Event-variant and Time-variant (max,+) systems

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***Abstract*—This paper deals with the input-output representation of a class of timed Discrete Event Systems. The systems considered are those that can be described using Timed Event Graphs extended with weights on the arcs and clock rate modifiers or time varying delays. The model relies on periodic expressions using six elementary operators: shift, multiplication and division of events and time. In this context, we show how to develop the transfer matrix computation based on a matrix decomposition called core decomposition.**

***Index Terms*—Discrete Event Systems, Petri nets, Weighted Timed Event Graphs, Time-variant systems, Dioids, Operators**

I. Introduction

The analysis of Discrete Event Systems (DESs) by a description in the (max,+) algebra has known important developments since the eighties [18]. In the work presented here, systems are modeled by combining elementary operators so that it is possible to obtain an input-output description. More precisely, we extend the approach detailed in [4],[1], where Timed Event Graphs (TEGs) are described by means of formal power series in a dioid denoted $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ whose variables correspond to the event-shift operator γ and the time-shift operator δ. Within this context, TEGs satisfy *event-invariance* and *time-invariance* properties that are formally expressed by: for all input u, $H(\gamma^1 u) = \gamma^1 H(u)$ and $H(\delta^1 u) = \delta^1 H(u)$. The systems considered in this paper are more general since they exhibit *event-variant* and *time-variant* properties characterized by : $\exists K, K', T, T' \in \mathbb{N}$ s.t. $\forall u, H(\gamma^K \delta^T u) = \gamma^{K'} \delta^{T'} H(u)$.

TEGs are adapted to model parallelism, event/time shifts and synchronizations. Some classes of DESs studied in the literature extend TEGs in order to describe additional phenomena. There are Weighted TEGs (WTEGs) [3],[12], Synchronous Dataflow Graphs (SDF) [10] and Cyclo-static SDF [11], which allow us to describe batching/unbatching operations. We can also mention phenomena such as partial synchronization [8] or time-varying holding times [17], which cannot be modeled with the semantic of ordinary TEGs either. For these different classes of systems, the shift operators γ and δ are not sufficient anymore, and new operators have to be introduced.

This contribution extends prior work by the authors. In [5], the behavior of WTEGs is described by formal series $s = \bigoplus_i w_i \delta^{t_i}$ in a dioid denoted $\mathcal{E}[\![\delta]\!]$, where the time-shift operator is denoted δ and coefficients w_i are event-operators composed of three basic operators denoted γ, μ and β. These basic event-operators can model, respectively, the phenomena of event-shift, event duplication and event batching. With this model, WTEGs are *event-variant/time-invariant* systems: $\exists K, K' \in \mathbb{N}$ s.t $\forall u, H(\gamma^K \delta^1 u) = \gamma^{K'} \delta^1 H(u)$. In [17], periodic time-variant TEGs are described by formal series $s = \bigoplus_j v_j \gamma^{n_j}$ in a dioid denoted $\mathcal{T}[\![\gamma]\!]$ where the event-shift operator is denoted γ and coefficients v_j are time-operators composed of two basic operators denoted δ and Δ_T. These time-operators can model respectively the time-shift and the synchronization on dates in $T\mathbb{Z}$ (integer multiples of T). The systems are therefore *time-variant/event-invariant*: $\exists T \in \mathbb{N}$ s.t $\forall u, H(\gamma^1 \delta^T u) = \gamma^1 \delta^T H(u)$. The symmetry between the models developed in [5], respectively [17] simply follows from the fact that dynamic phenomena in the event domain also have their counterpart in the time domain. For instance with WTEGs, it is possible to model a synchronization that does not operate on all firings of a transition, but only every K firings. This is a synchronization on events numbered in $K\mathbb{Z}$. The counterpart with periodic time-variant TEGs is to model the synchronization on dates that are in $T\mathbb{Z}$. Although the models developed in [5] and [17] have a strong symmetry, it is not straightforward to handle all basic event-operators γ, μ, β and basic time-operators δ, Δ_T together. For instance, for a series $s = \bigoplus w_i \delta^{t_i} \in \mathcal{E}[\![\delta]\!]$, there is a particular form where coefficients satisfy: $t_j > t_i \Rightarrow w_j \prec w_i$. The same applies for series in $\mathcal{T}[\![\gamma]\!]$. The formal simplifications obtained in $\mathcal{E}[\![\delta]\!]$ and $\mathcal{T}[\![\gamma]\!]$ rely heavily on this monotonicity property which is no longer identifiable when all operators are used simultaneously. In [15] and [16], a new representation of series in $\mathcal{E}[\![\delta]\!]$ and $\mathcal{T}[\![\gamma]\!]$ is given. It is written with a so-called *core decomposition* mQb where m, b are vectors and Q a matrix with entries in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. This viewpoint has led to new algorithms to compute $\oplus, \otimes$ and Kleene star operations either on $\mathcal{E}[\![\delta]\!]$ or on $\mathcal{T}[\![\gamma]\!]$. This approach detailed in [19] is necessary for the computation of formal expressions involving all the event and time operators together.

Regarding related work, WTEGs/SDFs have been studied in [10],[11],[2] where they are especially used to model communications between components with different rates in electronic design. In those papers, the main objective is to determine the maximum throughput or a schedule to achieve it. Compared to those contributions, the transfer approach adopted here gives a much more detailed representation since we obtain an exhaustive description of the system behavior, especially its transient part, for any input. This increases the computation time, but our model makes it possible to synthesize controllers for the considered class of time and event-variant systems adopting a similar approach as presented in [13] for "ordinary" (*i.e.,* time-

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invariant and event-invariant) TEGs. Regarding time-variant systems, to the best of our knowledge, the only related work addresses partial synchronization [8]. The time-operators used in that paper can be seen as a way to describe the effect of periodic partial synchronizations on events.

In this paper, the contribution is twofold. First, we provide a new interpretation of the operators of time multiplication, denoted ν_v, and time division, denoted ω_w. These operators can be interpreted as *clock rate modifiers*. This gives the possibility to model TEGs using several clocks (different time scales) or with *cyclic* holding times, *i.e.,* holding times that depend in a cyclic manner on the time when tokens are deposited in a place [17]. Second, we summarize relevant results from the PhD thesis [19, Chap.5] to compute the transfer matrix of a system using all the event-variant and time-variant operators together. This amounts to bringing together the symmetric and complementary approaches described in [5] and [17] within a single model, *i.e.,* a class of systems that are both *event-variant and time-variant*. However, the presentation here is slightly different from that given in [19, Chap.5] since the core matrix obtained here is not a matrix in $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]$. The results presented here have allowed us to complete the ETVO (Event-variant Time-Variant Operators) C++ library [7] devoted to the calculation of transfer functions of event-variant and time-variant (max,+) systems. In terms of application, this class is large enough to uniformly describe periodic routings [6], delays that depend on event numbers [6] and delays that are time-varying (eg., traffic lights) [17].

Since the presentation here is based on many results from our previous work, sometimes quite technical, we have chosen a presentation mode where some preliminary results are recalled without proof. In Section II, we introduce the 6 basic operators needed in the sequel and some notation. In Section III, we recall how different classes of TEGs can be modeled by operators. This section also recalls the main results about the modeling of WTEGs and time-variant TEGs, and shows the symmetry between the two models. Then, Section IV shows how to adapt the core decomposition to carry out calculations on event-variant and time-variant operators. Finally, an example is given in Section V.

II. Dioids of operators

A dioid [1][14] (or idempotent semiring) is an algebraic structure with two inner operations, addition, denoted $\oplus$, and multiplication, denoted $\otimes$. Addition is commutative, associative, idempotent ($a \oplus a = a$) and has a neutral element denoted ε, and multiplication is associative, distributive over addition and has a neutral element denoted e. Since addition is idempotent, a natural order can be associated with a dioid: $a \succeq b \iff a = a \oplus b$. For instance, $\mathbb{Z} \cup \{-\infty\}$ with $\oplus = \max$ and $\otimes = +$ is a dioid denoted $\mathbb{Z}_{\max}$, also called (max,+) algebra. Clearly, in the (max,+) algebra, the natural order $\succeq$ corresponds to the standard $\geq$. A dioid is said to be *complete* if infinite sums are defined and if the product is distributive over infinite sums. In complete dioids, a^*b is the least solution to the implicit equation $x = ax \oplus b$, where $a^* := e \oplus a \oplus a^2 \cdots = \bigoplus_{i\geq 0} a^i$ is the Kleene star.

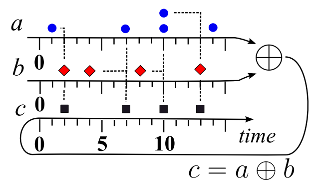

Figure 1. Synchronization of events

A dater function $x : \mathbb{Z} \to \mathbb{Z}_{\max}$ is introduced to describe a sequence of events spread over time. The date $x(k)$ is, by convention, the date of the $(k+1)-$st occurrence of event x, with $\forall k < 0, x(k) = -\infty$. Moreover dater functions $x : \mathbb{Z} \to \mathbb{Z}_{\max}$ are non-decreasing, *i.e.,* $k \geq k' \Rightarrow x(k) \succeq x(k')$. In other words, for $k \geq k'$, the $k-$th occurrence of an event cannot be before the $k'-$th. Since $x(k) \in \mathbb{Z}_{\max}$, time is discrete. In Fig.1, the occurrence of the events labeled a and b is depicted on a discrete-time axis. Expressed as dater functions, we obtain $a(0) = 1$, $a(1) = 7$, $b(0) = 2$, $b(1) = 4$, etc. The occurrence of events is supposed to be synchronized with the ticks of a clock, and the dates are then given as multiples of the clock time unit. In the same figure, the event labeled c corresponds to the synchronization of events a and b. The occurrence of event c is described by the dater $c = a \oplus b$, *i.e.,* the dater function s.t. $\forall k, c(k) = \max(a(k), b(k)) = a(k) \oplus b(k)$. Let us denote by Σ the set of non-decreasing functions from $\mathbb{Z}$ to $\mathbb{Z}_{\max}$ (the set of dater functions is a subset of Σ). An *operator* $\rho : \Sigma \to \Sigma$ is a map which is said to be *additive* if $\forall a, b \in \Sigma, \rho(a \oplus b) = \rho(a) \oplus \rho(b)$.

Definition 1 (Dioid $\mathcal{O}$): The set of additive operators on Σ, with operations $\oplus$ and $\otimes$ defined below, is a non commutative complete dioid denoted $\mathcal{O}$: $x \in \Sigma$, $\forall \rho_1, \rho_2 \in \mathcal{O}$,

$$(\rho_1 \oplus \rho_2)(x) = \rho_1(x) \oplus \rho_2(x), (\rho_1 \otimes \rho_2)(x) = \rho_1(\rho_2(x)).$$

The zero operator of $\mathcal{O}$ is denoted ε, and the unit operator (identity) is denoted e. To simplify notation, we usually write ρx instead of $\rho(x)$, and the symbol $\otimes$ is sometimes omitted. Let us note that $\mathbb{Z}_{\max}$ is a dioid of numbers whereas $\mathcal{O}$ is a dioid of maps (operators), and that we use the same symbols ($\oplus$ and $\otimes$) for inner operations in both dioids.

Definition 2 (Basic operators): The following operators in $\mathcal{O}$ are called basic operators: $x \in \Sigma, n, t \in \mathbb{Z}, m, b, v, w \in \mathbb{N}, \forall k \in \mathbb{Z}$,

$$(\gamma^n x)(k) = x(k-n) \qquad (\delta^t x)(k) = x(k) + t \qquad (1)$$
$$(\mu_m x)(k) = x(\lfloor k/m \rfloor) \qquad (\nu_v x)(k) = x(k) \times v \qquad (2)$$
$$(\beta_b x)(k) = x(bk + b - 1) \qquad (\omega_w x)(k) = \lceil x(k)/w \rceil \qquad (3)$$

where $\lfloor\ \rfloor$ and $\lceil\ \rceil$ represent respectively the floor and the ceiling function.

Operators γ^n, μ_m and β_b have an effect on event numbering and are refered to as E-operators (E for event), whereas δ^t, ν_v and ω_w imply modifications only in the date (time) of events and are refered to as T-operators (T for time). The identity operator e of $\mathcal{O}$ can be expressed differently: $e = \gamma^0 = \delta^0 = \mu_1 = \beta_1 = \nu_1 = \omega_1$.

Remark 1: The definition of operators μ_m and β_b is different from [5] because they operate here on dater functions, instead of on counter functions.

Definition 3 (Dioids $\mathcal{E}$, $\mathcal{T}$ and $\mathcal{ET}$): Let $\mathcal{E} \subset \mathcal{O}$ be the dioid of operators obtained by finite sums and products in $\{\varepsilon, \gamma^n, \mu_m, \beta_b\}$ (with $n \in \mathbb{Z}, m, b \in \mathbb{N}$) and let $\mathcal{T} \subset \mathcal{O}$ be the dioid of operators obtained by finite sums and products in $\{\varepsilon, \delta^t, \nu_v, \omega_w\}$ (with $t \in \mathbb{Z}, v, w \in \mathbb{N}$). Finally, we denote by $\mathcal{ET}$ the dioid of operators obtained by sums and products in $\mathcal{E} \cup \mathcal{T}$.

The product of operators in $\mathcal{ET}$ is not commutative, even though any E-operator ($\mathcal{E}$) commutes with any T-operator ($\mathcal{T}$). For instance, $\delta^1\mu_3\nu_2\beta^2 = \mu_3\beta^2\delta^1\nu_2$.

Proposition 1: In dioid $\mathcal{ET}$, the following equalities are satisfied:

$$\forall \rho \in \mathcal{E}, \forall \sigma \in \mathcal{T}, \rho\sigma = \sigma\rho \tag{4}$$

$$\gamma^n\gamma^{n'} = \gamma^{n+n'} \qquad \delta^t\delta^{t'} = \delta^{t+t'} \tag{5}$$
$$\gamma^n \oplus \gamma^{n'} = \gamma^{\min(n,n')} \qquad \delta^t \oplus \delta^{t'} = \delta^{\max(t,t')} \tag{6}$$
$$\mu_m\mu_{m'} = \mu_{m\times m'} \qquad \nu_v\nu_{v'} = \nu_{v\times v'} \tag{7}$$
$$\beta_b\beta_{b'} = \beta_{b\times b'} \qquad \omega_w\omega_{w'} = \omega_{w\times w'} \tag{8}$$
$$\mu_m\gamma^n = \gamma^{n\times m}\mu_m \qquad \nu_v\delta^t = \delta^{t\times v}\nu_v \tag{9}$$
$$\gamma^n\beta_b = \beta_b\gamma^{n\times b} \qquad \delta^t\omega_w = \omega_w\delta^{t\times w} \tag{10}$$
$$\beta_\alpha\gamma^n\mu_\alpha = \gamma^{\lfloor n/\alpha \rfloor} \qquad \omega_\alpha\delta^t\nu_\alpha = \delta^{\lceil t/\alpha \rceil} \tag{11}$$

Proof: These properties come from [4],[9],[5] and [16]. ■

Remark 2: An operator $\rho \in \mathcal{O}$ is called *event invariant* if, $\forall n$, $\rho\gamma^n = \gamma^n\rho$. It is called *time invariant* if $\forall t$, $\rho\delta^t = \delta^t\rho$. Because of (9) and (10), operators μ_m and β_b are event-variant and operators ν_v and ω_w are time-variant.

The operators introduced in Def.2 are used to model some dynamic phenomena arising in DESs. They are illustrated in Fig.2 and Fig.3. For instance, in Fig.2, the signals are described by dater functions denoted u, x, y and z, with $x = \gamma^2 u$, $y = \mu_2 u$, $z = \beta_3 u$. For $u(0) = -2$, $u(1) = u(2) = 1$, $u(3) = 4, \cdots$ (as shown), we have, according to Def.2, $x(0) = u(-2) = -\infty$, $x(1) = u(-1) = -\infty$, $x(2) = -2$, $x(3) = x(4) = 1, \cdots$ $y(0) = y(1) = u(0) = -2$, $y(2) = y(3) = u(1) = 1$, $y(4) = y(5) = u(2) = 1, \cdots$ $z(0) = u(2) = 1$, $z(1) = u(5) = 8, \cdots$

The operators γ^n and δ^t (1) simply describe a shift in the event numbering or in time. The operators μ_m and ν_v (2) model a multiplication in the event numbering or in time, and β_b and ω_w (3) describe a division, again in event numbering or in time.

Remark 3: Within one system, several clocks can be used to date/time events. For example in Fig.3, there are three different clocks with clock intervals C_1, $C_2 = C_1/2$ and $C_3 = 3C_1$. Depending on the clock used, dates/times are therefore multiples of C_1, C_2 or C_3. More precisely, dates/times for signals u' and x' are expressed as multiples of C_1, for signal y' as multiples of C_2 and for z' as multiples of C_3. In this context, operators ν_v and ω_w can be interpreted as *clock rate modifiers*, where the clock rate is the inverse of the clock interval. These operators describe how the clock interval (or clock rate) is changed. Operator ν_2 multiplies the clock rate by 2 (*i.e.,* clock intervals are divided by 2), and ω_3 divides the clock rate by

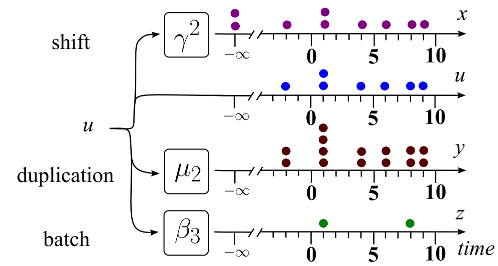

Figure 2. E-operators γ^n (event shift), μ_m (event multiplication or duplication) and β_b (event division or batch).

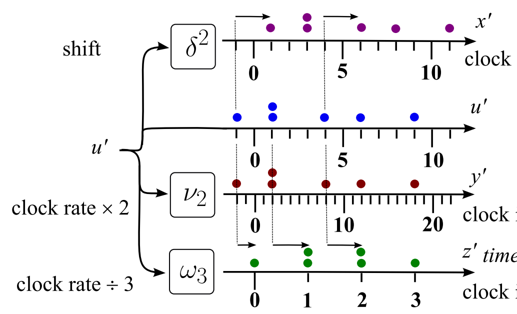

Figure 3. T-operators δ^t (time shift or delay), ν_v (clock rate multiplication) and ω_w (clock rate division).

3 (*i.e.,* clock intervals are multiplied by 3). It is worth noting that operator ω_3 has, in addition, an effect of *variant* delay. In Fig.3 we see that depending on the input date, the input-output delay induced by ω_3 is either 0 (no delay), or C_1 or $2C_1$. For instance, the first input event at time $-1C_1$ ($u'(0) = -1$) is delayed up to date 0 ($z'(0) = 0$), *i.e.,* this is a delay of $1C_1$, while the events at time $1C_1$ ($u'(1) = u'(2) = 1$) are delayed up to time $3C_1 = 1C_3$ ($z'(1) = z'(2) = 1$), *i.e.,* this is a delay of $2C_1$. Finally, the event at time $9C_1$ ($u'(5) = 9$) is not delayed at all ($z'(5) = 3$), since $9C_1 = 3C_3$.

Definition 4 (Gain): Let $\rho_a, \rho_b \in \mathcal{ET}$. The event gain $\Gamma_e : \mathcal{ET} \to \mathbb{Q}$ is defined by $\Gamma_e(\rho_a\rho_b) = \Gamma_e(\rho_a)\Gamma_e(\rho_b)$ and $\Gamma_e(\rho_a \oplus \rho_b) = \min(\Gamma_e(\rho_a), \Gamma_e(\rho_b))$, with $\Gamma_e(\gamma^n) = \Gamma_e(\delta^t) = \Gamma_e(\nu_v) = \Gamma_e(\omega_w) = 1$, $\Gamma_e(\mu_m) = m$ and $\Gamma_e(\beta_b) = 1/b$. Similarly, the clock rate gain $\Gamma_t : \mathcal{ET} \to \mathbb{Q}$ is defined by $\Gamma_t(\rho_a\rho_b) = \Gamma_t(\rho_a)\Gamma_t(\rho_b)$ and $\Gamma_t(\rho_a \oplus \rho_b) = \max(\Gamma_t(\rho_a), \Gamma_t(\rho_b))$, with $\Gamma_t(\gamma^n) = \Gamma_t(\delta^t) = \Gamma_t(\mu_m) = \Gamma_t(\beta_b) = 1$, $\Gamma_t(\nu_v) = v$ and $\Gamma_t(\omega_w) = 1/w$. Finally, the gain is defined by $\Gamma : \mathcal{ET} \to \mathbb{Q}^2$, $\Gamma(\rho) = (\Gamma_e(\rho), \Gamma_t(\rho))$.

Example 1: $\Gamma(\beta_2\nu_4\gamma^1\mu_3\omega_3\delta^2) = (3/2, 4/3)$.

For a given operator, the gain indicates how many output events are produced in average for each input event and how the clock rate is modified.

Definition 5 (Balanced operator): An operator $\rho = \bigoplus_i \rho_i \in \mathcal{ET}$ is said to be *balanced* if $\forall i, \Gamma(\rho) = \Gamma(\rho_i)$.

Example 2: Operator $\gamma^1\beta_2\mu_2\delta^2 \oplus \gamma^4\delta^5\nu_3\omega_3$ is balanced since $\Gamma(\gamma^1\beta_2\mu_2\delta^2) = (2/2, 1) = \Gamma(\gamma^4\delta^5\nu_3\omega_3) = (1, 3/3)$.

In this paper, we only consider balanced operators. In [5] and [17], the following composed operators are introduced to simplify notation:

$$\nabla_{m|b} := \mu_m\beta_b \text{ and } \Delta_{v|w} := \nu_v\omega_w. \tag{12}$$

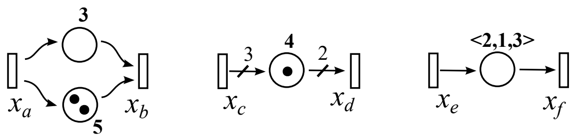

Figure 4. Elementary timed Petri nets (left: Timed Event Graph, middle: Weighted Timed Event Graph, right: Timed Event Graph with time-varying holding times)

Proposition 2 ([5][17]): Let $\rho \in \mathcal{E}$ and $\sigma \in \mathcal{T}$ be two operators. If ρ and σ are balanced then $\exists m, b, v, w \in \mathbb{N}$ s.t.

$$\rho = \bigoplus_{i=1}^{\alpha_e} \gamma^{n_i} \mu_m \beta_b \gamma^{n'_i} \text{ and } \sigma = \bigoplus_{j=1}^{\alpha_t} \delta^{t_j} \nu_v \omega_w \delta^{t'_j}, \tag{13}$$

with α_e, α_t finite integers and $n_i, n'_i, t_j, t'_j \in \mathbb{Z}$.

It follows that $\rho\gamma^b = \gamma^m \rho$ and $\sigma\delta^w = \delta^v \sigma$, and $\Gamma(\rho) = (m/b, 1)$ and $\Gamma(\sigma) = (1, v/w)$.

Corollary 1: If $\chi \in \mathcal{ET}$ is balanced, then $\exists m, b, v, w \in \mathbb{N}$ s.t.

$$\begin{array}{rcl} \chi & = & \bigoplus_i \gamma^{n_i} \mu_m \beta_b \gamma^{k_i} \delta^{t_i} \nu_v \omega_w \delta^{s_i}, \\ & = & \bigoplus_i \gamma^{n_i} \nabla_{m|b} \gamma^{k_i} \delta^{t_i} \Delta_{v|w} \delta^{s_i} \end{array} \tag{14}$$

with $n_i, k_i, t_i, s_i \in \mathbb{Z}$. Moreover, $\chi\gamma^b\delta^w = \gamma^m\delta^v\chi$ and $\Gamma(\chi) = (m/b, v/w)$.

III. Timed Event Graphs extensions

Timed Event Graphs (TEGs) constitute a subclass of timed Petri nets, *i.e.,* those whose places have one and only one upstream and downstream transition and whose arcs have weight 1. Places can have a holding time. This is the time a token in this place has to wait before contributing to firing the downstream transition. It is well known that a TEG can be transformed into a (max,+) linear model [14], and vice versa, provided that a transition fires as soon as it can fire, *i.e.,* the TEG operates under the earliest firing rule. To obtain an algebraic model, a dater function is associated to each transition in order to describe the firing sequence of this transition. Then, the operators introduced in the previous section are used to model the relations between transitions. For TEGs with l input transitions, p output transitions and q internal transitions[1], one can describe the different signals by vectors of daters $u \in \Sigma^l$, $y \in \Sigma^p$ and $x \in \Sigma^q$, and the relation between these signals are collected into matrices of operators $A \in \mathcal{O}^{q\times q}$, $B \in \mathcal{O}^{q\times l}$, and $C \in \mathcal{O}^{p\times q}$. The global evolution of the system is then described by

$$x = Ax \oplus Bu, \; y = Cx. \tag{15}$$

Solving the first equation provides $x = A^*Bu$, hence

$$y = CA^*Bu,$$

where matrix $H = CA^*B \in \mathcal{O}^{p\times l}$ is called the *transfer matrix.*

[1]a transition in a TEG is called input transition if it does not have any upstream place. It is called output transition, if it does not have any downstream place. All other transitions are called internal transitions.

A. Model of ordinary TEGs

Writing an ordinary TEG as (15) implies that A_{ij}, B_{ij} and C_{ij} are finite sums of operators $\gamma^n\delta^t$. The system obtained is then both time-invariant and event-invariant. For example, in Fig.4, the relation between daters x_a and x_b is expressed by $x_b = (\delta^3 \oplus \gamma^2\delta^5)x_a$. It is shown in [9] that for ordinary TEGs, each entry of $H = CA^*B$ can be expressed in a *ultimately periodic* form

$$\begin{array}{rcl} s & = & p \oplus q(\gamma^\kappa\delta^\tau)^* \\ & = & \underbrace{\bigoplus_{i=1}^{\alpha_1} \gamma^{n_i}\delta^{t_i}}_{p} \oplus \underbrace{\left(\bigoplus_{j=1}^{\alpha_2} \gamma^{N_j}\delta^{T_j}\right)}_{q} (\gamma^\kappa\delta^\tau)^*. \end{array} \tag{16}$$

Since only γ^n and δ^t are involved, the operators can be described as formal power series in two commutating variables γ and δ, *i.e.,* in a dioid called $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma, \delta]\!]$ (see [4],[1]).

Example 3: In Fig.5, the sub-graph including u_1, x_1, x_2 and y_1 is an ordinary TEG. Its transfer relation in $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma, \delta]\!]$ is $y_1 = \delta^2(\gamma^1\delta^1)^*u_1 = H_1u_1$. This can be seen easily by observing that $y_1 = \delta^1x_2$, $x_2 = \delta^1x_1$ and $x_1 = \gamma^1x_2 \oplus u_1$. The second and third equation provide $x_1 = \gamma^1\delta^1x_1 \oplus u_1$, which can be solved as $x_1 = (\gamma^1\delta^1)^*u_1$.

B. Model of Weighted TEGs

When considering Weighted TEGs such as in [5], the weights associated to arcs indicate how many tokens are produced/consumed by the transition firings. In Fig.4, when x_c is fired, three tokens are added to the downstream place, and the firing of x_d removes two tokens from the upstream place. We graphically indicate that an arc has a weight different from 1 by drawing a bar across the arrow representing the arc. In the depicted simple WTEG, for x_c, $x_d \in \Sigma$, the relation between x_c and x_d is given by $x_d = \beta_2\delta^4\gamma^1\mu_3x_c$.

Example 4: In Fig.5, the sub-graph including u_2, x_3, x_4, x_5, x_6 and y_2 is a Weighted TEG. The model in $\mathcal{ET}$ is $y_2 = \left(\delta^1\mu_2\delta^3(\gamma^1\delta^3)^*\beta_2\gamma^1\delta^1 \oplus \gamma^1\delta^1\mu_2\delta^4(\gamma^2\delta^4)^*\beta_2\right)u_2 = H_2u_2$.

C. Model of Time-variant TEGs

T-operators allow one to describe both clock rate modifications and cyclic holding times (time-variant delays) [17]. In Fig.4 (right), we denote by $\langle 2, 1, 3\rangle$ the fact that a token entering the place connecting x_e to x_f needs to reside for 2 clock intervals if its entry time is in $3\mathbb{Z}$, *i.e.,* $(\cdots, -3, 0, 3, 6, \cdots)$, 1 clock interval for entry times in $3\mathbb{Z}+1$, and 3 clock intervals for the other entry times. A place with a cyclic holding time can be modeled as a sum of T-operators. E.g., in Fig.4, $x_f = \delta^2\nu_3\omega_3\delta^{-1}x_e = \delta^2\Delta_3\delta^{-1}x_e$. This can be easily seen as, by definition, $x_f(k) = 2 + 3 \times \lceil (x_e(k) - 1)/3 \rceil$. Hence, if x_e fires (and deposites a token) at times $0, 1, 2, 3, 4, \cdots$, transition x_f fires at times $2, 2, 5, 5, 5, 8, \cdots$ and the holding time of the deposited token is $2, 1, 3, 2, 1, 3, \cdots$ time units.

Thanks to operators ν_v and ω_w we can also describe a multi-clock model (see Remark 3). Fig.5 describes a system whose the overall behavior is written $y_2 = H_2\nu_2\omega_3H_1u_1$. Although there is no standard graphical representation, we

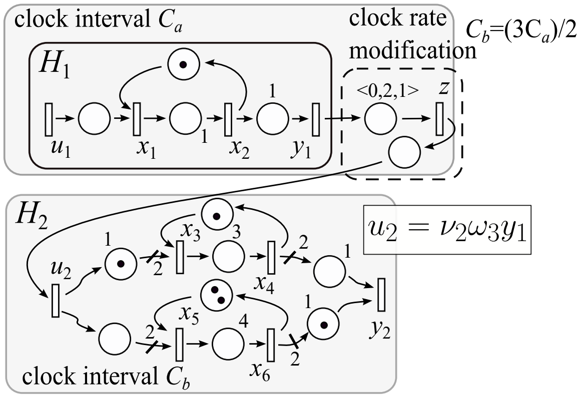

Figure 5. Time-variant WTEG

will nevertheless try to decompose the clock change mechanism described by $u_2 = \nu_2\omega_3 y_1$. In Fig.5 we assume that between transitions y_1 and u_2 there is a clock rate change. It means there are two different clocks whose clock intervals are denoted C_a and C_b, with $C_b = (3C_a)/2$. As, due to (11), $\omega_3\nu_3 = \delta^0 = e$, we can decompose the clock rate modification into: $z = \Delta_3 y_1 = \nu_3\omega_3 y_1$ and $u_2 = \nu_2\omega_3 z$. With the argument given before, one sees that the relation between y_1 and z can be represented, as shown in Fig.5, by a place with time-variant cyclic holding times $\langle 0, 2, 1\rangle$. This implies that transition z fires only at times that are multiples of $3C_a$, where C_a is the clock interval of the clock in the upper part of Fig.5. Then, from z to u_2, there is a clock rate change with the clock interval C_b of the clock governing the evolution in the lower part of Fig.5 given by $C_b = (3C_a)/2 = \frac{3}{2}C_a$. It follows that u_2 fires only at time that are multiples of $2C_b$. In other words, transition z and u_2 always fire simultaneously, but are timed by different clocks, z by a clock with interval C_a, and u_2 by a clock with interval C_b.

Note that the system between u_1 and y_2 is both event-variant (due to the weights) and time-variant (due to the clock rate change). The computation of its transfer function requires adapted tools that will be discussed in the sequel.

D. Review of results for WTEGs and Time-variant TEGs

The investigation of WTEGs and Time-variant TEGs, with the help of basic operators, was first conducted in [5],[15],[16]. In these studies, one considers event-variant/time-invariant systems (WTEGs) on the one hand, and time-variant/event-invariant systems (Time-variant TEGs) on the other hand, the two models being symmetrical. The preliminary results reported in these studies were largely influenced by the model of TEGs in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ [1],[9], in particular the need to manipulate operators described in ultimately periodic form such as (16). We now provide here a brief summary of the available results.

In [5], the modeling of WTEGs relies on a dioid denoted $\mathcal{E}[\![\delta]\!]$. It is the set of operators obtained by sums and products in $\{\varepsilon, \gamma^n, \mu_m, \beta_b, \delta^t\}$. In a symmetric way in [16], Time-variant TEGs are described in a dioid denoted $\mathcal{T}[\![\gamma]\!]$, which is the set of operators obtained by sums and products in $\{\varepsilon, \delta^t, \nu_v, \omega_w, \gamma^n\}$.

A balanced operator $\rho \in \mathcal{E}[\![\delta]\!]$ or $\sigma \in \mathcal{T}[\![\gamma]\!]$ can be written

$$\begin{array}{rcl|rcl} \rho & = & \bigoplus_i \gamma^{n_i}\mu_m\beta_b\gamma^{n'_i}\delta^{t_i} & \sigma & = & \bigoplus_i \delta^{t_i}\nu_v\omega_w\delta^{t'_i}\gamma^{n_i} \\ & = & \bigoplus_i \gamma^{n_i}\nabla_{m|b}\gamma^{n'_i}\delta^{t_i} & & = & \bigoplus_i \delta^{t_i}\Delta_{v|w}\delta^{t'_i}\gamma^{n_i} \end{array}$$

Definition 6 (Ultimately periodic): A balanced operator $s_e \in \mathcal{E}[\![\delta]\!]$ or $s_t \in \mathcal{T}[\![\gamma]\!]$ is said to be ultimately periodic (abbreviated u.p.) if it can be written

$$\begin{array}{l|l} s_e = p_e \oplus q_e(\gamma^\kappa\delta^\tau)^* \text{ with} & s_t = p_t \oplus q_t(\gamma^\kappa\delta^\tau)^* \text{ with} \\ p_e = \bigoplus_{i=1}^{\alpha_1} \gamma^{n_i}\nabla_{m|b}\gamma^{n'_i}\delta^{t_i} & p_t = \bigoplus_{i=1}^{\alpha_1} \delta^{t_i}\Delta_{v|w}\delta^{t'_i}\gamma^{n_i} \\ q_e = \bigoplus_{j=1}^{\alpha_2} \gamma^{N_j}\nabla_{m|b}\gamma^{N'_j}\delta^{T_j} & q_t = \bigoplus_{j=1}^{\alpha_2} \delta^{T_j}\Delta_{v|w}\delta^{T'_j}\gamma^{N_j}. \end{array}$$

In [5] and [16], calculations $(\oplus, \otimes, *)$ on operators in $\mathcal{E}[\![\delta]\!]$ and $\mathcal{T}[\![\gamma]\!]$ can only be carried out on u.p. operators. The conditions for the ultimate periodicity property to be preserved are recalled below.

Proposition 3 ([5][17]): Let $s_e, s'_e \in \mathcal{E}[\![\delta]\!]$ and $s_t, s'_t \in \mathcal{T}[\![\gamma]\!]$ be u.p. operators.

• $s_e \otimes s'_e$ is u.p.,	• $s_t \otimes s'_t$ is u.p.,
• $\Gamma(s_e) = \Gamma(s'_e)$ implies $s_e \oplus s'_e$ is u.p.,	• $\Gamma(s_t) = \Gamma(s'_t)$ implies $s_t \oplus s'_t$ is u.p.,
• $\Gamma(s_e) = (1, 1)$ implies $(s_e)^*$ is u.p..	• $\Gamma(s_t) = (1, 1)$ implies $(s_t)^*$ is u.p..

Recall from Def.4 that $\Gamma(\rho) = (\Gamma_e(\rho), \Gamma_t(\rho))$ with $\Gamma_e(\rho) = 1$ if $\rho \in \mathcal{T}[\![\gamma]\!]$, and $\Gamma_t(\rho) = 1$ if $\rho \in \mathcal{E}[\![\delta]\!]$.

Based on Prop.3, WTEGs and Time-variant TEGs are characterized by an ultimately periodic transfer matrix if all paths of the graph are modeled by balanced operators. Under this condition, the results proposed in [5][17] also provide algorithms to calculate the transfer matrix.

IV. Core decomposition of operators in $\mathcal{ET}$

Based on the results recalled in the previous section, we show how the manipulation of operators in $\mathcal{ET}$, both event-variant and time-variant, is possible. For this purpose, we use the techniques developped in [15] and [16]. In these papers, it is shown how calculations involving u.p. operators in $\mathcal{E}[\![\delta]\!]$ and $\mathcal{T}[\![\gamma]\!]$ can be performed by means of matrix calculations on u.p. series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. With the same techniques, we show here how computation with u.p. operators in $\mathcal{ET}$ reduces to matrix calculation in $\mathcal{E}[\![\delta]\!]$ or in $\mathcal{T}[\![\gamma]\!]$. Our approach relies on a matrix decomposition called *core decomposition*, which was first introduced in [15].

Notation 1: Let us denote by $\mathsf{m}_m \in \mathcal{E}[\![\delta]\!]^{1\times m}$, $\mathsf{b}_b \in \mathcal{E}[\![\delta]\!]^{b\times 1}$, $\mathsf{v}_v \in \mathcal{T}[\![\gamma]\!]^{1\times v}$, $\mathsf{w}_w \in \mathcal{T}[\![\gamma]\!]^{w\times 1}$, $m, b, v, w \in \mathbb{N}$ the following matrices

$$\begin{array}{rcl} \mathsf{m}_m & := & \begin{pmatrix}\mu_m & \gamma^1\mu_m & \cdots & \gamma^{m-1}\mu_m\end{pmatrix} \\ \mathsf{b}_b & := & \begin{pmatrix}\beta_b\gamma^{b-1} & \cdots & \beta_b\gamma^1 & \beta_b\end{pmatrix}^\intercal \\ \mathsf{v}_v & := & \begin{pmatrix}\nu_v & \delta^{-1}\nu_v & \cdots & \delta^{1-v}\nu_v\end{pmatrix} \\ \mathsf{w}_w & := & \begin{pmatrix}\omega_w\delta^{1-w} & \cdots & \omega_w\delta^{-1} & \omega_w\end{pmatrix}^\intercal \end{array}$$

Proposition 4 ([15] [16]): For a given $\alpha \in \mathbb{N}$, we have

$$\mathsf{m}_\alpha \otimes \mathsf{b}_\alpha = e = \mathsf{v}_\alpha \otimes \mathsf{w}_\alpha. \tag{17}$$

Matrices $\mathsf{E}_\alpha := \mathsf{b}_\alpha \otimes \mathsf{m}_\alpha$ and $\mathsf{T}_\alpha := \mathsf{w}_\alpha \otimes \mathsf{v}_\alpha$ belong to $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{\alpha\times\alpha}$ and are defined by : $i, j \in \{1, ..., \alpha\}$

$$(\mathsf{E}_\alpha)_{ij} = \begin{cases} e, & j \le i, \\ \gamma^1, & j > i. \end{cases} \quad (\mathsf{T}_\alpha)_{ij} = \begin{cases} e, & j \le i, \\ \delta^{-1}, & j > i. \end{cases}$$

This follows immediately from (11). Because of (17), $\mathsf{E}_\alpha \otimes \mathsf{E}_\alpha = \mathsf{b}_\alpha \mathsf{m}_\alpha \mathsf{b}_\alpha \mathsf{m}_\alpha = \mathsf{b}_\alpha \mathsf{m}_\alpha = \mathsf{E}_\alpha$, and $\mathsf{T}_\alpha \otimes \mathsf{T}_\alpha = \mathsf{T}_\alpha$.

Proposition 5: Let s be a balanced operator in $\mathcal{ET}$. Each $\oplus$-term can be factorized as follows

$$s = \bigoplus_i \gamma^{l_i} \mu_m \left(\delta^{t_i} \Delta_{v|w} \delta^{s_i} \gamma^{c_i}\right) \beta_b \gamma^{r_i} \quad \text{where } 0 \le l_i < m \text{ and } 0 \le r_i < b, \tag{18}$$

or,

$$s = \bigoplus_i \delta^{l_i} \nu_v \left(\gamma^{k_i} \nabla_{m|b} \gamma^{n_i} \delta^{c_i}\right) \omega_w \delta^{r_i} \quad \text{where } 0 \ge l_i > -v \text{ and } 0 \ge r_i > -w. \tag{19}$$

Proof: Because of (14), a balanced operator can be written $s = \bigoplus_i \gamma^{n_i} \mu_m \beta_b \gamma^{k_i} \delta^{t_i} \nu_v \omega_w \delta^{s_i} = \bigoplus_i \gamma^{n_i} \nabla_{m|b} \gamma^{k_i} \delta^{t_i} \Delta_{v|w} \delta^{s_i}$. By applying (9), $\gamma^\alpha \mu_m = \gamma^{\alpha - \lfloor \alpha/m \rfloor \times m} \mu_m \gamma^{\lfloor \alpha/m \rfloor}$, with $\alpha - \lfloor \alpha/m \rfloor \times m \in [0, m-1]$. For instance, $\gamma^5 \mu_3 = \gamma^2 \mu_3 \gamma^1$. Identically, with (10), $\beta_b \gamma^\alpha = \gamma^{\lfloor \alpha/b \rfloor} \beta_b \gamma^{\alpha - \lfloor \alpha/b \rfloor \times b}$ with $\alpha - \lfloor \alpha/b \rfloor \times b \in [0, b-1]$. Finally, as E-operators and T-operators commute, for each term we have $\gamma^{n_i} \mu_m \beta_b \gamma^{k_i} \delta^{t_i} \Delta_{v|w} \delta^{s_i} = \gamma^{n_i} \mu_m (\delta^{t_i} \Delta_{v|w} \delta^{s_i}) \beta_b \gamma^{k_i} = \gamma^{l_i} \mu_m (\delta^{t_i} \Delta_{v|w} \delta^{s_i} \gamma^{c_i}) \beta_b \gamma^{r_i}$, with $l_i = n_i - \lfloor n_i/m \rfloor \times m$, $r_i = k_i - \lfloor k_i/b \rfloor \times b$ and $c_i = \lfloor n_i/m \rfloor + \lfloor k_i/b \rfloor$. For instance, $\gamma^5 \mu_3 \beta_2 \gamma^4 \delta^5 \Delta_{2|3} \delta^5 = \gamma^2 \mu_3 (\delta^5 \Delta_{2|3} \delta^5 \gamma^3) \beta^2 \gamma^0$. We can use the same technique to obtain factorization (19). For instance, $\gamma^5 \mu_3 \beta_2 \gamma^4 \delta^5 \Delta_{2|3} \delta^5 = \gamma^5 \nabla_{3|2} \gamma^4 \delta^5 \nu_2 \omega_3 \delta^5 = \delta^{-1} \nu_2 (\gamma^5 \nabla_{3|2} \gamma^4 \delta^5) \omega_3 \delta^{-1}$, since $\delta^5 \nu_2 = \delta^{-1} \nu_2 \delta^3$ and $\omega_3 \delta^5 = \delta^2 \omega_3 \delta^{-1}$. ∎

We can remark that in (18), the left and right factors are in $\mathcal{E}[\![\delta]\!]$ whereas the central part in brackets is in $\mathcal{T}[\![\gamma]\!]$. Symmetrically, factorization (19) leads to a central part in $\mathcal{E}[\![\delta]\!]$ while the left and right factors are in $\mathcal{T}[\![\gamma]\!]$. Since there is a finite number of left and right factors in (19) and (18), these factorizations lead to two decompositions where the core is a matrix.

Proposition 6 (Core decomposition): A balanced operator $s \in \mathcal{ET}$ can be described by two equivalent decompositions

$$\begin{aligned} s &= \mathsf{m}_m \mathsf{Q}_\mathsf{t} \mathsf{b}_b \quad (\text{with } \mathsf{Q}_\mathsf{t} \in \mathcal{T}[\![\gamma]\!]^{m \times b}) \\ &= \mathsf{v}_v \mathsf{Q}_\mathsf{e} \mathsf{w}_w \quad (\text{with } \mathsf{Q}_\mathsf{e} \in \mathcal{E}[\![\delta]\!]^{v \times w}) \end{aligned}$$

Proof: Factorizations (18) and (19) are simply written as a matrix product. ∎

Proposition 7: Let $s \in \mathcal{ET}$ be an u.p. balanced operator, say

$$s = p \oplus q(\gamma^\kappa \delta^\tau)^*, \quad \text{with } p = \bigoplus_{i=1}^{\alpha_1} \gamma^{k_i} \nabla_{m|b} \gamma^{n_i} \delta^{t_i} \Delta_{v|w} \delta^{s_i}, \quad \text{and } q = \bigoplus_{j=1}^{\alpha_2} \gamma^{K_j} \nabla_{m|b} \gamma^{N_j} \delta^{T_j} \Delta_{v|w} \delta^{S_j}.$$

The core decompositions $s = \mathsf{m}_m \mathsf{Q}_\mathsf{t} \mathsf{b}_b = \mathsf{v}_v \mathsf{Q}_\mathsf{e} \mathsf{w}_w$ are such that all entries $(\mathsf{Q}_\mathsf{t})_{ij}$ are u.p. operators in $\mathcal{T}[\![\gamma]\!]$ and $(\mathsf{Q}_\mathsf{e})_{ij}$ are u.p. operators in $\mathcal{E}[\![\delta]\!]$.

Proof: First, we can write $q(\gamma^\kappa \delta^\tau)^*$ by developing $(\gamma^\kappa \delta^\tau)^* = (\bigoplus_{i=0}^{n-1} \gamma^{i\kappa} \delta^{i\tau})(\gamma^{n\kappa} \delta^{n\tau})^*$. Then, $q(\gamma^\kappa \delta^\tau)^* = (q(\bigoplus_{i=0}^{n-1} \gamma^{i\kappa} \delta^{i\tau}))(\gamma^{n\kappa} \delta^{n\tau})^* = q'(\gamma^{n\kappa} \delta^{n\tau})^*$. Moreover, we can choose n so that $n\kappa$ be a multiple of b. If now, as in (18), we write $q' = \bigoplus_j \gamma^{L'_j} \mu_m \left(\delta^{T'_j} \Delta_{v|w} \delta^{S'_j} \gamma^{C'_j}\right) \beta_b \gamma^{R'j}$, since $\beta_b (\gamma^{n\kappa} \delta^{n\tau})^* = (\gamma^{(n\kappa)/b} \delta^{n\tau})^* \beta_b$, then we obtain $\gamma^{L'_j} \mu_m \left(\delta^{T'_j} \Delta_{v|w} \delta^{S'_j} \gamma^{C'_j} (\gamma^{(n\kappa)/b} \delta^{n\tau})^*\right) \beta_b \gamma^{R'j}$, where the ultimate peridocity is applied to the central part. ∎

Because of Prop.6 and Prop.7, operations on u.p. operators in $\mathcal{ET}$ can be converted to operations on matrices in $\mathcal{E}[\![\delta]\!]$ or $\mathcal{T}[\![\gamma]\!]$.

Proposition 8 (Operations via core decomposition): Let $s, s' \in \mathcal{ET}$ be u.p. balanced operators.

- If $s = \mathsf{m}_m \mathsf{Q}_\mathsf{t} \mathsf{b}_b = \mathsf{v}_v \mathsf{Q}_\mathsf{e} \mathsf{w}_w$ and $s' = \mathsf{m}_m \mathsf{Q'}_\mathsf{t} \mathsf{b}_b = \mathsf{v}_v \mathsf{Q'}_\mathsf{e} \mathsf{w}_w$, then $s \oplus s' = \mathsf{m}_m (\mathsf{Q}_\mathsf{t} \oplus \mathsf{Q'}_\mathsf{t}) \mathsf{b}_b = \mathsf{v}_v (\mathsf{Q}_\mathsf{e} \oplus \mathsf{Q'}_\mathsf{e}) \mathsf{w}_w$.
- If $s = \mathsf{m}_m \mathsf{Q}_\mathsf{t} \mathsf{b}_{\alpha 1} = \mathsf{v}_v \mathsf{Q}_\mathsf{e} \mathsf{w}_{\alpha 2}$ and $s' = \mathsf{m}_{\alpha 1} \mathsf{Q'}_\mathsf{t} \mathsf{b}_b = \mathsf{v}_{\alpha 2} \mathsf{Q'}_\mathsf{e} \mathsf{w}_w$, then $s \otimes s' = \mathsf{m}_m (\mathsf{Q}_\mathsf{t} \mathsf{E}_{\alpha 1} \mathsf{Q'}_\mathsf{t}) \mathsf{b}_b = \mathsf{v}_v (\mathsf{Q}_\mathsf{e} \mathsf{T}_{\alpha 2} \mathsf{Q'}_\mathsf{e}) \mathsf{w}_w$.
- If $s = \mathsf{m}_{\alpha 1} \mathsf{Q}_\mathsf{t} \mathsf{b}_{\alpha 1} = \mathsf{v}_{\alpha 2} \mathsf{Q}_\mathsf{e} \mathsf{w}_{\alpha 2}$ then $s^* = \mathsf{m}_{\alpha 1} (\mathsf{Id} \oplus \mathsf{Q}_\mathsf{t} (\mathsf{E}_{\alpha 1} \mathsf{Q}_\mathsf{t})^*) \mathsf{b}_{\alpha 1} = \mathsf{v}_{\alpha 2} (\mathsf{Id} \oplus \mathsf{Q}_e (\mathsf{T}_{\alpha 2} \mathsf{Q}_\mathsf{e})^*) \mathsf{w}_{\alpha 2}$, where Id is the identity matrix of appropriate size with e on the diagonal and ε elsewhere.

Prop.8 requires that the core matrices are of compatible size. To solve a possible problem of matrix size, the following proposition provides a scaling operation.

Proposition 9 (Extension): Let $s = \mathsf{m}_m \mathsf{Q}_\mathsf{t} \mathsf{b}_b = \mathsf{v}_v \mathsf{Q}_\mathsf{e} \mathsf{w}_w$ be an u.p. balanced operator. The core decompositions can be extended $s = \mathsf{m}_{mn} \mathsf{Q'}_\mathsf{t} \mathsf{b}_{bn} = \mathsf{v}_{vn} \mathsf{Q'}_\mathsf{e} \mathsf{w}_{wn}$ where $\mathsf{Q'}_\mathsf{t} \in \mathcal{T}[\![\gamma]\!]^{mn \times bn}$ and $\mathsf{Q'}_\mathsf{e} \in \mathcal{E}[\![\delta]\!]^{vn \times wn}$.

Proof: Because of (17), we can write $s = \mathsf{m}_m \mathsf{Q}_\mathsf{t} \mathsf{b}_b = \mathsf{m}_{mn} \mathsf{b}_{mn} (\mathsf{m}_m \mathsf{Q}_\mathsf{t} \mathsf{b}_b) \mathsf{m}_{bn} \mathsf{b}_{bn} = \mathsf{m}_{mn} (\mathsf{b}_{mn} \mathsf{m}_m \mathsf{Q}_\mathsf{t} \mathsf{b}_b \mathsf{m}_{bn}) \mathsf{b}_{bn}$ and therefore $\mathsf{Q'}_\mathsf{t} = \mathsf{b}_{mn} \mathsf{m}_m \mathsf{Q}_\mathsf{t} \mathsf{b}_b \mathsf{m}_{bn}$. It is shown in [15, Prop.8], and in [16, Prop.6] that by defining $\hat{\mathsf{Q}}_\mathsf{t} = \mathsf{E}_m \mathsf{Q}_\mathsf{t} \mathsf{E}_b$ and $\hat{\mathsf{Q}}_\mathsf{e} = \mathsf{T}_v \mathsf{Q}_\mathsf{e} \mathsf{T}_w$, then

$$\mathsf{Q'}_\mathsf{t} = \begin{pmatrix} \beta_n \gamma^{n-1} \hat{\mathsf{Q}}_\mathsf{t} \mu_n & \cdots & \beta_n \gamma^{n-1} \hat{\mathsf{Q}}_\mathsf{t} \gamma^{n-1} \mu_n \\ \vdots & & \vdots \\ \beta_n \hat{\mathsf{Q}}_\mathsf{t} \mu_n & \cdots & \beta_n \hat{\mathsf{Q}}_\mathsf{t} \gamma^{n-1} \mu_n \end{pmatrix}$$

$$\mathsf{Q'}_\mathsf{e} = \begin{pmatrix} \omega_w \delta^{1-n} \hat{\mathsf{Q}}_\mathsf{e} \nu_v & \cdots & \omega_w \delta^{1-n} \hat{\mathsf{Q}}_\mathsf{e} \delta^{1-n} \nu_v \\ \vdots & & \vdots \\ \omega_w \hat{\mathsf{Q}}_\mathsf{e} \nu_v & \cdots & \omega_w \hat{\mathsf{Q}}_\mathsf{e} \delta^{1-n} \nu_v \end{pmatrix}$$

Finally, although some β_b and μ_m operators appear in the expression of Q'_t, because of (11), all entries in $\mathsf{Q'}_\mathsf{t}$ stay in $\mathcal{T}[\![\gamma]\!]$. For the same reason, all entries in $\mathsf{Q'}_\mathsf{e}$ stay in $\mathcal{E}[\![\delta]\!]$. ∎

Proposition 10: Let s, s' be u.p. (balanced) operators in $\mathcal{ET}$.

- $s \otimes s'$ is u.p.,
- $\Gamma(s) = \Gamma(s')$ implies $s \oplus s'$ is u.p.,
- $\Gamma(s) = (1, 1)$ implies s^* is u.p..

For an event-variant and time-variant system, typically a Time-variant WTEG, it follows from Prop.10 that if all the paths are described by balanced operators in $\mathcal{ET}$, then one can develop the transfer matrix computation. The practical computations then rely on operations with u.p. operators in $\mathcal{E}[\![\delta]\!]$ and $\mathcal{T}[\![\gamma]\!]$ for which we have software tools (ETVO [7]), with scaling operations on the core decompositions when necessary (see Prop.9).

Remark 4 (Balanced graphs): To satisfy the conditions of Prop.10, the considered graphs must be gain-balanced, *i.e.,* parallel paths must have the same event-gain (as for Weight-Balanced TEGs [5],[6]), and the same clock rate gain. In particular, loops must have an event-gain and a clock rate gain equal to 1. Considering unbalanced graphs describe either inconsistencies (synchronisation of signals with different clocks) or can asymptotically lead to deadlock problems. Restricting ourselves to the conditions of Prop.10 (gain-balanced graphs) is therefore not very restrictive.

V. Example

The transfer function of the Time-variant WTEG given in Fig.5 is to be computed. There are two parts. First, a Time-variant TEG from u_1 to u_2, and then a WTEG from u_2 to y_2. Previously, we obtained $u_2 = \nu_2 \omega_3 \delta^2 (\gamma^1 \delta^1)^* u_1 = H_a u_1$ and $y_2 = (\delta^1 \mu_2 \delta^3 (\gamma^1 \delta^3)^* \delta^1 \beta_2 \gamma^1 \oplus \gamma^1 \delta^1 \mu_2 \delta^4 (\gamma^2 \delta^4)^* \beta_2) u_2 = H_b u_2$. Therefore, the overall transfer function is $y_2 = H u_1 = (H_b \otimes H_a) u_1$. Arbitrarily, we choose to decompose the operators with core matrices in $\mathcal{T}[\![\gamma]\!]$. First, $H_b = \mathsf{m}_2 \mathsf{Q}_\mathsf{b} \mathsf{b}_2$ with $(\mathsf{Q}_\mathsf{b})_{11} = \delta^5 (\gamma^1 \delta^3)^*$, $(\mathsf{Q}_\mathsf{b})_{12} = \gamma^1 \delta^5 (\gamma^1 \delta^3)^*$, $(\mathsf{Q}_\mathsf{b})_{21} = \delta^5 (\gamma^1 \delta^3)^*$ and $(\mathsf{Q}_\mathsf{b})_{22} = \delta^5 \oplus \gamma^2 \delta^9 \oplus \gamma^3 \delta^{11} (\gamma^1 \delta^3)^*$. Then for H_a, the decomposition is $H_a = \mathsf{m}_1 (\nu_2 \omega_3 \delta^2 (\gamma^1 \delta^1)^*) \mathsf{b}_1$, but this decomposition is not adapted for the product $H_b \otimes H_a$. According to Prop.9, we can extend the core matrix as $H_a = \mathsf{m}_2 (\mathsf{b}_2 (\nu_2 \omega_3 \delta^2 (\gamma^1 \delta^1)^*) \mathsf{m}_2) \mathsf{b}_2$ and we obtain $H_a = \mathsf{m}_2 \mathsf{Q}_a \mathsf{b}_2$ with $(\mathsf{Q}_\mathsf{a})_{11} = \delta^2 \nu_2 \omega_3 \delta^{-1} (\gamma^1 \delta^2)^*$, $(\mathsf{Q}_\mathsf{a})_{12} = \delta^2 \nu_2 \omega_3 \gamma^1 (\gamma^1 \delta^2)^*$, $(\mathsf{Q}_\mathsf{a})_{21} = \delta^2 \nu_2 \omega_3 (\gamma^1 \delta^2)^*$ and $(\mathsf{Q}_\mathsf{a})_{21} = \delta^2 \nu_2 \omega_3 \delta^{-1} (\gamma^1 \delta^2)^*$. Finally, we obtain $H = H_b \otimes H_a = \mathsf{m}_2 \mathsf{Q}_\mathsf{b} \mathsf{b}_2 \mathsf{m}_2 \mathsf{Q}_\mathsf{a} \mathsf{b}_2 = \mathsf{m}_2 \mathsf{Q}_\mathsf{t} \mathsf{b}_2$ with $\mathsf{Q}_\mathsf{t} = \mathsf{Q}_\mathsf{b} \mathsf{E}_2 \mathsf{Q}_\mathsf{a} \in \mathcal{T}[\![\gamma]\!]^{2\times 2}$. This calculation can be processed in dioid $\mathcal{T}[\![\gamma]\!]$ and the result is

$$
\begin{aligned}
(\mathsf{Q}_\mathsf{t})_{11} &= (\delta^7 \Delta_{2|3} \delta^{-1} \oplus \delta^{10} \Delta_{2|3} \delta^{-1} \gamma^1)(\gamma^2 \delta^9)^* \\
(\mathsf{Q}_\mathsf{t})_{12} &= (\delta^7 \Delta_{2|3} \gamma^1 \oplus \delta^{10} \Delta_{2|3} \gamma^2)(\gamma^2 \delta^9)^* \\
(\mathsf{Q}_\mathsf{t})_{21} &= (\delta^7 \Delta_{2|3} \oplus \delta^{10} \Delta_{2|3} \delta^{-1} \gamma^1 \oplus \delta^{13} \Delta_{2|3} \delta^{-1} \gamma^2)(\gamma^2 \delta^9)^* \\
(\mathsf{Q}_\mathsf{t})_{22} &= \delta^7 \Delta_{2|3} \delta^{-1} \oplus \delta^9 \Delta_{2|3} \delta^{-2} \gamma^1 \oplus \delta^{10} \Delta_{2|3} \gamma^2 \\
&\quad \oplus \delta^{11} \Delta_{2|3} \delta^{-1} \gamma^2 \oplus \left(\delta^{13} \Delta_{2|3} \gamma^3 \oplus \delta^{16} \Delta_{2|3} \gamma^4\right) (\gamma^2 \delta^9)^*
\end{aligned}
$$

To obtain a flat version in $\mathcal{ET}$, we can expand $\mathsf{m}_2 \mathsf{Q}_\mathsf{t} \mathsf{b}_2$ and we obtain the transfer function as an ultimately periodic operator: $H = \nabla_{2|2} \gamma^1 \delta^{-1} \Delta_{2|3} \delta^{11} \oplus \gamma^1 \nabla_{2|2} \gamma^1 \delta^{-1} \Delta_{2|3} \delta^{12} \oplus \gamma^1 \nabla_{2|2} \delta^{-1} \Delta_{2|3} \delta^{11} \oplus \gamma^2 \nabla_{2|2} \delta^{-1} \Delta_{2|3} \delta^{12} \oplus \gamma^3 \nabla_{2|2} \delta^{-1} \Delta_{2|3} \delta^{13} \oplus \gamma^2 \nabla_{2|2} \gamma^1 \Delta_{2|3} \delta^{14} \oplus \gamma^4 \nabla_{2|2} \Delta_{2|3} \delta^{15} \oplus \gamma^5 \nabla_{2|2} \delta^{-1} \Delta_{2|3} \delta^{17} \oplus (\gamma^4 \nabla_{2|2} \gamma^1 \delta^{-1} \Delta_{2|3} \delta^{20} \oplus \gamma^6 \nabla_{2|2} \delta^{-1} \Delta_{2|3} \delta^{21} \oplus \gamma^6 \nabla_{2|2} \gamma^1 \Delta_{2|3} \delta^{23} \oplus \gamma^8 \nabla_{2|2} \Delta_{2|3} \delta^{24})(\gamma^4 \delta^9)^*$.

Remark 5: Let us note that this computation could be developped as well with core matrices in $\mathcal{E}[\![\delta]\!]$, say $H = \mathsf{v}_2 \mathsf{Q}_\mathsf{e} \mathsf{w}_3$ with $\mathsf{Q}_\mathsf{e} \in \mathcal{E}[\![\delta]\!]^{2\times 3}$ (see Prop.6).

VI. Conclusion

Weighted Timed Event Graphs (WTEGs) and Time-variant TEGs represent two well-known extensions of standard TEGs. Input-output models of WTEGs, respectively Time-variant TEGs, can be conveniently written in terms of the dioids $\mathcal{E}[\![\delta]\!]$, respectively $\mathcal{T}[\![\gamma]\!]$, see [5], respectively [16]. In this paper, we have shown how a more general class of event-variant and time-variant TEGs encompassing WTEGs and Time-variant TEGs as special cases can be modeled in a unified framework involving six elementary operators. If the system can be modelled by a balanced graph (the same event gain and clock rate gain on parallel paths), then the input-output relation can be calculated using a matrix decomposition. This framework allows us to perform the required calculations with adapted software tools. Possible applications are in the manufacturing domain, where our approach can be used for the evaluation of internal inventory levels or for the development of input flow controllers, such as the ones presented in [13] for standard TEGs. In particular, it was used to control a realistic production cell, modeled with ETVO; the details are given in the example section of the internal report [7] dedicated to the presentation of the software library.

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