Duality and interval analysis over idempotent semirings*

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Abstract

In this paper semirings with an idempotent addition are considered. These algebraic structures are endowed with a partial order. This allows to consider residuated maps to solve systems of inequalities $A \otimes X \preceq B$ (see [3]). The purpose of this paper is to consider a dual product, denoted $\odot$, and the dual residuation of matrices, in order to solve the following inequality $A \otimes X \preceq X \preceq B \odot X$. Sufficient conditions ensuring the existence of a non-linear projector in the solution set are proposed. The results are extended to semirings of intervals such as they were introduced in [25].

Key words: Max-plus algebra; Idempotent semiring; Interval analysis; Residuation theory;

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1. Introduction

Many problems in mathematics are non-linear in the traditional sense but appear to be linear over idempotent semirings. The max-plus algebra is a pop-

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ular semiring widely studied (see e.g., [7, 15, 5, 6, 27]). An idempotent semiring $\mathcal{S}$ can be endowed with a partial order relation. According to this order relation, and according to continuity assumptions, it is possible to obtain the greatest solution of inequality $A \otimes X \preceq B$ where A, X and B are matrices of proper dimension and $(A \otimes X)_{ij} = \bigoplus\limits_{k=1...n} (a_{ik} \otimes x_{kj})$. The greatest solution is obtained by considering residuation theory. In this paper we will consider the dual matrix product $A \odot X$ defined as $(A \odot X)_{ij} = \bigwedge\limits_{k=1...n} (a_{ik} \odot x_{kj})$, where $\wedge$ represents the greatest lower bound. Then we will consider the dual residuation to deal with computation of the smallest solution of inequality $A \odot X \succeq B$. The existence of a unique solution is not always ensured. Nevertheless if all elements of the semiring admit an inverse (*i.e.*, it is a semifield) then the smallest solution exists. This condition is fulfilled in (max-plus) algebra and it has allowed to deal with opposite semimodules in [10]. This condition is fulfilled neither in the semirings of non decreasing power series nor in the semirings of intervals such as introduced in [24, 23, 18], hence we will give some sufficient conditions to ensure the existence of this smallest solution.

From a practical point of view, it is useful to be able to solve systems such as $A \otimes X \preceq X \preceq B \odot X$, as they are involved in the study of dynamical discrete event systems subject to constraints (see [31, 4, 20]). Hence sufficient conditions for the existence of a projector in the set of solutions is given. Its computation is based on additive closure of matrices and on the dual residuation of the dual product. This projector is also given in semirings of intervals which allow us to deal with uncertainties.

This paper is organized as follows: in Section 2, algebraic preliminaries are recalled. More precisely, semiring defintion are first introduced and then some useful theorems about residuation theory are recalled. Next the section is devoted to the presentation of closure mapping properties. In Section 3, the dual product and its dual residuation are considered. Inequalities $A \otimes X \preceq X \preceq B \odot X$ is considered in Section 4, and in order to propose a projector in the solution set, some sufficient conditions are given. In Section 5, the previous results are applied in the semiring (max,plus) and in a semiring of non-decreasing power series. In Section 6, semirings of intervals are considered. Useful results initially presented in [24, 23, 18] are recalled, and the results of Section 3 are extended in this algebraic setting.

2. Preliminaries

2.1. Idempotent Semiring

In this section we recall useful results (for a more exhaustive presentation see reference [1]).

Definition 1 (Monoid). $(M, \cdot, e)$ *is a monoid if* $\cdot$ *is an internal law, associative and with an identity element* e*. If the law* $\cdot$ *is commutative,* $(M, \cdot, e)$ *is a commutative monoid.*

Definition 2 (Idempotent Semiring, semifield). *An idempotent semiring is a set,* $\mathcal{S}$*, endowed with two internal operations denoted by* $\oplus$ *(addition) and* $\otimes$ *(multiplication) such that :*

> $(\mathcal{S}, \oplus, \varepsilon)$ *is an idempotent commutative monoid, i.e.,* $\forall a \in \mathcal{S}, a \oplus a = a$,
>
> $(\mathcal{S}, \otimes, e)$ *is a monoid,*
>
> $\otimes$ *operation is distributive with respect to* $\oplus$,
>
> ε *is absorbing for the law* $\otimes$*, i.e.,* $\forall a, \quad \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$.
>
> *If* $\otimes$ *is commutative, the semiring is said to be commutative. A semifield is a semiring in which all elements except* ε *have a multiplicative inverse.*

An idempotent semiring[2] can be endowed with a canonical *order* defined by: $a \succeq b$ iff $a = a \oplus b$. Then it becomes a sup-semilattice, and $a \oplus b$ is the least upper bound of a and b. A semiring is *complete* if sums of infinite number of terms are always defined, and if multiplication distributes over infinite sums, too. In particular, the sum of all elements of a complete semiring is defined and

[2]In the following we will only refer to idempotent semirings and therefore drop the adjective

denoted by $\top$ (for "top"). A complete semiring becomes a complete lattice for which the greatest lower bound of a and b is denoted $a \wedge b$.

Definition 3 (Subsemiring). *A subset $\mathcal{C}$ of a semiring is called a subsemiring of $\mathcal{S}$ if*

$\varepsilon \in \mathcal{C}$ and $e \in \mathcal{C}$;

$\mathcal{C}$ is closed for $\oplus$ and $\otimes$, i.e, $\forall a, b \in \mathcal{C}$, $a \oplus b \in \mathcal{C}$ and $a \otimes b \in \mathcal{C}$.

Furthermore the subsemiring is complete if it is closed for infinite sums and if the product distributes over infinite sums.

Lemma 4 ([1, §4.3.4]). *Let $\mathcal{S}$ be a semiring. $\forall a, b, c \in \mathcal{S}$ the following inequality holds :*

$$c \otimes (a \wedge b) \preceq (c \otimes a) \wedge (c \otimes b).$$

Furthermore, if c admits a multiplicative inverse, i.e., if there exists a unique element, denoted c^{-1}, such that $c^{-1} \otimes c = c \otimes c^{-1} = e$, then

$$c \otimes (a \wedge b) = (c \otimes a) \wedge (c \otimes b).$$

Definition 5 (Formal power series). *A formal power series in p (commutative) variables, denoted z_1 to z_p, with coefficients in a semiring $\mathcal{S}$, is a mapping s defined from $\mathbb{Z}^p$ into $\mathcal{S}$: $\forall k = (k_1, ..., k_p) \in \mathbb{Z}^p$, $s(k)$ represents the coefficient of $z_1^{k_1} ... z_p^{k_p}$ and $(k_1, ..., k_p)$ are the exponents. Another equivalent representation is*

$$s(z_1, ..., z_p) = \bigoplus_{k \in \mathbb{Z}^p} s(k) z_1^{k_1} ... z_p^{k_p}.$$

Definition 6 (Semiring of series). *The set of formal power series with coefficients in a semiring $\mathcal{S}$ endowed with the following sum and Cauchy product:*

$$s \oplus s' : (s \oplus s')(k) = s(k) \oplus s'(k),$$
$$s \otimes s' : (s \otimes s')(k) = \bigoplus_{i+j=k} s(i) \otimes s'(j),$$

is a semiring denoted $\mathcal{S}[\![z_1, ..., z_p]\!]$. If $\mathcal{S}$ is complete, $\mathcal{S}[\![z_1, ..., z_p]\!]$ is complete. A series with a finite support is called a polynomial, and a monomial if there is *only one element in the series. The greatest lower bound of series is given by :*

$$s \wedge s' : (s \wedge s')(k) = s(k) \wedge s'(k).$$

2.2. Residuation Theory

Residuation theory allows to deal with the inverse of order preserving mappings defined over ordered sets, *i.e.* a set equipped with a partial order relation. This theory gives another point of view on Galois connection. Useful references are [14, 3, 2].

Definition 7 (Continuity). *An order preserving mapping $f : \mathcal{D} \to \mathcal{E}$, where $\mathcal{D}$ and $\mathcal{E}$ are complete ordered sets, is a mapping such that: $x \succeq y \Rightarrow f(x) \succeq f(y)$. It is said to be isotone in [1].*

A mapping f is lower-semicontinuous (l.s.c.), respectively, upper-semicontinuous (u.s.c.) if, for every (finite or infinite) subset $\mathcal{X}$ of $\mathcal{D}$,

$$f(\bigoplus_{x \in \mathcal{X}} x) = \bigoplus_{x \in \mathcal{X}} f(x),$$

respectively,

$$f(\bigwedge_{x \in \mathcal{X}} x) = \bigwedge_{x \in \mathcal{X}} f(x).$$

A mapping f is continuous if it is both l.s.c. and u.c.s.

Definition 8 (Image, Kernel). *Let $f : \mathcal{D} \to \mathcal{E}$ be a mapping, where $\mathcal{D}$ and $\mathcal{E}$ are semirings. The image of f, denoted $\mathsf{Im}f$, is classically defined as $\mathsf{Im}f = \{y \in \mathcal{E} | y = f(x) \text{ for some } x \in \mathcal{D}\}$. The equivalence kernel is defined as $\mathrm{ker}f := \{(x, x') \in \mathcal{D} \times \mathcal{D} \mid f(x) = f(x')\}$.*

Definition 9 (Residuated and dually residuated mapping). *An order preserving mapping $f : \mathcal{D} \to \mathcal{E}$, where $\mathcal{D}$ and $\mathcal{E}$ are ordered sets, is a residuated mapping if for all $y \in \mathcal{E}$, the least upper bound of the subset $\{x | f(x) \preceq y\}$ exists and belongs to this subset. It is then denoted by $f^\sharp(y)$. The mapping $f^\sharp$ is called the residual of f. When f is residuated, $f^\sharp$ is the unique order preserving mapping such that*

$$f \circ f^\sharp \preceq \mathsf{Id}_{\mathcal{E}} \qquad \text{and} \qquad f^\sharp \circ f \succeq \mathsf{Id}_{\mathcal{D}}, \tag{1}$$

where Id *is the identity mapping on* $\mathcal{D}$ *and* $\mathcal{E}$ *respectively.*
Mapping g *is a dually residuated mapping if for all* $y \in \mathcal{E}$*, the greatest lower bound of the subset* $\{x|g(x) \succeq y\}$ *exists and belongs to this subset. It is then denoted by* $g^\flat(y)$*. The mapping* $g^\flat$ *is called the dual residual of* g*. When* g *is dually residuated,* $g^\flat$ *is the unique order preserving mapping such that*

$$g \circ g^\flat \succeq \mathsf{Id}_{\mathcal{E}} \qquad \text{and} \qquad g^\flat \circ g \preceq \mathsf{Id}_{\mathcal{D}}. \tag{2}$$

Remark 10. *According to this definition, it is clear that* $f^\sharp$ *is dually residuated and that* $g^\flat$ *is residuated, furthermore,* $(f^\sharp)^\flat = f$ *and* $(g^\flat)^\sharp = g$*.*

Theorem 11 ([1, §4.4.2]). *Consider the order preserving mappings* $f : \mathcal{E} \to \mathcal{F}$ *and* $g : \mathcal{E} \to \mathcal{F}$ *where* $\mathcal{E}$ *and* $\mathcal{F}$ *are complete semirings. Their bottom elements are, respectively, denoted by* $\varepsilon_{\mathcal{E}}$ *and* $\varepsilon_{\mathcal{F}}$*. Their top elements are, respectively, denoted by* $\top_{\mathcal{E}}$ *and* $\top_{\mathcal{F}}$*.*
Mapping f *is residuated iff* $f(\varepsilon_{\mathcal{E}}) = \varepsilon_{\mathcal{F}}$ *and* $f(\bigoplus_{x\in\mathcal{X}} x) = \bigoplus_{x\in\mathcal{X}} f(x)$ *for each* $\mathcal{X} \subseteq \mathcal{E}$ *(i.e.,* f *is lower-semicontinuous), furthermore* $f^\sharp(\top_{\mathcal{F}}) = \top_{\mathcal{E}}$ *and* $f^\sharp(\bigwedge_{y\in\mathcal{Y}} y) = \bigwedge_{y\in\mathcal{Y}} f^\sharp(y)$ *for each* $\mathcal{Y} \subseteq \mathcal{F}$ *(i.e.,* $f^\sharp$ *is upper-semicontinuous).*
Mapping g *is dually residuated iff* $g(\top_{\mathcal{E}}) = \top_{\mathcal{F}}$ *and* $g(\bigwedge_{x\in\mathcal{X}} x) = \bigwedge_{x\in\mathcal{X}} g(x)$ *for each* $\mathcal{X} \subseteq \mathcal{E}$ *(i.e.,* g *is upper-semicontinuous), furthermore* $g^\flat(\varepsilon_{\mathcal{F}}) = \varepsilon_{\mathcal{E}}$ *and* $g^\flat(\bigoplus_{y\in\mathcal{Y}} y) = \bigoplus_{y\in\mathcal{Y}} g^\flat(y)$ *for each* $\mathcal{Y} \subseteq \mathcal{F}$ *(i.e.,* $g^\flat$ *is lower-semicontinuous).*

Theorem 12 ([1, Th. 4.56]). *Let* $\mathcal{D}$*,* $\mathcal{C}$*,* $\mathcal{B}$ *be three semirings. Let* $h : \mathcal{D} \to \mathcal{C}$ *and* $f : \mathcal{C} \to \mathcal{B}$ *be residuated mappings. The following properties hold :*

$$f \circ f^\sharp \circ f = f \text{ and } f^\sharp \circ f \circ f^\sharp = f^\sharp, \tag{3}$$

$$(f \circ h)^\sharp \quad = \quad h^\sharp \circ f^\sharp. \tag{4}$$

Let $h : \mathcal{D} \to \mathcal{C}$ *and* $g : \mathcal{C} \to \mathcal{B}$ *be dually residuated mappings. The following properties hold :*

$$g \circ g^\flat \circ g = g \text{ and } g^\flat \circ g \circ g^\flat = g^\flat, \tag{5}$$

$$(g \circ h)^\flat \quad = \quad h^\flat \circ g^\flat. \tag{6}$$

Theorem 13 ([1, Th. 4.56]). *Let $\mathcal{D}$, $\mathcal{C}$ be two semirings. Let $h : \mathcal{D} \to \mathcal{C}$ and $f : \mathcal{D} \to \mathcal{C}$ be residuated mappings. The following properties hold :*

$$f \preceq h \Leftrightarrow h^\sharp \preceq f^\sharp, \tag{7}$$

$$(f \oplus h)^\sharp = f^\sharp \wedge h^\sharp. \tag{8}$$

Let $h : \mathcal{D} \to \mathcal{C}$ and $g : \mathcal{D} \to \mathcal{C}$ be dually residuated mappings. The following properties hold :

$$g \preceq h \Leftrightarrow h^\flat \preceq g^\flat, \tag{9}$$

$$(g \wedge h)^\flat = g^\flat \oplus h^\flat. \tag{10}$$

Theorem 14 ([12]). *Let $\mathcal{S}$,$\mathcal{C}$ be semirings, $f : \mathcal{S} \to \mathcal{C}$ and $g : \mathcal{S} \to \mathcal{C}$ be two residuated mappings, then the following equivalence holds :*

$$\mathsf{Im}\ f \subset \mathsf{Im} g \Leftrightarrow g \circ g^\sharp \circ f = f.$$

Proof. If $\mathsf{Im}\ f \subset \mathsf{Im}\ g$ then there exists a mapping $h : \mathcal{S} \to \mathcal{S}$, s.t. $f = g \circ h$. According to Equation (3), $g \circ g^\sharp \circ f = g \circ g^\sharp \circ g \circ h = g \circ h = f$. If $g \circ g^\sharp \circ f = f$ then $\mathsf{Im} f \subset \mathsf{Im}\ g$. □

Proposition 15 ([12, 9], Projection on the image of a mapping). *Let $\mathcal{S}$, $\mathcal{C}$ be semirings. Let $f : \mathcal{S} \to \mathcal{C}$ be a residuated mapping, mapping $P_f = f \circ f^\sharp$ is a projector and $P_f(c)$ with $c \in \mathcal{C}$ is the greatest element in $\mathsf{Im} f$ less than or equal to c. Let $g : \mathcal{S} \to \mathcal{C}$ be a dually residuated mapping, mapping $P_g = g \circ g^\flat$ is a projector and $P_g(d)$ with $d \in \mathcal{C}$ is the lowest element in $\mathsf{Im} g$ greater than or equal to d.*

Proof. According to Definition 9, $P_f(c) = \{\bigoplus x | f(x) \preceq c\}$ and $P_g(d) = \{\bigwedge x | g(x) \succeq d\}$. According to Equations (3) and (5), $P_f \circ P_f = f \circ f^\sharp \circ f \circ f^\sharp = f \circ f^\sharp$, and $P_g \circ P_g = g \circ g^\flat \circ g \circ g^\flat = g \circ g^\flat$, hence they are both projectors. □

The problem of mapping restriction and its connection with residuation theory is now addressed.

Definition 16 (Restricted mapping). *Let $f : \mathcal{E} \to \mathcal{F}$ be a mapping and $\mathcal{A} \subseteq \mathcal{E}$. We will denote $f_{|\mathcal{A}} : \mathcal{A} \to \mathcal{F}$ the mapping defined by $f_{|\mathcal{A}} = f \circ \mathsf{Id}_{|\mathcal{A}}$ where $\mathsf{Id}_{|\mathcal{A}} : \mathcal{A} \to \mathcal{E}$ is the canonical injection from $\mathcal{A}$ to $\mathcal{E}$. Similarly, let $\mathcal{B} \subseteq \mathcal{F}$ with $\mathsf{Im} f \subseteq \mathcal{B}$. Mapping ${}_{\mathcal{B}|}f : \mathcal{E} \to \mathcal{B}$ is defined by $f = \mathsf{Id}_{|\mathcal{B}} \circ {}_{\mathcal{B}|}f$, where $\mathsf{Id}_{|\mathcal{B}} : \mathcal{B} \to \mathcal{F}$.*

Proposition 17 ([3]). *Let $\mathcal{S}_{sub}$ be a complete subsemiring of $\mathcal{S}$. Let $\mathsf{Id}_{|\mathcal{S}_{sub}} : \mathcal{S}_{sub} \to \mathcal{S}$, $x \mapsto x$ be the canonical injection. The injection $\mathsf{Id}_{|\mathcal{S}_{sub}}$ is both residuated and dually residuated and their residuals are projectors.*

Proof. According to Definition 7, mapping $\mathsf{Id}_{|\mathcal{S}_{sub}}$ is both *l.s.c.* and *u.s.c.*, *i.e.* continuous, and by assumption $\varepsilon \in \mathcal{S}_{sub}$ and $\top \in \mathcal{S}_{sub}$, hence $\mathsf{Id}_{|\mathcal{S}_{sub}}$ is both residuated and dually residuated (see Theorem 11). Furthermore, $\mathsf{Id}_{|\mathcal{S}_{sub}} = \mathsf{Id}_{|\mathcal{S}_{sub}} \circ \mathsf{Id}_{|\mathcal{S}_{sub}}$ hence $(\mathsf{Id}_{|\mathcal{S}_{sub}})^\sharp = (\mathsf{Id}_{|\mathcal{S}_{sub}} \circ \mathsf{Id}_{|\mathcal{S}_{sub}})^\sharp = (\mathsf{Id}_{|\mathcal{S}_{sub}})^\sharp \circ (\mathsf{Id}_{|\mathcal{S}_{sub}})^\sharp$ which proves that $(\mathsf{Id}_{|\mathcal{S}_{sub}})^\sharp$ is a projector. The same can be done for $(\mathsf{Id}_{|\mathcal{S}_{sub}})^\flat$. □

Proposition 18. *Let $f : \mathcal{D} \to \mathcal{E}$ be a residuated mapping, $g : \mathcal{D} \to \mathcal{E}$ be a dually residuated mapping and $\mathcal{D}_{sub}$ (resp. $\mathcal{E}_{sub}$) be a complete subsemiring of $\mathcal{D}$ (resp. $\mathcal{E}$):*

1. *mapping $f_{|\mathcal{D}_{sub}}$ is residuated and its residual is given by :*
$$(f_{|\mathcal{D}_{sub}})^\sharp = (f \circ \mathsf{Id}_{|\mathcal{D}_{sub}})^\sharp = (\mathsf{Id}_{|\mathcal{D}_{sub}})^\sharp \circ f^\sharp;$$
2. *if $\mathsf{Im} f \subset \mathcal{E}_{sub}$ then mapping ${}_{\mathcal{E}_{sub}|}f$ is residuated and its residual is given by:*
$$\left({}_{\mathcal{E}_{sub}|}f\right)^\sharp \quad = \quad f^\sharp \circ \mathsf{Id}_{|\mathcal{E}_{sub}} = \left(f^\sharp\right)_{|\mathcal{E}_{sub}};$$
3. *mapping $g_{|\mathcal{D}_{sub}}$ is dually residuated and its dual residual is given by :*
$$(g_{|\mathcal{D}_{sub}})^\flat = (g \circ \mathsf{Id}_{|\mathcal{D}_{sub}})^\flat = (\mathsf{Id}_{|\mathcal{D}_{sub}})^\flat \circ g^\flat;$$
4. *if $\mathsf{Im} g \subset \mathcal{E}_{sub}$ then mapping ${}_{\mathcal{E}_{sub}|}g$ is dually residuated and its dual residual is given by:*
$$\left({}_{\mathcal{E}_{sub}|}g\right)^\flat \quad = \quad g^\flat \circ \mathsf{Id}_{|\mathcal{E}_{sub}} = \left(g^\flat\right)_{|\mathcal{E}_{sub}}.$$

Proof. Statements 1 and 3 follow directly from Theorem 12 and Proposition 17. Statement 2 is obvious since f is residuated and $\mathsf{Im} f \subset \mathcal{E}_{sub} \subset \mathcal{E}$. Statement 4 can be prove in the same manner. □

2.3. Closure mappings

Definition 19 (Closure mapping). *Let $\mathcal{S}$ be a semiring and $h : \mathcal{S} \to \mathcal{S}$ be an isotone mapping. If $h \circ h = h \succeq \mathsf{Id}_{\mathcal{S}}$ then h is a closure mapping. If $h \circ h = h \preceq \mathsf{Id}_{\mathcal{S}}$ then h is a dual closure mapping.*

Remark 20. *According to this definition, it can be checked that the projector P_f (see Proposition 15) is a dual closure mapping, and the projector P_g is a closure mapping.*

Theorem 21 ([12, Th. 19 and Th. 20]). *Let $\mathcal{S}$ be a semiring, $h : \mathcal{S} \to \mathcal{S}$ be a residuated mapping and $g : \mathcal{S} \to \mathcal{S}$ be a dually residuated mapping, then the following equivalences hold:*

$$h \text{ is a closure mapping } \Leftrightarrow h^\sharp \text{ is a dual closure mapping} \Leftrightarrow h^\sharp \circ h = h \Leftrightarrow h \circ h^\sharp = h^\sharp, \tag{11}$$

$$g \text{ is a dual closure mapping} \Leftrightarrow g^\flat \text{ is a closure mapping} \Leftrightarrow g \circ g^\flat = g \Leftrightarrow g^\flat \circ g = g^\flat. \tag{12}$$

Proposition 22. *Let $\mathcal{S}$ be a semiring, $h : \mathcal{S} \to \mathcal{S}$, $g : \mathcal{S} \to \mathcal{S}$ and $f : \mathcal{S} \to \mathcal{S}$ be three mappings, and assume that g and f are two closure mappings which are residuated. The following equivalence holds*

$$\mathsf{Im} h \subset \mathsf{Im} f \Leftrightarrow f \circ h = h,$$

$$g \preceq f \Leftrightarrow f \circ g = f = g^\sharp \circ f \Leftrightarrow \mathsf{Im} f \subset \mathsf{Im}\ g \Leftrightarrow \mathsf{Im} f \subset \mathsf{Im}\ g^\sharp.$$

Proof. For the first statement, $\mathsf{Im} h \subset \mathsf{Im} f \Rightarrow \exists\ m$ such that $h = f \circ m \Rightarrow f \circ h = f \circ f \circ m = f \circ m = h$, since f is a closure mapping, and obviously $f \circ h = h \Rightarrow \mathsf{Im} h \subset \mathsf{Im} f$.
For the second statement, according to the closure mapping definition $g \succeq \mathsf{Id}_{\mathcal{S}}$, hence $g \succeq \mathsf{Id}_{\mathcal{S}} \Rightarrow f \circ g \succeq f$. Mapping f is assumed to be a closure mapping, this yields $g \preceq f \Rightarrow f \circ g \preceq f \circ f = f$. Hence $g \preceq f \Leftrightarrow f \circ g = f$.

According to Equivalences (11), $g^\sharp$ is a dual closure mapping, therefore according to Definition 19 $g^\sharp \preceq \mathsf{Id}_{\mathcal{S}}$, hence $g^\sharp \preceq \mathsf{Id}_{\mathcal{S}} \Rightarrow g^\sharp \circ f \preceq f$. According to the assumptions, f and g are residuated, hence Equation (7) yields

$f \succeq g \Leftrightarrow g^\sharp \succeq f^\sharp \Rightarrow g^\sharp \circ f \succeq f^\sharp \circ f = f$ (the last equality comes from Equivalences (11)), hence $g \preceq f \Leftrightarrow f = g^\sharp \circ f$.

By considering Equivalences (11) and Theorem 14, $f = g^\sharp \circ f = g \circ g^\sharp \circ f \Rightarrow \mathsf{Im} f \subset \mathsf{Im}\ g$, on the other hand $\mathsf{Im} f \subset \mathsf{Im}\ g \Rightarrow \exists\ m$ such that $f = g \circ m \Rightarrow g^\sharp \circ f = g^\sharp \circ g \circ m = g \circ m = f$.

In the same manner, Equivalences (11) and Theorem 14 yield : $f = g^\sharp \circ f \Rightarrow \mathsf{Im} f \subset \mathsf{Im}\ g^\sharp$, on the other hand $\mathsf{Im} f \subset \mathsf{Im}\ g^\sharp \Rightarrow \exists\ m$ such that $f = g^\sharp \circ m \Rightarrow g^\sharp \circ f = g^\sharp \circ g^\sharp \circ m = g^\sharp \circ m = f$ (indeed $g^\sharp = g^\sharp \circ g^\sharp$ since $g^\sharp$ is a dual closure mapping). □

2.4. Applications

Definition 23 (Left product, right product). *Let $\mathcal{S}$ be a complete semiring, $a, b \in \mathcal{S}$, and $L_a : \mathcal{S} \to \mathcal{S}, x \mapsto a \otimes x$ and $R_a : \mathcal{S} \to \mathcal{S}, x \mapsto x \otimes a$. Since ε is absorbing for the multiplicative law and according to distributivity of this law over the additive law, L_a and R_a are both lower semi-continuous, hence both mappings are residuated. In [1], their residuals are denoted, respectively, by $L_a^\sharp(x) = a \backslash\!\!\circ x$ and $R_a^\sharp(x) = x \circ\!\!/ a$. Therefore, $a \backslash\!\!\circ b$ (resp. $b \circ\!\!/ a$) is the greatest solution of $a \otimes x \preceq b$ (resp. $x \otimes a \preceq b$) and equality is achieved when $b \in \mathsf{Im} L_a$ (resp. $b \in \mathsf{Im} R_a$). It must be noted that $\varepsilon \backslash\!\!\circ \varepsilon = \top$ and $\top \backslash\!\!\circ \top = \top$. In the matrix case, mappings $L_A : \mathcal{S}^{p \times m} \to \mathcal{S}^{n \times m}, X \mapsto A \otimes X$ and $R_A : \mathcal{S}^{m \times n} \to \mathcal{S}^{m \times p}, X \mapsto X \otimes A$ where $A \in \mathcal{S}^{n \times p}$, are residuated mappings. The corresponding entries are obtained as follows,*

$$(A \backslash\!\!\circ B)_{ij} = \bigwedge_{k=1\ldots n} (a_{ki} \backslash\!\!\circ b_{kj}), \tag{13}$$

$$(C \circ\!\!/ A)_{ij} = \bigwedge_{k=1\ldots p} (c_{ik} \circ\!\!/ a_{jk}) \tag{14}$$

with $B \in \mathcal{S}^{n \times m}$ and $C \in \mathcal{S}^{m \times p}$.

Definition 24 (Kleene star). *Let $\mathcal{S}$ be a complete semiring. The additive closure of matrix $A \in \mathcal{S}^{n \times n}$ is defined as follows :*

$$K : \mathcal{S}^{n \times n} \to \mathcal{S}^{n \times n}, A \mapsto A^* = \bigoplus_{i \in \mathbb{N}_0} A^i,$$

where $A^0 = E$, $A^k = A \otimes A^{k-1}$ *and* E *is the identity matrix, i.e.* $\forall i, j \in [1, n]$, $E_{ii} = e$ *and* $E_{ij} = \varepsilon$ *if* $i \neq j$.
This mapping is a closure mapping (indeed $K \circ K = K$ *and* $K \succeq Id_{\mathcal{S}^{n\times n}}$*). It is sometimes called the Kleene star operator. Among many references about the Kleene star matrix we can cite [33], where the link between the Kleene star* A^* *and the subeigenvectors of* A *for an eigenvalue* λ*, i.e., vectors* x *s.t.* $A \otimes x \preceq \lambda \otimes x$*, was studied.*

Property 25. *Let* $A \in \mathcal{S}^{n\times n}$*, and* $X \in \mathcal{S}^{n\times p}$*. According to Definition 24 mapping* $L_{A^*} : \mathcal{S}^{n\times p} \to \mathcal{S}^{n\times p}, X \mapsto A^* \otimes X$ *is a closure mapping, (see Definition 19), hence :*

$$A^* \otimes A^* \otimes X = A^* \otimes X, \tag{15}$$

and as a consequence the following equivalence holds :

$$X = A^* X \Leftrightarrow X \in \mathsf{Im} L_{A^*}. \tag{16}$$

Furthermore according to Theorem 21, $L^{\sharp}_{A^*}$ *is a dual closure mapping, hence :*

$$A^* \backslash\!\!\circ A^* \backslash\!\!\circ X = A^* \backslash\!\!\circ X, \tag{17}$$

according to Equation (11), $L_{A^*} \circ L^{\sharp}_{A^*} = L^{\sharp}_{A^*}$ *and* $L^{\sharp}_{A^*} \circ L_{A^*} = L_{A^*}$ *hence :*

$$A^* \otimes (A^* \backslash\!\!\circ X) = A^* \backslash\!\!\circ X, \tag{18}$$

and

$$A^* \backslash\!\!\circ (A^* \otimes X) = A^* \otimes X. \tag{19}$$

According to Proposition 15, Equation (18) means that $L^{\sharp}_{A^*}$ *is a projector on* $\mathsf{Im} L_{A^*}$.
Let $B \in \mathcal{S}^{n\times n}$ *such that* $B^* \preceq A^*$*, i.e.,* $L_{B^*} \preceq L_{A^*}$*, then according to Proposition 22, the following equivalence holds :*

$$B^* \preceq A^* \Leftrightarrow A^* B^* X = A^* X = B^* \backslash\!\!\circ (A^* X) \Leftrightarrow \mathsf{Im} L_{A^*} \subset \mathsf{Im} L_{B^*} \Leftrightarrow \mathsf{Im} L_{A^*} \subset \mathsf{Im} L^{\sharp}_{B^*}. \tag{20}$$

Lemma 26 ([1], Lemma 4.77). *Let $A \in \mathcal{S}^{n\times n}$, and $X \in \mathcal{S}^{n\times p}$. The following equivalences hold :*

$$X \preceq A \backslash\!\!\circ X \Leftrightarrow X \succeq AX \Leftrightarrow X = A^*X \Leftrightarrow X = A^* \backslash\!\!\circ X.$$

3. Dual product over semirings

In this section a dual product is considered and its properties are explored.

Definition 27 (Dual product). *Given a semiring $\mathcal{S}$, the dual product in $\mathcal{S}$, denoted $\odot$, is a law assumed to be associative and to have e as neutral element, i.e., $(\mathcal{S}, \odot, e)$ is a monoid. Furthermore this dual product is assumed to distribute with respect to $\wedge$ of infinitely many elements, and element $\top$ is absorbing ($\forall a, \quad \top \odot a = a \odot \top = \top$).*

Definition 28 (Dual matrix product). *Let $\mathcal{S}$ be a semiring and $A \in \mathcal{S}^{n\times p}$, $B \in \mathcal{S}^{p\times m}$ and $C \in \mathcal{S}^{n\times m}$ matrices, then $C = A \odot B$ is defined as :*

$$C_{ij} = (A \odot B)_{ij} \quad = \bigwedge_{k=1\dots p} (a_{ik} \odot b_{kj}),$$

the identity matrix is denoted $E^{\odot}$ and is such that $E^{\odot}_{ii} = e$ and $E^{\odot}_{ij} = \top$ for $i \neq j$.

In the sequel, mapping $\Lambda_A : \mathcal{S}^{p\times m} \to \mathcal{S}^{n\times m}, X \mapsto A \odot X$ will be considered.

Proposition 29. *Let $\mathcal{S}$ be a semiring and $A \in \mathcal{S}^{p\times n}, X \in \mathcal{S}^{n\times m}$ be matrices, mapping $\Lambda_A : \mathcal{S}^{n\times m} \to \mathcal{S}^{p\times m}, X \mapsto A \odot X$ is upper-semicontinuous, i.e.,*

$$\Lambda_A(\bigwedge_{X\in\mathcal{X}} X) = \bigwedge_{X\in\mathcal{X}} \Lambda_A(X).$$

Proof. Let $\mathcal{X}$ be a subset of $\mathcal{S}^{n\times m}$, then according to the definition of $\odot$ the following equalities hold :

$$
\begin{aligned}
\Lambda_A\Big(\bigwedge_{X\in\mathcal{X}} X\Big) &= A\odot\Big(\bigwedge_{X\in\mathcal{X}} X\Big) \\
\Big(\Lambda_A\Big(\bigwedge_{x\in\mathcal{X}} X\Big)\Big)_{ij} &= \bigwedge_{k=1}^{n} a_{ik}\odot\Big(\bigwedge_{x\in\mathcal{X}} x_{kj}\Big) = \bigwedge_{k=1}^{n}\bigwedge_{x\in\mathcal{X}}(a_{ik}\odot x_{kj}) \\
&= \bigwedge_{x\in\mathcal{X}}\bigwedge_{k=1}^{n}(a_{ik}\odot x_{kj}) = \bigwedge_{x\in\mathcal{X}}(\Lambda_A(X))_{ij}.
\end{aligned}
$$

□

Corollary 30. *Let $\mathcal{S}$ be a semiring and $A\in\mathcal{S}^{n\times p}$ be a matrix. Mapping $\Lambda_A : \mathcal{S}^{p\times m}\to\mathcal{S}^{n\times m}, X\mapsto A\odot X$ is dually residuated, and its dual residual will be denoted*[3]*:*

$$\Lambda_A^{\flat} : \mathcal{S}^{n\times m}\to\mathcal{S}^{p\times m}, X\mapsto A\backslash\!\!\!\!\bullet X$$

with the following rules :

$$(A\backslash\!\!\!\!\bullet X)_{ij} = \bigoplus_{k=1}^{n} a_{ki}\backslash\!\!\!\!\bullet x_{kj}, \tag{21}$$

and : $\top\backslash\!\!\!\!\bullet x=\varepsilon$, $\varepsilon\backslash\!\!\!\!\bullet x=\top$ and $\varepsilon\backslash\!\!\!\!\bullet\varepsilon=\varepsilon$.

Proposition 31. *Let $\mathcal{S}$ be a complete semiring and $A\in\mathcal{S}^{n\times p}$, $B\in\mathcal{S}^{n\times r}$ and $X\in\mathcal{S}^{p\times q}$ be three matrices. If for each entry b_{ij} of B the following equality holds $b_{ij}\backslash\!\!\!\!\bullet(a\otimes x)=(b_{ij}\backslash\!\!\!\!\bullet a)\otimes x$, $\forall a,x\in\mathcal{S}$, then the following equality holds :*

$$B\backslash\!\!\!\!\bullet(A\otimes X) = (B\backslash\!\!\!\!\bullet A)\otimes X. \tag{22}$$

Proof.

$$
\begin{aligned}
(B\backslash\!\!\!\!\bullet(A\otimes X))_{ij} &= \bigoplus_{l=1}^{n} b_{li}\backslash\!\!\!\!\bullet(A\otimes X)_{lj} \\
&= \bigoplus_{l=1}^{n} b_{li}\backslash\!\!\!\!\bullet\Big(\bigoplus_{k=1}^{p} a_{lk}\otimes x_{kj}\Big) \\
&= \bigoplus_{l=1}^{n}\bigoplus_{k=1}^{p} b_{li}\backslash\!\!\!\!\bullet(a_{lk}\otimes x_{kj}) \text{ since } \Lambda_B^{\flat} \text{ is lower semi-continuous} \\
&= \bigoplus_{k=1}^{p}\bigoplus_{l=1}^{n}(b_{li}\backslash\!\!\!\!\bullet a_{lk})\otimes x_{kj} \text{ according to the assumption} \\
&= \bigoplus_{k=1}^{p}(B\backslash\!\!\!\!\bullet A)_{ik}\otimes x_{kj} = ((B\backslash\!\!\!\!\bullet A)\otimes X)_{ij}.
\end{aligned}
$$

[3] This notation was initially introduced in the talk entitled "Projective $max,+$ semi modules", given by G. Cohen during the International Workshop on $max,+$ Algebra (IWMA Birmingham 2003, in honor of Prof. Cuninghame-Green [8]).

$\square$

Definition 32. *Let $\mathcal{S}$ be a semiring. The $\wedge$-closure of $B \in \mathcal{S}^{n\times n}$ is defined as:*

$$B_* = \bigwedge_{k\in\mathbb{N}_0} B^{\odot k},$$

where $B^{\odot 0} = E^{\odot}$ and $B^{\odot k} = B \odot B^{\odot (k-1)}$.

Property 33. *Let $B \in \mathcal{S}^{n\times n}$, and $X \in \mathcal{S}^{n\times p}$. Since Λ_B is upper-semicontinuous and, according to Definition 32, mapping $\Lambda_{B_*} : \mathcal{S}^{n\times p} \to \mathcal{S}^{n\times p}, X \mapsto B_* \odot X$ is a dual closure mapping (see Definition 19), hence :*

$$B_* \odot B_* \odot X = B_* \odot X, \tag{23}$$

and as a consequence the following equivalence holds :

$$X = B_* \odot X \Leftrightarrow X \in \mathsf{Im}\Lambda_{B_*}. \tag{24}$$

Proposition 34. *Let $\mathcal{S}$ be a semiring and $B \in \mathcal{S}^{n\times n}$ and $X \in \mathcal{S}^{n\times p}$ be two matrices. The following statements are equivalent:*

1. $X \preceq B \odot X$;
2. $B \mathbin{\bullet\!\!\!\!\backslash} X \preceq X$;
3. $B_* \mathbin{\bullet\!\!\!\!\backslash} X = X$;
4. $B_* \odot X = X$.

Proof. (1) $\Rightarrow$ (2) According to Definition 9 mapping $\Lambda_B^\flat$ is order preserving, hence $X \preceq B \odot X \Rightarrow B \mathbin{\bullet\!\!\!\!\backslash} X \preceq B \mathbin{\bullet\!\!\!\!\backslash} (B \odot X)$, furthermore the same definition implies $B \mathbin{\bullet\!\!\!\!\backslash} X \preceq B \mathbin{\bullet\!\!\!\!\backslash} (B \odot X) \preceq X$. Hence $X \preceq B \odot X \Rightarrow B \mathbin{\bullet\!\!\!\!\backslash} X \preceq X$.

(2) $\Rightarrow$ (3) According to Equation (6), $(\Lambda_B \circ \Lambda_B)^\flat = (\Lambda_B)^\flat \circ (\Lambda_B)^\flat$, hence $B \mathbin{\bullet\!\!\!\!\backslash} (B \mathbin{\bullet\!\!\!\!\backslash} X) = B^{\odot 2} \mathbin{\bullet\!\!\!\!\backslash} X$, furthermore mapping $\Lambda_B^\flat$ is order preserving, then

$$X \succeq B \mathbin{\bullet\!\!\!\!\backslash} X \Rightarrow B \mathbin{\bullet\!\!\!\!\backslash} X \succeq B \mathbin{\bullet\!\!\!\!\backslash} (B \mathbin{\bullet\!\!\!\!\backslash} X) = B^{\odot 2} \mathbin{\bullet\!\!\!\!\backslash} X,$$

hence

$$X \succeq B \mathbin{\bullet\!\!\!\!\backslash} X \succeq B^{\odot 2} \mathbin{\bullet\!\!\!\!\backslash} X \succeq \ldots \Rightarrow X \succeq (E^{\odot} \mathbin{\bullet\!\!\!\!\backslash} X) \oplus (B \mathbin{\bullet\!\!\!\!\backslash} X) \oplus (B^{\odot 2} \mathbin{\bullet\!\!\!\!\backslash} X) \oplus ...$$

furthermore according to Equation (10) and to Definition 32,

$$(E^{\odot} \mathbin{\bullet\!\!\backslash} X) \oplus (B \mathbin{\bullet\!\!\backslash} X) \oplus (B^{\odot 2} \mathbin{\bullet\!\!\backslash} X) \oplus \ldots = (E^{\odot} \wedge B \wedge B^{\odot 2} \wedge \ldots) \mathbin{\bullet\!\!\backslash} X = B_* \mathbin{\bullet\!\!\backslash} X,$$

then, $X \succeq (B \mathbin{\bullet\!\!\backslash} X) \Rightarrow X \succeq B_* \mathbin{\bullet\!\!\backslash} X$. On the other hand $B_* \preceq E^{\odot}$ then $B_* \mathbin{\bullet\!\!\backslash} X \succeq X$, hence $X \succeq (B \mathbin{\bullet\!\!\backslash} X) \Rightarrow X = B_* \mathbin{\bullet\!\!\backslash} X$.

(3) $\Rightarrow$ (4) From Definition 9 (Equation (2)) the following inequality holds : $B_* \odot (B_* \mathbin{\bullet\!\!\backslash} X) \succeq X$, hence,

$$X = B_* \mathbin{\bullet\!\!\backslash} X \Rightarrow B_* \odot X = B_* \odot (B_* \mathbin{\bullet\!\!\backslash} X) \succeq X,$$

but the definition of the dual closure yields $B_* \odot X \preceq X$, hence

$$X = B_* \mathbin{\bullet\!\!\backslash} X \Rightarrow B_* \odot X = X.$$

(4) $\Rightarrow$ (1) According to Definitions 27 and 32, Mapping Λ_{B_*} is upper semi-continuous, then

$$B_* \odot X = (E^{\odot} \wedge B \wedge B^{\odot 2} \wedge \ldots) \odot X = (X \wedge B \odot X \wedge B^{\odot 2} \odot X \wedge \ldots),$$

hence $X = B_* \odot X \Rightarrow X \preceq B \odot X$. □

4. The Inequality $\boldsymbol{A \otimes X \preceq X \preceq B \odot X}$

Proposition 35. *Let $\mathcal{S}$ be a semiring and $A, B \in \mathcal{S}^{n\times n}$ and $X \in \mathcal{S}^{n\times m}$. The following equivalence holds :*

$$A \otimes X \preceq X \preceq B \odot X \Leftrightarrow X \in \mathsf{Im}L_{A^*} \cap \mathsf{Im}\Lambda_{B_*}. \tag{25}$$

Proof. Direct application of Equivalence (16) (see Property 25) and of Equivalence (24) (see Property 33). □

Proposition 36. *Let $\mathcal{S}$ be a semiring and $A, B \in \mathcal{S}^{n\times n}$ and $X \in \mathcal{S}^{n\times m}$. If $\forall X$, the equality $B_* \mathbin{\bullet\!\!\backslash} (A^* \otimes X) = (B_* \mathbin{\bullet\!\!\backslash} A^*) \otimes X$ holds, then the mapping*

$$P : \mathcal{S}^{n\times m} \to \mathcal{S}^{n\times m}, X \mapsto (B_* \mathbin{\bullet\!\!\backslash} A^*)^* \mathbin{\circ\!\!\backslash} X,$$

is a projector in $\mathsf{Im}L_{A^*} \cap \mathsf{Im}\Lambda_{B_*}$, formally

$$P(X) = \{\bigvee Y | Y \preceq X \text{ and } Y \in \mathsf{Im}L_{A^*} \cap \mathsf{Im}\Lambda_{B_*}\}.$$

Proof. First, according to Equations (17) and (18), P is a projector on the image of $L_{(B_*\bullet\!\!\!\backslash A^*)^*}$, and $P(X) \preceq X$.

According to Definition 32, $B_* \preceq E^{\odot}$, then $B_*\bullet\!\!\!\backslash A^* \succeq E^{\odot}\bullet\!\!\!\backslash A^* = A^*$ and $(B_*\bullet\!\!\!\backslash A^*)^* \succeq (A^*)^* = A^*$, which, according to Equation (20), implies that $\mathsf{Im}L_{(B_*\bullet\!\!\!\backslash A^*)^*} \subset \mathsf{Im}L_{A^*}$, hence $P(X) \in \mathsf{Im}L_{A^*}$.

Since $P(X) \in \mathsf{Im}L_{(B_*\bullet\!\!\!\backslash A^*)^*}$, equality $P(X) = (B_*\bullet\!\!\!\backslash A^*)^*P(X)$ holds, and according to Lemma 26, this is equivalent to $P(X) \succeq (B_*\bullet\!\!\!\backslash A^*) \otimes P(X)$.

Because of the assumption, the equality :

$(B_*\bullet\!\!\!\backslash A^*) \otimes P(X) = B_*\bullet\!\!\!\backslash(A^* \otimes P(X))$ holds, furthermore $P(X) \in \mathsf{Im}L_{A^*}$, therefore $A^* \otimes P(X) = P(X)$, hence

$$P(X) \succeq (B_*\bullet\!\!\!\backslash A^*) \otimes P(X) = B_*\bullet\!\!\!\backslash(A^* \otimes P(X)) = B_*\bullet\!\!\!\backslash P(X).$$

Otherwise, $B_* \preceq E^{\odot}$, then

$$B_*\bullet\!\!\!\backslash P(X) \succeq E^{\odot}\bullet\!\!\!\backslash P(X) = P(X).$$

Hence, $P(X) = B_*\bullet\!\!\!\backslash P(X)$.

Furthermore, Proposition 34 gives :

$$P(X) = B_*\bullet\!\!\!\backslash P(X) = B_* \odot P(X),$$

then, by considering Equivalence (24), this implies that $P(X) \in \mathsf{Im}\Lambda_{B_*}$.

Now we show that $P(X)$ is the greatest element in $\mathsf{Im}L_{A^*} \cap \mathsf{Im}\Lambda_{B_*}$ less or equal to X.

Let $Y \in \mathsf{Im}L_{A^*} \cap \mathsf{Im}\Lambda_{B_*}$ such that $Y \preceq X$, hence according to Lemma 26 and Proposition 34, the following equalities hold :

$$Y = A^* \otimes Y = B_* \odot Y = B_*\bullet\!\!\!\backslash Y = B_*\bullet\!\!\!\backslash(A^*Y),$$

and because of the assumption $B_*\bullet\!\!\!\backslash(A^*Y) = (B_*\bullet\!\!\!\backslash A^*)Y$.

From Definition 9, $Y = (B_*\bullet\!\!\!\backslash A^*)Y \Rightarrow Y \preceq (B_*\bullet\!\!\!\backslash A^*)\circ\!\!\!\backslash Y$, and from Lemma 26, this is equivalent to $Y = (B_*\bullet\!\!\!\backslash A^*)^*\circ\!\!\!\backslash Y$. Mapping $L^{\sharp}_{(B_*\bullet\!\!\!\backslash A^*)^*}$ being an isotone mapping, the following implication holds : $Y \preceq X \Rightarrow (B_*\bullet\!\!\!\backslash A^*)^*\circ\!\!\!\backslash Y \preceq (B_*\bullet\!\!\!\backslash A^*)^*\circ\!\!\!\backslash X$ which means that if $Y \preceq X$ then $Y = (B_*\bullet\!\!\!\backslash A^*)^*\circ\!\!\!\backslash Y \preceq P(X)$. $\square$

Remark 37. *The previous result shows that $P(X_0)$ is the greatest solution of the following system of inequalities*

$$A \otimes X \preceq X \preceq B \odot X \text{ and } X \preceq X_0,$$

which is equivalent to

$$A^* \otimes X = B_* \odot X = X \text{ and } X \preceq X_0.$$

This projector can be useful to synthesize a controller for manufacturing systems subject to constraints. This kind of problem can be seen as a model matching problem (see [34, 35]) and is of practical interest in many industrial applications (see e.g. [4] for an example from high-throughput-screening).

5. Examples

The results introduced in the previous section are illustrated in two semirings of practical interest in control theory of discrete event systems.

Definition 38 (Semiring $\overline{\mathbb{Z}}_{\max}$). *According to Definition 2, the set $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the max operator as $\oplus$ and the classical sum as $\otimes$ is a complete idempotent semiring, denoted $\overline{\mathbb{Z}}_{\max}$, where $\varepsilon = -\infty$, $e = 0$ and $\top = +\infty$. The greatest lower bound is $a \wedge b = min(a, b)$, and $b \circ\!\!\backslash a = a - b$. Furthermore $a \odot b = a + b$ and $b \bullet\!\!\backslash a = a - b$. Hence, except ε and $\top$, all elements admit a multiplicative inverse a^{-1}, i.e, $a \otimes a^{-1} = a^{-1} \otimes a = e$ and $a \odot a^{-1} = a^{-1} \odot a = e$. As a consequence, the following distributivity properties hold : $c \otimes (a \wedge b) = (c \otimes a) \wedge (c \otimes b)$, $c \odot (a \wedge b) = (c \odot a) \wedge (c \odot b)$ and $c \odot (a \oplus b) = (c \odot a) \oplus (c \odot b)$. Obviously, this is not true in the matrix case.*

Example 39. *Let $A = \begin{pmatrix} 1 & \top & 3 \\ 4 & \varepsilon & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ be matrices with entries in $\overline{\mathbb{Z}}_{\max}$. The product of these matrices is:*

$$A \otimes B = \begin{pmatrix} (1 \otimes 8) \oplus (\top \otimes 9) \oplus (3 \otimes 10) \\ (4 \otimes 8) \oplus (\varepsilon \otimes 9) \oplus (6 \otimes 10) \end{pmatrix} = \begin{pmatrix} \top \\ 16 \end{pmatrix},$$

and the dual product yields

$$A \odot B = \begin{pmatrix} (1 \odot 8) \wedge (\top \odot 9) \wedge (3 \odot 10) \\ (4 \odot 8) \wedge (\varepsilon \odot 9) \wedge (6 \odot 10) \end{pmatrix} = \begin{pmatrix} 9 \\ \varepsilon \end{pmatrix}.$$

The greatest solution of $C \otimes X \preceq B$ is given by

$$C \circ\!\!\!\backslash B = \begin{pmatrix} (1 \circ\!\!\!\backslash 8) \wedge (3 \circ\!\!\!\backslash 9) \wedge (5 \circ\!\!\!\backslash 10) \\ (2 \circ\!\!\!\backslash 8) \wedge (4 \circ\!\!\!\backslash 9) \wedge (6 \circ\!\!\!\backslash 10) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix},$$

and the smallest solution of $C \odot X \succeq B$ is given by

$$C \bullet\!\!\!\backslash B = \begin{pmatrix} (1 \bullet\!\!\!\backslash 8) \oplus (3 \bullet\!\!\!\backslash 9) \oplus (5 \bullet\!\!\!\backslash 10) \\ (2 \bullet\!\!\!\backslash 8) \oplus (4 \bullet\!\!\!\backslash 9) \oplus (6 \bullet\!\!\!\backslash 10) \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}.$$

Remark 40. *The dual product can be used to perform residuation of matrices in the $(max, plus)$ algebra (see [16]). More precisely, in this particular case, $A \circ\!\!\!\backslash B = -A^T \odot B$.*

Definition 41 (Semiring $\gamma^*\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$, [1], §5.3.2). *According to Definition 6, the set of non-decreasing formal power series in one variable γ with coefficients in the semiring $\overline{\mathbb{Z}}_{\max}$ and exponents in $\mathbb{Z}$ is a semiring denoted $\gamma^*\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$, where $\gamma^* = \bigoplus_{i \in \mathbb{N}_0} \gamma^i$ (see Definition 24). The neutral element of addition is the series $\varepsilon(\gamma) = \bigoplus_{k \in \mathbb{Z}} \varepsilon\gamma^k$ and the neutral element of multiplication is the series $e(\gamma) = \bigoplus_{k \in \mathbb{N}_0} e\gamma^k$, furthermore $\top(\gamma) = \bigoplus_{k \in \mathbb{Z}} \top\gamma^k$. The monomials are defined as $\gamma^*(t\gamma^n) = \bigoplus_{k \in \mathbb{N}_0} t\gamma^{n+k}$. In order to keep notation simple, this will be denoted $t\gamma^n$ in the sequel of this paper. In the same way, a series will be simply denoted $s = \bigoplus_{i \in I_S} t_i\gamma^{n_i}$, where $I_S \subset \mathbb{N}_0$. The computational rules between monomials are the following :*

$$t_1\gamma^n \oplus t_2\gamma^n = max(t_1, t_2)\gamma^n, \quad t_1\gamma^{n_1} \otimes t_2\gamma^{n_2} = (t_1 + t_2)\gamma^{n_1+n_2}, \tag{26}$$

$$t_1\gamma^{n_1} \wedge t_2\gamma^{n_2} = min(t_1, t_2)\gamma^{max(n_1,n_2)}, \quad t_1\gamma^{n_1} \odot t_2\gamma^{n_2} = (t_1 + t_2)\gamma^{n_1+n_2}, \tag{27}$$

$$(t_1\gamma^{n_1}) \circ\!\!\!\backslash (t_2\gamma^{n_2}) = (t_2 - t_1)\gamma^{n_2-n_1}, \quad (t_1\gamma^{n_1}) \bullet\!\!\!\backslash (t_2\gamma^{n_2}) = (t_2 - t_1)\gamma^{n_2-n_1}. \tag{28}$$

Furthermore, the order relation is such that $t_1\gamma^{n_1} \succeq t_2\gamma^{n_2} \Leftrightarrow n_1 \leq n_2$ and $t_1 \geq t_2$. According to these rules, a non decreasing series admits many representations (e.g., $2\gamma^2 \oplus 3\gamma^2 = 3\gamma^2$) and one of which is canonical. It is the representation whose $t_0 < t_1 < \ldots$ and $n_0 < n_1 < \ldots$. The computation rules between *two series $s = \bigoplus_{i \in I_S} t_i \gamma^{n_i}$ and $s' = \bigoplus_{j \in I_{S'}} t_j \gamma^{n_j}$ are given as follows :*

$$\begin{aligned}
s \oplus s' &= \bigoplus_{i \in I_S} t_i \gamma^{n_i} \oplus \bigoplus_{j \in I_{S'}} t_j \gamma^{n_j}, && (29)\\
s \otimes s' &= \bigoplus_{i \in I_S} \bigoplus_{j \in I_{S'}} (t_i + t_j) \gamma^{n_i + n_j}, && (30)\\
s \wedge s' &= \bigoplus_{i \in I_S} \bigoplus_{j \in I_{S'}} min(t_i, t_j) \gamma^{max(n_i, n_j)}, && (31)\\
s \circ\!\!\!/\, s' = s' \circ\!\!\!\backslash\, s &= \bigwedge_{j \in I_{S'}} \bigoplus_{i \in I_S} (t_i - t_j) \gamma^{n_i - n_j}. && (32)
\end{aligned}$$

According to Definition 27, the dual product has to distribute with respect to the operator $\wedge$, hence it is only defined between a monomial and a series in the following way :

$$t\gamma^n \odot s = \bigoplus_{i \in I_S} (t + t_i) \gamma^{n + n_i}. \quad (33)$$

It can be checked that $a \odot (s \wedge s') = (a \odot s) \wedge (a \odot s')$. The dual residual is then given by :

$$t\gamma^n \bullet\!\!\!\backslash\, s = \bigoplus_{i \in I_S} (t_i - t) \gamma^{n_i - n}. \quad (34)$$

In [11], periodic series were introduced. They are defined as $s = p \oplus q \otimes r^*$ where $p = \bigoplus_{i=1}^{m} t_i \gamma^{n_i}$ (respectively $q = \bigoplus_{j=1}^{l} t_j \gamma^{n_j}$) is a polynomial depicting the transient (resp. the periodic) behavior, and $r = \tau\gamma^\nu$ is a monomial depicting the periodicity allowing to define the asymptotic slope of the series as $\sigma_\infty(s) = \nu/\tau$. Sum, product, Kleene star and residuation of periodic series are periodic series (see [17]), and algorithms and software toolboxes are available in order to handle them (see [13]). In the same way, the dual product and its dual residual are well *defined. Below, only properties concerning asymptotic slopes are recalled:*

$$
\begin{array}{lcl}
\sigma_\infty(s \oplus s') & = & \min(\sigma_\infty(s), \sigma_\infty(s')), \\
\sigma_\infty(s \otimes s') & = & \min(\sigma_\infty(s), \sigma_\infty(s')), \\
\sigma_\infty(s \wedge s') & = & \max(\sigma_\infty(s), \sigma_\infty(s')), \\
\sigma_\infty(s^*) & = & \min(\underset{i=1..m}{min}(n_i/t_i), \underset{j=1..l}{min}(n_j/t_j), \sigma_\infty(s)), \\
\sigma_\infty(t\gamma^n \odot s) & = & \sigma_\infty(s) \\
\sigma_\infty((t\gamma^n) \bullet\!\!\!\backslash s) & = & \sigma_\infty(s) \\
\text{if } \sigma_\infty(s) \leq \sigma_\infty(s') & \text{then} & \sigma_\infty(s' \circ\!\!\!\backslash s) = \sigma_\infty(s), \text{ else } \ s' \circ\!\!\!\backslash s = \varepsilon.
\end{array}
$$

Example 42. *Let* $B = \begin{pmatrix} \top & 15\gamma^3 & 7\gamma^0 & \top \\ \top & \top & \top & \top \\ 3\gamma^0 & 8\gamma^4 & \top & \top \\ 6\gamma & 4\gamma^5 & \top & \top \end{pmatrix}$ *be a matrix where the entries are monomials in* $\gamma^*\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$*. According to Definitions 41 and 28. It can be checked that :*

$$
B^{\odot 2} = \begin{pmatrix} 10\gamma^0 & 15\gamma^4 & \top & \top \\ \top & \top & \top & \top \\ \top & 18\gamma^3 & 10\gamma^0 & \top \\ \top & 21\gamma^4 & 13\gamma & \top \end{pmatrix} \text{ and } B^{\odot 3} = \begin{pmatrix} \top & 25\gamma^3 & 17\gamma^0 & \top \\ \top & \top & \top & \top \\ 13\gamma^0 & 18\gamma^4 & \top & \top \\ 16\gamma & 21\gamma^5 & \top & \top \end{pmatrix}
$$

It can be also checked that $B^{\odot n} \succeq B^{\odot 3} \forall n > 3$*, hence :*

$$
B_* = E^\odot \wedge B \wedge B^{\odot 2} \wedge B^{\odot 3} \wedge \ldots = \begin{pmatrix} e & 15\gamma^4 & 7\gamma^0 & \top \\ \top & e & \top & \top \\ 3\gamma^0 & 8\gamma^4 & e & \top \\ 6\gamma & 4\gamma^5 & 13\gamma & e \end{pmatrix}.
$$

Note that, due to the computation rules (27), the entries of matrix B_* *are always monomials.*

Remark 43. *From these examples, it can be seen that the assumption of Proposition 31, i.e., that* $b_{ij} \bullet\!\!\!\backslash (a \otimes x) = (b_{ij} \bullet\!\!\!\backslash a) \otimes x$*, is clearly fulfilled in the semiring* $\overline{\mathbb{Z}}_{\max}$ *(indeed* $b_{ij} - (a + x) = (b_{ij} - a) + x$*). In* $\gamma^*\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$*, the dual product is only defined between monomials and series. Hence by considering monomial* $b_{ij} = t\gamma^n$*, series* $a = \bigoplus_{i \in I_A} t_i \gamma^{n_i}$ *and* $x = \bigoplus_{j \in I_X} t_j \gamma^{n_j}$*, and according to*

Equations (30) and (34) the following equalities hold :

$$\begin{aligned}(t\gamma^n)\lcirclearrowright(a\otimes x) &= (t\gamma^n)\lcirclearrowright(\bigoplus_{i\in I_A} t_i\gamma^{n_i}\otimes\bigoplus_{j\in I_X} t_j\gamma^{n_j}) = (t\gamma^n)\lcirclearrowright(\bigoplus_{i\in I_A}\bigoplus_{j\in I_X}(t_i+t_j)\gamma^{n_i+n_j})\\ &= \bigoplus_{i\in I_A}\bigoplus_{j\in I_X}(t_i+t_j-t)\gamma^{n_i+n_j-n} = (\bigoplus_{i\in I_A}(t_i-t)\gamma^{n_i-n})\otimes\bigoplus_{j\in I_X} t_j\gamma^{n_j}\\ &= ((t\gamma^n)\lcirclearrowright a)\otimes x.\end{aligned}$$

The assumption $B_*\lcirclearrowright(A^*\otimes X) = (B_*\lcirclearrowright A^*)\otimes X$ used in Proposition 36 is still valid in $\overline{\mathbb{Z}}_{\max}$ since B_* is with entries in the semiring $\overline{\mathbb{Z}}_{\max}$. In the same way, it also holds in $\gamma^*\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ since all entries of B are assumed to be monomials and, as noticed in Example 42, under this assumption all entries of B_* are monomials.

6. Interval Analysis over idempotent semirings

Interval mathematics was pioneered by R.E. Moore (see [32]) as a tool for bounding rounding errors in computer programs. Since then, interval mathematics has been developed into a general methodology for investigating numerical uncertainty in many problems and algorithms [21]. In [24] idempotent semirings were extended to interval arithmetic (see also [29]). Below some preliminary statements are recalled from this reference.

Definition 44 (Interval). *Let $\mathcal{S}$ be a semiring. A (closed) interval is a set of the form $\mathbf{x} = [\underline{x}, \overline{x}] = \{t \in \mathcal{S} | \underline{x} \preceq t \preceq \overline{x}\}$, where $\underline{x} \in \mathcal{S}$ and $\overline{x} \in \mathcal{S}$ (with $\underline{x} \preceq \overline{x}$) are called the lower and the upper bounds of the interval $\mathbf{x}$, respectively.*

Definition 45 (Semiring of intervals). *The set of intervals denoted by $\mathrm{I}\mathcal{S}$, endowed with the following element-wise algebraic operations*

$$\mathbf{x}\mathbin{\bar{\oplus}}\mathbf{y} \triangleq [\underline{x}\oplus\underline{y}, \overline{x}\oplus\overline{y}] \qquad \text{and} \qquad \mathbf{x}\mathbin{\bar{\otimes}}\mathbf{y} \triangleq [\underline{x}\otimes\underline{y}, \overline{x}\otimes\overline{y}] \tag{35}$$

is a semiring, where the intervals $\boldsymbol{\varepsilon} = [\varepsilon, \varepsilon]$ and $\mathbf{e} = [e, e]$ are the neutral elements of $\mathrm{I}\mathcal{S}$. The canonical order $\preceq_{\mathrm{I}\mathcal{S}}$ induced by the additive law is such that $\mathbf{x}\mathbin{\bar{\oplus}}\mathbf{y} = [\underline{x}\oplus\underline{y}, \overline{x}\oplus\overline{y}] \Leftrightarrow \mathbf{x}\preceq_{\mathrm{I}\mathcal{S}}\mathbf{y} \Leftrightarrow \underline{x}\preceq_{\mathcal{S}}\underline{y}$ and $\overline{x}\preceq_{\mathcal{S}}\overline{y}$, where $\preceq_{\mathcal{S}}$ is the order relation in $\mathcal{S}$.

Remark 46. *In the sequel, in the absence of ambiguity, the order relation in* $\mathrm{I}\mathcal{S}$ *will be denoted* $\preceq$. *Operations (35) give the tightest intervals containing all results of the same operations to arbitrary elements of its interval operands.*

Remark 47. *Let* $\mathcal{S}$ *be a complete semiring and* $\{\mathbf{x}_\alpha\}$ *be an infinite subset of* $\mathrm{I}\mathcal{S}$, *the infinite sum of elements of this subset is :*

$$\overline{\bigoplus_\alpha} \mathbf{x}_\alpha = \left[\bigoplus_\alpha \underline{x}_\alpha, \bigoplus_\alpha \overline{x}_\alpha\right].$$

The top element is given by $\mathsf{T} = [\top, \top]$.

Remark 48. *Note that if* $\mathbf{x}$ *and* $\mathbf{y}$ *are intervals in* $\mathrm{I}\mathcal{S}$, *then* $\mathbf{x} \subset \mathbf{y}$ *iff* $\underline{y} \preceq \underline{x} \preceq \overline{x} \preceq \overline{y}$. *In particular,* $\mathbf{x} = \mathbf{y}$ *iff* $\underline{x} = \underline{y}$ *and* $\overline{x} = \overline{y}$.

Remark 49. *An interval for which* $\underline{x} = \overline{x}$ *is called degenerate. Degenerate intervals allow to represent numbers without uncertainty. In this case* $\mathbf{x}$ *will be simply denoted* x.

Remark 50. $\mathrm{I}\mathcal{S}$ *is not a semifield even if* $\mathcal{S}$ *is one. Indeed, except for degenerate intervals, an interval does not admit a multiplicative inverse.*

Definition 51 (Dual product over semiring $\mathrm{I}\mathcal{S}$). *In a semiring of intervals, the dual product* $\odot$ *is defined as :* $\mathbf{x} \mathbin{\bar{\odot}} \mathbf{y} \triangleq \left[\underline{x} \odot \underline{y}, \overline{x} \odot \overline{y}\right]$, *where* $\odot$ *is the dual product in* $\mathcal{S}$.

In [26] (see also [23]), it has been shown that order preserving mappings admit a natural extension over the semirings of intervals by considering the image of the interval bounds in an independent way. Especially the additive closure and $\wedge$-closure can be computed in an efficient way and are defined as follows.

Proposition 52 ([26],[23]). *Let* $\mathrm{I}\mathcal{S}$ *be a semiring of intervals. The additive closure of matrix* $\mathbf{A} \in \mathrm{I}\mathcal{S}^{n\times n}$ *is given by :*

$$\mathbf{A}^* = [\underline{A}, \overline{A}]^* = [\underline{A}^*, \overline{A}^*],$$

and its $\wedge$*-closure is :*

$$\mathbf{A}_* = [\underline{A}, \overline{A}]_* = [\underline{A}_*, \overline{A}_*].$$

Notation 53 (Semiring of pairs). *Let $\mathcal{S}$ be a complete semiring. The set of pairs (x', x'') with $x' \in \mathcal{S}$ and $x'' \in \mathcal{S}$ is a complete semiring denoted by $\mathcal{C}(\mathcal{S})$ with $(\varepsilon, \varepsilon)$ as the zero element, (e, e) as the identity element and $(\top, \top)$ as top element (see Definition 2). The set of pairs (x', x'') such that $x' \preceq x''$ is a complete subsemiring of $\mathcal{C}(\mathcal{S})$ (see Definition 3). It will be denoted $\mathcal{C}_0(\mathcal{S})$.*

Proposition 54. *The canonical injection $\mathsf{Id}_{|\mathcal{C}_0(\mathcal{S})} : \mathcal{C}_0(\mathcal{S}) \to \mathcal{C}(\mathcal{S})$ is both residuated and dually residuated. Its residual $(\mathsf{Id}_{|\mathcal{C}_0(\mathcal{S})})^\sharp$ is a projector. Its practical computation is given by :*

$$(\mathsf{Id}_{|\mathcal{C}_0(\mathcal{S})})^\sharp((x', x'')) \quad = \quad (x' \wedge x'', x'') = (\widetilde{x}', \widetilde{x}''). \tag{36}$$

Its dual residual $(\mathsf{Id}_{|\mathcal{C}_0(\mathcal{S})})^\flat$ is a projector. Its practical computation is given by :

$$(\mathsf{Id}_{|\mathcal{C}_0(\mathcal{S})})^\flat((x', x'')) \quad = \quad (x', x' \oplus x'') = (\widetilde{x}', \widetilde{x}''). \tag{37}$$

Proof. This theorem is a direct application of Proposition 17, since $\mathcal{C}_0(\mathcal{S})$ is a subsemiring of $\mathcal{C}(\mathcal{S})$. Practically, let us consider $(x', x'') \in \mathcal{C}(\mathcal{S})$, we have $(\mathsf{Id}_{|\mathcal{C}_0(\mathcal{S})})^\sharp((x', x'')) = (\widetilde{x}', \widetilde{x}'') = (x' \wedge x'', x'')$, which is the greatest pair such that :

$$\widetilde{x}' \preceq x', \quad \widetilde{x}'' \preceq x'' \ \text{ and } \ \widetilde{x}' \preceq \widetilde{x}''.$$

On the other hand, we have $(\mathsf{Id}_{|\mathcal{C}_0(\mathcal{S})})^\flat((x', x'')) = (\widetilde{x}', \widetilde{x}'') = (x', x' \oplus x'')$, which is the smallest pair such that :

$$\widetilde{x}' \succeq x', \quad \widetilde{x}'' \succeq x'' \ \text{ and } \ \widetilde{x}'' \succeq \widetilde{x}'.$$

$\square$

Proposition 55 ([18]). *Mapping $L_{(a', a'')} : \mathcal{C}_0(\mathcal{S}) \to \mathcal{C}_0(\mathcal{S}), (x', x'') \mapsto (a' \otimes x', a'' \otimes x'')$ with $(a', a'') \in \mathcal{C}_0(\mathcal{S})$ is residuated. Its residual is equal to*

$$L^\sharp_{(a', a'')} : \mathcal{C}_0(\mathcal{S}) \to \mathcal{C}_0(\mathcal{S}), (x', x'') \mapsto (a' \diagdown x' \wedge a'' \diagdown x'', a'' \diagdown x''). \tag{38}$$

Proposition 56 ([18]). *Let* $\mathrm{I}\mathcal{S}$ *be a semiring of intervals. Mapping* $L_{\mathbf{a}} : \mathrm{I}\mathcal{S} \to \mathrm{I}\mathcal{S}, \mathbf{x} \mapsto \mathbf{a} \bar{\otimes} \mathbf{x}$ *is residuated. Its residual is equal to*

$$L_{\mathbf{a}}^{\sharp} : \mathrm{I}\mathcal{S} \to \mathrm{I}\mathcal{S}, \mathbf{x} \mapsto \mathbf{a} \overline{\circ\!\!\!\backslash} \mathbf{x} = [\underline{a} \circ\!\!\!\backslash \underline{x} \wedge \overline{a} \circ\!\!\!\backslash \overline{x}, \overline{a} \circ\!\!\!\backslash \overline{x}].$$

Therefore, $\mathbf{a} \overline{\circ\!\!\!\backslash} \mathbf{b}$ *is the greatest solution of* $\mathbf{a} \bar{\otimes} \mathbf{x} \preceq \mathbf{b}$*, and equality is achieved if* $\mathbf{b} \in \mathsf{Im} L_{\mathbf{a}}$*.*

Remark 57. *In the same manner, it can be shown that mapping* $R_{\mathbf{a}} : \mathrm{I}\mathcal{S} \to \mathrm{I}\mathcal{S}, \mathbf{x} \mapsto \mathbf{x} \bar{\otimes} \mathbf{a}$ *is residuated.*

Proposition 58. *Mapping* $\Lambda_{(a',a'')} : \mathcal{C}_0(\mathcal{S}) \to \mathcal{C}_0(\mathcal{S}), (x', x'') \mapsto (a' \odot x', a'' \odot x'')$ *with* $(a', a'') \in \mathcal{C}_0(\mathcal{S})$ *is dually residuated. Its dual residual is equal to*

$$\Lambda_{(a',a'')}^{\flat} : \mathcal{C}_0(\mathcal{S}) \to \mathcal{C}_0(\mathcal{S}), (x', x'') \mapsto (a' \bullet\!\!\!\backslash x', a' \bullet\!\!\!\backslash x' \oplus a'' \bullet\!\!\!\backslash x''). \tag{39}$$

Proof. According to Corollary 30, mapping $\Lambda_{(a',a'')} : \mathcal{C}(\mathcal{S}) \to \mathcal{C}(\mathcal{S}), (x', x'') \mapsto (a' \odot x', a'' \odot x'')$ is dually residuated and its dual residual is $\Lambda_{(a',a'')}^{\flat} : \mathcal{C}(\mathcal{S}) \to \mathcal{C}(\mathcal{S}), (x', x'') \mapsto (a' \bullet\!\!\!\backslash x', a'' \bullet\!\!\!\backslash x'')$. Mapping $\Lambda_{(a',a'')}$ is order preserving, hence $\mathsf{Im}\Lambda_{(a',a'')|\mathcal{C}_0(\mathcal{S})} \in \mathcal{C}_0(\mathcal{S})$. Furthermore, the canonical injection $\mathsf{Id}_{|\mathcal{C}_0(\mathcal{S})} : \mathcal{C}_0(\mathcal{S}) \to \mathcal{C}(\mathcal{S})$ is dually residuated. Hence Proposition 18 yields

$$({}_{\mathcal{C}_0(\mathcal{S})|}\Lambda_{(a',a'')|\mathcal{C}_0(\mathcal{S})})^{\flat} = ({}_{\mathcal{C}_0(\mathcal{S})|}\Lambda_{(a',a'')} \circ \mathsf{Id}_{|\mathcal{C}_0(\mathcal{S})})^{\flat} = (\mathsf{Id}_{|\mathcal{C}_0(\mathcal{S})})^{\flat} \circ (\Lambda_{(a',a'')})^{\flat} \circ \mathsf{Id}_{|\mathcal{C}_0(\mathcal{S})}.$$

To conclude, Equation (37) of Proposition 54 yields equation (39). □

Proposition 59. *Let* $\mathcal{S}$ *be a semiring and* $\mathrm{I}\mathcal{S}$ *be a semiring of intervals. Mapping* $\Lambda_{\mathbf{a}} : \mathrm{I}\mathcal{S} \to \mathrm{I}\mathcal{S}, \mathbf{x} \mapsto \mathbf{a} \bar{\odot} \mathbf{x}$ *is dually residuated. Its dual residual is equal to*

$$\Lambda_{\mathbf{a}}^{\flat} : \mathrm{I}\mathcal{S} \to \mathrm{I}\mathcal{S}, \mathbf{x} \mapsto \mathbf{a} \overline{\bullet\!\!\!\backslash} \mathbf{x} = [\underline{a} \bullet\!\!\!\backslash \underline{x}, \underline{a} \bullet\!\!\!\backslash \underline{x} \oplus \overline{a} \bullet\!\!\!\backslash \overline{x}].$$

Therefore, $\mathbf{a} \overline{\bullet\!\!\!\backslash} \mathbf{b}$ *is the smallest solution of* $\mathbf{a} \bar{\odot} \mathbf{x} \succeq \mathbf{b}$*, and equality is achieved if* $\mathbf{b} \in \mathsf{Im}\Lambda_{\mathbf{a}}$*.*

Proof. Let $\Psi : \mathcal{C}_0(\mathcal{S}) \to \mathrm{I}\mathcal{S}, (\widetilde{x}', \widetilde{x}'') \mapsto [\underline{x}, \overline{x}] = [\widetilde{x}', \widetilde{x}'']$ be the mapping which maps an ordered pair to an interval. This mapping defines an isomorphism, since it is sufficient to deal with the bounds to handle an interval. Then the result follows directly from Proposition 58. □

Corollary 60. *Let $\mathcal{S}$ be a semiring and $\mathbf{A} \in \mathrm{I}\mathcal{S}^{n\times p}$, $\mathbf{X} \in \mathrm{I}\mathcal{S}^{p\times q}$ and $\mathbf{Y} \in \mathrm{I}\mathcal{S}^{n\times q}$ be matrices. According to Corollary 30, mapping $\Lambda_{\mathbf{A}} : \mathrm{I}\mathcal{S}^{p\times q} \to \mathrm{I}\mathcal{S}^{n\times q}, \mathbf{X} \mapsto \mathbf{A} \mathbin{\bar{\odot}} \mathbf{X}$ is dually residuated. Its dual residual is equal to*

$$\Lambda^\flat_{\mathbf{A}} : \mathrm{I}\mathcal{S}^{n\times q} \to \mathrm{I}\mathcal{S}^{p\times q}, \mathbf{Y} \mapsto \mathbf{A} \overline{\bullet\!\!\!\!\backslash} \mathbf{Y} = [\underline{A} \bullet\!\!\!\!\backslash \underline{Y}, \underline{A} \bullet\!\!\!\!\backslash \underline{Y} \oplus \overline{A} \bullet\!\!\!\!\backslash \overline{Y}]. \tag{40}$$

Additive closure and residuation being well defined over a semiring of intervals the Properties 25 can be translated as follows.

Property 61. *Let $\mathbf{A} \in \mathrm{I}\mathcal{S}^{n\times n}$, $\mathbf{B} \in \mathrm{I}\mathcal{S}^{n\times n}$, $\mathbf{C} \in \mathrm{I}\mathcal{S}^{n\times n}$, and $\mathbf{X} \in \mathrm{I}\mathcal{S}^{n\times p}$ be four matrices. The following statements hold:*

$$\mathbf{A}^* \overline{\otimes} \mathbf{A}^* \overline{\otimes} \mathbf{X} = \mathbf{A}^* \overline{\otimes} \mathbf{X}, \tag{41}$$

$$\mathbf{A}^* \overline{\circ\!\!\!\!\backslash} \mathbf{A}^* \overline{\circ\!\!\!\!\backslash} \mathbf{X} = \mathbf{A}^* \overline{\circ\!\!\!\!\backslash} \mathbf{X}, \tag{42}$$

$$\mathbf{A}^* \overline{\otimes} (\mathbf{A}^* \overline{\circ\!\!\!\!\backslash} \mathbf{X}) = \mathbf{A}^* \overline{\circ\!\!\!\!\backslash} \mathbf{X}, \tag{43}$$

$$\mathbf{A}^* \overline{\circ\!\!\!\!\backslash} (\mathbf{A}^* \overline{\otimes} \mathbf{X}) = \mathbf{A}^* \overline{\otimes} \mathbf{X}, \tag{44}$$

$$\mathbf{C}^* \preceq \mathbf{A}^* \Leftrightarrow \mathbf{A}^*\mathbf{C}^*\mathbf{X} = \mathbf{A}^*\mathbf{X} = \mathbf{C}^* \circ\!\!\!\!\backslash (\mathbf{A}^*\mathbf{X}) \Leftrightarrow \mathsf{Im} L_{\mathbf{A}^*} \subset \mathsf{Im} L_{\mathbf{C}^*} \Leftrightarrow \mathsf{Im} L_{\mathbf{A}^*} \subset \mathsf{Im} L^\sharp_{\mathbf{C}^*}. \tag{45}$$

For the dual product the following property can be stated :

$$\mathbf{B}_* \overline{\odot} \mathbf{B}_* \overline{\odot} \mathbf{X} = \mathbf{B}_* \overline{\odot} \mathbf{X}, \tag{46}$$

and the following equivalences hold

$$\begin{array}{lclclcl} \mathbf{A} \overline{\otimes} \mathbf{X} \preceq \mathbf{X} & \Leftrightarrow & \mathbf{X} = \mathbf{A}^* \overline{\otimes} \mathbf{X} & \Leftrightarrow & \mathbf{A}^* \overline{\circ\!\!\!\!\backslash} \mathbf{X} & \Leftrightarrow & \mathbf{X} \in \mathsf{Im} L_{\mathbf{A}^*}, \\ \mathbf{X} \preceq \mathbf{B} \overline{\odot} \mathbf{X} & \Leftrightarrow & \mathbf{X} = \mathbf{B}_* \overline{\odot} \mathbf{X} & \Leftrightarrow & \mathbf{B}_* \overline{\bullet\!\!\!\!\backslash} \mathbf{X} & \Leftrightarrow & \mathbf{X} \in \mathsf{Im} \Lambda_{\mathbf{B}_*}. \end{array} \tag{47}$$

Remark 62. *According to Proposition 56 and 59, the following implications hold :*

$$\begin{array}{rcl} \mathbf{X} \in \mathsf{Im} L_{\mathbf{A}^*} \Rightarrow \mathbf{X} & = & [\underline{A}^* \underline{X}, \overline{A}^* \overline{X}] = [\underline{A}^* \circ\!\!\!\!\backslash \underline{X} \wedge \overline{A}^* \circ\!\!\!\!\backslash \overline{X}, \overline{A}^* \circ\!\!\!\!\backslash \overline{X}] \\ & = & [\underline{A}^* \circ\!\!\!\!\backslash \underline{X}, \overline{A}^* \circ\!\!\!\!\backslash \overline{X}] \text{ since } \underline{A}^* \underline{X} \preceq \overline{A}^* \overline{X}, \\ \mathbf{X} \in \mathsf{Im} \Lambda_{\mathbf{B}_*} \Rightarrow \mathbf{X} & = & [\underline{B}_* \odot \underline{X}, \overline{B}_* \odot \overline{X}] = [\underline{B}_* \bullet\!\!\!\!\backslash \underline{X}, \underline{B}_* \circ\!\!\!\!\backslash \underline{X} \oplus \overline{B}_* \bullet\!\!\!\!\backslash \overline{X}] \\ & = & [\underline{B}_* \bullet\!\!\!\!\backslash \underline{X}, \overline{B}_* \bullet\!\!\!\!\backslash \overline{X}] \text{ since } \underline{B}_* \odot \underline{X} \preceq \overline{B}_* \odot \overline{X}. \end{array}$$

Below, the extension of Proposition 36 to a semiring of intervals is given.

Proposition 63. *Let $\mathcal{S}$ be a semiring and $\mathbf{A}, \mathbf{B} \in \mathrm{I}\mathcal{S}^{n\times n}$ and $\mathbf{X} \in \mathcal{S}^{n\times m}$. If $\forall \mathbf{X}$ the equality $\mathbf{B}_* \overline{\bullet\!\!\!\!\backslash} (\mathbf{A}^* \overline{\otimes} \mathbf{X}) = (\mathbf{B}_* \overline{\bullet\!\!\!\!\backslash} \mathbf{A}^*) \overline{\otimes} \mathbf{X}$ holds, mapping*

$$\mathbf{P} \quad : \quad \mathrm{I}\mathcal{S}^{n\times m} \quad \to \quad \mathrm{I}\mathcal{S}^{n\times m}, \mathbf{X} \quad \mapsto \quad (\mathbf{B}_* \overline{\bullet\!\!\!\!\backslash} \mathbf{A}^*)^* \overline{\circ\!\!\!\!\backslash} \mathbf{X}$$

with

$$(\mathbf{B}_* \overline{\bullet\!\!\!\!\backslash} \mathbf{A}^*)^* \overline{\circ\!\!\!\!\backslash} \mathbf{X} \quad = \quad [((\underline{B}_* \bullet\!\!\!\!\backslash \underline{A}^*)^* \circ\!\!\!\!\backslash \underline{X}) \wedge (((\underline{B}_* \bullet\!\!\!\!\backslash \underline{A}^*) \oplus (\overline{B}_* \bullet\!\!\!\!\backslash \overline{A}^*))^* \circ\!\!\!\!\backslash \overline{X}), ((\underline{B}_* \bullet\!\!\!\!\backslash \underline{A}^*) \oplus (\overline{B}_* \bullet\!\!\!\!\backslash \overline{A}^*))^* \circ\!\!\!\!\backslash \overline{X}],$$

is a projector in $\mathsf{Im}L_{\mathbf{A}^*} \cap \mathsf{Im}\Lambda_{\mathbf{B}_*}$, formally

$$\mathbf{P}(\mathbf{X}) = \{\bigvee \mathbf{Y} | \mathbf{Y} \preceq_{\mathrm{I}\mathcal{S}} \mathbf{X} \text{ and } \mathbf{Y} \in \mathsf{Im}L_{\mathbf{A}^*} \cap \mathsf{Im}\Lambda_{\mathbf{B}_*}\}.$$

PROOF. It is a direct application of Proposition 36. For the practical computation, from Proposition 56, we get :

$$(\mathbf{B}_* \overline{\bullet\!\!\!\!\backslash} \mathbf{A}^*)^* \overline{\circ\!\!\!\!\backslash} \mathbf{X} = [((\underline{\mathbf{B}_* \overline{\bullet\!\!\!\!\backslash} \mathbf{A}^*})^* \circ\!\!\!\!\backslash \underline{X}) \wedge ((\overline{\mathbf{B}_* \overline{\bullet\!\!\!\!\backslash} \mathbf{A}^*})^* \circ\!\!\!\!\backslash \overline{X}), ((\overline{\mathbf{B}_* \overline{\bullet\!\!\!\!\backslash} \mathbf{A}^*})^* \circ\!\!\!\!\backslash \overline{X})]$$

with, according to Propositions 59 and 52,

$$(\underline{\mathbf{B}_* \overline{\bullet\!\!\!\!\backslash} \mathbf{A}^*})^* = (\underline{B}_* \bullet\!\!\!\!\backslash \underline{A}^*)^*$$

and

$$(\overline{\mathbf{B}_* \overline{\bullet\!\!\!\!\backslash} \mathbf{A}^*})^* = ((\underline{B}_* \bullet\!\!\!\!\backslash \underline{A}^*) \oplus (\overline{B}_* \bullet\!\!\!\!\backslash \overline{A}^*))^*.$$

□

Example 64. *Below, we compute the greatest interval vector which satisfies :*

$$\mathbf{A} \overline{\otimes} \mathbf{X} \preceq \mathbf{X} \preceq \mathbf{B} \overline{\odot} \mathbf{X}$$
$$\mathbf{X} \preceq \mathbf{X_0},$$

where

$$\mathbf{A} = \begin{pmatrix} [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] \\ [7,11] & [\varepsilon,\varepsilon] & [8,14] & [\varepsilon,\varepsilon] & [2,7] \\ [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] \\ [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] & [4,12] & [\varepsilon,\varepsilon] & [1,5] \\ [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} [\top,\top] & [\top,\top] & [\top,\top] & [\top,\top] & [\top,\top] \\ [11,16] & [\top,\top] & [15,19] & [\top,\top] & [7,10] \\ [\top,\top] & [\top,\top] & [\top,\top] & [\top,\top] & [\top,\top] \\ [\top,\top] & [\top,\top] & [13,18] & [\top,\top] & [5,9] \\ [\top,\top] & [\top,\top] & [\top,\top] & [\top,\top] & [\top,\top] \end{pmatrix}$$

and $\mathbf{X_0} = \begin{pmatrix} [10,14] & [10,14] & [10,14] & [10,14] & [10,14] \end{pmatrix}^T$.
We get :

$$(\underline{B}_* \backslash\!\!\!\bullet\, \underline{A}^*)^* = \begin{pmatrix} e & -11 & -3 & -14 & -9 \\ 7 & e & 8 & -3 & 2 \\ -8 & -15 & e & -13 & -12 \\ 1 & -6 & 4 & e & 1 \\ e & -7 & 1 & -5 & e \end{pmatrix}, \ (\overline{B}_* \backslash\!\!\!\bullet\, \overline{A}^*)^* = \begin{pmatrix} e & -16 & -2 & -18 & -9 \\ 11 & e & 14 & -2 & 7 \\ -8 & -19 & e & -18 & -12 \\ 6 & -5 & 12 & e & 5 \\ 1 & -10 & 4 & -9 & e \end{pmatrix}.$$

This yields $\mathbf{X} = \mathbf{P}(\mathbf{X_0}) = \begin{pmatrix} [3,3] & [10,14] & [0,0] & [10,12] & [7,7] \end{pmatrix}^T$ *as greatest interval vector.*

Example 65. *We provide also an example in the semiring* $\gamma^*\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$. *We consider :*

$$\mathbf{A} = \begin{pmatrix} [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] & [8\gamma^2, 8\gamma] \\ [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] \\ [7\gamma \oplus 9\gamma^2, 10 \oplus 11\gamma^3] & [2\gamma \oplus 4\gamma^3, 4\gamma \oplus 6\gamma^2] & [\varepsilon,\varepsilon] \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} [\top,\top] & [\top,\top] & [15\gamma, 18] \\ [\top,\top] & [\top,\top] & [\top,\top] \\ [\top,\top] & [5\gamma, 7] & [\top,\top] \end{pmatrix}$$

$$\text{and } \mathbf{X_0} = \begin{pmatrix} [4\gamma \oplus 7\gamma^4(18\gamma)^*, 7 \oplus 8\gamma^3(18\gamma)^*] \\ [5\gamma^2 \oplus 8\gamma^5(18\gamma)^*, 8\gamma \oplus 9\gamma^4(18\gamma)^*] \\ [6\gamma^3 \oplus 9\gamma^6(18\gamma)^*, 9\gamma^2 \oplus 10\gamma^5(18\gamma)^*] \end{pmatrix}.$$

According to the computation rules given in Definition 41 (see also [17, 13] for algorithmic issues and software tools), the following vector is obtained :

$$\mathbf{X} = \mathbf{P}(\mathbf{X_0}) = \begin{pmatrix} [21\gamma^4(18\gamma)^*, 17\gamma^3(18\gamma)^*] \\ [4\gamma^2(18\gamma)^*, 5\gamma(18\gamma)^*] \\ [6\gamma^3(18\gamma)^*, 9\gamma^2(18\gamma)^*] \end{pmatrix}.$$

7. Conclusion

This work deals with a dual product in a semiring and its extension to semirings of intervals. Sufficient conditions are given in order to ensure the existence of a projector in the solution set of the following system : $\mathbf{A} \overline{\otimes} \mathbf{X} \preceq \mathbf{X} \preceq \mathbf{B} \overline{\odot} \mathbf{X}$, where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{X}$ are interval matrices. This projector can be useful to solve control problems for timed discrete event systems. More precisely, control for uncertain systems with parameters that are only known to be in an interval, and where the state evolution is subject to constraints (see e.g. [30, 31, 22, 19, 28, 4]).

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