

Interval systems over idempotent semiring

Laurent Hardouin*

Bertrand Cottenceau

Mehdi Lhommeau

Euriell Le Corrond

*Laboratoire d'Ingénierie des Systèmes Automatisés,
Université d'Angers,
62, avenue Notre Dame du Lac, 49000 Angers, France.*

Abstract

This paper deals with solution of inequality $\mathbf{A} \otimes \mathbf{x} \preceq \mathbf{b}$, where \mathbf{A} , \mathbf{x} and \mathbf{b} are interval matrices with entries defined over idempotent semiring. It deals also with the computation of a pair of intervals, (\mathbf{x}, \mathbf{y}) which satisfies the equation $\mathbf{A} \otimes \mathbf{x} = \mathbf{B} \otimes \mathbf{y}$. It will be shown that this equation may be solved by considering the interval version of the iterative scheme proposed in [7].

Key words: Max algebra; Idempotent semiring; Interval analysis; Residuation theory;

AMS classification : Primary 65G40 Secondary : 06F05, 15A09.

1. Introduction

Many problems in the optimization theory and other fields of mathematics are non-linear in the traditional sense but appear to be linear over idempotent semirings (e.g., see [1, 3, 6, 10]). Idempotency of the additive law induces

*Corresponding author

Email addresses: laurent.hardouin@univ-angers.fr (Laurent Hardouin),
bertrand.cottenceau@univ-angers.fr (Bertrand Cottenceau),
mehdi.lhommeau@univ-angers.fr (Mehdi Lhommeau), euriell.lecorrond@univ-angers.fr
(Euriell Le Corrond)

that idempotent semirings are (partially) ordered sets. The Residuation theory [2, 5, 8] is a suitable tool to deal with inverse problems of order preserving mappings. It is usually used to solve equations defined over idempotent semiring [1, 6, 7], for instance the greatest solution of inequality $Ax \preceq b$ may be computed by means of this theory.

Interval mathematics was pioneered by R.E. Moore (see [16]) as a tool for bounding rounding errors in computer programs. Since then, interval mathematics has been developed into a general methodology for investigating numerical uncertainty in numerous problems and algorithms. In [14] the idempotent version is addressed. The authors show that idempotent interval mathematics appears to be remarkably simpler than its traditional analog. For example, in the traditional interval arithmetic, multiplication of intervals is not distributive with respect to addition of intervals, while idempotent interval arithmetic keeps this distributivity. This paper deals first with solution of inequality $\mathbf{A} \otimes \mathbf{x} \preceq \mathbf{b}$, where \mathbf{A} , \mathbf{x} and \mathbf{b} are interval matrices (see proposition 31). When equality is achieved, according to definition given in [4], the equations system is said weakly solvable since at least one of its subsystems is solvable. In a second step, the paper deals with the computation of a pair of intervals, (\mathbf{x}, \mathbf{y}) which satisfies the equation $\mathbf{A} \otimes \mathbf{x} = \mathbf{B} \otimes \mathbf{y}$. It will be shown that this equation may be solved by considering the interval version of the iterative scheme proposed in [7].

2. Preliminaries

Definition 1. *A semiring \mathcal{S} is a set endowed with two internal operations denoted by \oplus (addition) and \otimes (multiplication), both associative and both having neutral elements denoted by ε and e respectively, such that \oplus is also commutative and idempotent (i.e. $a \oplus a = a$). The \otimes operation is distributive with respect to \oplus , and ε is absorbing for the product (i.e. $\forall a, \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$). When \otimes is commutative, the semiring is said to be commutative.*

Semirings can be endowed with a canonical *order* defined by : $a \succeq b$ iff $a = a \oplus b$. Then they become sup-semilattices and $a \oplus b$ is the least upper bound

of a and b . A semiring is *complete* if sums of infinite number of terms are always defined, and if multiplication distributes over infinite sums too. In particular, the sum of all elements of a complete semiring is defined and denoted by \top (for 'top'). A complete semiring (sup-semilattice) becomes a complete lattice for which the greatest lower bound of a and b is denoted $a \wedge b$, i.e., the least upper bound of the (nonempty) subset of all elements which are less than a and b (see [1, §4]).

Example 2 ((max, +) algebra). *The set $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the max operator as \oplus and the classical sum as \otimes is a complete idempotent semiring of which $\varepsilon = -\infty$, $e = 0$ and $\top = +\infty$ and the greatest lower bound $a \wedge b = \min(a, b)$.*

Example 3 ((max, min) algebra). *The set $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the max operator as \oplus and the min operator as \otimes is a complete idempotent semiring of which $\varepsilon = -\infty$, $e = +\infty$ and $\top = +\infty$ and the greatest lower bound $a \wedge b = \min(a, b)$.*

Definition 4 (Subsemiring). *A subset \mathcal{C} of a semiring is called a subsemiring of \mathcal{S} if*

- $\varepsilon \in \mathcal{C}$ and $e \in \mathcal{C}$;
- \mathcal{C} is closed for \oplus and \otimes , i.e., $\forall a, b \in \mathcal{C}$, $a \oplus b \in \mathcal{C}$ and $a \otimes b \in \mathcal{C}$.

Definition 5 (Principal order ideal). *Let \mathcal{S} be an idempotent semiring. An order ideal set is a nonempty subset \mathcal{X} of \mathcal{S} such that $(x \in \mathcal{X} \text{ and } y \preceq x) \Rightarrow y \in \mathcal{X}$. A principal order ideal (generated by x) is an order ideal, denoted $\downarrow \mathcal{X}_x$, of the form $\downarrow \mathcal{X}_x := \{y \in \mathcal{S} \mid y \preceq x\}$.*

The residuation theory provides, under some assumptions, *greatest* solutions to inequalities such as $f(x) \preceq b$ where f is an order preserving mapping (i.e., $a \preceq b \Rightarrow f(a) \preceq f(b)$) defined over ordered sets.

Definition 6 (Residual and residuated mapping). *An order preserving mapping $f : \mathcal{D} \rightarrow \mathcal{E}$, where \mathcal{D} and \mathcal{E} are ordered sets, is a residuated mapping if for*

all $y \in \mathcal{E}$, the least upper bound of the subset $\{x | f(x) \preceq y\}$ exists and belongs to this subset. It is then denoted by $f^\sharp(y)$. Mapping f^\sharp is called the residual of f . When f is residuated, f^\sharp is the unique order preserving mapping such that

$$f \circ f^\sharp \preceq \text{Id}_{\mathcal{E}} \quad \text{and} \quad f^\sharp \circ f \succeq \text{Id}_{\mathcal{D}}, \quad (1)$$

where Id is the identity mapping respectively on \mathcal{D} and \mathcal{E} .

Property 7. Let $f : \mathcal{D} \rightarrow \mathcal{E}$ be a residuated mapping, then

$$y \in \text{Im} f \Leftrightarrow f(f^\sharp(y)) = y.$$

Property 8 ([1, Th. 4.56]). If $h : \mathcal{D} \rightarrow \mathcal{C}$ and $f : \mathcal{C} \rightarrow \mathcal{B}$ are residuated mappings, then $f \circ h$ is also residuated and

$$(f \circ h)^\sharp = h^\sharp \circ f^\sharp. \quad (2)$$

Theorem 9 ([1, §4.4.2]). Consider the mapping $f : \mathcal{E} \rightarrow \mathcal{F}$ where \mathcal{E} and \mathcal{F} are complete semirings. Their bottom elements are, respectively, denoted by $\varepsilon_{\mathcal{E}}$ and $\varepsilon_{\mathcal{F}}$. Then, f is residuated iff $f(\varepsilon_{\mathcal{E}}) = \varepsilon_{\mathcal{F}}$ and $f(\bigoplus_{x \in \mathcal{G}} x) = \bigoplus_{x \in \mathcal{G}} f(x)$ for each $\mathcal{G} \subseteq \mathcal{E}$ (i.e., f is lower-semicontinuous).

Corollary 10. The mappings $L_a : x \mapsto a \otimes x$ and $R_a : x \mapsto x \otimes a$ defined over a complete semiring \mathcal{S} are both residuated. Their residuals are usually denoted, respectively, by $L_a^\sharp(x) = a \backslash x$ and $R_a^\sharp(x) = x / a$ in literature. Therefore, $a \backslash b$ (resp. b / a) is the greatest solution of $a \otimes x \preceq b$ (resp. $x \otimes a \preceq b$). For matrices, the practical computation is obtained as follows,

$$(A \backslash B)_{ij} = \bigwedge_{k=1 \dots n} (A_{ki} \backslash B_{kj}), \quad (3)$$

$$(B / C)_{ij} = \bigwedge_{k=1 \dots m} (B_{ik} / C_{jk}) \quad (4)$$

with $A \in \mathcal{S}^{n \times p}$, $B \in \mathcal{S}^{n \times m}$ and $C \in \mathcal{S}^{p \times m}$.

Example 11. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix}$ be matrices with entries in $(\text{max}, +)$ algebra.

In $(max, +)$ algebra $a_{ij} \dot{\setminus} b_j = b_j - a_{ij}$ then the greatest x such that $A \otimes x \preceq b$ is given by :

$$x = A \dot{\setminus} B = \begin{pmatrix} (1 \dot{\setminus} 8) \wedge (3 \dot{\setminus} 9) \wedge (5 \dot{\setminus} 10) \\ (2 \dot{\setminus} 8) \wedge (4 \dot{\setminus} 9) \wedge (6 \dot{\setminus} 10) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

Example 12. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 8 \\ 1 \\ 10 \end{pmatrix}$ be matrices with entries in (max, min) algebra. In (max, min) algebra, if $b_j \succeq a_{ij}$ then $a_{ij} \dot{\setminus} b_j = b_j$ else $a_{ij} \dot{\setminus} b_j = \top$.

$$A \dot{\setminus} B = \begin{pmatrix} (1 \dot{\setminus} 8) \wedge (3 \dot{\setminus} 1) \wedge (5 \dot{\setminus} 10) \\ (2 \dot{\setminus} 8) \wedge (4 \dot{\setminus} 1) \wedge (6 \dot{\setminus} 10) \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix}$$

The problem of mapping restriction and its connection with the residuation theory is now addressed.

Property 13 ([2]). Let $\text{Id}_{|\mathcal{S}_{sub}} : \mathcal{S}_{sub} \rightarrow \mathcal{S}$, $x \mapsto x$ be the canonical injection from a complete subsemiring into a complete semiring. The injection $\text{Id}_{|\mathcal{S}_{sub}}$ is residuated and its residual is a projector which will be denoted by $\text{Pr}_{\mathcal{S}_{sub}}$, with :

$$\text{Pr}_{\mathcal{S}_{sub}} = (\text{Id}_{|\mathcal{S}_{sub}})^\# = \text{Pr}_{\mathcal{S}_{sub}} \circ \text{Pr}_{\mathcal{S}_{sub}}.$$

Definition 14 (Restricted mapping). Let $f : \mathcal{E} \rightarrow \mathcal{F}$ be a mapping and $\mathcal{A} \subseteq \mathcal{E}$. We will denote $f_{|\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{F}$ the mapping defined by $f_{|\mathcal{A}} = f \circ \text{Id}_{|\mathcal{A}}$ where $\text{Id}_{|\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{E}$ is the canonical injection from \mathcal{A} to \mathcal{E} . Similarly, let $\mathcal{B} \subseteq \mathcal{F}$ with $\text{Im} f \subseteq \mathcal{B}$. Mapping ${}_{\mathcal{B}}f : \mathcal{E} \rightarrow \mathcal{B}$ is defined by $f = \text{Id}_{|\mathcal{B}} \circ {}_{\mathcal{B}}f$, where $\text{Id}_{|\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{F}$.

Property 15. Let $f : \mathcal{D} \rightarrow \mathcal{E}$ be a residuated mapping and \mathcal{D}_{sub} (resp. \mathcal{E}_{sub}) be a complete subsemiring of \mathcal{D} (resp. \mathcal{E}).

1. Mapping $f_{|\mathcal{D}_{sub}}$ is residuated and its residual is given by :

$$(f_{|\mathcal{D}_{sub}})^\# = (f \circ \text{Id}_{|\mathcal{D}_{sub}})^\# = \text{Pr}_{\mathcal{S}_{sub}} \circ f^\#.$$

2. If $\text{Im} f \subseteq \mathcal{E}_{sub}$ then mapping ${}_{\mathcal{E}_{sub}}f$ is residuated and its residual is given by:

$$({}_{\mathcal{E}_{sub}}f)^\# = f^\# \circ \text{Id}_{|\mathcal{E}_{sub}} = (f^\#)_{|\mathcal{E}_{sub}}.$$

PROOF. Statement 1 follows directly from property 8 and proposition 13. Statement 2 is obvious since f is residuated and $\text{Im} f \subset \mathcal{E}_{sub} \subset \mathcal{E}$. \square

In [7], the authors propose to compute a pair (x, y) satisfying the following equation :

$$a \otimes x = b \otimes y. \quad (5)$$

Theorems 16 and 17 given below recall how to compute such a pair, called simply a solution hereafter.

Theorem 16. *Let*

$$\Pi : x \mapsto L_a^\sharp \circ L_b \circ L_b^\sharp \circ L_a(x) = a \backslash (b \otimes (b \backslash (a \otimes x)))$$

be a mapping defined over a semiring \mathcal{S} and consider the following iterative scheme :

Let $x_0 \in \mathcal{S}$ be an arbitrary element

$$\text{do } x_{n+1} = \Pi(x_n)$$

until $x_{m+1} = x_m$ for $m \in \mathbb{N}$

If function Π admits a fixed point x such that $x \preceq \Pi(x_0)$ and $x \neq \varepsilon$ then the previous algorithm converges toward the greatest finite fixed point in the following principal order ideal $\downarrow \mathcal{X}_{\Pi(x_0)} = \{x \mid x \preceq \Pi(x_0)\}$.

PROOF. First let us recall that a lower-bounded non increasing integer sequence converges in a finite number of steps to the greatest fixed point. Then if function Π admits a fixed point x such that $x \preceq \Pi(x_0)$ and $x \neq \varepsilon$ then it is sufficient to show that the sequence x_n ($n = 1, 2, \dots$) is non increasing. Definition 6 yields that $L_b \circ L_b^\sharp \preceq \text{Id}$ and $L_a \circ L_a^\sharp \preceq \text{Id}$, then $\Pi(x) \preceq L_a^\sharp \circ L_a(x)$ and

$$\Pi \circ \Pi = L_a^\sharp \circ L_b \circ L_b^\sharp \circ L_a \circ L_a^\sharp \circ L_b \circ L_b^\sharp \circ L_a \quad (6)$$

$$\Pi \circ \Pi \preceq L_a^\sharp \circ L_b \circ L_b^\sharp \circ L_a = \Pi \quad (7)$$

therefore $x_{n+1} \preceq x_n$, *i.e.* the sequence is non increasing.

Obviously if $x_{n+1} = \Pi(x_n)$ then $x_{n+1} \preceq \Pi(x_0)$, and it is the greatest in $\downarrow \mathcal{X}_{\Pi(x_0)}$.

\square

Theorem 17. Define $y = L_b^\sharp \circ L_a(x) = b \backslash (a \otimes x)$. If the function Π defined in theorem 16 admits a fixed point $x \neq \varepsilon$, then the pair (x, y) is a solution of (5).

PROOF. If $x = \Pi(x)$ and $y = L_b^\sharp \circ L_a(x)$ then, by using twice equation (1), we obtain

$$L_a(x) = L_a \circ L_a^\sharp \circ L_b \circ L_b^\sharp \circ L_a(x) \preceq L_b \circ L_b^\sharp \circ L_a(x) = L_b(y) \preceq L_a(x).$$

Therefore all these terms are equal, and $a \otimes x = L_a(x) = L_b(y) = b \otimes y$. \square

3. Semiring of intervals

A semiring of interval may be constructed by considering a semiring of pairs. The set of pairs (x', x'') with $x' \in \mathcal{S}$ and $x'' \in \mathcal{S}$ endowed with the following coordinate-wise algebraic operations :

$$(x', x'') \oplus (y', y'') \triangleq (x' \oplus y', x'' \oplus y'')$$

and $(x', x'') \otimes (y', y'') \triangleq (x' \otimes y', x'' \otimes y'')$,

is a semiring denoted by $\mathcal{C}(\mathcal{S})$ with $(\varepsilon, \varepsilon)$ as the zero element and (e, e) as the identity element (see definition 1). Some of the preliminary statements of this section are adapted from [14].

Remark 18. The operation \oplus generates the corresponding canonical partial order $\preceq_{\mathcal{C}}$ in $\mathcal{C}(\mathcal{S})$:

$(x', x'') \oplus (y', y'') = (y', y'') \Leftrightarrow (x', x'') \preceq_{\mathcal{C}} (y', y'') \Leftrightarrow x' \preceq_{\mathcal{S}} y'$ and $x'' \preceq_{\mathcal{S}} y''$ where $\preceq_{\mathcal{S}}$ is the order relation in \mathcal{S} .

Property 19. If the semiring \mathcal{S} is complete, then the semiring $\mathcal{C}(\mathcal{S})$ is complete and its top element is given by (\top, \top) .

Notation 20. Consider the following mappings over $\mathcal{C}(\mathcal{S})$:

$$\begin{aligned} L_{(a', a'')} &: (x', x'') \mapsto (a', a'') \otimes (x', x'') && \text{(left multiplication by } (a', a'')) \\ R_{(a', a'')} &: (x', x'') \mapsto (x', x'') \otimes (a', a'') && \text{(right multiplication by } (a', a'')). \end{aligned}$$

Property 21. The mappings $L_{(a', a'')}$ and $R_{(a', a'')}$ defined over $\mathcal{C}(\mathcal{S})$ are both residuated. Their residuals are equal to $L_{(a', a'')}^\sharp(b', b'') = (a', a'') \backslash (b', b'') = (a' \backslash b', a'' \backslash b'')$ and $R_{(a', a'')}^\sharp(b', b'') = (b', b'') \phi (a', a'') = (b' \phi a', b'' \phi a'')$.

PROOF. Observe that $L_{(a', a'')} \left(\bigoplus_{(x', x'') \in X} (x', x'') \right) = \bigoplus_{(x', x'') \in X} L_{(a', a'')} (x', x'')$, (for every subset X of $\mathcal{C}(\mathcal{S})$), moreover $L_{(a', a'')}(\varepsilon, \varepsilon) = (a' \otimes \varepsilon, a'' \otimes \varepsilon) = (\varepsilon, \varepsilon)$. Then $L_{(a', a'')}$ is residuated (due to Theorem 9). Therefore, we have to find, for given (b', b'') and (a', a'') , the greatest solution (x', x'') for inequality $(a', a'') \otimes (x', x'') \preceq_{\mathcal{C}} (b', b'') \Leftrightarrow (a' \otimes x', a'' \otimes x'') \preceq_{\mathcal{C}} (b', b'')$, moreover according to Remark 18 on the order relation induced by \oplus on $\mathcal{C}(\mathcal{S})$ we have,

$$a' \otimes x' \preceq_{\mathcal{S}} b' \quad \text{and} \quad a'' \otimes x'' \preceq_{\mathcal{S}} b''.$$

Since the mappings $x' \mapsto a' \otimes x'$ and $x'' \mapsto a'' \otimes x''$ are residuated over \mathcal{S} (cf. Corollary 10), we have $x' \preceq_{\mathcal{S}} a' \backslash b'$ and $x'' \preceq_{\mathcal{S}} a'' \backslash b''$. Then, we obtain $L_{(a', a'')}^{\sharp}(b', b'') = (a' \backslash b', a'' \backslash b'')$. \square

Notation 22. The set of pairs $(\tilde{x}', \tilde{x}'')$ s.t. $\tilde{x}' \preceq \tilde{x}''$ is denoted by $\mathcal{C}_0(\mathcal{S})$.

Property 23. Let \mathcal{S} be a complete semiring. The set $\mathcal{C}_0(\mathcal{S})$ is a complete subsemiring of $\mathcal{C}(\mathcal{S})$.

PROOF. Clearly $\mathcal{C}_0(\mathcal{S}) \subset \mathcal{C}(\mathcal{S})$ and it is closed for \oplus and \otimes since $\tilde{x}' \oplus \tilde{y}' \preceq \tilde{x}'' \oplus \tilde{y}''$ and $\tilde{x}' \otimes \tilde{y}' \preceq \tilde{x}'' \otimes \tilde{y}''$ whenever $\tilde{x}' \preceq \tilde{x}''$ and $\tilde{y}' \preceq \tilde{y}''$. Moreover zero element $(\varepsilon, \varepsilon)$, unit element (e, e) and top element (\top, \top) of $\mathcal{C}(\mathcal{S})$ are in $\mathcal{C}_0(\mathcal{S})$. Therefore definition 4 yields the result. \square

Property 24. The canonical injection $\text{Id}_{|\mathcal{C}_0(\mathcal{S})} : \mathcal{C}_0(\mathcal{S}) \rightarrow \mathcal{C}(\mathcal{S})$ is residuated. Its residual $(\text{Id}_{|\mathcal{C}_0(\mathcal{S})})^{\sharp}$ is a projector denoted by $\text{Pr}_{\mathcal{C}_0(\mathcal{S})}$. Its practical computation is given by :

$$\text{Pr}_{\mathcal{C}_0(\mathcal{S})}((x', x'')) = (x' \wedge x'', x'') = (\tilde{x}', \tilde{x}''). \quad (8)$$

PROOF. It is a direct application of proposition 13, since $\mathcal{C}_0(\mathcal{S})$ is a subsemiring of $\mathcal{C}(\mathcal{S})$. Practically, let $(x', x'') \in \mathcal{C}(\mathcal{S})$, we have $\text{Pr}_{\mathcal{C}_0(\mathcal{S})}((x', x'')) = (\tilde{x}', \tilde{x}'') = (x' \wedge x'', x'')$, which is the greatest pair such that :

$$\tilde{x}' \preceq x', \quad \tilde{x}'' \preceq x'' \quad \text{and} \quad \tilde{x}' \preceq \tilde{x}''.$$

\square

Property 25. Let $(\tilde{a}', \tilde{a}'') \in \mathcal{C}_0(\mathcal{S})$, then the mapping ${}_{\mathcal{C}_0(\mathcal{S})}L_{(\tilde{a}', \tilde{a}'')|_{\mathcal{C}_0(\mathcal{S})}} : \mathcal{C}_0(\mathcal{S}) \rightarrow \mathcal{C}_0(\mathcal{S})$ is residuated. Its residual is given by

$$\left({}_{\mathcal{C}_0(\mathcal{S})}L_{(\tilde{a}', \tilde{a}'')|_{\mathcal{C}_0(\mathcal{S})}} \right)^\# = \text{Pr}_{\mathcal{C}_0(\mathcal{S})} \circ (L_{(\tilde{a}', \tilde{a}'')})^\# \circ \text{Id}_{|\mathcal{C}_0(\mathcal{S})}.$$

PROOF. Since $(\tilde{a}', \tilde{a}'') \in \mathcal{C}_0(\mathcal{S}) \subset \mathcal{C}(\mathcal{S})$, it follows directly from proposition 21 that mapping $L_{(\tilde{a}', \tilde{a}'')}$ defined over $\mathcal{C}(\mathcal{S})$ is residuated. Furthermore, $\mathcal{C}_0(\mathcal{S})$ being closed for \otimes we have $\text{Im}L_{(\tilde{a}', \tilde{a}'')|_{\mathcal{C}_0(\mathcal{S})}} \subset \mathcal{C}_0(\mathcal{S})$, it follows from definition 14 and proposition 15 that :

$$\begin{aligned} \left({}_{\mathcal{C}_0(\mathcal{S})}L_{(\tilde{a}', \tilde{a}'')|_{\mathcal{C}_0(\mathcal{S})}} \right)^\# &= (L_{(\tilde{a}', \tilde{a}'')} \circ \text{Id}_{|\mathcal{C}_0(\mathcal{S})})^\# \circ \text{Id}_{|\mathcal{C}_0(\mathcal{S})} \\ &= \text{Pr}_{\mathcal{C}_0(\mathcal{S})} \circ (L_{(\tilde{a}', \tilde{a}'')})^\# \circ \text{Id}_{|\mathcal{C}_0(\mathcal{S})}. \end{aligned}$$

Then, by considering $(\tilde{b}', \tilde{b}'') \in \mathcal{C}_0(\mathcal{S}) \subset \mathcal{C}(\mathcal{S})$, the greatest solution in $\mathcal{C}_0(\mathcal{S})$ of $L_{(\tilde{a}', \tilde{a}'')}((\tilde{x}', \tilde{x}'')) = (\tilde{a}', \tilde{a}'') \otimes (\tilde{x}', \tilde{x}'') \preceq (\tilde{b}', \tilde{b}'')$ is $L_{(\tilde{a}', \tilde{a}'')}^\#((\tilde{b}', \tilde{b}'')) = (\tilde{x}', \tilde{x}'') = (\tilde{a}', \tilde{a}'') \mathbin{\text{\textcircled{&}}} (\tilde{b}', \tilde{b}'') = \text{Pr}_{\mathcal{C}_0(\mathcal{S})}((\tilde{a}' \mathbin{\text{\textcircled{&}}} \tilde{b}', \tilde{a}'' \mathbin{\text{\textcircled{&}}} \tilde{b}'')) = (\tilde{a}' \mathbin{\text{\textcircled{&}}} \tilde{b}' \wedge \tilde{a}'' \mathbin{\text{\textcircled{&}}} \tilde{b}'', \tilde{a}'' \mathbin{\text{\textcircled{&}}} \tilde{b}'')$. \square

Definition 26. A (closed) interval in semiring \mathcal{S} is a set of the form $\mathbf{x} = [x, \bar{x}] = \{t \in \mathcal{S} | x \preceq t \preceq \bar{x}\}$, where $(x, \bar{x}) \in \mathcal{C}_0(\mathcal{S})$, x (respectively, \bar{x}) is said to be the lower (respectively, upper) bound of the interval \mathbf{x} .

Property 27. The set of intervals, denoted by $\mathbf{I}(\mathcal{S})$, endowed with the following coordinate-wise algebraic operations :

$$\mathbf{x} \bar{\oplus} \mathbf{y} \triangleq [x \oplus y, \bar{x} \oplus \bar{y}] \quad \text{and} \quad \mathbf{x} \bar{\otimes} \mathbf{y} \triangleq [x \otimes y, \bar{x} \otimes \bar{y}] \quad (9)$$

is a semiring, where the interval $\boldsymbol{\varepsilon} = [\varepsilon, \varepsilon]$ (respectively, $\mathbf{e} = [e, e]$) is zero (respectively, unit) element of $\mathbf{I}(\mathcal{S})$. Moreover, the semiring $\mathbf{I}(\mathcal{S})$ is isomorphic to $\mathcal{C}_0(\mathcal{S})$.

PROOF. First, $x \oplus y \preceq \bar{x} \oplus \bar{y}$ and $x \otimes y \preceq \bar{x} \otimes \bar{y}$ whenever $x \preceq \bar{x}$ and $y \preceq \bar{y}$, then $\mathbf{I}(\mathcal{S})$ is closed with respect to the operations $\bar{\oplus}, \bar{\otimes}$. From definition 1, it follows directly that it is a semiring. Obviously, it is isomorphic to $\mathcal{C}_0(\mathcal{S})$, indeed let $\Psi : \mathcal{C}_0(\mathcal{S}) \rightarrow \mathbf{I}(\mathcal{S}), (\tilde{x}', \tilde{x}'') \mapsto [x, \bar{x}] = [\tilde{x}', \tilde{x}'']$ be the mapping which maps an interval to an ordered pair. Obviously Ψ^{-1} is well defined and the both mappings are homomorphisms. \square

Remark 28. Operations (9) give the tightest interval containing all results of the same operations to arbitrary elements of its interval operands.

Remark 29. Let \mathcal{S} be a complete semiring and $\{\mathbf{x}_\alpha\}$ be an infinite subset of $\mathbf{I}(\mathcal{S})$, the infinite sum of elements of this subset is :

$$\overline{\bigoplus_{\alpha} \mathbf{x}_\alpha} = \left[\bigoplus_{\alpha} \underline{x}_\alpha, \bigoplus_{\alpha} \overline{x}_\alpha \right].$$

Remark 30. If \mathcal{S} is a complete semiring then $\mathbf{I}(\mathcal{S})$ is a complete semiring. Its top element is given by $\top = [\top, \top]$.

Note that if \mathbf{x} and \mathbf{y} are intervals in $\mathbf{I}(\mathcal{S})$, then $\mathbf{x} \subset \mathbf{y}$ iff $\underline{y} \preceq \underline{x} \preceq \overline{x} \preceq \overline{y}$. In particular, $\mathbf{x} = \mathbf{y}$ iff $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$.

An interval for which $\underline{x} = \overline{x}$ is called degenerate. Degenerate intervals allow to represent numbers without uncertainty. In this case we identify \mathbf{x} with its element by writing $\mathbf{x} \equiv x$.

Property 31. Mapping $L_{\mathbf{a}} : \mathbf{I}(\mathcal{S}) \rightarrow \mathbf{I}(\mathcal{S}), \mathbf{x} \mapsto \mathbf{a} \otimes \overline{\mathbf{x}}$ is residuated. Its residual is equal to

$$L_{\mathbf{a}}^{\sharp} : \mathbf{I}(\mathcal{S}) \rightarrow \mathbf{I}(\mathcal{S}), (\mathbf{x}) \mapsto \overline{\mathbf{a} \backslash \mathbf{x}} = [\underline{\mathbf{a} \backslash \underline{x}}, \overline{\mathbf{a} \backslash \overline{x}}].$$

Therefore, $\overline{\mathbf{a} \backslash \mathbf{b}}$ is the greatest solution of $\mathbf{a} \otimes \overline{\mathbf{x}} \preceq \mathbf{b}$, and the equality is achieved if $\mathbf{b} \in \text{Im} L_{\mathbf{a}}$.

PROOF. Let $\Psi : \mathcal{C}_0(\mathcal{S}) \rightarrow \mathbf{I}(\mathcal{S}), (\tilde{x}', \tilde{x}'') \mapsto [\underline{x}, \overline{x}] = [\tilde{x}', \tilde{x}'']$ be the mapping which maps an interval to an ordered pair. This mapping defines an isomorphism, since it is sufficient to handle the bounds to handle an interval. Then the result follows directly from proposition 25. \square

Remark 32. We would show in the same manner that mapping $R_{\mathbf{a}} : \mathbf{I}(\mathcal{S}) \rightarrow \mathbf{I}(\mathcal{S}), \mathbf{x} \mapsto \overline{\mathbf{x}} \otimes \mathbf{a}$ is residuated.

Remark 33. These results show that it is not sufficient to consider independently the intervals bounds to compute the solution of the equation $\mathbf{a} \otimes \overline{\mathbf{x}} \preceq \mathbf{b}$. Therefore solution (\mathbf{x}, \mathbf{y}) of equality $\mathbf{a} \otimes \overline{\mathbf{x}} = \overline{\mathbf{b}} \otimes \mathbf{y}$ can not be obtained by

using the algorithm proposed in theorem 16 independently for each bounds. The following interval version needs to be used with the following mapping

$$\mathbf{\Pi} : \mathbf{x} \mapsto \mathbf{a} \bar{\bowtie} (\mathbf{b} \bar{\otimes} (\mathbf{b} \bar{\bowtie} (\mathbf{a} \bar{\otimes} \mathbf{x}))). \quad (10)$$

According to theorem 17, if $\mathbf{\Pi}$ admits a fixed point $\mathbf{x} \neq [\varepsilon, \varepsilon]$, the pair (\mathbf{x}, \mathbf{y}) with $\mathbf{y} = \mathbf{b} \bar{\bowtie} (\mathbf{a} \bar{\otimes} \mathbf{x})$ is a solution of equation $\mathbf{a} \bar{\otimes} \mathbf{x} = \mathbf{b} \bar{\otimes} \mathbf{y}$ and obviously the iterative scheme given in theorem 16 still valid.

Example 34. Let $\mathbf{A} = \begin{pmatrix} [2, 3] & [5, 9] \\ [7, 8] & [3, 6] \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} [1, 9] & [2, 5] & [3, 4] \\ [1, 13] & [3, 10] & [9, 10] \end{pmatrix}$ be matrices with entries in $(\max, +)$ semiring. By considering interval version of algorithm proposed in theorem 16, with $\mathbf{x}_0 = \begin{pmatrix} [4, 7] \\ [3, 5] \end{pmatrix}$ the convergence is achieved with the following value of $\mathbf{x} = \begin{pmatrix} [4, 7] \\ [2, 2] \end{pmatrix}$, and then by applying proposi-

tion 31 we obtain $\mathbf{y} = \begin{pmatrix} [2, 2] \\ [5, 5] \\ [2, 5] \end{pmatrix}$. These computations may be obtained thanks to the $(\max, +)$ toolboxes interfaced with Scilab (see [11, 12]) and this specific example may be obtained at <http://www.istia-angers.fr/~hardouin/LAAA.html>.

Example 35. By considering these same matrices in (\max, \min) algebra, the same algorithm with the rules given in example 12, the vectors obtained are :

$$x = \begin{pmatrix} [3, 7] \\ [2, 5] \end{pmatrix} \text{ and } y = \begin{pmatrix} [5, 5] \\ [7, 7] \\ [7, 7] \end{pmatrix}.$$

4. Conclusion

This work shows that Residuation theory and resolution of matrix systems of the form $\mathbf{Ax} \preceq \mathbf{B}$ over interval semirings is not so straightforward as interval extension of simpler linear-algebraic operations considered in [14]. Nevertheless, by considering the right calculus rule the classical algorithm (see [7]) remains

efficient to solve some interval systems. Example are given both in $(max, +)$ semiring and (max, min) semiring but may be extended to other semirings such as semiring of series introduced in [1, 9] and can be useful to solve control problems such as the one considered in [13, 15].

References

- [1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity : An Algebra for Discrete Event Systems*. Wiley and Sons, 1992.
- [2] T.S. Blyth and M.F. Janowitz. *Residuation Theory*. Pergamon press, 1972.
- [3] B.A. Carré. *Graphs and Networks*. Oxford University Press, 1979.
- [4] K. Cechlárova and R.A. Cuninghame-Green. Interval systems of max-separable linear equations. *Linear Algebra and its Applications*, (340):215–224, 2002.
- [5] R. Croisot. Applications résiduées. *Annales Scientifiques, Paris Ecole Normale Supérieure*, 73:453–474, 1956.
- [6] R.A. Cuninghame-Green. *Minimax Algebra*. Number 166 in Lecture notes in Economics and Mathematical Systems. Springer, 1979.
- [7] R.A. Cuninghame-Green and P. Butkovič. The equation $ax = by$ over $(max, +)$. *Theoretical Computer Science*, 293:3–12, 2003.
- [8] J. Derderian. Residuated mappings. *Pacific Journal of Mathematics*, 20:35–44, 1967.
- [9] S. Gaubert. *Théorie des Systèmes Linéaires dans les Dioïdes*. Thèse, École des Mines de Paris, July 1992.
- [10] S. Gaubert and R. Katz. Maxplus convex geometry. *Lecture Notes in Computer Science*, LNCIS 4136:192–206, 2006.

- [11] L. Hardouin, B. Cottenceau, and M. Lhommeau. Minmaxgd a library for computation in semiring of formal series. <http://www.istia-angers.fr/~hardouin/outils.html>, 2006.
- [12] M. Lhommeau, L. Hardouin, B. Cottenceau, and J.-L. Boimond. Data processing tool for calculation in dioid. In *WODES'2000, Workshop on Discrete Event Systems*, Ghent, Belgique, Août 2000.
- [13] M. Lhommeau, L. Hardouin, J.L. Ferrier, and I. Ouerghi. Interval analysis in dioid : Application to robust open loop control for timed event graphs. *44th CDC-ECC'05*, Sevilla, 2005.
- [14] G.L. Litvinov and A.N. Sobolevski. Idempotent Interval Analysis and Optimization Problems. *Kluwer, Reliable Computing*, 7(5):353–377, 2001.
- [15] I. Ouerghi and L. Hardouin. A precompensator synthesis for p-temporal event graphs. In *LNCIS 341, Springer, POSTA'06*, Grenoble, France, 2006.
- [16] Moore R.E. *Methods and Applications of Interval Analysis*. SIAM, 1979.