On the dual product and the dual residuation over idempotent semiring of intervals

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Outline

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- Idempotent Semirings in few words
- Interval analysis over idempotent semirings
- Residuation and dual residuation of isotone mappings
- Residuation and interval analysis
- Conclusion

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Idempotent Semiring in few words

Idempotent Semiring $\mathcal{S}$

- Sum $\oplus$, associative,commutative, neutral element denoted ε,
- Product $\otimes$, associative, neutral element denoted e,
- Product $\otimes$ distributes with respect to the sum, $(a \oplus b) \otimes c = a \otimes c \oplus b \otimes c$,
- Neutral element ε is absorbing, $a \otimes \varepsilon = \varepsilon$
- The sum is idempotent, $a \oplus a = a$.
- $a \oplus b = a \Leftrightarrow b \preceq a \Leftrightarrow a \wedge b = b$
  hence a semiring has a complete lattice structure, with (ε) as bottom element and $(T = \bigoplus_{x \in \mathcal{S}} x)$ as top element. Operator $\oplus$ corresponds to operator $\vee$.

Subsemiring

A subset $\mathcal{C} \subset \mathcal{S}$ is called a subsemiring of $\mathcal{S}$ if

- $\varepsilon \in \mathcal{C}$ and $e \in \mathcal{C}$;
- $\mathcal{C}$ is closed for $\oplus$ and $\otimes$, i.e, $\forall a, b \in \mathcal{C}$, $a \oplus b \in \mathcal{C}$ and $a \otimes b \in \mathcal{C}$.

Idempotent Semiring Examples

Max-plus algebra $\overline{\mathbb{Z}}_{\max}$

Set $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the *max* operator as $\oplus$ and the classical sum $+$ as $\otimes$ is a complete idempotent semiring of which $\varepsilon = -\infty$, $e = 0$ and $T = +\infty$ and the greatest lower bound $a \wedge b = min(a, b)$.

Min-plus algebra $\overline{\mathbb{Z}}_{\min}$

Set $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the *min* operator as $\oplus$ and the classical sum as $\otimes$ is a complete idempotent semiring of which $\varepsilon = +\infty$, $e = 0$ and $T = -\infty$ and the greatest lower bound $a \wedge b = max(a, b)$.

Max-min algebra

The set $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the *max* operator as $\oplus$ and the *min* operator as $\otimes$ is a complete idempotent semiring of which $\varepsilon = -\infty$, $e = +\infty$ and $\top = +\infty$ and the greatest lower bound $a \wedge b = min(a, b)$.

Idempotent Semiring of formal series

Semiring of formal series $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ (Cohen, Quadrat et al. IEEE TAC 89)

Let $s = \bigoplus_{k\in\mathbb{Z}} s(k)\gamma^k$ a formal series where $s(k) \in \overline{\mathbb{Z}}_{\max}$. The set of formal series endowed with the following sum and Cauchy product :

$$s \oplus s' : (s \oplus s')(k) = s(k) \oplus s'(k),$$
$$s \otimes s' : (s \otimes s')(k) = \bigoplus_{i+j=k} s(i) \otimes s'(j),$$

is a semiring denoted $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$.
A series with a finite support is called a polynomial, and a monomial if there is only one element.

Idempotent Semiring of Intervals $\mathcal{IS}$ (Litvinov 2001, Lhommeau 2003, Hardouin 2010)

A (closed) interval

it is a set of the form $\mathbf{x} = [\underline{x}, \overline{x}] = \{t \in \mathcal{S} | \underline{x} \preceq t \preceq \overline{x}\}$, where , $\underline{x} \in \mathcal{S}$ (respectively, $\overline{x} \in \mathcal{S}$) is said to be the lower (respectively, upper) bound of the interval $\mathbf{x}$. If $\underline{x} = \overline{x}$ the interval is said to be degenerated.

Semiring of Interval $\mathcal{IS}$

The set of intervals, denoted by $\mathcal{IS}$, endowed with the following coordinate-wise algebraic operations :

$$\mathbf{x} \,\bar{\oplus}\, \mathbf{y} \triangleq [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}] \quad \text{and} \quad \mathbf{x} \,\bar{\otimes}\, \mathbf{y} \triangleq [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}] \tag{1}$$

is an idempotent semiring, denoted $\mathcal{IS}$, where interval $\boldsymbol{\varepsilon} = [\varepsilon, \varepsilon]$ is the neutral element of the sum, and $\mathbf{e} = [e, e]$ is the identity element.

Idempotent Semiring of Interval $\mathcal{IS}$ (Litvinov 2001, Lhommeau 2003, Hardouin 2010)

Order Relation

Let $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$ two intervals with bounds in $\mathcal{S}$

$$\mathbf{x} \preceq_{\mathcal{IS}} \mathbf{y} \Leftrightarrow \underline{x} \preceq_{\mathcal{S}} \underline{y} \text{ and } \overline{x} \preceq_{\mathcal{S}} \overline{y}$$

Residuation Theory, Mapping inversion

Definition (Croisot 56, Blyth 72, Cuninghame-Green 79, Baccelli 92)

Let $\mathcal{S}, \preceq$ and $\mathcal{T}, \preceq$ be two complete lattices, $f : \mathcal{S} \rightarrow \mathcal{T}$ an order preserving mapping is residuated if $\exists f^\sharp : \mathcal{T} \rightarrow \mathcal{S}$ an order preserving mapping such that

$$f \circ f^\sharp \preceq Id_{\mathcal{T}}, \qquad f^\sharp \circ f \succeq Id_{\mathcal{S}}$$

$f^\sharp$ is the residual mapping.

Necessary and Sufficient Condition

- f is residuated iff $f(\bigvee_{x \in \mathcal{T}} x) = \bigvee_{x \in \mathcal{T}} f(x)$ (f is lower semi continuous).

Properties

- $f \circ f^\sharp \circ f = f$
- $f^\sharp \circ f \circ f^\sharp = f^\sharp$
- $(f \circ g)^\sharp = g^\sharp \circ f^\sharp$ with $g : \mathcal{U} \rightarrow \mathcal{S}$ another residuated mapping.

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Dual Residuation

Definition

Let $\mathcal{S}, \preceq$ and Let $\mathcal{T}, \preceq$ be two complete lattices, $f : \mathcal{S} \to \mathcal{T}$ an order preserving mapping is dually residuated if $\exists f^\flat : \mathcal{T} \to \mathcal{S}$ an order preserving mapping such that

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$f^\flat$ is the dual residual mapping.

Necessary and Sufficient Condition

- f is dually residuated iff $f(\bigwedge_{x\in\mathcal{T}} x) = \bigwedge_{x\in\mathcal{T}} f(x)$ (f is upper semi continuous).

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Residuated Mapping

Example : Mapping $L_a : x \mapsto a \otimes x$ (Baccelli et al. 92)

Mapping $L_a : x \mapsto a \otimes x$ defined over semiring $\mathcal{S}$ is $l.s.c$, then $(L_a)^\sharp$ exists, i.e. inequality $a \otimes x \preceq b$ admits a greatest solution , denoted, $x = a \circ\!\!\!\! \backslash b$.

For matrices

Practical computation is obtained as follows,

$$C_{ij} = (A \circ\!\!\!\! \backslash B)_{ij} = \bigwedge_{k=1 \ldots n} (A_{ki} \circ\!\!\!\! \backslash B_{kj}),$$

with $A \in \mathcal{S}^{n \times p}$, $B \in \mathcal{S}^{n \times m}$ and $C \in \mathcal{S}^{p \times m}$.

Residuated Mapping $(L_a)^\sharp$ in $(max, +)$ algebra

$A \otimes x \preceq B$

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix}$ be matrices with entries in $(max, +)$ algebra.

In $(max, +)$ algebra $a_{ij} \circ\!\!\!\backslash\, b_j = b_j - a_{ij}$ then the greatest x such that $A \otimes x \preceq B$ is given by :

$$\begin{aligned} x = A \circ\!\!\!\backslash\, B &= \begin{pmatrix} (1 \circ\!\!\!\backslash\, 8) \wedge (3 \circ\!\!\!\backslash\, 9) \wedge (5 \circ\!\!\!\backslash\, 10) \\ (2 \circ\!\!\!\backslash\, 8) \wedge (4 \circ\!\!\!\backslash\, 9) \wedge (6 \circ\!\!\!\backslash\, 10) \end{pmatrix} \\ &= \begin{pmatrix} min((8-1), (9-3), (10-5)) \\ min((8-2), (9-4), (10-6)) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \end{aligned}$$

$(L_{\mathsf{a}})^\sharp$ in semiring of intervals (Lhommeau 2004, Hardouin 2010)

$\mathbf{a}\overline{\otimes}\mathbf{x} \preceq \mathbf{b}$ over semiring of intervals $\mathcal{IS}$

The greatest solution of $\mathbf{a}\overline{\otimes}\mathbf{x} \preceq \mathbf{b}$ with $\mathbf{a}, \mathbf{x}, \mathbf{b}$ in semiring of intervals $\mathcal{IS}$ is given by :

$$\mathbf{x} = \mathbf{a}\,\overline{\circ\!\!\!\backslash}\,\mathbf{b} = [\underline{a} \circ\!\!\!\backslash \underline{b} \wedge \overline{a} \circ\!\!\!\backslash \overline{b}, \overline{a} \circ\!\!\!\backslash \overline{b}]$$

where bounds of intervals, $\underline{a}, \underline{b}, \overline{a}, \overline{b}$ are in $\mathcal{S}$.

Example in $\mathcal{I}\overline{\mathbb{Z}}_{\mathsf{max}}$

$$[5, 10] \otimes [\underline{x}, \overline{x}] \preceq [20, 21]$$

admits a greatest solution in $\mathcal{I}\overline{\mathbb{Z}}_{\mathsf{max}}$, it is given by :

$$[\underline{x}, \overline{x}] \preceq [5 \circ\!\!\!\backslash 20 \wedge 10 \circ\!\!\!\backslash 21, 10 \circ\!\!\!\backslash 21] = [11, 11]$$

Skecth of proof

Dual Residuation

Mapping (L_a) is not $u.s.c$

Due to the lack of distributivity of operators $\wedge$ over $\otimes$ mapping (L_a) is not $u.s.c$ hence is not dually residuated. Only sub-distributivity holds :

$$a \otimes (b \wedge c) \preceq (a \otimes b) \wedge (a \otimes c)$$

Sufficient condition

If a admits an inverse, ($i.e.$ $\exists d$ s.t. $a \otimes d = e$) then $a \otimes (b \wedge c) = (a \otimes b) \wedge (a \otimes c)$ hence L_a is $u.s.c.$, $i.e.$ dually residuated : $a \otimes x \succeq b$ admits a lowest solution denoted $(L_a)^\flat(b)$.

Particular Case : $L_a : \overline{\mathbb{Z}}_{\max} \rightarrow \overline{\mathbb{Z}}_{\max}$ is dually residuated

$\forall a \in \overline{\mathbb{Z}}_{\max}$ it exists an inverse, hence $L_a : \overline{\mathbb{Z}}_{\max} \rightarrow \overline{\mathbb{Z}}_{\max}$ is dually residuated , $a \otimes x \succeq b$ admits a lowest solution, $x \succeq b - a$.

Dual Product

Dual product $\Lambda_A : \mathcal{S}^{n\times q} \rightarrow \mathcal{S}^{p\times q}, x \mapsto A \odot x$

Let $A \in \mathcal{S}^{p\times n}$ and $B \in \mathcal{S}^{n\times q}$ be matrices, and the following product $A \odot B$ defined as follows :

$$(A \odot B)_{ij} = \bigwedge_{k=1}^{n} A_{ik} \odot B_{kj}$$

with the following rules $A_{ik} \odot B_{kj} = A_{ik} \otimes B_{kj}$, $x \odot \top = \top \odot x = \top$ and $\varepsilon \odot \top = \top \odot \varepsilon = \top$.

Particular case, max-plus algebra

$\Lambda_A : \overline{\mathbb{Z}}_{\max}^{n\times q} \rightarrow \overline{\mathbb{Z}}_{\max}^{p\times q}, x \mapsto A \odot x$ corresponds to the (min,plus) product.

Dual Residuation of Dual Product, $A \odot x \succeq B$

Sufficient Condition

Let $A \in \mathcal{S}^{p\times n}$ be a matrix . If each entry of A admits an inverse, mapping Λ_A is *u.s.c* and then is dually residuated, we denote

$$\Lambda_A^\flat : x \mapsto A \backslash\!\!\!\bullet\, x.$$

$$(A \backslash\!\!\!\bullet\, x)_{ij} = \bigoplus_{l=1}^{k=n} A_{ki} \backslash\!\!\!\bullet\, x_{kj}.$$

with the following rules : $\top \backslash\!\!\!\bullet\, x = \varepsilon$, $\varepsilon \backslash\!\!\!\bullet\, x = \top$ and $\varepsilon \backslash\!\!\!\bullet\, \varepsilon = \varepsilon$.
Hence, $A \backslash\!\!\!\bullet\, B$ is the lowest solution of $A \odot x \succeq B$.

Particular case, max-plus algebra

$\Lambda_A^\flat : \overline{\mathbb{Z}}_{\max}^{p\times q} \rightarrow \overline{\mathbb{Z}}_{\max}^{n\times q}, x \mapsto A \backslash\!\!\!\bullet\, x$ is a (max,plus) linear operator.

$\Lambda^\flat_{\mathbf{A}}$ in semiring of intervals $\mathcal{IS}$

What is happen for intervals ?

Intervals don't admit inverse.

Sufficient Condition

Let $\mathbf{a} = [\underline{a}, \overline{a}] \in \mathcal{IS}$ be an interval. If each bound of the interval admits an inverse, mapping $\Lambda_{\mathbf{a}}$ is dually residuated, and

$$\Lambda^\flat_{\mathbf{a}}(\mathbf{b}) = [\Lambda^\flat_{\underline{a}}(\underline{b}), \Lambda^\flat_{\underline{a}}(\underline{b}) \oplus \Lambda^\flat_{\overline{a}}(\overline{b})]$$

with $\mathbf{b} = [\underline{b}, \overline{b}]$ an interval. Hence, $\mathbf{a}\overline{\odot}\mathbf{x} \succeq \mathbf{b}$ admits a lowest solution :

$$\mathbf{a} \overline{\blacklozenge} \mathbf{b} = [\underline{a} \blacklozenge \underline{b}, \underline{a} \blacklozenge \underline{b} \oplus \overline{a} \blacklozenge \overline{b}].$$

$\Lambda^{\flat}_{\mathbf{a}}$ in semiring of intervals $\mathcal{IS}$

Sufficient Condition

Let $\mathbf{a} = [\underline{a}, \overline{a}] \in \mathcal{IS}$ be an interval. If each bound of the interval admits an inverse, mapping $\Lambda_{\mathbf{a}}$ is residuated, and

$$\Lambda^{\flat}_{\mathbf{a}}(\mathbf{b}) = [\Lambda^{\flat}_{\underline{a}}(\underline{b}), \Lambda^{\flat}_{\underline{a}}(\underline{b}) \oplus \Lambda^{\flat}_{\overline{a}}(\overline{b})]$$

with $\mathbf{b} = [\underline{b}, \overline{b}]$ an interval. Hence, $\mathbf{a} \overline{\odot} \mathbf{x} \succeq \mathbf{b}$ admits a lowest solution :

$$\mathbf{a} \overline{\bullet\!\!\setminus} \mathbf{b} = [\underline{a} \bullet\!\!\setminus \underline{b}, \underline{a} \bullet\!\!\setminus \underline{b} \oplus \overline{a} \bullet\!\!\setminus \overline{b}].$$

Sketch of proof

Semiring of intervals $\mathcal{IS}$ is a subsemiring of $\mathcal{S} \times \mathcal{S}$. The canonical injection from a subsemiring into a semiring is dually residuated (Blyth 72, Gaubert 92), *i.e.* $\mathrm{Id}_{|\mathcal{IS}} : \mathcal{IS} \to \mathcal{S} \times \mathcal{S}, x \mapsto x$ is dually residuated. Its dual residual $(\mathrm{Id}_{|\mathcal{IS}})^{\flat}$ is a projector :

$$(\mathrm{Id}_{|\mathcal{IS}})^{\flat}(x', x'') = (x', x' \oplus x'') = [\underline{x}, \overline{x}]$$

Hence $(\Lambda_{\mathbf{a}} \circ \mathrm{Id}_{|\mathcal{IS}})^{\flat} = (\mathrm{Id}_{|\mathcal{IS}})^{\flat} \circ (\Lambda_{\mathbf{a}})^{\flat}$ which yields the result.

$\Lambda^{\flat}_{\mathbf{A}}$ in semiring of intervals $\mathcal{I}\overline{\mathbb{Z}}_{\max}$

Illustration in $\mathcal{I}\overline{\mathbb{Z}}_{\max}$

In $\mathcal{I}\overline{\mathbb{Z}}_{\max}$ each bound of the interval admits an inverse. Let $\mathbf{a} = [5, 9]$ and $\mathbf{b} = [8, 20]$ be intervals in $\mathcal{I}\overline{\mathbb{Z}}_{\max}$.

The lowest solution of $[5, 9]\overline{\odot}\mathbf{x} \succeq [8, 20]$ is given by :

$$\mathbf{a}\overline{\lozenge\!\!\!\setminus}\mathbf{b} = [5\lozenge\!\!\!\setminus 8, 5\lozenge\!\!\!\setminus 8 \oplus 9\lozenge\!\!\!\setminus 20] = [\ 3,11\ ].$$

$\mathbf{A}\overline{\odot}\mathbf{x} \succeq \mathbf{B}$

Let $A = \begin{pmatrix} [1,3] & [2,5] \\ [3,7] & [4,6] \\ [5,8] & [6,7] \end{pmatrix}$ and $B = \begin{pmatrix} [4,9] \\ [5,10] \\ [3,8] \end{pmatrix}$ be matrices with entries in $\mathcal{I}\mathbb{Z}_{\max}$.

Greatest $\mathbf{x}$ such that $\mathbf{A}\overline{\odot}\mathbf{x} \succeq \mathbf{B}$ is given by : $\mathbf{x} = \mathbf{A}\lozenge\!\!\!\setminus\mathbf{B} = \begin{pmatrix} [3,6] \\ [2,4] \end{pmatrix}$

obtained by applying the following rules $(A\lozenge\!\!\!\setminus x)_{ij} = \bigoplus_{l=1}^{k=n} A_{ki}\lozenge\!\!\!\setminus x_{kj}$.

Conclusion

Conclusion

- Residuation and dual residuation of product law in semiring of intervals $\mathbf{L_a} : \mathbf{x} \mapsto \mathbf{a} \otimes \mathbf{x}$
- Useful in control theory to characterize state space achieving :

$$\mathbf{A} \otimes \mathbf{x} \preceq \mathbf{x} \preceq \mathbf{A} \odot \mathbf{x}$$

  where the entries are intervals, *i.e.* the system is known in an uncertain way.
- Illustration are given in $\mathcal{I}\mathbb{Z}_{\mathrm{max}}$ but it runs also in semiring of series $\mathcal{I}\overline{\mathbb{Z}}_{\mathrm{max}}[\![\gamma]\!]$ and all semirings of intervals.
- Questions ?

Bibliography

References

- T.S. Blyth and M.F. Janowitz,
  Residuation Theory, Pergamon press, 1972.
- G. Cohen, P. Moller, J.P. Quadrat and M. Viot,
  Algebraic Tools for the Performance Evaluation of Discrete Event Systems,
  IEEE Proceedings : Special issue on Discrete Event Systems,
  vol. 77 :1,pp 39-58, 1989.
- R. Croisot,
  Applications Résiduées,
  Annales Scientifiques, Paris Ecole Normale Supérieure, 73, 1956, pp 453-474.
- R.A. Cuninghame-Green,
  Minimax Algebra,
  Lecture notes in Economics and Mathematical Systems, 166, 1979.

Bibliography

References

- S. Gaubert,
  Théorie des Systèmes Linéaires dans les Dioïdes,
  PhD, 1992.
- L. Hardouin, B. Cottenceau, M. Lhommeau and E. LeCorronc,
  Interval systems over idempotent semiring,
  Linear Algebra and Applications, 2009, 431, 5-7, pp 855-862,
  dx.doi.org/10.1016/j.laa.2009.03.039.
- G.L. Litvinov and A.N. Sobolevski,
  Idempotent Interval Analysis and Optimization Problems,
  Reliable Computing, 2001, 7 :5, pp 353-377.
- M. Lhommeau, L. Hardouin, B. Cottenceau and L. Jaulin,
  Interval Analysis and Dioid : Application to Robust Controller Design for Timed Event graphs,
  Automatica, 2004, 40, pp 1923-1930,
  doi:10.1016/j.automatica.2004.05.013.