur simultaneously at date 6 etc. Alternatively, we can also describe an event list by a *counter function* which is a function of time that gives the (cumulated) number of events occurred up to a date  $t$ . Fig.1 depicts the counter function  $b(t)$  associated to events  $b$ :  $b(t < 2) = 0$  (no event  $b$  before  $t = 2$ ),  $b(2 \leq t < 6) = 1$ , etc.

For timed DESs, event lists or counter functions are analogous to *signals* in the classical system theory. By extending this analogy further, a timed DES can be viewed as a combination of elementary subsystems that transform signals; such subsystems are called *operators* hereafter. For instance, synchronizations, time shifts and event shifts can be seen as specific transformations on signals. To be homogeneous with [4], we denote by  $\delta^\tau$  a time shift of  $\tau$  time units, by  $\gamma^\nu$  an event shift of  $\nu$  event occurrences and by  $\oplus$  the

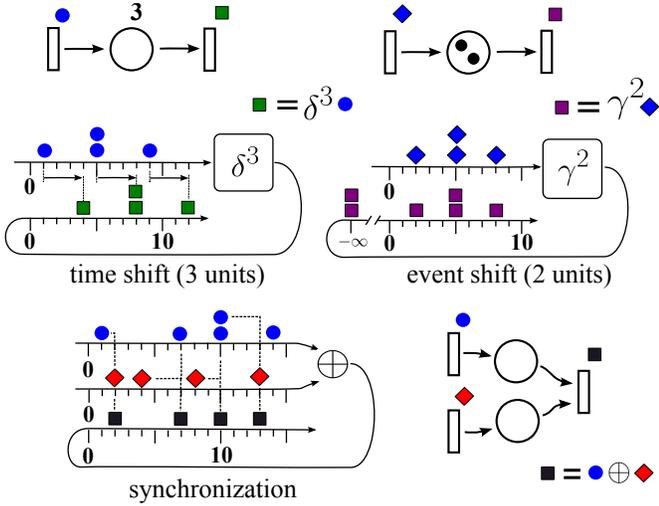


Figure 2. Time Shift, Event Shift and Synchronization

synchronization of two kinds of events. Fig.2 illustrates how signals are transformed by the operators  $\delta^\tau$ ,  $\gamma^\nu$  and  $\oplus$ . For the  $\delta^3$  operator, an input event at date  $t$  leads to an output event at date  $t + 3$ . The  $\gamma^2$  operator describes a system such that there are permanently 2 more output events than the number of input events. Therefore, 2 output events have occurred at date  $-\infty$ . Translated into relations between counter functions, for  $x, y$  two counter functions, we have  $\forall t, (\delta^\tau x)(t) = x(t - \tau)$ ,  $(\gamma^\nu x)(t) = x(t) + \nu$  and  $(x \oplus y)(t) = \min(x(t), y(t))$ .

Timed Petri Nets constitute a powerful graphical and mathematical model to describe how the events of a system are related to each other [16]. Among timed Petri nets, Timed Event Graphs (TEG) is a subclass where it is only possible to model synchronizations, time delays and event shifts. From an operatorial point of view, TEGs correspond to systems generated by a finite combination of  $\gamma^\nu$  and  $\delta^\tau$  operators (with  $\nu, \tau \in \mathbb{N}$ ) and synchronizations of events. For instance, Fig.2 describes how operators  $\gamma^\nu$ ,  $\delta^\tau$  and  $\oplus$  are transposed into TEG models. For TEGs, the events are the firing of transitions.

Since the early eighties, it is known that TEGs can be modeled as linear systems by considering some specific idempotent semirings (or dioids) such as the  $(\min, +)$  algebra (see [1],[10]). In particular, TEGs have an input-output model (transfer function) in a dioid of formal series denoted  $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$  [4], where variable  $\gamma$  (resp.  $\delta$ ) can be assimilated to the event-shift (resp. time-shift) operator introduced before.

In this paper, we consider the modeling of manufacturing systems by using TEGs where non unitary weights can be attached to the edges. This class of models is called Weighted Timed Event Graphs (WTEGs) or Weighted T-systems [15] [8] and corresponds also to Synchronous Data Flow (SDF) graphs used in computer science [13] [7]. WTEGs encompass the class of (ordinary) TEGs and have more expressiveness since they can model lot making (or palletization) and cutting operations. For instance, Fig.3 describes a manufacturing cell where raw parts (events  $u$ ) are first cut to produce 2 sub-parts (events  $x_2$ ) and conveyed to a palletizer where a pallet (events  $y$ ) is filled with 3 subparts. In summary, 3 raw parts are needed

to fill 2 pallets that corresponds to an input-output gain of  $2/3$ . This system can be modeled by a Weighted TEG where the weight of  $\langle 2 \rangle$  in the output edge of  $x_1$  is needed to model the cutting operation, and the weight of  $\langle 3 \rangle$  on the input edge of  $x_3$  describes the palletization.

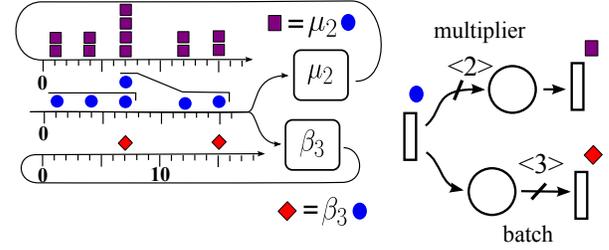


Figure 4. Event multiplication and batch

When one considers *weighted* instead of *ordinary* TEGs, operators  $\gamma^\nu$  and  $\delta^\tau$  are not sufficient anymore. Two supplementary operators denoted  $\mu_m$  (multiplication of events) and  $\beta_b$  (batch of events) have to be introduced to describe the effect of the weighted edges. For instance, Fig.4 illustrates how  $\mu_2$  and  $\beta_3$  operate on a given signal. For operator  $\mu_2$ , each input event induces immediately two output events. For operator  $\beta_3$ , one output event is released after three successive input events have occurred. The modeling of WTEGs on the basis of operators  $\gamma^\nu, \mu_m, \beta_b, \delta^\tau$  is investigated in [5] and [11]. This prior work widens the class of timed DES that can be studied with the tools stemming from the  $(\min, +)$  linear system theory, even if there is a restriction on the class of weighted TEGs that can be handled with this approach. The main result of [5] can be stated as : when a Single Input Single Output (SISO) WTEG is *weight-balanced* (parallel paths have the same gain), then it can be described by an ultimately periodic transfer function. This transfer function allows us to address control problems for Weight-Balanced TEGs (WB-TEGs) since the controller synthesis problem is analogous to the one solved for ordinary TEGs. For instance, the controller strategies obtained in [14], [9] for TEGs can be applied for WB-TEGs as well.

We focus in this paper on some modeling issues, concerning manufacturing systems, that can be handled thanks to WB-TEGs. More precisely, we show that WB-TEGs are well suited to describe a family of routing policies with a periodic dispatching rule. In other words, we can model systems where parts are routed into subsystems. In order to simplify the description of these phenomena, we introduce a class of WB-TEGs with varying weights : a weight can change after the firing of a transition according to a periodic sequence. This class of WB-TEGs is named Cyclo-Weighted Timed Event Graphs (CW-TEGs) and is inspired by Cyclo-Static Dataflow Graphs (CSDF) [18] [2]. For instance, CW-TEGs are useful to simplify the description of periodic phenomena such as periodic multiplexer or demultiplexer (routing). As a consequence, a family of varying  $(\max, +)$  systems already studied in [12] can be described by CW-TEGs.

The paper is organized as follows. First, we give some recalls on the operatorial modeling of WB-TEGs coming from [5]. Then we introduce two new event operators to

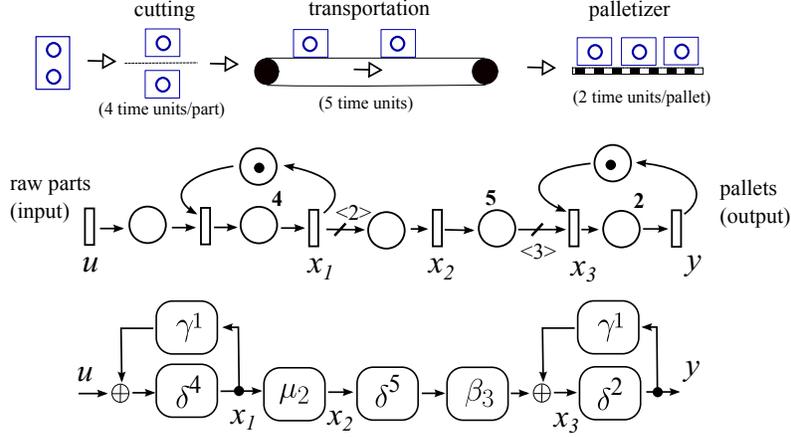


Figure 3. Weighted TEG to model cutting and palletizer

describe multiplier and batch operations the weight of which can change according to a periodic sequence. This leads to the introduction of WTEGs where the edges can be valued with periodic sequences. Such models are called Cyclo-Weighted TEGs (CW-TEG) and under some conditions a CW-TEG is equivalent to a WB-TEG. Finally, some elements of modeling based on CW-TEGs are given : periodic routing, multiplexing and time-variation (with a periodic sequence of delays).

## II. WEIGHT-BALANCED TIMED EVENT GRAPHS

### A. Operatorial description of WB-TEGs

A Timed Event Graph (TEG) is a timed Petri net - with  $P$  the set of places,  $T$  the set of transitions and  $A \subset (P \times T) \cup (T \times P)$  the set of edges - such that each place has exactly one upstream and one downstream transition. A place  $p_k \in P$  can have a positive holding time value  $\tau \in \mathbb{N}$  and an initial marking denoted  $M_0(p_k) \in \mathbb{N}$ . For  $p_k \in P$ , the edge  $t_i \rightarrow p_k$  (resp.  $p_k \rightarrow t_o$ ) is valued by a strictly positive integer denoted  $\omega_i(p_k)$  (resp.  $\omega_o(p_k)$ ) (the weights of the edges). In order to avoid confusion with holding times, weights of edges are denoted between brackets, e.g.  $\langle 2 \rangle$ . Moreover,  $t_i \rightarrow p_k \rightarrow t_o$  defines an elementary path denoted  $\pi_k$  the gain of which is given by  $\Gamma(\pi_k) \triangleq \omega_i(p_k)/\omega_o(p_k) \in \mathbb{Q}$ .

When one considers only the *earliest firing rule*, a transition  $t_j$  fires as soon as each input place  $p_l$  of  $t_j$  contains at least  $\omega_o(p_l)$  available token(s). Then  $\omega_o(p_l)$  token(s) is(are) removed from each input place  $p_l$  of  $t_j$ , and  $\omega_i(p_k)$  token(s) is(are) added to each output place  $p_k$  of  $t_j$ .

**Definition 1 (Gain  $\Gamma$ ):** For a path  $\pi = \pi_a \rightarrow \pi_b \rightarrow \dots \rightarrow \pi_k$ , the gain is  $\Gamma(\pi) = \Gamma(\pi_a) \times \Gamma(\pi_b) \times \dots \times \Gamma(\pi_k)$ . For the WTEG depicted in Fig.5,  $\omega_i(p_1) = 3$ ,  $\omega_o(p_1) = 2$ ,  $\Gamma(\pi_1) = 3/2$ ,  $\Gamma(\pi_2 \rightarrow \pi_3) = 6/4$  and  $\Gamma(\pi_4) = 1$ .

In [5], a weighted TEG is said to be *weight-balanced* if  $\forall t_a, t_b \in T$ , all the paths from  $t_a$  to  $t_b$  have the same gain. For instance, the WTEG of Fig.5 is weight-balanced since every paths from  $x_1$  to  $x_3$  have the same gain:  $\Gamma(\pi_1) = 3/2 = \Gamma(\pi_2 \rightarrow \pi_3) = 6/4 = \Gamma(\pi_2 \rightarrow \pi_4 \rightarrow \pi_3) = 1/4 \times 1 \times 6$ . Clearly, a WTEG is weight-balanced only if any circuit has a gain equal to 1.

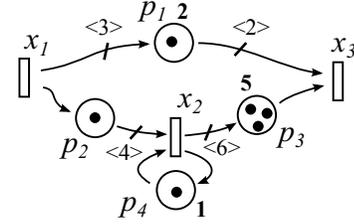


Figure 5. Weight-Balanced TEG.

The operators needed to model Weight-Balanced TEGs (WB-TEGs) are now recalled in a more formal way. The set of counter functions (denoted  $\Sigma$ ) corresponds to the set of signals on which the operators are defined. By denoting  $\mathbb{Z}_{\min} \triangleq \{\mathbb{Z} \cup \pm\infty\}$ , a counter function  $x \in \Sigma$ ,  $x : \mathbb{Z} \rightarrow \mathbb{Z}_{\min}$ ,  $t \mapsto x(t)$  gives the number of events of type  $x$  occurred up to time  $t$  (see Fig.1) and is naturally a non-decreasing function. An *operator* is a map  $\mathcal{H} : \Sigma \rightarrow \Sigma$  which is said to be *linear* if  $\forall x, y \in \Sigma$ , a)  $\mathcal{H}(\min(x, y)) = \min(\mathcal{H}(x), \mathcal{H}(y))$  and b)  $\mathcal{H}(\lambda + x) = \lambda + \mathcal{H}(x)$ . An operator is said to be *additive* if at least a) is satisfied.

**Definition 2 (Dioid  $\mathcal{O}$ ):** The set of additive operators on  $\Sigma$  is a dioid (idempotent semiring) denoted  $\mathcal{O}$  with  $x \in \Sigma$ ,  $\forall \mathcal{H}_1, \mathcal{H}_2 \in \mathcal{O}$

$$\begin{aligned} \mathcal{H}_1 \oplus \mathcal{H}_2 &\triangleq \forall x, (\mathcal{H}_1 \oplus \mathcal{H}_2)(x) = \min(\mathcal{H}_1(x), \mathcal{H}_2(x)), \\ \mathcal{H}_1 \circ \mathcal{H}_2 &\triangleq \forall x, (\mathcal{H}_1 \circ \mathcal{H}_2)(x) = \mathcal{H}_1(\mathcal{H}_2(x)). \end{aligned}$$

The null operator (neutral for  $\oplus$  and absorbing for  $\circ$ ) is denoted  $\varepsilon : \forall x \in \Sigma, \forall t, (\varepsilon x)(t) = +\infty$  and the unit operator (neutral for  $\circ$ ) is denoted  $e : \forall x \in \Sigma, \forall t, (ex)(t) = x(t)$ .

WB-TEGs operating under the earliest firing rule can be modeled in dioid  $\mathcal{O}$  by combining the operators given hereafter.

**Definition 3 (Basic operators):** Operators  $\delta^t, \gamma^n, \mu_m, \beta_b \in \mathcal{O}$  are defined by  $x \in \Sigma$ ,

$$\begin{aligned} \tau \in \mathbb{Z}, \delta^\tau &: \forall x, \forall t, (\delta^\tau x)(t) = x(t - \tau), \\ \nu \in \mathbb{Z}, \gamma^\nu &: \forall x, \forall t, (\gamma^\nu x)(t) = x(t) + \nu, \\ b \in \mathbb{N}, \beta_b &: \forall x, \forall t, (\beta_b x)(t) = \lfloor x(t)/b \rfloor, \\ m \in \mathbb{N}, \mu_m &: \forall x, \forall t, (\mu_m x)(t) = x(t) \times m, \end{aligned}$$

where  $\lfloor a \rfloor \in \mathbb{Z}$  denotes the greatest integer less than or equal to  $a \in \mathbb{Q}$ .

In order to lighten formal expressions, symbol  $\circ$  (composition) is omitted, e.g., operator  $\gamma^3 \circ \delta^2 \circ \mu_2$  is written  $\gamma^3 \delta^2 \mu_2$ .

*Example 1:* For the WB-TEG in Fig.3, we associate counter functions to transitions, therefore we obtain:  $x_i \in \Sigma$ ,  $x_1 = \delta^4(u \oplus \gamma^1 x_1)$ ,  $x_2 = \mu_2 x_1$ ,  $x_3 = \beta_3 \delta^5 x_2 \oplus \gamma^1 y$  and  $y = \delta^2 x_3$ . These relations can be described by a block diagram (see Fig.3).

We define the gain of an operator  $\Gamma : \mathcal{O} \rightarrow \mathbb{Q}$  by  $v_a, v_b \in \mathcal{O}$ ,  $\Gamma(v_a \circ v_b) = \Gamma(v_a) \times \Gamma(v_b)$  and  $\Gamma(v_a \oplus v_b) = \min(\Gamma(v_a), \Gamma(v_b))$ , where  $\Gamma(\gamma^n) = \Gamma(\delta^t) = 1$ ,  $\Gamma(\mu_m) = m$  and  $\Gamma(\beta_b) = 1/b$ . We intentionally keep the same notation for the gain of a path in a WB-TEG (see Def.1) and the gain of an operator, because it has the same meaning. For instance,  $\Gamma(\gamma^1 \mu_2 \beta_3 \delta^2 \oplus \beta_6 \gamma^1 \mu_4 \delta^5) = 2/3$ . Moreover, an operator  $v = \bigoplus_i v_i$  is said to be *weight-balanced* if  $\forall i, \Gamma(v_i) = \Gamma(v)$  where  $v_i$  is obtained only by products of basic operators in  $\{\delta^t, \gamma^n, \mu_m, \beta_b\}$ . In other words, an operator is weight-balanced if all its  $\oplus$  terms have the same gain.

Describing WB-TEGs with the operators given in Def.3 leads to an input-output representation. For a given WB-TEG with  $m$  inputs,  $p$  outputs, and  $n$  internal transitions, by associating a counter function to each input transition (in a vector  $u$ ), to each output transition (in a vector  $y$ ), and to each internal transition (in a vector  $x$ ), the earliest behavior is described by

$$\begin{cases} x = Ax \oplus Bu \\ y = Cx \end{cases}$$

where  $A \in \mathcal{O}^{n \times n}$ ,  $B \in \mathcal{O}^{n \times m}$  and  $C \in \mathcal{O}^{p \times n}$  are matrices.

Since equation  $x = Ax \oplus Bu$  has  $x = A^*B$  as least solution (see [1, Th.4.75]), with  $A^* = \bigoplus_{n \geq 0} A^n = e \oplus A \oplus A^2 \oplus \dots$  the Kleene star operator, the input-output behavior is

$$y = CA^*Bu = C(e \oplus A \oplus A^2 \oplus \dots)Bu = Hu. \quad (1)$$

The matrix  $H \in \mathcal{O}^{p \times m}$  is called the *transfer function matrix* of the WB-TEG.

*Example 2:* For the WB-TEG in Fig.3, the input-output behavior is expressed by  $y = Hu$  where the transfer function is given by  $H = \delta^2(\gamma^1 \delta^2)^* \beta_3 \delta^5 \mu_2 \delta^4 (\gamma^1 \delta^4)^*$ .

The transfer function of a WB-TEG can always be expressed in a dioid of formal series denoted  $\mathcal{E}[[\delta]]$  which encompasses  $\mathcal{M}_{\text{in}}^{\text{ax}}[[\gamma, \delta]]$  [5]. Dioid  $\mathcal{E}[[\delta]]$  is a set of formal series in one variable  $\delta$  (time shift operator) with coefficients in the dioid of event operators denoted  $\mathcal{E}$  (see Def.4 hereafter). The coefficients are obtained by a finite combination of operators in  $\{\gamma^n, \mu_m, \beta_b, e, \varepsilon\}$ . The formal identities recalled below apply in  $\mathcal{E}[[\delta]]$ :

$$\gamma^1 \delta^1 = \delta^1 \gamma^1; \quad \mu_m \delta^1 = \delta^1 \mu_m; \quad \beta_b \delta^1 = \delta^1 \beta_b \quad (2)$$

$$\gamma^n \gamma^{n'} = \gamma^{n+n'}; \quad \delta^t \delta^{t'} = \delta^{t+t'} \quad (3)$$

$$\gamma^n \oplus \gamma^{n'} = \gamma^{\min(n, n')}; \quad \delta^t \oplus \delta^{t'} = \delta^{\max(t, t')} \quad (4)$$

$$\mu_m \gamma^n = \gamma^{m \times n} \mu_m; \quad \gamma^n \beta_b = \beta_b \gamma^{n \times b}. \quad (5)$$

## B. Periodic Event Operators

According to (2), the delay operator  $\delta^t$  can commute with all event operators (E-operators), i.e., operators obtained by

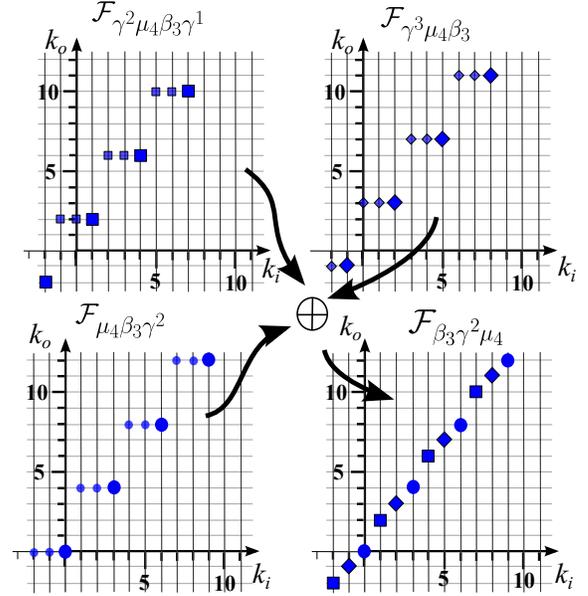


Figure 6. Graphical representation of  $\mathcal{F}_{\beta_3 \gamma^2 \mu_4} = \min(\mathcal{F}_{\mu_4 \beta_3 \gamma^2}, \mathcal{F}_{\gamma^2 \mu_4 \beta_3 \gamma^1}, \mathcal{F}_{\gamma^3 \mu_4 \beta_3})$

a finite number of sums and compositions of operators in  $\{\gamma^n, \mu_m, \beta_b\}$ . Therefore, we can always express the behavior of WB-TEGs as formal expressions  $\bigoplus_i w_i \delta^{t_i}$  where  $w_i$  are weight-balanced E-operators. Compared to ordinary TEGs, where the only possible E-operator is the  $\gamma^n$  operator, E-operators are more complex for WB-TEGs, and thus are investigated hereafter.

*Definition 4 (Dioid  $\mathcal{E}$ ):* We denote by  $(\mathcal{E}, \oplus, \circ) \subset \mathcal{O}$  the dioid of E-operators, i.e., those obtained by finite sums and products of operators in  $\{\gamma^n, \mu_m, \beta_b, \varepsilon, e\}$ . For instance,  $\gamma^1 \mu_2 \oplus \beta_2 \gamma^1 \in \mathcal{E}$  whereas  $\gamma^1 \delta^2 \notin \mathcal{E}$ .

An E-operator can be considered as an *instantaneous*<sup>1</sup> system. If  $w \in \mathcal{E}$ , for all counter  $x \in \Sigma$ , then  $(wx)(t)$  depends only on  $k_i = x(t)$ , and not on  $t$ . Therefore, an E-operator can be described by a Counter-value to Counter-value (C/C) function  $\mathcal{F}_w : \mathbb{Z}_{\min} \rightarrow \mathbb{Z}_{\min}, k_i \mapsto k_o$ . It is an untimed representation where  $k_i$  (resp.  $k_o$ ) is an input (resp. output) counter value. For a given E-operator  $w$ , its C/C function  $\mathcal{F}_w$  is obtained by replacing  $x(t)$  by  $k_i$  in the expression of  $(wx)(t)$ . For instance, if one considers  $w = \beta_3 \gamma^2 \mu_4 \in \mathcal{E}$ , then  $(wx)(t) = \lfloor (4 \times x(t) + 2)/3 \rfloor$  and  $\mathcal{F}_{\beta_3 \gamma^2 \mu_4}(k_i) = \lfloor (4 \times k_i + 2)/3 \rfloor$ . This C/C function is depicted in the bottom right part of Fig.6. Function  $\mathcal{F}_{\beta_3 \gamma^2 \mu_4}$  is characterized by a finite set of values:  $\mathcal{F}_{\beta_3 \gamma^2 \mu_4}(0) = 0, \mathcal{F}_{\beta_3 \gamma^2 \mu_4}(1) = 2, \mathcal{F}_{\beta_3 \gamma^2 \mu_4}(2) = 3$  and its periodicity  $\mathcal{F}_{\beta_3 \gamma^2 \mu_4}(k_i + 3) = 4 + \mathcal{F}_{\beta_3 \gamma^2 \mu_4}(k_i)$ .

*Remark 1:* The left and the right multiplication of  $w \in \mathcal{E}$  by  $\gamma^1$  leads to a shift of the associate function  $\mathcal{F}_w$  since  $\mathcal{F}_{w \gamma^1}(k_i) = \mathcal{F}_w(k_i + 1)$  and  $\mathcal{F}_{\gamma^1 w}(k_i) = \mathcal{F}_w(k_i) + 1$ . The left (resp. right) multiplication by  $\gamma^1$  corresponds to a shift of the C/C function to the top (resp. to the left).

*Remark 2:* Let us recall (see [5]) that  $\forall w_1, w_2 \in \mathcal{E}$ , then  $\mathcal{F}_{w_1 \oplus w_2} = \min(\mathcal{F}_{w_1}, \mathcal{F}_{w_2})$  and  $\mathcal{F}_{w_1 \circ w_2} = \mathcal{F}_{w_1} \circ \mathcal{F}_{w_2}$ . There

<sup>1</sup>or, equivalently, a memoryless system.

is an isomorphism between the dioid  $\mathcal{E}$  and the set of C/C functions.

**Definition 5 (Periodicity):** An E-operator  $w$  is said to be  $(n, n')$ -periodic if  $\forall k_i \in \mathbb{Z}, \mathcal{F}_w(k_i + n) = \mathcal{F}_w(k_i) + n'$  with  $n, n' \in \mathbb{N}$ :

$$w \text{ is } (n, n')\text{-periodic} \iff w\gamma^n = \gamma^{n'}w. \quad (6)$$

The set of periodic E-operators is denoted  $\mathcal{E}_{per}$ .

For instance, the operator  $\gamma^n$  is  $(1, 1)$ -periodic ( $\gamma^n\gamma^1 = \gamma^1\gamma^n$ ),  $\mu_m$  is  $(1, m)$ -periodic ( $\mu_m\gamma^1 = \gamma^m\mu_m$ ) and  $\beta_b$  is  $(b, 1)$ -periodic ( $\beta_b\gamma^b = \gamma^1\beta_b$ ). Let  $w \in \mathcal{E}_{per}$  be a  $(n, n')$ -periodic E-operator, then  $n'/n \in \mathbb{Q}$  is equal to its gain  $\Gamma(w)$  and is also equal to the average slope of  $\mathcal{F}_w$ . Let us note that if  $w \in \mathcal{E}_{per}$  is  $(n, n')$ -periodic, then it is also  $(k \times n, k \times n')$ -periodic, with  $k \in \mathbb{N}$ .

For the sequel, it is important to remark that  $\forall \nu, \nu' \in \mathbb{N}$ , the composed operator  $\gamma^\nu \mu_m \beta_b \gamma^{\nu'}$  is a  $(b, m)$ -periodic operator since (see (2)-(5)):  $(\gamma^\nu \mu_m \beta_b \gamma^{\nu'}) \gamma^b = \gamma^\nu \mu_m \beta_b \gamma^b \gamma^{\nu'} = \gamma^\nu \mu_m \gamma^1 \beta_b \gamma^{\nu'} = \gamma^m (\gamma^\nu \mu_m \beta_b \gamma^{\nu'})$ . The associated C/C function is depicted as a staircase function,  $\mathcal{F}_{\gamma^\nu \mu_m \beta_b \gamma^{\nu'}}(k_i) = (\lfloor (k_i + \nu')/b \rfloor \times m) + \nu$ . For instance, the C/C functions associated to  $\mu_4 \beta_3 \gamma^2$ ,  $\gamma^2 \mu_4 \beta_3 \gamma^1$  and  $\gamma^3 \mu_4 \beta_3$  are depicted in Fig.6. According to Rem.1, each of them is deduced from  $\mathcal{F}_{\mu_4 \beta_3}$  by shifts.

**Proposition 1 ([5]):** All weight-balanced E-operators are periodic.

Prop.1 amounts to saying that all E-operators arising in the modeling of WB-TEGs are periodic. Moreover, for a Single-Input Single-Output (SISO) WB-TEG with a transfer function  $H \in \mathcal{E}_{per}[[\delta]]$ , there exist  $n, n' \in \mathbb{N}$  such that

$$H\gamma^n = \gamma^{n'}H \text{ and } H\delta^1 = \delta^1H,$$

where  $\Gamma(H) = n'/n$  is the gain of the SISO system. Let us note that for ordinary TEGs,  $n = n' = 1$ .

**Proposition 2:** All  $(b, m)$ -periodic E-operators  $w \in \mathcal{E}_{per}$  can be written as a finite sum

$$w = \bigoplus_{j=0}^{b-1} \gamma^{\mathcal{F}_w(j)} \mu_m \beta_b \gamma^{b-1-j}. \quad (7)$$

*Proof:* Let us consider the operator defined by  $v = \bigoplus_{j=0}^{b-1} v_j$  with  $v_j = \gamma^{\mathcal{F}_w(j)} \mu_m \beta_b \gamma^{b-1-j}$ . We have to show that  $v = w$ , or equivalently,  $\forall k, \mathcal{F}_v(k) = \mathcal{F}_w(k)$ . First,

$$\mathcal{F}_{v_j}(k) = \left\lfloor \frac{k + (b-1) - j}{b} \right\rfloor m + \mathcal{F}_w(j).$$

Then, we can express  $\mathcal{F}_v(k) = \min_j \mathcal{F}_{v_j}(k) = \min(\lfloor (k + (b-1))/b \rfloor m + \mathcal{F}_w(0), \lfloor (k + (b-2))/b \rfloor m + \mathcal{F}_w(1), \dots, \lfloor k/b \rfloor m + \mathcal{F}_w(b-1))$ . Since  $v$  is  $(b, m)$ -periodic and  $w$  is  $(b, m)$ -periodic, it is sufficient to show that  $\mathcal{F}_v(k) = \mathcal{F}_w(k)$  for all  $k \in \{0, \dots, b-1\}$ . Let us remark that  $\mathcal{F}_w$  is isotone and satisfies

$$\mathcal{F}_w(0) \leq \mathcal{F}_w(1) \leq \dots \leq \mathcal{F}_w(b-1) \leq m + \mathcal{F}_w(0) \leq \dots$$

We can now evaluate  $\mathcal{F}_v(k)$  for  $k \in \{0, \dots, b-1\}$ . We have  $\mathcal{F}_v(0) = \min(\lfloor (b-1)/b \rfloor m + \mathcal{F}_w(0), \lfloor (b-2)/b \rfloor m + \mathcal{F}_w(1), \dots, \lfloor 0/b \rfloor m + \mathcal{F}_w(b-1)) =$

$\min(\mathcal{F}_w(0), \mathcal{F}_w(1), \dots, \mathcal{F}_w(b-1)) = \mathcal{F}_w(0)$  (because of the isotony of  $\mathcal{F}_w$ ). Similarly, after simplifications,

$$\begin{aligned} \mathcal{F}_v(1) &= \min(m + \mathcal{F}_w(0), \mathcal{F}_w(1), \dots, \mathcal{F}_w(b-1)) \\ &= \mathcal{F}_w(1), \\ \mathcal{F}_v(2) &= \min(m + \mathcal{F}_w(0), m + \mathcal{F}_w(1), \dots, \mathcal{F}_w(b-1)) \\ &= \mathcal{F}_w(2), \\ &\dots \\ \mathcal{F}_v(b-1) &= \min(\dots, m + \mathcal{F}_w(b-2), \mathcal{F}_w(b-1)) \\ &= \mathcal{F}_w(b-1). \end{aligned}$$

■

**Example 3:** Fig. 6 illustrates the decomposition of Prop.2 with the  $(3, 4)$ -periodic E-operator  $w = \beta_3 \gamma^2 \mu_4$ . We have,  $\mathcal{F}_w(0) = 0, \mathcal{F}_w(1) = 2, \mathcal{F}_w(2) = 3$ . Due to (7), we can write  $w = \beta_3 \gamma^2 \mu_4 = \gamma^0 \mu_4 \beta_3 \gamma^2 \oplus \gamma^2 \mu_4 \beta_3 \gamma^1 \oplus \gamma^3 \mu_4 \beta_3 \gamma^0$ .

**Corollary 1 (Realization):** For all  $(n, n')$ -periodic C/C function  $f$ , we can find  $w \in \mathcal{E}_{per}$  realizing  $f$ , i.e., satisfying  $\mathcal{F}_w = f$ .

### C. C/C functions and residuation

**Definition 6 ( $\mathbf{C}_{per}$ ):** Let us denote by  $\mathbf{C}_{per}$  the set of periodic C/C functions, i.e., the set of non-decreasing functions  $f: \mathbb{Z}_{\min} \rightarrow \mathbb{Z}_{\min}$ , such that  $\exists n, n' \in \mathbb{N}, f(k+n) = f(k) + n'$  and  $f(-\infty) = -\infty$  (resp.  $f(+\infty) = +\infty$ ), with the next operations:  $f, g \in \mathbf{C}_{per}$

$$\begin{aligned} f \oplus g &\triangleq \min(f, g), \\ f \otimes g &\triangleq f \circ g. \end{aligned}$$

The set  $(\mathbf{C}_{per}, \oplus, \otimes)$  is isomorphic to the set of periodic E-operators  $\mathcal{E}_{per}$ , i.e.,  $w \in \mathcal{E}_{per} \iff \mathcal{F}_w \in \mathbf{C}_{per}$ . In  $\mathbf{C}_{per}$ , a partial order relation can be defined<sup>2</sup> by

$$f \preceq g \iff g = f \oplus g \iff g = \min(f, g) \iff f \geq g.$$

On partially ordered sets and in particular in dioids, the residuation theory introduced in [3], and developed in [1] for dioids, allows one to tackle some mapping inversion problems. In [3], a mapping  $f: (A, \leq) \rightarrow (B, \leq)$  defined over ordered sets is said to be residuated (resp. dually residuated) if it is isotone ( $x \leq y \Rightarrow f(x) \leq f(y)$ ) and if  $\forall b \in B, f(x) \leq b$  (resp.  $f(x) \geq b$ ) has a greatest (resp. least) solution in  $A$  that is denoted  $f^\sharp(b)$  (resp.  $f^\flat(b)$ ).

When  $f$  is residuated (resp. dually residuated), then  $f^\sharp$  (resp.  $f^\flat$ ) is called its residual (resp. dual residual). Moreover, a residuated (resp. dually residuated) mapping is dually residuated (resp. residuated) with

$$(f^\sharp)^\flat = f \text{ (resp. } (f^\flat)^\sharp = f).$$

**Theorem 1 ([1]):** On a complete dioid, mappings  $L_a: x \mapsto a \otimes x$  and  $R_a: x \mapsto x \otimes a$  are residuated and their residual mappings are denoted by  $L_a^\sharp(x) = a \bowtie x$  and  $R_a^\sharp(x) = x \phi a$ .

**Remark 3:** Some formula to apply  $\bowtie$  and  $\phi$  operations on periodic operators in  $\mathcal{E}_{per}$  can be found in [5].

**Proposition 3:** A periodic C/C function  $\mathcal{F}_w \in \mathbf{C}_{per}$ , with  $w \in \mathcal{E}_{per}$ , is both residuated and dually residuated. The residual and the dual residual are defined by:  $\forall k_o \in \mathbb{Z}_{\min}$ ,

$$\begin{aligned} \mathcal{F}_w^\sharp(k_o) &= \min\{k_i \in \mathbb{Z}_{\min} \mid \mathcal{F}_w(k_i) \geq k_o\} \\ \mathcal{F}_w^\flat(k_o) &= \max\{k_i \in \mathbb{Z}_{\min} \mid \mathcal{F}_w(k_i) \leq k_o\} \end{aligned}$$

<sup>2</sup>It is the opposite of the classical order on functions.

If  $\mathcal{F}_w$  is  $(n, n')$ -periodic then  $\mathcal{F}_w^\sharp$  and  $\mathcal{F}_w^\flat$  are  $(n', n)$ -periodic.

*Proof:* Let us note that the order  $\preceq$  in  $(\mathbb{Z}_{\min}, \preceq)$  is the opposite of the classical order  $\leq$  in  $\mathbb{Z}$  ( $a \preceq b \iff a \geq b$ ). Function  $\mathcal{F}_w$  is residuated if for all  $K \in (\mathbb{Z}_{\min}, \preceq)$ , it is always possible to find a greatest solution  $k_i \in (\mathbb{Z}_{\min}, \preceq)$  to  $\mathcal{F}_w(k_i) \preceq K$ . Expressed differently, with the classical order, does the set  $\{k_i \in \mathbb{Z}_{\min} | \mathcal{F}_w(k_i) \geq K\}$  have a minimum value? Indeed, since  $\mathbb{Z}_{\min}$  is totally ordered, there is a unique minimum value. Therefore,  $\mathcal{F}_w^\sharp(K) = \min\{k_i \in \mathbb{Z}_{\min} | \mathcal{F}_w(k_i) \geq K\}$ . Analogously we can show that  $\mathcal{F}_w$  is also dually residuated. ■

*Remark 4:* It is important to note that  $\mathcal{F}_w$  is residuated and dually residuated but not invertible. And, in general,  $\mathcal{F}_w^\sharp$  is different from  $\mathcal{F}_w^\flat$ .

*Notation 1* ( $w^\sharp, w^\flat$ ): For  $w \in \mathcal{E}_{per}$ , we denote by  $w^\sharp$  (resp.  $w^\flat$ ) the operator such that  $\mathcal{F}_w^\sharp = \mathcal{F}_{w^\sharp}$  (resp.  $\mathcal{F}_w^\flat = \mathcal{F}_{w^\flat}$ ).

*Example 4:* Let us consider for instance  $w = \beta_3\gamma^2\mu_4$ , its corresponding C/C function is depicted in Fig.6. Since  $w$  is  $(3, 4)$ -periodic,  $w^\sharp$  and  $w^\flat$  are  $(4, 3)$ -periodic and we just have to compute the values of  $\mathcal{F}_w^\sharp(k)$  and  $\mathcal{F}_w^\flat(k)$  for  $0 \leq k \leq 3$ . We have,  $\mathcal{F}_w^\sharp(0) = \min\{k | \mathcal{F}_w(k) \geq 0\} = 0$ ,  $\mathcal{F}_w^\sharp(1) = \min\{k | \mathcal{F}_w(k) \geq 1\} = 1$  (let us note that the least value of  $\mathcal{F}_w(k)$  greater than 1 is  $\mathcal{F}_w(1) = 2$ ),  $\mathcal{F}_w^\sharp(2) = 1$  and  $\mathcal{F}_w^\sharp(3) = 2$ . Then,  $\mathcal{F}_w^\sharp(k+4) = \mathcal{F}_w^\sharp(k) + 3$ . Analogously,  $\mathcal{F}_w^\flat(0) = \max\{k | \mathcal{F}_w(k) \leq 0\} = 0$ ,  $\mathcal{F}_w^\flat(1) = 0$ ,  $\mathcal{F}_w^\flat(2) = 1$ ,  $\mathcal{F}_w^\flat(3) = 2$ , and then  $\mathcal{F}_w^\flat(k+4) = \mathcal{F}_w^\flat(k) + 3$ . The realization (see Cor.1) of these C/C functions by periodic operators gives  $w^\sharp = \gamma^0\mu_3\beta_4\gamma^3 \oplus \gamma^1\mu_3\beta_4\gamma^2 \oplus \gamma^1\mu_3\beta_4\gamma^1 \oplus \gamma^2\mu_3\beta_4$ , which can be simplified to  $w^\sharp = \mu_3\beta_4\gamma^3 \oplus (\gamma^1\mu_3\beta_4)(\gamma^2 \oplus \gamma^1) \oplus \gamma^2\mu_3\beta_4 = \mu_3\beta_4\gamma^3 \oplus \gamma^1\mu_3\beta_4\gamma^1 \oplus \gamma^2\mu_3\beta_4$  since  $\gamma^1 \oplus \gamma^2 = \gamma^{\min(1,2)}$  (see (4)). In a similar way,  $w^\flat = \gamma^0\mu_3\beta_4\gamma^3 \oplus \gamma^0\mu_3\beta_4\gamma^2 \oplus \gamma^1\mu_3\beta_4\gamma^1 \oplus \gamma^2\mu_3\beta_4 = \mu_3\beta_4\gamma^2 \oplus \gamma^1\mu_3\beta_4\gamma^1 \oplus \gamma^2\mu_3\beta_4$ .

*Remark 5:* Finding  $\mathcal{F}_w^\sharp(k_o)$  for each  $k_o \in \mathbb{Z}_{\min}$  is equivalent to find a function in  $\mathbf{C}_{per}$  defined by  $\mathcal{F}_w^\sharp = \bigoplus \{g \in \mathbf{C}_{per} | \mathcal{F}_w \circ g \preceq \text{Id}\}$ . Due to the isomorphism between  $\mathbf{C}_{per}$  and  $\mathcal{E}_{per}$ , this problem is equivalent to find the greatest operator  $x \in \mathcal{E}_{per}$  such that  $wx \preceq e$  where  $e$  is the identity operator. This problem is clearly linked to the residuation of the product in  $\mathcal{E}_{per}$  (see Th.1). The greatest solution in  $\mathcal{E}_{per}$  is obtained by  $w \backslash e$ . Therefore,  $\mathcal{F}_w^\sharp$  is the C/C function associated to operator  $w \backslash e$ .

*Definition 7 (Function  $\mathcal{D}_w$ ):* Let  $w \in \mathcal{E}_{per}$ . We define  $\mathcal{D}_w : \mathbb{Z}_{\min} \rightarrow \mathbb{N}$  by

$$\mathcal{D}_w(k_i) \triangleq \mathcal{F}_w(k_i + 1) - \mathcal{F}_w(k_i).$$

The function  $\mathcal{D}_w$  gives a differential analysis of the function  $\mathcal{F}_w$ . Let us remark that since  $\mathcal{F}_w$  is a non-decreasing function, then  $\mathcal{D}_w(k_i) \geq 0$  but  $\mathcal{D}_w$  is not anymore monotonous. Moreover, when  $w \in \mathcal{E}_{per}$  is a  $(n, n')$ -periodic E-operator, then  $\mathcal{F}_w$  is  $(n, n')$ -periodic too and  $\mathcal{D}_w$  is a  $n$ -periodic sequence, i.e.,  $\mathcal{D}_w(k_i + n) = \mathcal{D}_w(k_i)$ .

*Notation 2 (Periodic sequence  $\langle s \rangle_n$ ):* We denote by

$$\langle s \rangle_n = \langle s(0), \dots, s(n-1) \rangle$$

a  $n$ -periodic sequence  $s : \mathbb{Z} \rightarrow \mathbb{N}, k \mapsto s(k)$  where  $\forall k \in \mathbb{Z}, s(k+n) = s(k)$ .

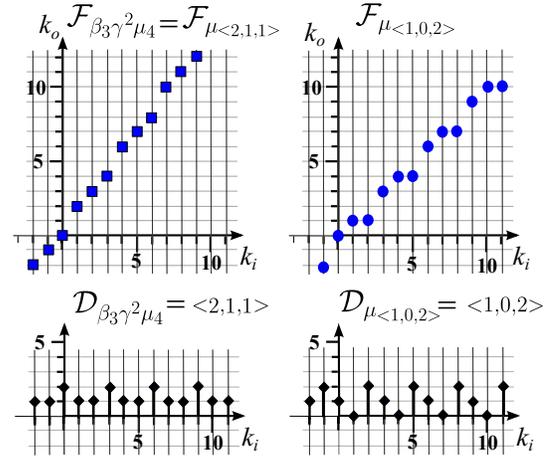


Figure 7. Graphical representation of  $\mathcal{D}_{\beta_3\gamma^2\mu_4}$  and  $\mathcal{F}_{\mu_{\langle 1,0,2 \rangle}}$

If  $\mathcal{F}_w$  is a  $(n, n')$ -periodic C/C function then  $\mathcal{D}_w = \langle s \rangle_n$  is a  $n$ -periodic sequence.

*Example 5:* Let us consider the operator  $w = \beta_3\gamma^2\mu_4$  the C/C function of which is depicted on Fig.7. We have  $\mathcal{F}_w(k_i) = \lfloor (4 \times k_i + 2)/3 \rfloor$ , therefore  $\mathcal{F}_w(k_i + 3) = 4 + \mathcal{F}_w(k_i)$ , with  $\mathcal{F}_w(0) = 0, \mathcal{F}_w(1) = 2$ , and  $\mathcal{F}_w(2) = 3$ . Therefore,  $\mathcal{D}_{\beta_3\gamma^2\mu_4}$  is the 3-periodic sequence  $\langle 2, 1, 1 \rangle$ .

*Remark 6:* A  $(n, n')$ -periodic C/C function  $\mathcal{F}_w$  can be defined by its values on a period, or by its value  $\mathcal{F}_w(0)$  and the  $n$ -periodic sequence  $\mathcal{D}_w = \langle s \rangle_n$ .

### III. CYCLO-WEIGHTED TIMED EVENT GRAPHS

#### A. Weight-Variable Operators

All the event operators arising in WB-TEGs are periodic and can be described by their associated periodic C/C functions. Conversely, we can always realize a periodic C/C function by a finite combination of basic periodic E-operators  $\gamma^n, \mu_m$  and  $\beta_b$  (see Cor.1), or equivalently by a (non timed) WB-TEG.

We propose here to reconsider periodic E-operators as *weight-variable* operators. This means that the weight of an operator, such as  $\mu_m$  or  $\beta_b$ , can change after the occurrence of an event, according to a periodic sequence. Therefore we introduce new operators denoted  $\mu_{\langle m \rangle_n}$  and  $\beta_{\langle b \rangle_{n'}}$  which are defined by periodic sequences of weights. For instance, Fig.8 illustrates the behavior of the weight-variable operators  $\mu_{\langle 1,3,2 \rangle}$  and  $\beta_{\langle 1,2 \rangle}$ . For  $\mu_{\langle 1,3,2 \rangle}$ , the multiplier weight changes after each occurrence of an event according to the sequence  $\langle 1, 3, 2 \rangle$ . For  $\beta_{\langle 1,2 \rangle}$ , the consumption weight changes according to the sequence  $\langle 1, 2 \rangle$ .

For this purpose, we consider the practical meaning of function  $\mathcal{D}_w$  (see Def.7). For a given E-operator  $w \in \mathcal{E}_{per}$ ,  $\mathcal{D}_w(k_i)$  gives the number of output events instantaneously released when the input event number  $k_i$  occurs. For instance,  $\mathcal{D}_{\beta_3\gamma^2\mu_4} = \langle 2, 1, 1 \rangle$  (see Fig.7). Therefore, operator  $\beta_3\gamma^2\mu_4$  is equivalent to a weight-variable multiplier where the weight changes according to the sequence  $\langle 2, 1, 1 \rangle$ . The first occurrence of an input event releases 2 output events. For the first event, the weight is 2. The next two ones induce only 1 output event each. The weight of the variable multiplier

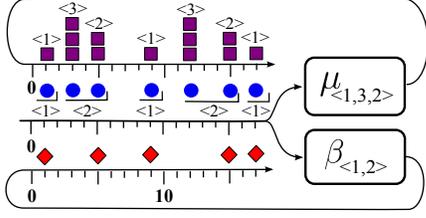


Figure 8. Weight-variable operators

is 1 for the next two events. This sequence is periodically repeated for the next occurrences of input events. We will denote  $\beta_3\gamma^2\mu_4 = \mu_{\langle 2,1,1 \rangle}$  to express this behavior.

**Definition 8 (Operator  $\mu_{\langle m \rangle_n}$ ):** We define the weight-variable multiplier  $\mu_{\langle m \rangle_n} \in \mathcal{E}_{per}$  as the operator  $w$  such that the input event number  $k_i$  induces instantaneously  $m(k_i)$  output events, and such that  $\mathcal{F}_w(0) = 0$ .

$$\mu_{\langle m \rangle_n} \triangleq w \in \mathcal{E}_{per}, \mathcal{D}_w = \langle m \rangle_n \text{ and } \mathcal{F}_w(0) = 0.$$

Fig.7 gives an illustration of the C/C function associated to  $\mu_{\langle 1,0,2 \rangle}$ . By definition, this E-operator is defined such that  $\mathcal{D}_{\mu_{\langle 1,0,2 \rangle}} = \langle 1,0,2 \rangle$ .

**Proposition 4 (Realization of  $\mu_{\langle m \rangle_n}$ ):** Let  $\langle m_0, \dots, m_{n-1} \rangle$  be a  $n$ -periodic sequence of weights, with  $m_i \in \mathbb{N}$ . By setting  $\nu = \sum_{i=0}^{n-1} m_i$ , we have

$$\begin{aligned} \mu_{\langle m \rangle_n} &= \gamma^0 \mu_\nu \beta_n \gamma^{n-1} \oplus \gamma^{m_0} \mu_\nu \beta_n \gamma^{n-2} \\ &\oplus \gamma^{m_0+m_1} \mu_\nu \beta_n \gamma^{n-3} \oplus \dots \\ &\oplus \gamma^{m_0+\dots+m_{n-2}} \mu_\nu \beta_n. \end{aligned}$$

*Proof:* According to the definition of  $\mathcal{D}_w$  (see Def.7), we can write  $\mathcal{F}_w(k_i+1) = \mathcal{F}_w(k_i) + \mathcal{D}_w(k_i)$ . Then, by denoting  $\langle m \rangle_n = \langle m_0, \dots, m_{n-1} \rangle$  the sequence of weights, the C/C function  $\mathcal{F}_{\mu_{\langle m \rangle_n}}$  is defined by

$$\begin{aligned} \mathcal{F}_{\mu_{\langle m \rangle_n}}(0) &= 0, \\ \mathcal{F}_{\mu_{\langle m \rangle_n}}(1) &= \mathcal{F}_{\mu_{\langle m \rangle_n}}(0) + \mathcal{D}_{\mu_{\langle m \rangle_n}}(0) = m_0, \\ \mathcal{F}_{\mu_{\langle m \rangle_n}}(2) &= \mathcal{F}_{\mu_{\langle m \rangle_n}}(1) + \mathcal{D}_{\mu_{\langle m \rangle_n}}(1) = m_0 + m_1, \\ &\dots, \end{aligned}$$

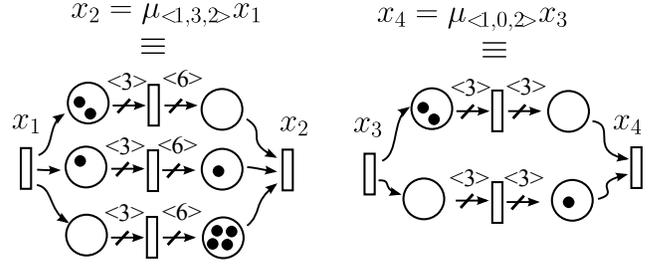
$$\mathcal{F}_{\mu_{\langle m \rangle_n}}(n-1) = m_0 + \dots + m_{n-2}.$$

Finally, we can obtain the realization of this periodic C/C function  $\mathcal{F}_{\mu_{\langle m \rangle_n}}$  by applying Prop.2. ■

This proposition means that we can always realize a weight-variable operator  $\mu_{\langle m \rangle_n}$  by a Weight-Balanced TEG.

**Example 6:** By applying Prop.4 for  $\mu_{\langle 1,3,2 \rangle}$  (see Fig.9), we have  $\nu = 1 + 3 + 2 = 6$ ,  $n = 3$  and then  $\mu_{\langle 1,3,2 \rangle} = \gamma^0 \mu_6 \beta_3 \gamma^2 \oplus \gamma^1 \mu_6 \beta_3 \gamma^1 \oplus \gamma^{1+3} \mu_6 \beta_3$ . For the operator  $\mu_{\langle 1,0,2 \rangle}$  (see Fig.7), we obtain  $\mu_{\langle 1,0,2 \rangle} = \gamma^0 \mu_3 \beta_3 \gamma^2 \oplus \gamma^1 \mu_3 \beta_3 \gamma^1 \oplus \gamma^{1+0} \mu_3 \beta_3 \gamma^0$  which can be simplified to  $\mu_{\langle 1,0,2 \rangle} = \mu_3 \beta_3 \gamma^2 \oplus \gamma^1 \mu_3 \beta_3$  because  $\gamma^1 \mu_3 \beta_3 \gamma^1 \oplus \gamma^1 \mu_3 \beta_3 \gamma^0 = (\gamma^1 \mu_3 \beta_3)(\gamma^1 \oplus \gamma^0)$  and  $\gamma^1 \oplus \gamma^0 = \gamma^0$  (see eq. (4)).

The interpretation of a periodic E-operator  $w$  as a weight-variable multiplier is direct due to the associated function  $\mathcal{D}_w$ . Now, we will express an operator as a weight-variable batch operator. The duality between multiplier and batch operators will be expressed as follows. We first introduce a new C/C function which describes, for a given E-operator, how many input events are necessary to obtain at least  $k_o$  output events. This C/C function that maps an output counter value to an

Figure 9. Realization of  $\mu_{\langle 1,3,2 \rangle}$  and  $\mu_{\langle 1,0,2 \rangle}$ 

input counter value is simply  $\mathcal{F}_w^\# = \mathcal{F}_{w^\#}$  (see Prop.3). Indeed, function  $\mathcal{F}_{w^\#}(k_o)$  gives the minimal number of (cumulated) input events necessary for the E-operator  $w$  to induce (at least)  $k_o$  output events. By considering then the  $\mathcal{D}_{w^\#}$  function,

$$\begin{aligned} \mathcal{D}_{w^\#}(k_o) &= \mathcal{F}_{w^\#}(k_o+1) - \mathcal{F}_{w^\#}(k_o) \\ &= \mathcal{F}_w^\#(k_o+1) - \mathcal{F}_w^\#(k_o), \end{aligned}$$

we obtain that for operator  $w$ , the output event number  $k_o$  consumes  $\mathcal{D}_{w^\#}(k_o)$  input events. If  $w \in \mathcal{E}_{per}$ , the sequence  $\mathcal{D}_{w^\#} = \langle b \rangle_{n'}$  is periodic. Then operator  $w$  can be considered as a batch operator the consumption weight of which can vary according to the sequence  $\langle b \rangle_{n'}$ .

**Definition 9 (Operator  $\beta_{\langle b \rangle_{n'}}$ ):** For a sequence  $\langle b \rangle_{n'}$ , the weight-variable batch operator  $\beta_{\langle b \rangle_{n'}}$  is defined as

$$\beta_{\langle b \rangle_{n'}} \triangleq w \in \mathcal{E}_{per}, \mathcal{D}_{w^\#} = \langle b \rangle_{n'} \text{ and } \mathcal{F}_{w^\#}(0) = 0.$$

Since the residuation builds a bridge between the expression of weight-variable multipliers and weight-variable batches, we can obtain the realization of a batch operator  $\beta_{\langle b \rangle_{n'}}$  in the following way. First we look for an operator  $w$  such that  $\mathcal{D}_{w^\#} = \langle b \rangle_{n'}$ . We define  $w^\# = \mu_{\langle b \rangle_{n'}}$  (see Prop.2 for its realization) and then  $w = (w^\#)^b$ .

**Example 7:** As an example, we develop here the expression of a weight-variable multiplier as an equivalent weight-variable batch, and conversely. Let us consider the operator  $\mu_{\langle 1,0,2 \rangle}$ . We obtained (see Ex.6) that  $\mu_{\langle 1,0,2 \rangle} = \mu_3 \beta_3 \gamma^2 \oplus \gamma^1 \mu_3 \beta_3$ , or equivalently  $\mathcal{F}_{\mu_{\langle 1,0,2 \rangle}}(0) = 0, \mathcal{F}_{\mu_{\langle 1,0,2 \rangle}}(1) = 1, \mathcal{F}_{\mu_{\langle 1,0,2 \rangle}}(2) = 1$  and  $\mathcal{F}_{\mu_{\langle 1,0,2 \rangle}}(k_i+3) = \mathcal{F}_{\mu_{\langle 1,0,2 \rangle}}(k_i) + 3$ . By applying Prop.3,  $\mathcal{F}_{\mu_{\langle 1,0,2 \rangle}}^\#$  is defined by

$$\begin{aligned} \mathcal{F}_{\mu_{\langle 1,0,2 \rangle}}^\#(0) &= 0, \mathcal{F}_{\mu_{\langle 1,0,2 \rangle}}^\#(1) = 1, \mathcal{F}_{\mu_{\langle 1,0,2 \rangle}}^\#(2) = 3, \\ \mathcal{F}_{\mu_{\langle 1,0,2 \rangle}}^\#(k_i+3) &= \mathcal{F}_{\mu_{\langle 1,0,2 \rangle}}^\#(k_i) + 3. \end{aligned}$$

The realization of this C/C function is the operator  $\mu_3 \beta_3 \gamma^2 \oplus \gamma^1 \mu_3 \beta_3 \gamma^1$  which is  $(\mu_{\langle 1,0,2 \rangle})^\#$ . For this E-operator, we obtain  $\mathcal{D}_{(\mu_{\langle 1,0,2 \rangle})^\#} = \langle 1,2,0 \rangle$ , which is finally equivalent to say that  $\mu_{\langle 1,0,2 \rangle} = \beta_{\langle 1,2,0 \rangle}$ : the first output event of  $\mu_{\langle 1,0,2 \rangle}$  consumes 1 input event and the next one consumes 2 input events. Finally, the third output event consumes no input events. With the earliest firing rule, it means that the third output event is released as soon as possible after (actually simultaneously with) the second output.

Conversely, we can express for instance  $\beta_{\langle 1,2 \rangle}$  (see Fig.8 for its behavior) as a weight-variable multiplier. First, we set  $w^\# = \mu_{\langle 1,2 \rangle} = \mu_3 \beta_2 \gamma^1 \oplus \gamma^1 \mu_3 \beta_2$  (due to Prop.4). Therefore, the corresponding operator  $w$  is  $w = (w^\#)^b =$

$\mu_2\beta_3\gamma^2 \oplus \gamma^1\mu_2\beta_3$ . Since  $\mathcal{F}_{\mu_2\beta_3\gamma^2 \oplus \gamma^1\mu_2\beta_3}(0) = 0$  and  $\mathcal{D}_{\mu_2\beta_3\gamma^2 \oplus \gamma^1\mu_2\beta_3} = \langle 1, 0, 1 \rangle$ , then we obtain  $\beta_{\langle 1, 2 \rangle} = \mu_{\langle 1, 0, 1 \rangle}$ .

Since  $\mu_{\langle m \rangle}$  and  $\beta_{\langle b \rangle}$  are periodic operators in  $\mathcal{E}_{per}$ , we can deduce their gain from the sequence of weights.

**Proposition 5 (Gain of  $\mu_{\langle m \rangle}$  and  $\beta_{\langle b \rangle}$ ):** For weight-variable operators we have

$$\Gamma(\mu_{\langle m \rangle_n}) = \frac{\sum_{i=0}^{n-1} m(i)}{n}$$

$$\Gamma(\beta_{\langle b \rangle_{n'}}) = \frac{n'}{\sum_{i=0}^{n'-1} b(i)}$$

*Proof:* Direct from Prop.4 and Prop.3. ■

### B. TEG with cyclo-weighted edges

In order to give a Petri net correspondance to weight-variable operators, we define the class of Cyclo-Weighted Timed Event Graphs (CW-TEGs) as the class of Weighted TEGs for which the edges can be valued by constant or periodic sequences of weights. This extension of Weighted TEGs is an immediate transposition of Cyclo-Static Dataflow Graphs used to model specific dataflow process networks [18]. For CW-TEGs, the number of tokens produced or consumed by a transition can change after it is fired. It means that for a given place  $p \in P$ , the input and the output weights  $\omega_i(p) = \langle W_i \rangle_m$  and  $\omega_o(p) = \langle W_o \rangle_b$  are now defined as periodic sequences. The first firing of the input transition of  $p$  adds  $W_i(0)$  tokens to  $p$ , the next one adds  $W_i(1)$  tokens, and so on according to the sequence  $\langle W_i \rangle_m$ . Conversely, the first firing of the output transition of  $p$  consumes  $W_o(0)$  tokens from  $p$ , the next one consumes  $W_o(1)$  tokens and so on. For the CW-

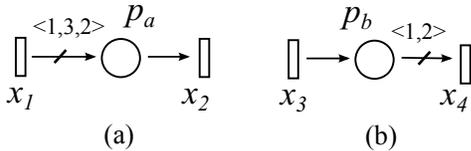


Figure 10. Edges with periodic weights

TEG depicted in Fig.10(a), we have  $\omega_i(p_a) = \langle 1, 3, 2 \rangle$ . It means that the first firing of transition  $x_1$  adds one token to  $p_a$  and induces instantaneously 1 firing of transition  $x_2$ . The second one induces 3 firings of transition  $x_2$ , the third induces 2 firings and so on. When no number is shown, the weight is implicitly a constant weight of 1,  $\omega_o(p_a) = \langle 1 \rangle$ . From an operatorial point of view,  $x_2 = \mu_{\langle 1, 3, 2 \rangle} x_1$ . For Fig.10(b), we have  $\omega_o(p_b) = \langle 1, 2 \rangle$ . The first firing of transition  $x_3$  adds one token to place  $p_b$  that can be consumed immediately by the firing of transition  $x_4$ , since the first weight of  $\omega_o(p_b)$  is 1. But then two supplementary firings of  $x_3$  are required to allow the next firing of  $x_4$ , because the second firing of  $x_4$  consumes 2 tokens. The behavior of the system between  $x_3$  and  $x_4$  is described by the operator  $x_4 = \beta_{\langle 1, 2 \rangle} x_3$ .

Due to section III-A, it is clear that a CW-TEG can always be described by a finite number of operators among the operators  $\gamma^\nu$ ,  $\delta^\tau$  and the weight-variable operators  $\mu_{\langle m \rangle}$  and

$\beta_{\langle b \rangle}$ . By extension, a CW-TEG will be said to be *weight-balanced* if for all pair of transitions  $t_a, t_b \in T$ , all paths from  $t_a$  to  $t_b$  have the same gain. As for Weighted TEGs, a Cyclo-Weighted TEG is weight-balanced only if all circuits have a gain of 1.

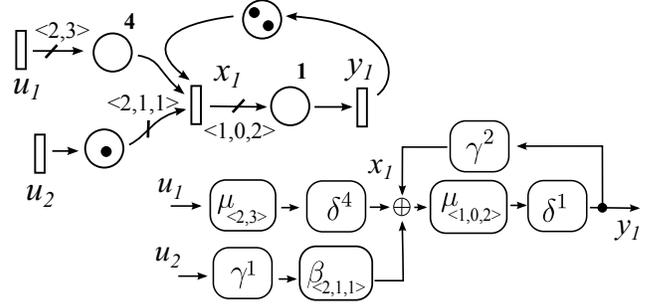


Figure 11. Cyclo-Weighted TEG

Fig.11 gives an example of a Weight-Balanced CW-TEG. Let us remark that the gain of the circuit containing  $x_1$  and  $y_1$  is  $\Gamma(\mu_{\langle 1, 0, 2 \rangle}) = 3/3 = 1$  (cf. Prop.5), the gain of the path from  $u_1$  to  $y_1$  is  $\Gamma(\mu_{\langle 2, 3 \rangle}) = 5/2$ , and the gain of the path from  $u_2$  to  $y_1$  is  $\Gamma(\beta_{\langle 2, 1, 1 \rangle}) = 3/4$ . This CW-TEG is equivalently described as a block diagram with a finite number of operators and synchronizations.

**Proposition 6:** A Weight-Balanced Cyclo-Weighted Timed Event Graph can be described by an equivalent Weight-Balanced TEG.

*Proof:* Direct since every weight-variable operator can be realized by a WB-TEG. ■

The previous proposition indicates that Weight-Balanced CW-TEGs are not more expressive than WB-TEGs. They can express exactly the same kind of phenomena. But, as presented in the next section, CW-TEGs can describe some phenomena in a more intuitive way or in a more compact form. As a consequence, the behavior of Weight-Balanced CW-TEGs can be described by transfer functions expressed as rational expressions with operators  $\gamma^\nu$ ,  $\delta^\tau$ ,  $\mu_m$  and  $\beta_b$ .

**Example 8:** For the CW-TEG of Fig.11, we have the next equivalences

$$\begin{aligned} \mu_{\langle 2, 3 \rangle} &= \mu_5 \beta_2 \gamma^1 \oplus \gamma^2 \mu_5 \beta_2 \\ \mu_{\langle 2, 1, 1 \rangle} \gamma^1 &= \mu_3 \beta_4 \gamma^3 \oplus \gamma^1 \mu_3 \beta_4 \gamma^2 \oplus \gamma^2 \mu_3 \beta_4 \gamma^1 \\ \mu_{\langle 1, 0, 2 \rangle} &= \mu_3 \beta_3 \gamma^2 \oplus \gamma^1 \mu_3 \beta_3 \end{aligned}$$

It means that the WB-TEG depicted in Fig.12 is equivalent to the CW-TEG of Fig.11. It means that by operating under the earliest firing rule, the same inputs  $u_1$  and  $u_2$  will lead to the same output  $y_1$  for both graphs. Replacing cyclo-weighted edges by constant weight edges only requires to introduce supplementary internal transitions. It is then possible to obtain the input-output transfer matrix of the CW-TEG of Fig.11 which is described by the next transfer function matrix

$$y_1 = H \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with

$$\begin{aligned} H_{11} &= (\gamma^3 \delta^2)^* (\mu_{\langle 1, 3, 3, 3, 2, 3 \rangle} \delta^5 \oplus \gamma^1 \mu_{\langle 2, 3, 3, 3, 1, 3 \rangle} \delta^6), \\ H_{12} &= (\gamma^3 \delta^2)^* (\mu_{\langle 1, 0, 2, 0 \rangle} \delta^1 \oplus \gamma^1 \mu_{\langle 1, 0, 2, 0 \rangle} \delta^2). \end{aligned}$$

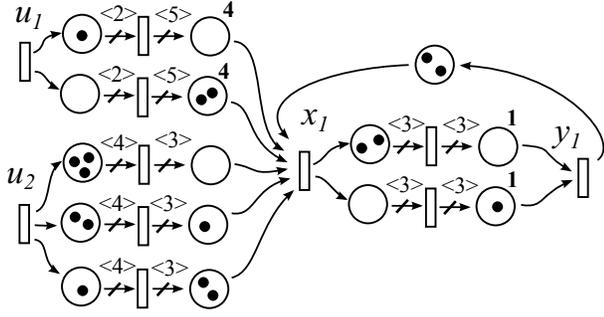


Figure 12. WB-TEG equivalent to the CWTEG of Fig.11

For this 2-input 1-output system, the (event) periodicity is expressed by

$$H_{11}\gamma^6 = \gamma^{15}H_{11} \quad H_{12}\gamma^4 = \gamma^3H_{12},$$

or  $\Gamma(H_{11}) = 15/6 = 5/2$  and  $\Gamma(H_{12}) = 3/4$ .

Finally, the commutation of a weight-variable operator with the  $\gamma^1$  operator can be interpreted as a shift in the sequence of weights.

*Proposition 7:* Let  $\langle m_0, \dots, m_{n-1} \rangle$  and  $\langle b_0, \dots, b_{n'-1} \rangle$  be two sequences of weights.

$$\mu_{\langle m_0, \dots, m_{n-1} \rangle} \gamma^1 = \gamma^{m_0} \mu_{\langle m_1, \dots, m_{n-1}, m_0 \rangle}$$

$$\beta_{\langle b_0, b_1, \dots, b_{n'-1} \rangle} \gamma^{b_0} = \gamma^1 \beta_{\langle b_1, \dots, b_{n'-1}, b_0 \rangle}$$

*Proof:* These formal equalities can be obtained by analyzing the corresponding CW-TEGs with the earliest firing rule. In Fig.13, for the CW-TEG given on the left (a realization of  $\mu_{\langle m_0, \dots, m_{n-1} \rangle} \gamma^1$ ), transition  $t_1$  is enabled at date  $-\infty$ . Therefore,  $t_1$  is fired at date  $-\infty$  and adds  $m_0$  tokens in its output place. In this new state, the next weight to apply is now  $m_1$ . It operates a shift on the sequence of weights. The situation is comparable (see CW-TEG on the right) for weight-variable batch operators. ■

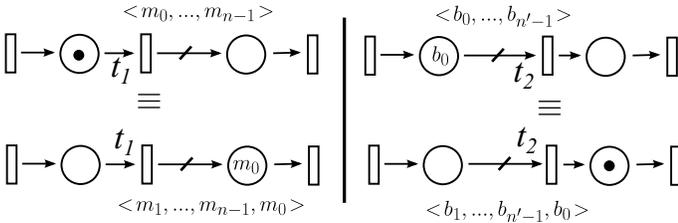


Figure 13. Equivalences

*Example 9:* For instance, we have:  $\mu_{\langle 2,1,3 \rangle} \gamma^2 = \gamma^2 \mu_{\langle 1,3,2 \rangle} \gamma^1 = \gamma^3 \mu_{\langle 3,2,1 \rangle}$ . Similarly,  $\gamma^1 \beta_{\langle 2,1,3 \rangle} = \beta_{\langle 3,2,1 \rangle} \gamma^3$ .

*Corollary 2:* Any periodic operator in  $\mathcal{E}_{per}$  can be written as  $\gamma^\nu \mu_{\langle m \rangle_n}$  or  $\beta_{\langle b \rangle_{n'}} \gamma^{\nu'}$ .

Note that we can express any periodic E-operator either as a sum of basic operators  $\gamma^{n_i} \mu_m \beta_b \gamma^{n'_i}$  (as developed in [5]), or as a specific weight-variable multiplier  $\gamma^\nu \mu_{\langle m \rangle_n}$ , or as a weight-variable batch operator  $\beta_{\langle b \rangle_{n'}} \gamma^{\nu'}$ .

#### IV. MODELING WITH CYCLO-WEIGHTED TEGS

We give here some modeling issues that can be handled by Cyclo-Weighted TEGs, or equivalently by Weight-Balanced TEG. The interest of Cyclo-Weighted TEGs is to express some routing problems in a more natural way.

##### A. Routing and multiplexing parts

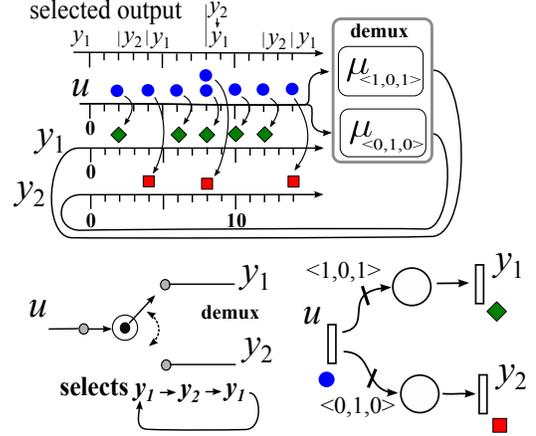


Figure 14. Demultiplexer modeled by a CW-TEG

1) *Routing (demultiplexing):* In manufacturing systems, the flow of parts is sometimes split in order to feed different subsystems, for instance different production cells. In this case, the routing policy can be defined by a periodic dispatching rule. Fig.14 illustrates how a weight-variable multiplier can be used to select some events into a given signal. It shows that operators  $\mu_{\langle 1,0,1 \rangle}$  and  $\mu_{\langle 0,1,0 \rangle}$  can be interpreted as event sampling operators. Signal  $\mu_{\langle 0,1,0 \rangle} u$  corresponds to one event out of three among events  $u$ , more exactly the events the occurrence number of which is  $3p+1$  with  $p \in \mathbb{N}$ . Signal  $\mu_{\langle 1,0,1 \rangle} u$  is complementary and selects only the events  $u$  the occurrence number of which is  $3p$  and  $3p+2$  with  $p \in \mathbb{N}$ . Used together, these operators model a system able to route instantaneously some events on one way and the other ones on the other way, with a 3-periodic rule. This system can be compared to a demultiplexer (demux) where events are separated according to their occurrence number. At a given step, input  $u$  is connected to one and only one output and the selected output can toggle to another output ( $y_1$  or  $y_2$ ) only as a result of an input event, according to the 3-periodic sequence  $y_1 \rightarrow y_2 \rightarrow y_1$ . This behavior is different from a time-division demultiplexer since the selected output is switched on events and not based on a time driven mechanism. In practice, the system described jointly by  $\mu_{\langle W_1 \rangle}$  and  $\mu_{\langle W_2 \rangle}$  corresponds to a periodic routing (or demultiplexer) if  $\langle W_1 \rangle + \langle W_2 \rangle = \langle 1, 1, \dots, 1 \rangle = \langle 1 \rangle$ , which means that each input event is dispatched into one and only one output.

2) *Multiplexing:* The role of a multiplexer (mux) is to recombine different input signals into only one output signal. By symmetry with the problem studied previously, such a system is defined here by synchronizing different weight-variable batch operators. Let us analyse the signal  $y =$

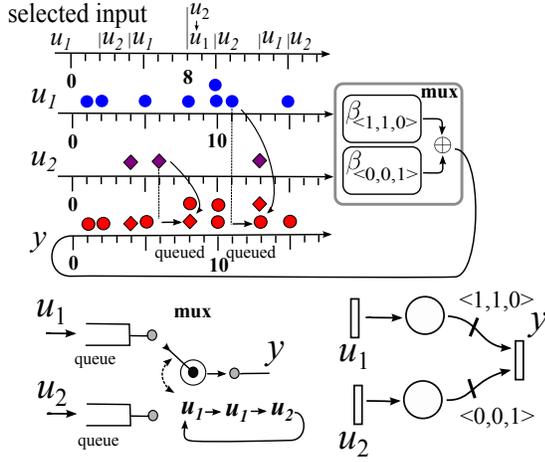


Figure 15. Multiplexer with two inputs modeled by a CW-TEG.

$\beta_{\langle 1,1,0 \rangle} u_1 \oplus \beta_{\langle 0,0,1 \rangle} u_2$  (see Fig.15) with  $u_1$  and  $u_2$  two input signals. The first and the second event  $y$  depend only on the first two events  $u_1$ , and the third event  $y$  depends only on the first event  $u_2$ . More generally, the output events number  $3p$  and  $3p + 1$ , with  $p \in \mathbb{N}$ , depend on events  $u_1$  while the output events number  $3p+2$  depend on events  $u_2$ . This system is comparable to a multiplexer where only one input ( $u_1$  or  $u_2$ ) is connected to the output  $y$  and the selected input can toggle to another input ( $u_1$  or  $u_2$ ) according to the 3-periodic sequence  $u_1 \rightarrow u_1 \rightarrow u_2 \rightarrow \dots$ . It is worth noting that an input event ( $u_1$  or  $u_2$ ) occurring on an input which is not currently selected is queued until the next selection of this input. For example on Fig.15, the second event  $u_2$  can not induce immediately an output event because at this step of the multiplexer, the selected input is  $u_1$ . This event is therefore queued until the fourth event  $u_1$  at date 8. And at date 8, the selected input changes twice (to finally select again input  $u_1$ ). Analogously, the seventh event  $u_1$  is queued until the third event  $u_2$ .

3) *Multiplexing vs. merging*: The multiplexer described previously is a specific way of mixing different input signals into one output signal. In the original sense, merging two kinds of signal consists in adding their events as illustrated by the block *merge* in Fig.16. When signals are given by counter functions, we have  $\forall t$ ,  $\text{Merge}(a, b)(t) = a(t) + b(t)$ . In other words, each input event  $a$  or  $b$  induces instantaneously one output event, which is not necessarily true for the multiplexer. Therefore, for a multiplexer defined by  $\beta_{\langle W_1 \rangle} a \oplus \beta_{\langle W_2 \rangle} b$ , with  $\langle W_1 \rangle + \langle W_2 \rangle = \langle 1 \rangle$ , we have  $\forall t$ ,  $\text{Merge}(a, b)(t) \geq (\beta_{\langle W_1 \rangle} a \oplus \beta_{\langle W_2 \rangle} b)(t)$ . Let us note that the standard merging operation can be described by a Petri net (with a place with two upstream transitions), as described in Fig.16, whereas it can not be described by a CW-TEG. The main difference of behavior is that a multiplexer can lead to a deadlock. Indeed, if one of the input signals ( $a$  or  $b$ ) stops, then the system is blocked because of the output synchronization. This is not the case for the classical merging.

4) *Example of a flow split into parallel TEGs*: As an application, we consider here the model of a SISO production

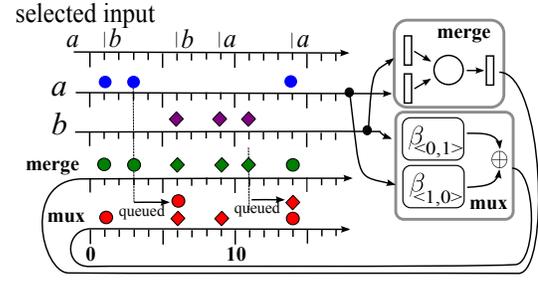


Figure 16. Merging vs. multiplexing two inputs

cell where raw parts are separated and processed into two different parallel subsystems and the processed parts are then multiplexed into a single output. The subsystems are modeled by ordinary TEGs<sup>3</sup>, the input flow is split by a demultiplexer (with a periodic rule) and the output flows are recombined by a multiplexer (with the same rule). The system is depicted in Fig.17. The demultiplexer dispatches the input events according to the 5-periodic sequence  $u_1 \rightarrow u_2 \rightarrow u_1 \rightarrow u_2 \rightarrow u_2$ . The whole system is described by a CW-TEG and can be described by a transfer function in dioid  $\mathcal{E}[\delta]$ . First, we can express:

$$u_1 = \mu_{\langle 1,0,1,0,0 \rangle} u, \quad u_2 = \mu_{\langle 0,1,0,1,1 \rangle} u, \\ y_1 = H_1 u_1, \quad y_2 = H_2 u_2,$$

$$y = \beta_{\langle 1,0,1,0,0 \rangle} y_1 \oplus \beta_{\langle 0,1,0,1,1 \rangle} y_2,$$

with  $H_1 = \delta^4 (\gamma^2 \delta^3)^*$  and  $H_2 = (\delta^5 \oplus \gamma^2 \delta^6) (\gamma^4 \delta^4)^*$  the transfer functions of the subsystems. Finally, the input-output behavior is given by  $y = G_a u$ , with  $G_a \in \mathcal{E}[\delta]$  given by

$$G_a = (\gamma^{20} \delta^{12})^* (\gamma^0 \delta^4 \oplus \gamma^1 \mu_{\langle 0,2,0,1,2 \rangle} \delta^5 \\ \oplus \gamma^4 \mu_{\langle 0,2,0,2,1 \rangle} \delta^6 \oplus \gamma^5 \mu_{\langle 2,0,2,1,0 \rangle} \delta^7 \\ \oplus \gamma^8 \mu_{\langle 0,1,0,2,2 \rangle} \delta^9 \oplus \gamma^{10} \delta^{10} \\ \oplus \gamma^{14} \mu_{\langle 0,2,0,2,1 \rangle} \delta^{13} \oplus \gamma^{18} \mu_{\langle 0,1,0,2,2 \rangle} \delta^{14})$$

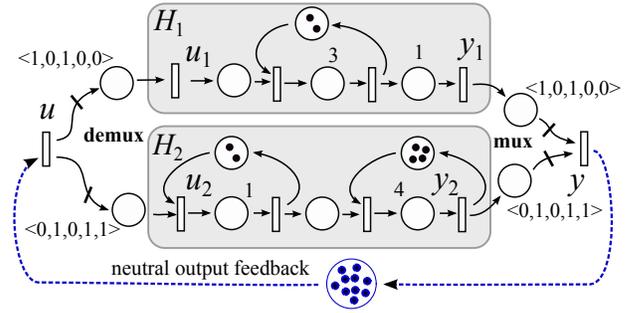


Figure 17. Demux/Mux with parallel Timed Event Graphs

*Remark 7 (Feedback control)*: On the basis of this transfer relation, it is possible to apply some results on controller synthesis. For instance, by applying [14], we can compute an optimal *neutral feedback loop*, i.e. an output feedback such that the closed loop and the open loop behavior are the same.

<sup>3</sup>but we could deal as well with systems modeled by WB-TEGs or CW-TEGs

This feedback is obtained by  $F_a = G_a \setminus G_a / G_a$ . Applied to the system of Fig.17, we can show that the output feedback with an initial marking of 10 tokens ( $F = \gamma^{10}$ ) is neutral (this feedback is depicted with dashed lines in Fig.17), and it is the minimum number of tokens to put in the feedback to keep the behavior unchanged. This feedback leads to bound the Work In Process in the production cell without decreasing its throughput. For manufacturing systems, this computation is useful for instance to estimate the number of transportation means (pallets) in a cell, or to make a flow regulation by limiting the size of buffers into a cell.

*Remark 8:* The system depicted in Fig.17 is similar to a system studied in [17]. The difference in [17] is that output flows are mixed with a classical merging operation, such as depicted in Fig.18. And subsystems are necessarily  $(\min, +)$  linear systems (ordinary TEGs) described by their transfer function, for instance in  $\mathcal{M}_{in}^{ax}[\gamma, \delta]$ . In this case, the system is not  $(\min, +)$  linear anymore, it cannot be described neither by an ordinary TEG nor by a Weighted TEG. Nevertheless, the authors show in [17] that, if  $H_1$  and  $H_2$  are TEGs, then the behavior can be framed by two  $(\min, +)$ -linear systems denoted respectively  $\underline{H}$  and  $\overline{H}$ , with  $\underline{H} \preceq \overline{H}$ , and  $\underline{H}, \overline{H} \in \mathcal{M}_{in}^{ax}[\gamma, \delta]$ . It means that for a given input sequence  $u$ , the output  $z$  is in the interval  $[\underline{y}, \overline{y}]$  where signals are given by  $\underline{y} = \underline{H}u$  and  $\overline{y} = \overline{H}u$ . The multiplexing operation is more complex to implement on an automated system than the merging operation, but the benefit is to keep the possibility of describing this mixing operation in the theory of  $(\min, +)$  linear systems, even if subsystems are WB-TEGs or CW-TEGs.

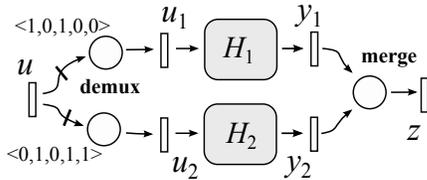


Figure 18. Demux/Merge with parallel Timed Event Graphs in [17]

### B. Systems with varying holding times

The problem considered here can be seen as a particular case of the previous subsection. We consider a routing problem (demux/mux) where subsystems are simply timed places with constant holding times. In this way, we are able to model delays the value of which can change according to a periodic sequence. This is useful to model manufacturing operations where durations can vary according to periodic scheme.

1) *FIFO place with varying holding times:* Such a problem has already been studied in [12] with the max-plus approach but without Weighted TEGs. We propose here to reconsider the problem thanks to Cyclo-Weighted TEGs and periodic operators in  $\mathcal{E}_{per}[\delta]$ .

In Fig.19, we describe (on the left part) a CW-TEG with an input sequence of events  $x_1$  which is demultiplexed into 3 different places, with different holding times, and then recombined with a multiplexing operation. This system is equivalent to a single place with a holding time that can

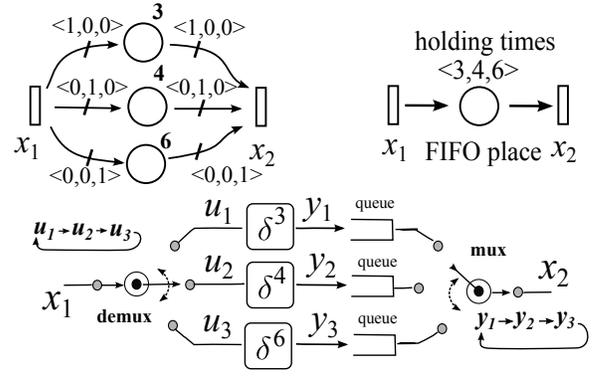


Figure 19. FIFO Place with a holding time that changes periodically modeled by a CW-TEG

vary according to the 3-periodic sequence  $\langle 3, 4, 6 \rangle$ . The first token spends at least 3 time units between transitions  $x_1$  and  $x_2$ , the next one 4 time units and so on. But the multiplexing adds another constraint: tokens cannot overtake each other. A token cannot leave the place before the previous one has left. In other words, the place has a First In First Out (FIFO) behavior. In summary, we can model a FIFO place with a varying holding time (with periodic variations) by an equivalent CW-TEG. Thanks to this equivalence, we can reconsider the systems with varying holding times studied in [12]. In general, a FIFO place with the sequence of holding times given by  $\langle \tau_1, \tau_2, \dots, \tau_n \rangle$  can be described by

$$w = \beta_{\langle 1,0,\dots,0 \rangle} \delta^{\tau_1} \mu_{\langle 1,0,\dots,0 \rangle} \oplus \beta_{\langle 0,1,0,\dots,0 \rangle} \delta^{\tau_2} \mu_{\langle 0,1,0,\dots,0 \rangle} \oplus \dots \oplus \beta_{\langle 0,\dots,0,1 \rangle} \delta^{\tau_n} \mu_{\langle 0,\dots,0,1 \rangle}$$

2) *Example:* The example studied in [6] with WB-TEGs is reconsidered here with CW-TEGs and depicted in Fig.20. For this system, the place  $x_1 \rightarrow x_2$  is a FIFO place with holding times  $\langle 3, 4, 6 \rangle$  (see Fig.19), and the place  $x_2 \rightarrow y$  is a FIFO place with holding times  $\langle 2, 5 \rangle$ . The system is defined by

$$\begin{aligned} x_1 &= u \oplus \gamma^3 x_2, \\ x_2 &= (\beta_{\langle 1,0,0 \rangle} \delta^3 \mu_{\langle 1,0,0 \rangle} \oplus \beta_{\langle 0,1,0 \rangle} \delta^4 \mu_{\langle 0,1,0 \rangle} \oplus \beta_{\langle 0,0,1 \rangle} \delta^6 \mu_{\langle 0,0,1 \rangle}) x_1, \\ y &= (\beta_{\langle 1,0 \rangle} \delta^2 \mu_{\langle 1,0 \rangle} \oplus \beta_{\langle 0,1 \rangle} \delta^5 \mu_{\langle 0,1 \rangle}) x_2. \end{aligned}$$

The transfer function of this system  $y = G_b u$  is computed in [6] with constant-weight operators and can be given also with weight-variable operators:

$$\begin{aligned} G_b &= \gamma^0 \delta^5 \oplus \gamma^1 \mu_{\langle 0,1,1,1,1,2 \rangle} \delta^6 \oplus \gamma^1 \mu_{\langle 0,1,1,2,0,2 \rangle} \delta^8 \\ &\oplus \gamma^1 \mu_{\langle 0,2,2,0,0,2 \rangle} \delta^9 \oplus \gamma^3 \mu_{\langle 0,0,2,0,0,4 \rangle} \delta^{11} \\ &\oplus \gamma^5 \mu_{\langle 0,0,2,0,1,3 \rangle} \delta^{13} \\ &\oplus (\gamma^6 \delta^{12}) * (\gamma^5 \mu_3 \beta_3 \delta^{14} \oplus \gamma^5 \mu_{\langle 0,0,4,0,0,2 \rangle} \delta^{17} \\ &\quad \oplus \gamma^8 \mu_3 \beta_3 \delta^{20} \oplus \gamma^9 \mu_{\langle 0,0,2,0,0,4 \rangle} \delta^{23}) \end{aligned}$$

## V. CONCLUSION

In this paper, we consider the class of Weight-Balanced Timed Event Graphs to model manufacturing systems. They are well suited to model batch and split operations such as

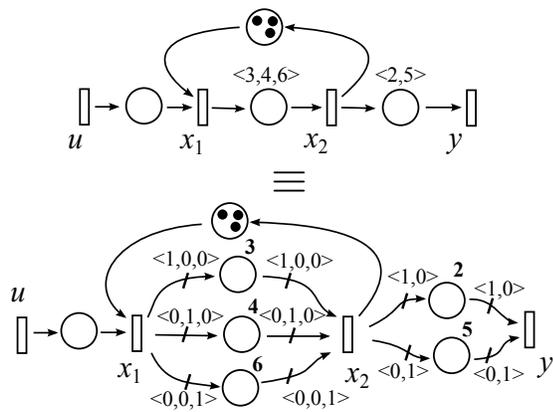


Figure 20. TEG with varying holding times and the equivalent CW-TEG

cutting, lot making or palletizing. But this class is also well adapted to describe some routing phenomena that can be seen as analogous to multiplexing and demultiplexing operations. In order to describe these phenomena in a more natural way, we introduce an extension of Weighted TEGs with weight-variable edges which we call Cyclo-Weighted TEGs since their weights change according to periodic sequences. Moreover, the behavior of Cyclo-Weighted TEGs can be described by both weight-variable operators and constant weight operators already used to model WB-TEGs. All these tools are useful to model a family of phenomena arising in manufacturing systems that can finally be modeled by input-output transfer relations. These algebraic tools give a mathematical framework to deal with flow control in manufacturing process, in particular by making it possible to compute some feedback loop controllers in the spirit of the conventional control theory.

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