Max-plus Linear Observer: Application to Manufacturing Systems ⋆

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Abstract: This paper deals with the observer design for max-plus linear systems. The approach is based on the residuation theory which is suitable to deal with linear mapping inversion in idempotent semiring. An illustrative example allows to discuss about a practical implementation.

Keywords: Discrete Event Dynamics Systems, Idempotent Semirings, Max-Plus Algebra, Residuation Theory, Timed Event Graphs, Dioid, Observer, State Estimation.

1. INTRODUCTION

This paper deals with observer synthesis for (max,plus) linear systems. These linear systems are useful to describe discrete event systems characterized by synchronization and delay phenomena. Among these systems we can cite the manufacturing systems, (see Cohen et al. (1985)), the transportation networks, (see Heidergott et al. (2006)), the computer networks (Bouillard et al. (2007)). In the first setting, the delay can be due to the duration of tasks, or due to the transportation of parts between machines, the synchronization phenomena occur when the system has to perform assembly tasks of many parts or also when the number of resources (e.g. the machines capacity) is limited, so some parts are obliged to wait before to be processed.

After a modeling step, all these systems can be described by a linear model in (max-plus) algebra, the state variables represent the date of events occurrence. A specific theory has been developed in order to solve some control problems very reminiscent to those of the control theory for linear systems in classical algebra. Among the problems considered we can mention: the optimal and the model predictive control (Cohen et al. (1989),Schutter and van den Boom (2001)), the control in order to optimize a just in time criterion (Cottenceau et al. (1999),Maia et al. (2005)), the problems of control with some uncontrollable inputs which can represent disturbances such as system breakdowns(Lhommeau et al. (2002)), and also the robust control of systems involving some uncertain parameters which are characterized by intervals (*e.g.* tasks duration or number of resources imperfectly known but assumed to be bounded, Lhommeau et al. (2004)). Some developments about characterization of invariant semimodules (which play an analogous role to the vector spaces in classical algebra) allowed to solve some specific control problems which involves constraint on the states vector (Gaubert and Katz (2003), Katz (2007), Houssin (2006), Ouerghi and Hardouin (2006)). The control can also be computed in order to optimize the energy consumption of the system, in (Mao and Cassandras (2008)) the criterion considered is a decreasing convex function of task duration (see also Li et al. (2004), for a close problem). More recently observer synthesis has been considered (see DiLoreto et al. (2009)), the authors consider a duality principle which allows to compute an observer matrix thanks to the characterization of invariant subsemimodule. The disturbances considered by the authors are given thanks to an implicit system which allows to modelize uncertainties on the delay assumed to be in a known interval. The estimated state can then be computed by using the available measure. In (Hardouin et al. (2010)), the synthesis of the observer matrix is considered thanks to residuation theory and leads to an estimation of the state as close as possible, from below, to the real state. This paper deals with an application of this observer to the example considered by (DiLoreto et al. (2009)), but instead of considering that disturbances belong to intervals, external uncontrollable inputs are added in order to model disturbances or unknown initial conditions, and no assumptions on disturbances and delays are done. The only assumptions are the following, the system is assumed to be linear in this algebraic setting (only delay and synchronization phenomena are considered) and the fastest behavior is assumed to be known (the ideal system behavior is known, the one where the duration and/or transportation task are minimal, hence the disturbances can only decrease the system performance, *i.e.* delay the events occurrence). We invite the reader to consult the web site given in (Hardouin et al. (2010)), to discover a dynamic illustration of the observer behavior.

⋆ This work has been supported by the CAPES (Brazil)/COFECUB (France), Ph 642/09.

2. ALGEBRAIC SETTING

An idempotent semiring $\mathcal{S}$ is an algebraic structure with two internal operations denoted by $\oplus$ and $\otimes$. The operation $\oplus$ is associative, commutative and idempotent, that is, $a \oplus a = a$. The operation $\otimes$ is associative (but not necessarily commutative) and distributive on the left and on the right with respect to $\oplus$. The neutral elements of $\oplus$ and $\otimes$ are represented by ε and e respectively, and ε is an absorbing element for the law $\otimes$ ($\forall a \in \mathcal{S}, \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$). As in classical algebra, the operator $\otimes$ will be often omitted in the equations, moreover, $a^i = a \otimes a^{i-1}$ and $a^0 = e$. In this algebraic structure, a partial order relation is defined by $a \succeq b \Leftrightarrow a = a \oplus b \Leftrightarrow \; b = a \wedge b$ (where $a \wedge b$ is the greatest lower bound of a and b), therefore an idempotent semiring $\mathcal{S}$ is a partially ordered set (see Baccelli et al. (1992) for an exhaustive introduction). An idempotent semiring $\mathcal{S}$ is said to be complete if it is closed for infinite $\oplus$-sums and if $\otimes$ distributes over infinite $\oplus$-sums. In particular $\top = \bigoplus_{x \in \mathcal{S}} x$ is the greatest element of $\mathcal{S}$ ($\top$ is called the top element of $\mathcal{S}$).

Example 1. ($\overline{\mathbb{Z}}_{\max}$). Set $\overline{\mathbb{Z}}_{\max} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the max operator as sum and the classical sum $+$ as product is a complete idempotent semiring, usually denoted $\overline{\mathbb{Z}}_{\max}$, of which $\varepsilon = -\infty$ and $e = 0$.

Theorem 2. (see Baccelli et al. (1992), th. 4.75). The implicit inequality $x \succeq ax \oplus b$ as well as the equation $x = ax \oplus b$ defined over $\mathcal{S}$, admit $x = a^* b$ as the least solution, where $a^* = \bigoplus_{i \in \mathbb{N}} a^i$ and $a^0 = e$, (Kleene star operator).

Properties 3. The Kleene star operator satisfies the following well known properties (see Gaubert (1992) for proofs, and Krob (1991) for more general results):

$$a^* = (a^*)^*, \qquad a^* a^* = a^*, \tag{1}$$

$$(a \oplus b)^* = a^*(ba^*)^* = (a^*b)^* a^*, b(ab)^* = (ba)^* b. \tag{2}$$

Thereafter, the operator $a^+ = \bigoplus_{i \in \mathbb{N}^+} a^i = aa^* = a^* a$ is also considered, it satisfies the following properties:

$$a^+ = (a^+)^+, \qquad a^* = e \oplus a^+, \tag{3}$$

$$(a^*)^+ = (a^+)^* = a^*, a^+ \preceq a^*. \tag{4}$$

Definition 4. (Residual and residuated mapping). An order preserving mapping $f : \mathcal{D} \to \mathcal{E}$, where $\mathcal{D}$ and $\mathcal{E}$ are partially ordered sets, is a *residuated mapping* if for all $y \in \mathcal{E}$ there exists a greatest solution for the inequality $f(x) \preceq y$ (hereafter denoted $f^\sharp(y)$). Obviously, if equality $f(x) = y$ is solvable, $f^\sharp(y)$ yields the greatest solution. The mapping $f^\sharp$ is called the *residual* of f and $f^\sharp(y)$ is the greatest solution of the inequality and can be seen as the optimal solution of the following constrained optimization problem :

$$x_{opt} = \bigoplus_{\{x | f(x) \preceq y\}} x \tag{5}$$

Theorem 5. (see Blyth and Janowitz (1972)). Let $f : (\mathcal{D}, \preceq) \to (\mathcal{C}, \preceq)$ be an order preserving mapping. The following statements are equivalent

(i) f is residuated.
(ii) there exists an unique order preserving mapping $f^\sharp : \mathcal{C} \to \mathcal{D}$ such that $f \circ f^\sharp \preceq \mathsf{Id}_{\mathcal{C}}$ and $f^\sharp \circ f \succeq \mathsf{Id}_{\mathcal{D}}$.

Example 6. Mappings $\Lambda_a : x \mapsto a \otimes x$ and $\Psi_a : x \mapsto x \otimes a$ defined over an idempotent semiring $\mathcal{S}$ are both residuated (Baccelli et al. (1992), p. 181). Their residuals are order preserving mappings denoted respectively by $\Lambda_a^\sharp(x) = a \backslash\!\!\circ x$ and $\Psi_a^\sharp(x) = x \circ\!\!/ a$. This means that $a \backslash\!\!\circ b$ (resp. $b \circ\!\!/ a$) is the greatest solution of the inequality $a \otimes x \preceq b$ (resp. $x \otimes a \preceq b$).

Definition 7. (Restricted mapping). Let $f : \mathcal{D} \to \mathcal{C}$ be a mapping and $\mathcal{B} \subseteq \mathcal{D}$. We will denote by $f_{|\mathcal{B}} : \mathcal{B} \to \mathcal{C}$ the mapping defined by $f_{|\mathcal{B}} = f \circ \mathsf{Id}_{|\mathcal{B}}$ where $\mathsf{Id}_{|\mathcal{B}} : \mathcal{B} \to \mathcal{D}, x \mapsto x$ is the canonical injection. Identically, let $\mathcal{E} \subseteq \mathcal{C}$ be a set such that $\mathsf{Im} f \subseteq \mathcal{E}$. Mapping ${}_{\mathcal{E}|}f : \mathcal{D} \to \mathcal{E}$ is defined by $f = \mathsf{Id}_{|\mathcal{E}} \circ {}_{\mathcal{E}|}f$, where $\mathsf{Id}_{|\mathcal{E}} : \mathcal{E} \to \mathcal{C}, x \mapsto x$.

Definition 8. (Closure mapping). A closure mapping is an order preserving mapping $f : \mathcal{D} \to \mathcal{D}$ defined on an ordered set $\mathcal{D}$ such that $f \succeq \mathsf{Id}_{\mathcal{D}}$ and $f \circ f = f$.

Proposition 9. (see Cottenceau et al. (2001)). Let $f : \mathcal{D} \to \mathcal{D}$ be a closure mapping. Then, ${}_{\mathsf{Im} f|}f$ is a residuated mapping whose residual is the canonical injection $\mathsf{Id}_{|\mathsf{Im} f}$.

Example 10. Mapping $K : \mathcal{S} \to \mathcal{S}, x \mapsto x^*$ is a closure mapping , (indeed by definition $a \preceq a^*$ and equation (1) gives $a^* = (a^*)^*$). The closure mappings restricted to their image are residuated (see Blyth and Janowitz (1972) and Cottenceau et al. (1999) for proof), this means that $({}_{\mathsf{Im} K|}K)$ is residuated and its residual is $({}_{\mathsf{Im} K|}K)^\sharp = \mathsf{Id}_{|\mathsf{Im} K}$. Practically this means $x = a^*$ is the greatest solution of inequality $x^* \preceq a$ if $a \in \mathsf{Im} K$, that is $x \preceq a^* \Leftrightarrow x^* \preceq a^*$.

Example 11. Mapping $P : \mathcal{S} \to \mathcal{S}, x \mapsto x^+$ is a closure mapping (indeed $a \preceq a^+$ and $a^+ = (a^+)^+$ see equation (3)). Then $({}_{\mathsf{Im} P|}P)$ is residuated and its residual is $({}_{\mathsf{Im} P|}P)^\sharp = \mathsf{Id}_{|\mathsf{Im} P}$. In other words, $x = a^+$ is the greatest solution of inequality $x^+ \preceq a$ if $a \in \mathsf{Im} P$, that is $x \preceq a^+ \Leftrightarrow x^+ \preceq a^+$.

Remark 12. According to equation (4), $(a^*)^+ = a^*$, therefore $\mathsf{Im} K \subset \mathsf{Im} P$.

Properties 13. Some useful results involving these residuals are presented below (see Baccelli et al. (1992) for proofs and more complete results).

$$a \backslash\!\!\circ a = (a \backslash\!\!\circ a)^* \qquad a \circ\!\!/ a = (a \circ\!\!/ a)^* \tag{6}$$

$$a(a \backslash\!\!\circ (ax)) = ax \qquad ((xa) \circ\!\!/ a)a = xa \tag{7}$$

$$b \backslash\!\!\circ a \backslash\!\!\circ x = (ab) \backslash\!\!\circ x \qquad x \circ\!\!/ a \circ\!\!/ b = x \circ\!\!/ (ba) \tag{8}$$

$$a^* \backslash\!\!\circ (a^* x) = a^* x \qquad (a^* x) \circ\!\!/ a^* = a^* x \tag{9}$$

$$(a \backslash\!\!\circ x) \wedge (a \backslash\!\!\circ y) = a \backslash\!\!\circ (x \wedge y) \qquad (x \circ\!\!/ a) \wedge (y \circ\!\!/ a) = (x \wedge y) \circ\!\!/ a \tag{10}$$

The set of $n \times n$ matrices with entries in $\mathcal{S}$ is an idempotent semiring. The sum, the product and the residuation of matrices are defined after the sum, the product and the residuation of scalars in $\mathcal{S}$, *i.e.*,

$$(A \otimes B)_{ik} = \bigoplus_{j=1 \ldots n} (A_{ij} \otimes B_{jk}), \tag{11}$$

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}, \tag{12}$$

$$(A \backslash\!\!\circ B)_{ij} = \bigwedge_{k=1..n} (A_{ki} \backslash\!\!\circ B_{kj}) \; , \tag{13}$$

$$(B \circ\!\!/ A)_{ij} = \bigwedge_{k=1..n} (B_{ik} \circ\!\!/ A_{jk}). \tag{14}$$

The identity matrix of $\mathcal{S}^{n \times n}$ is the matrix with entries equal to e on the diagonal and to ε elsewhere. This identity matrix will also be denoted e, and the matrix with all its entries equal to ε will also be denoted ε.

3. TEG DESCRIPTION IN IDEMPOTENT SEMIRING

Timed event graphs constitute a subclass of timed Petri nets *i.e.* those whose places have one and only one upstream and downstream transition. A timed event graph (TEG) description can be transformed into a $(max, +)$ or a $(min, +)$ linear model and *vice versa* (see Cohen et al. (1984), Baccelli et al. (1992), Heidergott et al. (2006)). To obtain an algebraic model in $\overline{\mathbb{Z}}_{\max}$, a "dater" function is associated to each transition. For transition labelled x_i, $x_i(k) \in \overline{\mathbb{Z}}_{\max}$ represents the date of the k^{th} firing. By considering suitable transformation (see Baccelli et al. (1992) for details), it is always possible to obtain an explicit dynamical system as follows :

$$\begin{aligned} x(k) &= \bar{A}x(k-1) \oplus Bu(k) \oplus Rw(k) \\ y(k) &= Cx(k), \end{aligned} \tag{15}$$

where $u \in (\overline{\mathbb{Z}}_{\max})^p$, $y \in (\overline{\mathbb{Z}}_{\max})^m$ and $x \in (\overline{\mathbb{Z}}_{\max})^n$ are respectively the controllable input, output and state vector. Matrices $\bar{A} \in (\overline{\mathbb{Z}}_{\max})^{n\times n}$, $B \in (\overline{\mathbb{Z}}_{\max})^{n\times p}$, $C \in (\overline{\mathbb{Z}}_{\max})^{m\times n}$. Vector $w \in (\overline{\mathbb{Z}}_{\max})^l$ represents uncontrollable inputs (*i.e.* disturbances [1]). Each entry of w corresponds to a transition which can disable the firing of internal transition of the graph, and so can decrease the performance of the system. This vector is bound to the graph through matrix $R \in (\overline{\mathbb{Z}}_{\max})^{n\times l}$. Afterwards, each input transition u_i (respectively w_i) is assumed to be connected to one and only one internal transition x_j, this means that each column of matrix B (resp. R) has one entry equal to e and the others equal to ε and at most one entry equal to e on each row. Furthermore, each output transition y_i is assumed to be linked to one and only one internal transition x_j, *i.e* each row of matrix C has one entry equal to e and the others equal to ε and at most one entry equal to e on each column. These requirements are satisfied without loss of generality, since it is sufficient to add extra input and output transition. Note that if R is equal to the identity matrix, w can represent initial state of the system $x(0)$ (see Baccelli et al. (1992), p. 245, for a discussion about compatible initial conditions). In the following we will consider the γ-transform defined as follows : $x_i(\gamma) = \bigoplus_{k\in\mathbb{Z}} x_i(k) \otimes \gamma^k$ where $x_i(k) \in \overline{\mathbb{Z}}_{\max}$ and γ is a backward shift operator [2] in the event domain (formally $\gamma x(k) = x(k-1)$). $x_i(\gamma)$ is a formal series representing the firing date sequence of the transition labelled x_i. The set of formal series in γ is denoted by $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ and constitutes a complete idempotent semiring. The previous dynamic system equation (15) can then be described equivalently as follows :

$$\begin{aligned} x(\gamma) &= \gamma\bar{A}x(\gamma) \oplus Bu(\gamma) \oplus Rw(\gamma) \\ y(\gamma) &= Cx(\gamma), \end{aligned} \tag{16}$$

In the sequel, in order to lighten the notation, $x(\gamma)$ will be denoted simply x and matrix $A = \gamma\bar{A}$ will be considered.

[1] In manufacturing context, w may represent machine breakdowns or failures in component supply.

[2] Operator γ plays a role similar to operator z^{-1} in the $\mathcal{Z}-$transform for the conventional linear systems theory.

By considering theorem 2, this system can be rewritten as:

$$x = A^*Bu \oplus A^*Rw \tag{17}$$

$$y = CA^*Bu \oplus CA^*Rw, \tag{18}$$

where $(CA^*B) \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{m\times p}$ (respectively $(CA^*R) \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{m\times l}$) is the input/output (resp. disturbance/output) transfer matrix. Matrix (CA^*B) represents the earliest behavior of the system, therefore it must be underlined that the uncontrollable inputs vector w (initial conditions or disturbances) is only able to delay the transition firings, *i.e.* , according to the order relation of the semiring, to increase the vectors x and y. According to assumptions about matrices C, B, and R, the matrices (CA^*B) and (CA^*R) are composed of some entries of the matrix A^*. Each entry is a periodic series in the $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ semiring. A series $s = \bigoplus_{k\in\mathbb{Z}} s(k)\gamma^k$, where $s(k)$ is a dater function, is periodic if it can be written as $s = p \oplus qr^*$, where $p = \bigoplus_{i=1}^{m} t_i\gamma^{n_i}$ (respectively $q = \bigoplus_{j=1}^{h} t_j\gamma^{n_j}$) is a polynomial depicting the transient (resp. the periodic) behavior, and $r = \tau\gamma^\nu$ is a monomial depicting the periodicity allowing to define the asymptotic slope of the series as $\sigma_\infty(s) = \nu/\tau$ (it is homogenous to the production rate of the series). Sum, product, and residuation of periodic series are well defined and algorithms and software toolboxes are available in order to handle periodic series and compute transfer relations (see (Cottenceau et al. (2000))). In the sequel counters of event are considered, more precisely a counter function $\mathcal{C}_s(t)$ will be associated to a given series $s(\gamma)$, being defined by means of the relation $s = \bigoplus_{t\in\mathbb{Z}} t\gamma^{\mathcal{C}_s(t)}$. In (MaxPlus (1991)) and in (Santos-Mendes et al. (2005)), it has been shown that the residuation of two series $s_1 \phi s_2$ can be used to compute bounds for the difference between the corresponding counter functions. More precisely, by considering two trajectories $x_1 = s_1u$ and $x_2 = s_2u$ where u is an input and s_1 and s_2 are transfer functions, the stock function is defined as $S_{x_1x_2}(t) = \mathcal{C}_{x_1}(t) - \mathcal{C}_{x_2}(t)$ and it characterizes the difference between the two trajectories associated to transitions x_1 and x_2. If $s_2 \succeq s_1$, then this stock function can be bounded as follows:

$$-\mathcal{C}_{s_2\phi s_1}(0) \le S_{x_1x_2}(t) \le \mathcal{C}_{s_1\phi s_2}(0).$$

In this paper the example introduced in (DiLoreto et al. (2009)) is considered, and the corresponding TEG is depicted figure 1, the size of the system are : $p = 0$, *i.e.* no controllable inputs, $m = 3$, *i.e.* 3 measured outputs, $n = 9$ is the state size, and $l = 2$ is the number of disturbances acting on the system. Below, due to the lack of place, only the entries different of ε are given for each matrix of system (16):

$$\begin{aligned} &\bar{A}_{1,3} = 4,\ \bar{A}_{1,7} = 2,\ \bar{A}_{2,1} = 1,\ \bar{A}_{2,8} = 3,\ \bar{A}_{3,2} = 5,\ \bar{A}_{3,9} = 1, \\ &\bar{A}_{4,1} = 4,\ \bar{A}_{4,6} = 3,\ \bar{A}_{5,2} = 3,\ \bar{A}_{5,4} = 1,\ \bar{A}_{6,3} = 5,\ \bar{A}_{6,5} = 4, \\ &\bar{A}_{7,4} = 4,\ \bar{A}_{7,9} = 3,\ \bar{A}_{8,7} = 5,\ \bar{A}_{8,5} = 3,\ \bar{A}_{9,8} = 4,\ \bar{A}_{9,6} = 2, \end{aligned}$$

$$C_{1,3} = e,\ C_{2,6} = e,\ C_{3,8} = e,$$

$$R_{2,1} = e,\ R_{5,2} = e.$$

Let us note that matrix $B = \varepsilon$, that means no controllable inputs are considered. The two entries $R_{2,1}$ and $R_{5,2}$ represent the links between vectors w and x. It means that

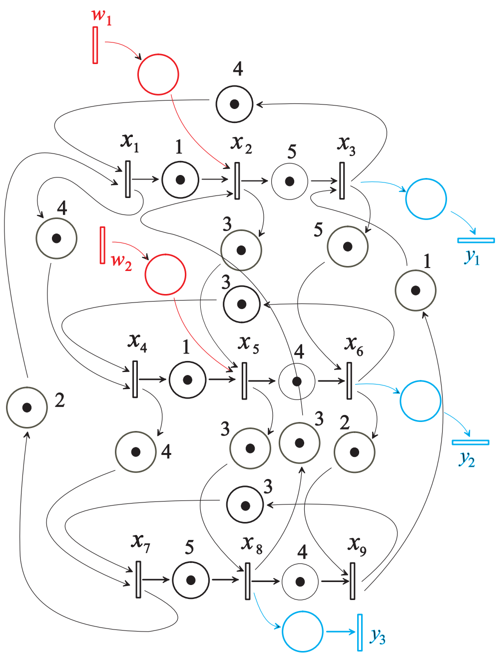

Fig. 1. Timed event graph, x_i internal transitions, y_i measured outputs and w_i uncontrollable inputs.

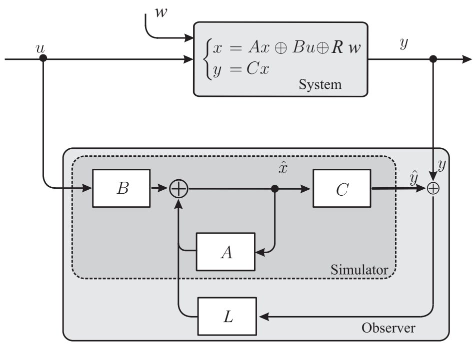

Fig. 2. Observer structure.

the sojourn time in the upstream places of transitions x_5 and x_2 can vary between a minimal value, respectively given by $A_{2,1}, A_{2,8}, A_{5,2}$ and $A_{5,4}$, and a maximal value which can depend of the firing dates of transitions w_1 and w_2. In (DiLoreto et al. (2009)) the sojourn time in theses upstream places is assumed to be in an interval, that means the entries of matrix A are in a semiring of interval, (see Lhommeau et al. (2005), Lhommeau et al. (2004), Hardouin et al. (2009) for details about this specific semiring). Precisely the authors choose, $\mathbf{A_{2,1}} = [1,7]$, $\mathbf{A_{2,8}} = [3,3]$, $\mathbf{A_{5,4}} = [1,3]$, $\mathbf{A_{5,2}} = [3,5]$, which means that the minimal sojourn time of the respective places are included in the corresponding interval. This can be seen as a particular case which can be achieved by choosing the lower bound of each interval for the minimal sojourn time and by building the particular vector w given below, rather than a completely free disturbance :

$$\begin{pmatrix} w_1(k) \\ w_2(k) \end{pmatrix} = \begin{pmatrix} w_1(k-1) \oplus x_1(k-1) \otimes \tilde{A}_{2,1} \\ w_2(k-1) \oplus x_4(k-1) \otimes \tilde{A}_{5,4} \oplus x_2(k-1) \otimes \tilde{A}_{5,2} \end{pmatrix}$$

where $\tilde{A}_{2,1}$, $\tilde{A}_{5,4}$, $\tilde{A}_{5,2}$ are random values in their corresponding interval $\mathbf{A_{2,1}}$, $\mathbf{A_{2,8}}$, $\mathbf{A_{5,4}}$, $\mathbf{A_{5,2}}$.

In the sequel we will not consider this additive input, but a completely free vector w, indeed we will show that the observer synthesis is done whatever be the disturbance. Actually we will compute the greatest observer matrix (in the sense of the semiring order) taking into account the disturbance.

4. MAX-PLUS OBSERVER

The observer structure depicted figure 2 is directly inspired from the classical linear system theory (see Luenberger (1971)). The observer matrix L aims at providing information from the system output into the simulator, in order to take the disturbances w acting on the system into account.

The simulator is described by the model [3] (matrices A, B, C) which is assumed to represent the fastest behavior of the real system, furthermore the simulator is initialized by the canonical initial conditions (*i.e.* $\hat{x}_i(k) = \varepsilon, \forall k \leq 0$). These assumptions induce that $y \succeq \hat{y}$ since disturbances and initial conditions, depicted by w, are only able to increase the system output. As in the development proposed in conventional linear systems theory, matrices A, B, C and R are assumed to be known and the system is assumed to be structurally observable, then the system transfer is given by equations (17) and (18). According to figure 2 the observer equations are given by:

$$\begin{aligned} \hat{x} &= A\hat{x} \oplus Bu \oplus L(\hat{y} \oplus y) \\ &= A\hat{x} \oplus Bu \oplus LC\hat{x} \oplus LCx \qquad (19) \\ \hat{y} &= C\hat{x}. \end{aligned}$$

By applying Theorem 2 and by considering equation (17), equation (19) becomes:

$$\begin{aligned} \hat{x} = &(A \oplus LC)^* Bu \oplus (A \oplus LC)^* LCA^* Bu \\ &\oplus (A \oplus LC)^* LCA^* Rw. \qquad (20) \end{aligned}$$

By applying equation (2) the following equality is obtained:

$$(A \oplus LC)^* = A^*(LCA^*)^*, \qquad (21)$$

by replacing in equation (20) :

$$\begin{aligned} \hat{x} = &A^*(LCA^*)^* Bu \oplus A^*(LCA^*)^* LCA^* Bu \\ &\oplus A^*(LCA^*)^* LCA^* Rw, \end{aligned}$$

and by recalling that $(LCA^*)^* LCA^* = (LCA^*)^+$, this equation may be written as follows :

$$\begin{aligned} \hat{x} = &A^*(LCA^*)^* Bu \oplus A^*(LCA^*)^+ Bu \\ &\oplus A^*(LCA^*)^+ Rw. \qquad (22) \end{aligned}$$

Equation (4) yields $(LCA^*)^* \succeq (LCA^*)^+$, then the observer model may be written as follows :

$$\begin{aligned} \hat{x} &= A^*(LCA^*)^* Bu \oplus A^*(LCA^*)^+ Rw \\ &= (A \oplus LC)^* Bu \oplus (A \oplus LC)^* LCA^* Rw. \qquad (23) \end{aligned}$$

As said previously the objective considered is to compute the greatest observation matrix L such that the estimated

[3] Disturbances are uncontrollable and *a priori* unknown, then the simulator does not take them into account.

state vector $\hat{x}$ be as close as possible to state x, under the constraint $\hat{x} \preceq x$ which has to hold for all u, w, formally this can be expressed as the two following inequalities:

$$(A \oplus LC)^* B \preceq A^* B, \quad (24)$$

$$(A \oplus LC)^* LCA^* R \preceq A^* R. \quad (25)$$

Lemma 14. (Hardouin et al. (2010)). The greatest matrix L such that $(A \oplus LC)^* B = A^* B$ is given by:

$$L_1 = (A^* B) \phi (CA^* B).$$

Lemma 15. (Hardouin et al. (2010)). The greatest matrix L that satisfies $(A \oplus LC)^* LCA^* R \preceq A^* R$ is given by:

$$L_2 = (A^* R) \phi (CA^* R). \quad (26)$$

Proposition 16. (Hardouin et al. (2010)). $L_x = L_1 \wedge L_2$ is the greatest observer matrix such that:

$$\hat{x} = A\hat{x} \oplus Bu \oplus L(\hat{y} \oplus y) \preceq x = Ax \oplus Bu \oplus Rw \quad \forall (u, w).$$

Corollary 17. By considering matrix $\overline{B} = (B \;\; R) \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{n \times q}$ with $q = p + l$, matrix L_x may be written as : $L_x = (A^* \overline{B}) \phi (CA^* \overline{B})$.

Proof. According to the definition of right residuation of matrices (see equation 14). It is obvious to check :

$$\begin{aligned}(A^* \overline{B}) \phi (CA^* \overline{B}) &= (A^* B \quad A^* R) \phi (CA^* B \quad CA^* R) \\ &= (A^* B) \phi (CA^* B) \wedge (A^* R) \phi (CA^* R) \\ &= L_x. \end{aligned} \quad (27)$$

Proposition 18. The matrix L_x ensures the equality between estimated output $\hat{y}$ and measured output y, *i.e.*

$$C(A \oplus L_x C)^* B = CA^* B, \quad (28)$$

$$C(A \oplus L_x C)^* L_x CA^* R = CA^* R. \quad (29)$$

Proof. Let $\tilde{L} = e \phi C$ be a particular observer matrix. Definition 4 yields $\tilde{L}C \preceq e$ then $(A \oplus \tilde{L}C)^* = A^*$. This equality implies $(A \oplus \tilde{L}C)^* B = A^* B$, therefore according to lemma 14 $\tilde{L} \preceq L_1$, since L_1 is the greatest solution. That implies also that L_1 is solution of equation (28). Equality $(A \oplus \tilde{L}C)^* = A^*$ and inequality $\tilde{L}C \preceq e$ yield $(A \oplus \tilde{L}C)^* \tilde{L}CA^* R = A^* \tilde{L}CA^* R \preceq A^* R$ then according to lemma 15 $\tilde{L} \preceq L_2$ since L_2 is the greatest solution. That implies also that $\tilde{L}$ and L_2 are such that $C(A \oplus \tilde{L}C)^* \tilde{L}CA^* R \preceq C(A \oplus L_2 C)^* L_2 CA^* R \preceq CA^* R$. The assumption about matrix C (see section 3) yields $CC^T = e$ and $\tilde{L} = e \phi C = C^T$, therefore $C(A \oplus \tilde{L}C)^* \tilde{L}CA^* R = CA^* \tilde{L}CA^* R = (C\tilde{L} \oplus CA\tilde{L} \oplus ...)CA^* R \succeq C\tilde{L}CA^* R = CC^T CA^* R = CA^* R$. Therefore, since $\tilde{L} \preceq L_2$, we have $C(A \oplus \tilde{L}C)^* \tilde{L}CA^* R = C(A \oplus L_2 C)^* L_2 CA^* R = CA^* R$ and both $\tilde{L}$ and L_2 yield equality (29). To conclude $\tilde{L} \preceq L_1 \wedge L_2 = L_x$, hence, $L_x \preceq L_1$ yields the equality (28) and $L_x \preceq L_2$ yields (29). Therefore equality $\hat{y} = y$ is ensured.

Remark 19. According to the residuation theory (see definition 4), L_x yields $x = \hat{x}$ if possible. Nevertheless, two questions arise, firstly is it possible to ensure equality between the asymptotic behavior of each state vector entries ? Secondly is it possible to ensure equality between these vectors ? The answer to the first question is positive and it is given in (Hardouin et al. (2010)). The key point is the following, if you are able to get measurement about all strongly connected components you have information on all the eigenvalues. The proposition below gives a sufficient condition related to the second question

Proposition 20. If $L_x CA^* \overline{B} = A^* \overline{B}$ then the observer of equation 19 ensures that $\hat{x} = x$.

Proof. By recalling that $\overline{B} = (B \;\; R)$, equality $L_x CA^* \overline{B} = A^* \overline{B}$ can be written

$$(L_x CA^* B \;\; L_x CA^* R) = (A^* B \;\; A^* R).$$

According to equation (20) and by using equation(21) the following equalities hold :

$$\begin{aligned}(A \oplus L_x C)^* L_x CA^* R &= A^* (L_x CA^*)^* L_x CA^* R \\ &= A^* (L_x CA^*)^+ R \\ &= A^* (L_x CA^* R \oplus (L_x CA^*)^2 R \\ &\quad \oplus (L_x CA^*)^3 R \oplus ...).\end{aligned}$$

Since $L_x CA^* R = A^* R$, the following equality is satisfied $(L_x CA^*)^2 R = L_x CA^* A^* R = L_x CA^* R = A^* R$ and more generally $(L_x CA^*)^i R = A^* R$, therefore L_x ensures equality $(A \oplus L_x C)^* L_x CA^* R = A^* (L_x CA^*)^+ R = A^* R$. On the other hand lemma 14 yields the equality $(A \oplus L_x C)^* B = A^* B$, which concludes the proof.

This sufficient condition gives an interesting test to know if the number of sensors is sufficient and if they are well localized to allow an exact estimation. Obviously, this condition is fulfilled if matrix C is equal to the identity.

By considering example of figure 1, thanks to the software given in (Cottenceau et al. (2000)) we have computed the matrix L_x. According to proposition 18 we have $\hat{y} \preceq y$, and since $B = \varepsilon$ equation (19) is given by :

$$\hat{x} = A\hat{x} \oplus Ly$$

by considering $d = Ly$ we can obtain the dynamical equation in $\overline{\mathbb{Z}}_{\max}$

$$\hat{x}(k) = A\hat{x}(k-1) \oplus d(k)$$

where $d(k) \in (\overline{\mathbb{Z}}_{\max})^n$ is the dynamical realization of $d \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^n$. Below a practical method to get this realization is given. By recalling that $L_x \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{n \times m}$ each entry of vector d can be written as $d_i = d_{i1} \oplus d_{i2} \oplus ... \oplus d_{im}$ where $d_{ij} = (L_x)_{ij} y_j$. Furthermore it must be recalled that each entry of matrix L_x can be expressed as a periodic series, *i.e.*, $(L_x)_{ij} = p_{ij} \oplus q_{ij} r_{ij}^*$ where p_{ij} and q_{ij} are polynomial of $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ and r_{ij} is a monomial of $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$. Hence, each d_{ij} can be expressed as follows :

$$\begin{aligned} d_{ij} &= p_{ij} y_j \oplus \delta_{ij}, \\ \delta_{ij} &= q_{ij} r_{ij}^* y_j = q_{ij} y_j \oplus r_{ij} \delta_{ij}. \end{aligned}$$

Then by using the definition of the backward shift operator γ, it is easy to obtain the expression of $d_{ij}(k) \in \overline{\mathbb{Z}}_{\max}$. Practically, the TEG of figure 1 is such that $n = 9$ and $m = 3$. And matrix L_x is given by :

$$
\begin{aligned}
(L_x)_{11} &= 4\gamma^1 \oplus 14\gamma^4 \oplus (29\gamma^7 \oplus 41\gamma^{10})(25\gamma^6)^* \\
(L_x)_{12} &= 9\gamma^3 \oplus (24\gamma^6 \oplus 36\gamma^9)(25\gamma^6)^* \\
(L_x)_{13} &= (12\gamma^3 \oplus 24\gamma^6)(25\gamma^6)^* \\
(L_x)_{21} &= 5\gamma^2 \oplus (20\gamma^5 \oplus 32\gamma^8)(25\gamma^6)^* \\
(L_x)_{22} &= \gamma^1 \oplus (15\gamma^4 \oplus 27\gamma^7)(25\gamma^6)^* \\
(L_x)_{23} &= (3\gamma^1 \oplus 15\gamma^4)(25\gamma^6)^* \\
(L_x)_{31} &= \gamma^0 \oplus 10\gamma^3 \oplus (25\gamma^6 \oplus 37\gamma^9)[25\gamma^6)^* \\
(L_x)_{32} &= 5\gamma^2 \oplus (20\gamma^5 \oplus 32\gamma^8)(25\gamma^6)^* \\
(L_x)_{33} &= (8\gamma^2 \oplus 20\gamma^5)(25\gamma^6)^* \\
(L_x)_{41} &= 8\gamma^2 \oplus 18\gamma^5 \oplus (33\gamma^8 \oplus 45\gamma^1 1)(25\gamma^6)^* \\
(L_x)_{42} &= 3\gamma^1 \oplus 13\gamma^4 \oplus (28\gamma^7 \oplus 40\gamma^1 0)(25\gamma^6)^* \\
(L_x)_{43} &= 1\gamma^1 \oplus (16\gamma^4 \oplus 28\gamma^7)(25\gamma^6)^* \\
(L_x)_{51} &= 9\gamma^3 \oplus (23\gamma^6 \oplus 35\gamma^9)(25\gamma^6)^* \\
(L_x)_{52} &= 4\gamma^2 \oplus (18\gamma^5 \oplus 30\gamma^8)(\gamma^6 d^2 5)^* \\
(L_x)_{53} &= (6\gamma^2 \oplus 18\gamma^5)(25\gamma^6)^* \\
(L_x)_{61} &= 5\gamma^1 \oplus 15\gamma^4 \oplus (30\gamma^7 \oplus 42\gamma^{10})(25\gamma^6)^* \\
(L_x)_{62} &= \gamma^0 \oplus 10\gamma^3 \oplus (25\gamma^6 \oplus 37\gamma^9)(25\gamma^6)^* \\
(L_x)_{63} &= (13\gamma^3 \oplus 25\gamma^6 d^2 5)(25\gamma^6)^* \\
(L_x)_{71} &= (12\gamma^3 \oplus 24\gamma^6)(25\gamma^6)^* \\
(L_x)_{72} &= (7\gamma^2 \oplus 19\gamma^5)(25\gamma^6)^* \\
(L_x)_{73} &= (7\gamma^2 \oplus 20\gamma^5)(25\gamma^6)^* \\
(L_x)_{81} &= 1\gamma^1 \oplus (17\gamma^4 \oplus 29\gamma^7)(25\gamma^6)^* \\
(L_x)_{82} &= (12\gamma^3 \oplus 24\gamma^6)(25\gamma^6)^* \\
(L_x)_{83} &= (\gamma^0 \oplus 12\gamma^3)(25\gamma^6)^* \\
(L_x)_{91} &= 7\gamma^2 \oplus (21\gamma^5 \oplus 33\gamma^8)(25\gamma^6)^* \\
(L_x)_{92} &= 2\gamma^1 \oplus (16\gamma^4 \oplus 28\gamma^7)(25\gamma^6)^* \\
(L_x)_{93} &= (4\gamma^1 \oplus 16\gamma^4)(25\gamma^6)^*
\end{aligned}
\tag{30}
$$

Below, the practical realization of the estimation of transition $\hat{x_7}$ is given which is the same that the one presented in (DiLoreto et al. (2009)).

$$(L_x)_{71} = (12\gamma^3 \oplus 24\gamma^6)(25\gamma^6)^*,$$

i.e., $p_{71} = \varepsilon$, $q_{71} = 12\gamma^3 \oplus 24\gamma^6$ and $r_{71} = 25\gamma^6$, and,

$$
\begin{aligned}
(L_x)_{72} &= (7\gamma^2 \oplus 19\gamma^5)(25\gamma^6)^*, \\
(L_x)_{73} &= (7\gamma^2 \oplus 20\gamma^5)(25\gamma^6)^*.
\end{aligned}
$$

Hence, the dynamic equation of $d_7(k)$ is given by :

$$
\begin{aligned}
d_7(k) &= d_{71}(k) \oplus d_{72}(k) \oplus d_{73}(k), \\
d_{71}(k) &= 12y_1(k-3) \oplus 24y_1(k-6) \oplus 25d_{71}(k-6), \\
d_{72}(k) &= 7y_2(k-2) \oplus 19y_2(k-5) \oplus 25d_{72}(k-6), \\
d_{73}(k) &= 7y_3(k-2) \oplus 20y_3(k-5) \oplus 25d_{73}(k-6),
\end{aligned}
$$

then, according to equation (30), the estimation of state $\hat{x}_7$ is given by :

$$\hat{x}_7(k) = 4\hat{x}_4(k-1) \oplus 3\hat{x}_9(k-1) \oplus d_7(k).$$

We can also easily check that the sufficient condition of proposition 20 is not achieved, indeed $L_x C A^* \overline{B} \neq A^* \overline{B}$ hence $x \neq \hat{x}$. By considering the expressions of x (equation (17)) and $\hat{x}$ (equation (23)), matrices (A^*R) and $(A \oplus LC)^* LCA^*R = A^*(LCA^*)^+R$ can be computed (let us recall that $B = \varepsilon$ in this example), it appears that the only difference is on the entries $(A^*R)_{21}$, $(A^*R)_{51}$ and $(A^*R)_{52}$ so, excepted transitions x_2 and x_5, all transitions will be perfectly observable in spite of disturbances. This result is consistent with the illustration proposed in (DiLoreto et al. (2009)). Below we give the mismatching entries:

$$
\begin{aligned}
(A^*R)_{21} &= e \oplus 10\gamma^3 \oplus (25\gamma^6 \oplus 37\gamma^9)(25\gamma^6)^*, \\
(A^*(LCA^*)^+R)_{21} &= 10\gamma^3 \oplus (25\gamma^6 \oplus 37\gamma^9)(25\gamma^6)^*, \\
(A^*R)_{51} &= 3\gamma \oplus 14\gamma^4 \oplus (28\gamma^7 \oplus 40\gamma^{10})(25\gamma^6)^*, \\
(A^*(LCA^*)^+R)_{51} &= 14\gamma^4 \oplus (28\gamma^7 \oplus 40\gamma^{10})(25\gamma^6)^*, \\
(A^*R)_{52} &= e \oplus (9\gamma^3 \oplus 22\gamma^6)(25\gamma^6)^*, \\
(A^*(LCA^*)^+R)_{52} &= (9\gamma^3 \oplus 22\gamma^6)(25\gamma^6)^*.
\end{aligned}
$$

In order to evaluate bounds for the difference between the mismatching trajectories, we can use the results recalled in section 3 since by assumption $x_i \succeq \hat{x}_i$, *e.g.* for transition x_2 :

$$
\begin{aligned}
(A^*R)_{21} \oslash (A^*(LCA^*)^+R)_{21} &= -15\gamma^{-3} \oplus (e \oplus 12\gamma^3)(25\gamma^6)^*, \\
(A^*(LCA^*)^+R)_{21} \oslash (A^*R)_{21} &= 10\gamma^3 \oplus (25\gamma^6 \oplus 37\gamma^9)(25\gamma^6)^*,
\end{aligned}
$$

hence, the lower bound is obtained from the first equation and the upper bound is obtained from the second one :

$$
\begin{aligned}
-\mathcal{C}_{(A^*R)_{21} \oslash (A^*(LCA^*)^+R)_{21}}(0) &= 0, \\
\mathcal{C}_{(A^*(LCA^*)^+R)_{21} \oslash (A^*R)_{21}}(0) &= 3,
\end{aligned}
$$

that means the difference of events number occurred between transitions x_2 and $\hat{x}_2$ will be bounded as follows, whatever be disturbance w :

$$0 \leq S_{\hat{x}_2 x_2}(t) \leq 3.$$

For transition x_5, 4 residuations have to be done, and yield the following values :

$$
\begin{aligned}
-\mathcal{C}_{(A^*R)_{51} \oslash (A^*(LCA^*)^+R)_{51}}(0) &= 0, \\
\mathcal{C}_{(A^*(LCA^*)^+R)_{51} \oslash (A^*R)_{51}}(0) &= 3, \\
-\mathcal{C}_{(A^*R)_{52} \oslash (A^*(LCA^*)^+R)_{52}}(0) &= 0, \\
\mathcal{C}_{(A^*(LCA^*)^+R)_{52} \oslash (A^*R)_{52}}(0) &= 3,
\end{aligned}
$$

hence the difference between the state x_5 and its estimation $\hat{x}_5$ at each time t will be:

$$0 = min(0,0) \leq S_{\hat{x}_5 x_5}(t) \leq max(3,3) = 3.$$

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