

Extended version with source code of the paper **Observer-based Controllers for Max-plus Linear Systems**

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Abstract

Max-plus linear systems are often used to model transportation networks, communications systems and manufacturing systems. One way of controlling this kind of systems consists in choosing the date of input events in order to achieve the desired performances, e.g., to obtain output events in order to respect given dates. This kind of control is optimal, according to a just-in-time criterion, if the input-event dates are delayed as much as possible while ensuring the output events to occur before a desired reference date. In this context, this paper presents an observed-based controller approach, where only a linear combination of the states, obtained from measurement, is available for the controller. As in the classical sense, this is a state-feedback control problem, which is solved in two steps: first an observer computes an estimation of the state by using the input and the output measurements, then this estimated state is used to compute the state-feedback control action. As a main result, it is shown that the optimal solution of this observer-based control problem leads to a greater control input than the one obtained with the output feedback strategy. A high throughput screening example in drug discovery illustrates this main result by showing that the scheduling obtained from the observer-based controller is better than the scheduling obtained from the output feedback controller.

I. INTRODUCTION

Many discrete event dynamic systems, such as transportation networks ([14], [20]), communication networks, manufacturing assembly lines [6], are subject to synchronization phenomena. These systems behavior can be represented by timed-event graphs (TEGs) which are timed Petri nets where each place has one upstream transition and one downstream transition. Its description can be transformed into a max-plus or a min-plus linear model and *vice versa* ([2], [10]). This property has advantaged the emergence

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of a specific control theory for these systems, and several control strategies have been proposed, *e.g.*, optimal open loop control ([9], [23], [29]) and optimal state-feedback control in order to solve the model matching problem ([11], [22], [25], [26]). Control strategies allowing the state to stay in a specific subset are proposed in ([1], [21], [26]). State estimation has also been considered in some recent works ([13], [18], [19], [27]). The observer aims at estimating the state for a given plant by using input and output measurements. The state trajectories correspond to the transition firings of the corresponding TEG. Their estimation is worthy of interest because it provides insight into internal properties of the system. For example, these state estimations are sufficient to reconstruct the markings of the graph, as it is done in [15] for Petri nets without temporization. Moreover, the state estimation has many potential applications, such as fault detection, diagnosis, and state-feedback control. In general, the complete measurement of all the state variables is not possible due to the lack of sensors. It is then classical to use the state estimation provided by the observer to feed the controller.

This paper proposes to use the observer introduced in ([18], [19]) to design the observer-based controllers for max-plus linear systems. As in the classical sense, this is a state-feedback control problem, which is solved in two steps: first an observer computes an estimation of the state by using the input and the output measurements, then this estimated state is used to compute the state-feedback control action. The observer is designed by analogy with the classical Luenberger observer [24] for linear systems. As a main result, it is shown that the optimal solution of this observer-based control problem leads to a greater control input than the one obtained with the output feedback strategy. For instance, in a manufacturing setting, the observer-based controller would provide a better scheduling by starting the process later than the output feedback control, while ensuring the same output parts finishing time. This scheduling would allow users to load the raw parts later rather than earlier to avoid unnecessary congestions in the manufacturing line.

This paper is organized as the following. The max-plus algebra is a particular idempotent semiring, therefore, section II reviews some algebraic tools concerning these algebraic structures. Some results about the Residuation theory and its applications over semirings are also given. Section III recalls the description of TEGs in a semiring of formal series. Section IV presents the concepts of output controllability and state observability. Section V presents the observer for max-plus linear systems, it is designed by analogy with the classical Luenberger observer for linear systems. If the max-plus linear systems modeled in TEGs are observable, then the observer ensures equality between the state and the estimated state in spite of possible disturbances. Section VI reviews the output feedback and state-feedback controllers as introduced in ([25], [26]). Section VII presents the observer-based controller and compares its differences between the output feedback controller and the observer-based state-feedback controller. In particular, it is shown

that this new strategy increases the performances. Section VIII illustrates the main results using a high throughput screening example in drug discovery. An observer-based controller is constructed and proved to have a better performance comparing with an output feedback controller.

II. ALGEBRAIC SETTING

An idempotent semiring \mathcal{S} is an algebraic structure with two internal operations denoted by \oplus and \otimes . The operation \oplus is associative, commutative and idempotent, that is, $a \oplus a = a$. The operation \otimes is associative (but not necessarily commutative) and distributive on the left and on the right with respect to \oplus . The neutral elements of \oplus and \otimes are represented by ε and e respectively, and ε is an absorbing element for the law \otimes ($\forall a \in \mathcal{S}, \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$). As in classical algebra, the operator \otimes will be often omitted in the equations, moreover, $a^i = a \otimes a^{i-1}$ and $a^0 = e$. In this algebraic structure, a partial order relation is defined by $a \succeq b \Leftrightarrow a = a \oplus b \Leftrightarrow b = a \wedge b$ (where $a \wedge b$ is the greatest lower bound of a and b), therefore, an idempotent semiring \mathcal{S} is a partially ordered set (see [2], [20] for an exhaustive introduction). An idempotent semiring \mathcal{S} is said to be complete if it is closed for infinite \oplus -sums and if \otimes distributes over infinite \oplus -sums. In particular, $\top = \bigoplus_{x \in \mathcal{S}} x$ is the greatest element of \mathcal{S} (\top is called the top element of \mathcal{S}).

Example 1 ($\overline{\mathbb{Z}}_{\max}$): Set $\overline{\mathbb{Z}}_{\max} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the \max operator as sum and the classical sum $+$ as product is a complete idempotent semiring, usually denoted $\overline{\mathbb{Z}}_{\max}$, of which $\varepsilon = -\infty$ and $e = 0$.

Theorem 1 (see [2], Th. 4.75): The implicit inequality $x \succeq ax \oplus b$ as well as $x = ax \oplus b$ defined over \mathcal{S} , admit $x = a^*b$ as the least solution, where $a^* = \bigoplus_{i \in \mathbb{N}} a^i$ (Kleene star operator).

Definition 1 (Residual and residuated mapping): An order preserving mapping $f : \mathcal{D} \rightarrow \mathcal{E}$, where \mathcal{D} and \mathcal{E} are partially ordered sets, is a *residuated mapping* if for all $y \in \mathcal{E}$ there exists a greatest solution to the inequality $f(x) \preceq y$ (hereafter denoted $f^\sharp(y)$). Obviously, if equality $f(x) = y$ is solvable, $f^\sharp(y)$ yields the greatest solution. The mapping f^\sharp is called the *residual* of f and $f^\sharp(y)$ is the optimal solution of the inequality.

Theorem 2 (see [2] Th. 4.50,[3]): Let $f : \mathcal{D} \rightarrow \mathcal{C}$ be an order preserving mapping. The following statements are equivalent:

- (i) f is residuated.
- (ii) there exists a unique order preserving mapping $f^\sharp : \mathcal{C} \rightarrow \mathcal{D}$ such that $f \circ f^\sharp \preceq \text{Id}_{\mathcal{C}}$ and $f^\sharp \circ f \succeq \text{Id}_{\mathcal{D}}$.

Theorem 3 ([2, Th. 4.56]): Let \mathcal{C} , \mathcal{B} be two semirings, and $f : \mathcal{C} \rightarrow \mathcal{B}$ a residuated mapping. The following properties hold:

$$f \circ f^\# \circ f = f \text{ and } f^\# \circ f \circ f^\# = f^\#. \quad (1)$$

Example 2: Mappings $\Lambda_a : x \mapsto a \otimes x$ and $\Psi_a : x \mapsto x \otimes a$ defined over an idempotent semiring \mathcal{S} are both residuated (see [2], Section 4.4.4). Their residuals are order preserving mappings, denoted respectively by $\Lambda_a^\#(x) = a \backslash x$ and $\Psi_a^\#(x) = x \not\! / a$. This means that $a \backslash b$ (resp. $b \not\! / a$) is the greatest solution of the inequality $a \otimes x \preceq b$ (resp. $x \otimes a \preceq b$).

In the Appendix, useful properties about left and right residuation are recalled. The proofs are given in ([2], Chapter 4). Below an original property is given:

Lemma 1: If $y \in \mathcal{D}$ admits a right inverse w and a left inverse z , then the two following statements hold

- $w = z$ and this unique inverse is denoted y^{-1} ;
- moreover, $\forall c, b, (b \backslash c)y = b \backslash (cy)$ and $y(c \not\! / b) = (yc) \not\! / b$.

Proof:

- Due to associativity of the product law, we have $w = w(yz) = (wy)z = z$, which proves uniqueness of a right and left inverse (see Lemma 4.36 in [2]).
- According to Theorem 2 and Example 2, $b \backslash c$ is the greatest solution of $bx \preceq c$, then $b(b \backslash c) \preceq c$ which implies $b(b \backslash c)y \preceq cy$, since the product is isotone. Moreover $b \backslash (cy)$ is the greatest solution of $bx \preceq cy$, then $(b \backslash c)y \preceq b \backslash (cy)$. On the other way, due to (f.13), $b \backslash (cy) = (b \backslash (cy))y^{-1}y \preceq (b \backslash (cyy^{-1}))y = (b \backslash c)y$. Hence, equality $(b \backslash c)y = b \backslash (cy)$ holds throughout. The equality $y(c \not\! / b) = (yc) \not\! / b$ can be proved in a similar way. ■

Definition 2 (Restricted mapping): Let $f : \mathcal{D} \rightarrow \mathcal{C}$ be a mapping and $\mathcal{B} \subset \mathcal{D}$. We will denote by $f|_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{C}$ the mapping defined by $f|_{\mathcal{B}} = f \circ \text{Id}|_{\mathcal{B}}$ where $\text{Id}|_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{D}, x \mapsto x$ is the canonical injection. Identically, let $\mathcal{E} \subset \mathcal{C}$ be a set such that $\text{Im} f \subset \mathcal{E}$. Mapping $\varepsilon|f : \mathcal{D} \rightarrow \mathcal{E}$ is defined by $\varepsilon|f = \text{Id}|_{\mathcal{E}} \circ \varepsilon|f$, where $\text{Id}|_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{C}, x \mapsto x$.

Definition 3 (Closure mapping): A closure mapping is an order preserving mapping $f : \mathcal{D} \rightarrow \mathcal{D}$ defined on an ordered set \mathcal{D} such that $f \succeq \text{Id}_{\mathcal{D}}$ and $f \circ f = f$.

Proposition 1 (see [11]): Let $f : \mathcal{D} \rightarrow \mathcal{D}$ be a closure mapping. Then, $\text{Im} f|f$ is a residuated mapping whose residual is the canonical injection $\text{Id}|_{\text{Im} f}$.

Example 3: Mapping $K : \mathcal{S} \rightarrow \mathcal{S}, x \mapsto x^*$ is a closure mapping (indeed $a \preceq a^*$ and $a^* = (a^*)^*$ see (f.2) in Appendix). Then, $(\text{Im}K|K)$ is residuated and its residual is $(\text{Im}K|K)^\sharp = \text{Id}_{|\text{Im}K}$. In other words, $x = a^*$ is the greatest solution of inequality $x^* \preceq a$ if $a \in \text{Im}K$, that is $x \preceq a^* \Leftrightarrow x^* \preceq a$.

Example 4: Mapping $P : \mathcal{S} \rightarrow \mathcal{S}, x \mapsto x^+ = \bigoplus_{i \in \mathbb{N}^+} x^i = xx^* = x^*x$ is a closure mapping (indeed $a \preceq a^+$ and $a^+ = (a^+)^+$ see (f.2) in Appendix). Then $(\text{Im}P|P)$ is residuated and its residual is $(\text{Im}P|P)^\sharp = \text{Id}_{|\text{Im}P}$. In other words, $x = a^+$ is the greatest solution of inequality $x^+ \preceq a$ if $a \in \text{Im}P$, that is $x \preceq a^+ \Leftrightarrow x^+ \preceq a$.

Remark 1: According to (f.5) in Appendix, $(a^*)^+ = a^*$, therefore, $\text{Im}K \subset \text{Im}P$.

The set of $n \times n$ matrices with entries in \mathcal{S} is an idempotent semiring. The sum, the product and the residuation of matrices are defined after the sum, the product and the residuation of scalars in \mathcal{S} , *i.e.*,

$$(A \otimes B)_{ik} = \bigoplus_{j=1..n} (a_{ij} \otimes b_{jk}) \quad (2)$$

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}, \quad (3)$$

$$(A \backslash B)_{ij} = \bigwedge_{k=1..n} (a_{ki} \backslash b_{kj}), \quad (B \not\! / A)_{ij} = \bigwedge_{k=1..n} (b_{ik} \not\! / a_{jk}). \quad (4)$$

The identity matrix of $\mathcal{S}^{n \times n}$ is a matrix with entries equal to e on the diagonal and to ε elsewhere. This identity matrix will also be denoted e , and the matrix with all its entries equal to ε will also be denoted ε .

Properties 1: ([16], [28]) Given a complete semiring \mathcal{S} , and four matrices $A \in \mathcal{S}^{p \times n}$, $B \in \mathcal{S}^{n \times p}$, $M \in \mathcal{S}^{p \times p}$, and $N \in \mathcal{S}^{n \times n}$, the following equalities are satisfied:

$$A \backslash A = (A \backslash A)^*, \quad B \not\! / B = (B \not\! / B)^*, \quad (5)$$

$$A \backslash (M^* A) = (M^* A) \backslash (M^* A) = (A \backslash (M^* A))^* \quad (6)$$

$$(AN^*) \not\! / A = (AN^*) \not\! / (AN^*) = ((AN^*) \not\! / A)^*. \quad (7)$$

From now on, to enlighten notation, as in classical algebra mapping Λ_C (see Example 2) will be denoted C .

Definition 4 (Kernel [7], [8]): Let \mathcal{S} be a complete idempotent semiring and let C be a $n \times p$ matrix with entries in \mathcal{S} , the *kernel* of C (denoted by $\ker C$) is the subset of all pairs of elements of \mathcal{S}^p whose components are both mapped by C to the same element in \mathcal{S}^n , *i.e.*, the following definition

$$\ker C := \left\{ (s, s') \in (\mathcal{S}^p)^2 \mid Cs = Cs' \right\}. \quad (8)$$

Clearly, $\ker C$ is an equivalence relation on \mathcal{X} , *i.e.*, $Cs = Cs' \iff s' \equiv s \pmod{\ker C}$. Furthermore, it is a congruence and then we can define the quotient $\mathcal{S}/\ker C$.

Remark 2: The subset of elements $s' \in \mathcal{S}^p$ that are equivalent to s modulo $\ker C$ is denoted $[s]_C$, i.e.,

$$[s]_C = \{s' \in \mathcal{S}^p \mid s' \equiv s \pmod{\ker C}\} \subset \mathcal{S}^p.$$

Lemma 2 ([7], [8]): For matrices $A \in \mathcal{S}^{n \times p}$, $B \in \mathcal{S}^{n \times m}$ the following statements are equivalent:

1. $\text{Im} B \subset \text{Im} A$;
2. $B = A(A \setminus B)$.

Proof:

- “Item 1 \Rightarrow Item 2”: If $\text{Im} B \subset \text{Im} A$, then, for any element $y = Bx \in \text{Im} B$, $\forall x$, there exists a z such that $y = Az = Bx$. Due to Property in (f.8), we have

$$y = Az = A(A \setminus Az) = A(A \setminus Bx),$$

for any $x \in \mathcal{X}$. Therefore, $Bx = A(A \setminus Bx)$. It implies $A(A \setminus B) = B$.

- “Item 2 \Rightarrow Item 1”: If $B = A(A \setminus B)$, then $\text{Im} B \subset \text{Im} A$. ■

Lemma 3 ([7], [8]): For matrices $C \in \mathcal{S}^{n \times p}$, $B \in \mathcal{S}^{m \times p}$ the following statements are equivalent:

1. $\ker C \subset \ker B$;
2. there exists a linear map $L : \mathcal{S}^n \rightarrow \mathcal{S}^m$, s.t. $L \circ C = B$.

Proof:

- “Item 1 \Rightarrow Item 2”: According to Definition 4, $\ker C \subset \ker B$ can be written as

$$Cx = Cx' \Rightarrow Bx = Bx',$$

where $(x, x') \in (\mathcal{S}^p)^2$. According to Definition 1, $C \setminus (Cx)$ is the greatest element in the equivalence class $[x]_C$, and Theorem 3 yields $C(C \setminus (Cx)) = Cx$. Hence, the following assertion holds,

$$\ker C \subset \ker B \Rightarrow B(C \setminus (Cx)) = Bx, \forall x \in \mathcal{S}^p.$$

By considering mapping $L : \mathcal{S}^n \rightarrow \mathcal{S}^m, y \mapsto B(C \setminus y)$, the following equality holds $L \circ C = B$. Because B and C are linear mappings, L is linear also, i.e., $L(\bigoplus y) = \bigoplus L(y)$ and $L(\lambda y) = \lambda L(y)$ where $\lambda \in \mathcal{S}$.

- “Item 2 \Rightarrow Item 1”: If a linear mapping $L \circ C = B$ exists, then the following assertion holds :
 $Cx = Cx' \Rightarrow L \circ Cx = L \circ Cx' = Bx = Bx'$, i.e., $\ker C \subset \ker B$. ■

Corollary 1: If there exists a linear mapping $L \circ C = B$, then

$$B = (B \setminus C)C. \tag{9}$$

Proof: If there exists a linear mapping $L \circ C = B$ then $L \preceq B \not\prec C$ due to the right residuation definition. Hence $B = L \circ C \preceq (B \not\prec C)C \preceq B$, due to property (f.14). Hence, the equality $B = (B \not\prec C)C$ holds. Of course, it implies $\ker C \subset \ker B$. ■

III. THE TEG DESCRIPTION IN AN IDEMPOTENT SEMIRING

TEGs constitute a subclass of timed Petri nets, *i.e.*, in which each place has one upstream and one downstream transition. A TEG description can be transformed into a $(max, plus)$ or a $(min, plus)$ linear model and *vice versa*. To obtain an algebraic model in $\overline{\mathbb{Z}}_{max}$, a “dater” function is associated to each transition. For transition labelled x_i , $x_i(k)$ represents the date of the k^{th} firing (see [2],[20]). A trajectory of a TEG transition is then a firing date sequence of this transition. This collection of dates can be represented by a formal series $x(\gamma) = \bigoplus_{k \in \mathbb{Z}} x_i(k) \otimes \gamma^k$ where $x_i(k) \in \overline{\mathbb{Z}}_{max}$ and γ is a backward shift operator¹ in the event domain (formally $\gamma x(k) = x(k-1)$). The set of formal series in γ is denoted by $\overline{\mathbb{Z}}_{max}[[\gamma]]$ and constitutes a complete idempotent semiring.

In this paper, TEGs are defined in this setting by the following model:

$$\begin{aligned} x &= Ax \oplus Bu \oplus Rw, \\ y &= Cx, \end{aligned} \tag{10}$$

where $u \in \mathcal{U} = (\overline{\mathbb{Z}}_{max}[[\gamma]])^p$, $y \in \mathcal{Y} = (\overline{\mathbb{Z}}_{max}[[\gamma]])^m$ and $x \in \mathcal{X} = (\overline{\mathbb{Z}}_{max}[[\gamma]])^n$ are respectively the controllable inputs, outputs and state vectors, *i.e.*, each of their entries is a trajectory which represents the collection of firing dates of the corresponding transition. Matrices $A \in (\overline{\mathbb{Z}}_{max}[[\gamma]])^{n \times n}$, $B \in (\overline{\mathbb{Z}}_{max}[[\gamma]])^{n \times p}$, and $C \in (\overline{\mathbb{Z}}_{max}[[\gamma]])^{m \times n}$ represent the links between each transition, and then describe the structure of the graph. Vector $w \in \mathcal{W} = (\overline{\mathbb{Z}}_{max}[[\gamma]])^l$ represents uncontrollable inputs (*i.e.* disturbances). Each entry of w corresponds to a transition which disables the firing of internal transition of the graph, and then decreases the performance of the system. This vector is bounded to the transitions through matrix $R \in (\overline{\mathbb{Z}}_{max}[[\gamma]])^{n \times l}$. Matrix R , that equals to the identity matrix, is a particular case where each transition is directly affected by independent disturbances.

From now on, each uncontrollable input transition w_i is assumed to be connected to one and only one internal transition x_j , this means that each column of matrix R has one entry equal to e and the others equal to ε and at most one entry equal to e on each row. Furthermore, each output transition y_i is assumed to be linked to one and only one internal transition x_j , *i.e.*, each row of matrix C has one entry equal to e and the others equal to ε and at most one entry equal to e on each column. These

¹Operator γ plays a role similar to operator z^{-1} in the \mathcal{Z} -transform for the conventional linear systems theory.

requirements are satisfied without loss of generality, since it is sufficient to add extra input and output transitions. Note that if R is equal to the identity matrix, w can represent initial state of the system $x(0)$ by considering $w = x(0)\gamma^0 \oplus \dots$ (see [2], p. 245, for a discussion about compatible initial conditions). By considering Theorem 1, this system can be rewritten as:

$$\begin{aligned} x &= A^*Bu \oplus A^*Rw \\ y &= CA^*Bu \oplus CA^*Rw, \end{aligned} \quad (11)$$

where $CA^*B \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{m \times p}$ (respectively, $CA^*R \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{m \times l}$) is the input/output (respectively, the disturbance/output) transfer matrix. Matrix CA^*B represents the earliest behavior of the system, therefore, it must be underlined that the uncontrollable input vector w (initial conditions or disturbances) is only able to delay the transition firings, *i.e.*, according to the order relation of the semiring, to increase the vectors x and y . Consequently, it is assumed that the model and the initial state correspond to the fastest behavior (e.g. ideal behavior of a manufacturing system without extra delays) and that disturbances only delay the occurrence of events (e.g. disturbance due to machine breakdown or delay due to an unexpected failure in component supply).

IV. OUTPUT CONTROLLABILITY AND STATE OBSERVABILITY

Definition 5: (Structural Controllability [2]) A TEG is said to be *structurally controllable* if every internal transition can be reached by a path from at least one input transition.

Theorem 4: ([16], [30]) A TEG is structurally controllable if the corresponding matrix A^*B is such that at least one entry on each row is different from ε .

Definition 6: (Output Controllability) A system described by

$$\begin{aligned} x &= Ax \oplus Bu \\ y &= Cx, \end{aligned} \quad (12)$$

is said to be *output controllable* if $\forall x$ and $y = CA^*x$, there exists an input u s.t. $y = CA^*Bu$.

Theorem 5: For a system given by Eq. (12), the following statements are equivalent :

1. $\forall x$, and $y = CA^*x \Rightarrow \exists u$ s.t. $y = CA^*Bu$, *i.e.*, the system is output controllable;
2. $\text{Im}CA^* \subset \text{Im}CA^*B$;
3. $CA^*B((CA^*B) \setminus (CA^*)) = CA^*$.

Proof:

- “Item 1 \Rightarrow Item 2”: Because Item 1 is satisfied, for any output $y = CA^*x$, $\forall x$, there exists an input u such that $y = CA^*Bu$. It implies $\text{Im}CA^* \subset \text{Im}CA^*B$.

- “Item 2 \Rightarrow Item 3”: According to Lemma 2, $\text{Im}CA^* \subset \text{Im}CA^*B$ is equivalent to

$$CA^*B((CA^*B)\setminus(CA^*)) = CA^*.$$

- “Item 3 \Rightarrow Item 1”’: If $CA^*B((CA^*B)\setminus(CA^*)) = CA^*$, then any element $y = CA^*x \in \text{Im}CA^*$ can be written as $y = CA^*B((CA^*B)\setminus(CA^*x))$. Define $u = (CA^*B)\setminus(CA^*x)$, then for any $y = CA^*x \in \text{Im}CA^*$, there exists such an u , such that $y = CA^*Bu$.

■

Definition 7: (Structural Observability [2]) A TEG is said to be *structurally observable* if, from every internal transition, there exists a path to at least one output transition.

Theorem 6: ([16], [30]) A TEG is structurally observable if the corresponding matrix CA^* is such that at least one entry on each column is different from ε .

Definition 8: (State Observability) A system over an idempotent semiring described by (12) is said to be *state observable* if, for any pair of inputs u and u' satisfying $CA^*Bu = CA^*Bu'$, the equality $A^*Bu = A^*Bu'$ holds.

Theorem 7: For a system given by Eq. (12), the following statements are equivalent :

1. $\forall u, u'$, the equality $CA^*Bu = CA^*Bu'$ implies $A^*Bu = A^*Bu'$, i.e. the system is state observable;
2. $\ker CA^*B = \ker A^*B$;
3. $A^*B = ((A^*B)\setminus(CA^*B))CA^*B$.

Proof:

- “Item 1 \Leftrightarrow Item 2”’: According to Definition 4, Item 1 means that $\ker CA^*B \subset \ker A^*B$. On the other hand, $\forall (u, u')$, we have $A^*Bu = A^*Bu' \Rightarrow CA^*Bu = CA^*Bu'$ i.e., $\ker A^*B \subset \ker CA^*B$, hence the equality holds.
- “Item 2 \Rightarrow Item 3”’: According to Lemma 3 and Corollary 1, $\ker CA^*B \subset \ker A^*B$ is equivalent to $((A^*B)\setminus(CA^*B))CA^*B = A^*B$.
- “Item 3 \Rightarrow Item 1”’: For all (u, u') , we have $CA^*Bu = CA^*Bu'$. After applying $((A^*B)\setminus(CA^*B))$ on both sides, we obtain $((A^*B)\setminus(CA^*B))CA^*Bu = ((A^*B)\setminus(CA^*B))CA^*Bu'$. The previous equality implies $A^*Bu = A^*Bu'$, because $((A^*B)\setminus(CA^*B))CA^*B = A^*B$.

■

Remark 3: All the results in this section can be extended to the system given by Eq. (10), by rewriting

the system as

$$\begin{aligned} x &= Ax \oplus \bar{B}\bar{u}, \text{ with } \bar{B} = \begin{bmatrix} B & R \end{bmatrix}, \bar{u} = \begin{bmatrix} u \\ w \end{bmatrix}, \\ y &= Cx, \end{aligned} \quad (13)$$

where the extended input is $\bar{u} \in (\bar{\mathbb{Z}}_{\max}[\gamma])^{p+l}$, the output is $y \in (\bar{\mathbb{Z}}_{\max}[\gamma])^m$, and the state is $x \in (\bar{\mathbb{Z}}_{\max}[\gamma])^n$.

V. MAX-PLUS OBSERVER

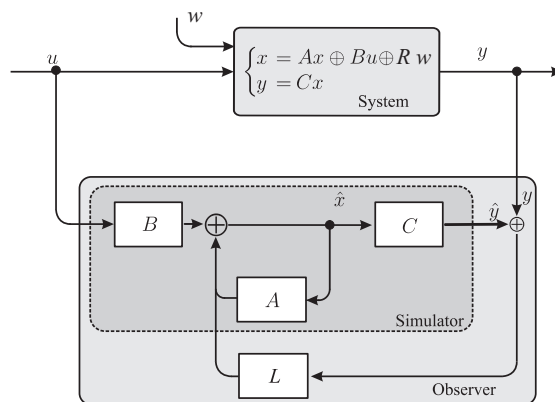


Fig. 1. The observer structure of max-plus linear systems.

Fig. 1 depicts the observer structure directly inspired from the Luenberger observer in classical linear system theory ([18],[19],[24]). The observer matrix L is used to provide information from the system output into the simulator in order to take the disturbances w acting on the system into account. The simulator is described by the model²(matrices A, B, C) which is assumed to represent the fastest behavior of the real system in a guaranteed way³, furthermore the simulator is initialized by the canonical initial conditions (*i.e.* $\hat{x}_i(k) = \varepsilon, \forall k \leq 0$). These assumptions induce that $\hat{y} \preceq y$ since disturbances and initial conditions, depicted by w , are only able to increase the system outputs. By considering the configuration in Fig. 1 and these assumptions, the computation of the optimal observer matrix L will be proposed in order to achieve the constraint $\hat{x} \preceq x$. Optimality means that the obtained observer matrix L is the greatest one due to the residuation theory (see Definition 1). Therefore, the estimated state \hat{x} is the

²Disturbances are uncontrollable and *a priori* unknown, then the simulator does not take them into account.

³Unlike in the conventional linear system theory, this assumption means that the fastest behavior of the system is assumed to be known and that the disturbances can only delay its behavior.

greatest which achieves the objective, so as close as possible to x . Obviously this optimality is only ensured under the assumptions considered (*i.e.* $\hat{y} \preceq y$). As in the development proposed in conventional linear systems theory, matrices A , B , C and R are assumed to be known, then the system trajectories are given by Eq. (11). According to Fig. 1, the observer equations, similarly as the Luenberger observer, are given by:

$$\begin{aligned}\hat{x} &= A\hat{x} \oplus Bu \oplus L(\hat{y} \oplus y) = A\hat{x} \oplus Bu \oplus Ly, \quad \text{since by assumption } \hat{y} \preceq y, \\ &= A\hat{x} \oplus Bu \oplus LCx = A\hat{x} \oplus Bu \oplus LC(A^*Bu \oplus A^*Rw) \\ \hat{y} &= C\hat{x}.\end{aligned}\tag{14}$$

By applying Theorem 1 and by considering Eq. (11), Eq.(14) becomes:

$$\hat{x} = (A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCA^*Bu \oplus (A \oplus LC)^*LCA^*Rw.\tag{15}$$

By applying (f.1) the following equality is obtained:

$$(A \oplus LC)^* = A^*(LCA^*)^*,\tag{16}$$

by replacing in Eq. (15):

$$\begin{aligned}\hat{x} &= A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^*LCA^*Bu \\ &\quad \oplus A^*(LCA^*)^*LCA^*Rw,\end{aligned}$$

and by recalling that $(LCA^*)^*LCA^* = (LCA^*)^+$, this equation may be written as follows:

$$\hat{x} = A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^+Bu \oplus A^*(LCA^*)^+Rw.$$

Eqs. (f.2) and (f.5) yield $(LCA^*)^* \succeq (LCA^*)^+$, then the observer model may be written as follows:

$$\begin{aligned}\hat{x} &= A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^+Rw \\ &= (A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCA^*Rw.\end{aligned}\tag{17}$$

As said previously, the objective considered is to compute the greatest observation matrix L , denoted as L_{opt} , such that the estimated state vector \hat{x} be as close as possible to state x , under the constraint $\hat{x} \preceq x$, formally it can be written as, finding the greatest L satisfying the following inequality:

$$(A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCA^*Rw \preceq A^*Bu \oplus A^*Rw, \quad \forall(u, w),$$

or equivalently :

$$(A \oplus LC)^*B \preceq A^*B\tag{18}$$

$$\text{and, } (A \oplus LC)^*LCA^*R \preceq A^*R.\tag{19}$$

Lemma 4 ([19]): The following equivalence holds :

$$(A \oplus LC)^*B = A^*B \iff L \preceq L_1 = (A^*B)\phi(CA^*B).$$

Lemma 5 ([19]): The following equivalence holds

$$(A \oplus LC)^*LCA^*R \preceq A^*R \iff L \preceq L_2 = (A^*R)\phi(CA^*R).$$

Proposition 2 ([19]): $L_{opt} = L_1 \wedge L_2$ is the greatest observer matrix L such that:

$$\hat{x} = A\hat{x} \oplus Bu \oplus Ly \preceq x = Ax \oplus Bu \oplus Rw, \quad \forall(u, w).$$

Corollary 2 ([18]): The matrix L_{opt} ensures the equality between estimated output \hat{y} and measured output y , *i.e.* the two equality holds:

$$C(A \oplus L_{opt}C)^*B = CA^*B, \quad (20)$$

$$C(A \oplus L_{opt}C)^*L_{opt}CA^*R = CA^*R. \quad (21)$$

Remark 4: According to this Corollary $\hat{y} = y$, so the state estimation can be computed by considering matrix $\bar{B} = [B \ R]$, matrix L_{opt} may be written as: $L_{opt} = (A^*\bar{B})\phi(CA^*\bar{B})$.

Theorem 8: If the system given in Eq. (13) is state observable then the observed state is the same as the real state of the system, *i.e.* $\hat{x} = x$.

Proof: The observed state is given by $\hat{x} = (A \oplus L_{opt}C)^*Bu \oplus (A \oplus L_{opt}C)^*L_{opt}CA^*Rw$ (see Eq. (17)). The real state is given by $x = A^*Bu \oplus A^*Rw$ (see Eq. (11)). According to Theorem 7, the system in Eq. (13) is state observable if and only if $((A^*\bar{B})\phi(CA^*\bar{B}))CA^*\bar{B} = CA^*\bar{B}$, which can be written $L_{opt}CA^*\bar{B} = CA^*\bar{B}$. Hence the following equivalences hold :

$$\begin{aligned} L_{opt}CA^*\bar{B} = A^*\bar{B} &\iff \begin{bmatrix} L_{opt}CA^*B & L_{opt}CA^*R \end{bmatrix} = \begin{bmatrix} A^*B & A^*R \end{bmatrix} \\ &\iff L_{opt}CA^*B = A^*B \text{ and } L_{opt}CA^*R = A^*R. \end{aligned}$$

First, by Lemma 4, L_{opt} ensures the equality

$$(A \oplus L_{opt}C)^*B = A^*B.$$

Secondly, due to (f.1), we have

$$\begin{aligned} (A \oplus L_{opt}C)^*L_{opt}CA^*R &= A^*(L_{opt}CA^*)^*L_{opt}CA^*R = A^*(L_{opt}CA^*)^+R \\ &= A^*(L_{opt}CA^*R \oplus (L_{opt}CA^*R)^2R \oplus \dots). \end{aligned}$$

Since $L_{opt}CA^*R = A^*R$, the following equality is satisfied

$$(L_{opt}CA^*)^2R = L_{opt}CA^*A^*R = L_{opt}CA^*R = A^*R.$$

More generally, $A^*(L_{opt}CA^*)^+R = A^*A^*R = A^*R$ is true as well. Therefore, L_{opt} ensures the equality

$$(A \oplus L_{opt}C)^*L_{opt}CA^*R = A^*(L_{opt}CA^*)^+R = A^*R.$$

Therefore, the observed state is the same as the real state of the system, that is,

$$\begin{aligned} \hat{x} &= (A \oplus L_{opt}C)^*Bu \oplus (A \oplus L_{opt}C)^*L_{opt}CA^*Rw \\ &= A^*Bu \oplus A^*Rw = x. \end{aligned}$$

■

VI. OUTPUT FEEDBACK AND STATE-FEEDBACK CONTROLLERS

This section presents how to synthesize output feedback and state-feedback controllers in order to solve the model matching problem (or model reference control problem, see [17], [25], [26]). First, an output feedback control $u_F = P(v \oplus Fy)$ is considered, where $P \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{p \times p}$ and $F \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{p \times m}$, and is synthesized in order to reach a given specification $G_{ref} \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{m \times p}$. Because the transfer matrix of the system in Eq. (13) is $CA^*B \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{m \times p}$, then the output of the controlled system is $y = CA^*BP(v \oplus Fy)$, which yields, by applying Theorem 1, the following transfer relation between v and y :

$$y = (CA^*BPF)^*CA^*BPv. \quad (22)$$

The synthesis aim is to get the greatest closed-loop transfer relation smaller than the one specified by G_{ref} , *i.e.*, $(CA^*BPF)^*CA^*BP \preceq G_{ref}$. Practically, this means that the actual output date of the closed-loop system $y(k) = (CA^*BPF)^*CA^*BPv(k)$ must be smaller than the desired output given by the specification $(G_{ref}v)(k)$ for each event k . This output $y(k)$ is optimal according to the just-in-time criterion, if it is the greatest solution satisfying this constraint, *i.e.*, the system input $u(k)$ is delayed as much as possible. This section presents the optimal output feedback controller $u_{Fopt} = P_{opt}(v \oplus F_{opt}y)$, for any v and y .

Proposition 3: If the specification $(CA^*BPF)^*CA^*BPv \preceq G_{ref}v, \forall v$, is satisfied, then the matrix $P \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{p \times p}$ satisfies the following inequality:

$$P \preceq (CA^*B) \setminus G_{ref} = P_{opt}. \quad (23)$$

Proof: According to Kleene star operator definition ($a^* = e \oplus a \oplus a^2 \oplus \dots$), we have

$$(CA^*BPF)^*CA^*BP \preceq G_{ref} \Rightarrow CA^*BP \preceq G_{ref}.$$

According to the residuation definition of the left product (see Example 2), the right hand side of this inequality is equivalent to $P \preceq (CA^*B)\backslash G_{ref}$. ■

Proposition 4: Considering an output feedback control $u_F = P_{opt}(v \oplus Fy)$, the greatest output feedback matrix $F \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{p \times m}$, denoted as F_{opt} , ensuring that the closed-loop behavior of the system respects the constraint $y = (CA^*BP_{opt}F)^*CA^*BP_{opt}v \preceq G_{ref}v, \forall v$, is given by :

$$\begin{aligned} F_{opt} &= \bigoplus \{F \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{p \times m} \mid (CA^*BP_{opt}F)^*CA^*BP_{opt}v \preceq G_{ref}v, \forall v\} \\ &= P_{opt}\backslash P_{opt}\phi(CA^*BP_{opt}). \end{aligned} \quad (24)$$

Proof:

$$\begin{aligned} &(CA^*BP_{opt}F)^*CA^*BP_{opt}v \preceq G_{ref}v, \quad \forall v, \\ \Leftrightarrow &(CA^*BP_{opt}F)^*CA^*BP_{opt} \preceq G_{ref}, \\ \Leftrightarrow &CA^*BP_{opt}(FCA^*BP_{opt})^* \preceq G_{ref}, \quad \text{thanks to (f.3),} \\ \Leftrightarrow &P_{opt}(FCA^*BP_{opt})^* \preceq (CA^*B)\backslash G_{ref} = P_{opt}, \quad \text{see Example 2,} \\ \Leftrightarrow &(FCA^*BP_{opt})^* \preceq P_{opt}\backslash P_{opt}, \quad \text{see Example 2,} \\ \Leftrightarrow &FCA^*BP_{opt} \preceq P_{opt}\backslash P_{opt}, \quad \text{see Example 3,} \\ \Leftrightarrow &F \preceq P_{opt}\backslash P_{opt}\phi(CA^*BP_{opt}), \quad \text{see Example 2.} \end{aligned}$$

Therefore, the output feedback control law $u_{Fopt} = P_{opt}(v \oplus F_{opt}y)$ is the optimal solution to the model matching problem according to the just-in-time criterion. Next a state-feedback control is considered, *i.e.* $u_K = P(v \oplus Kx)$, with $K \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{p \times n}$. Hence, the state of the corrected system is $x = A^*BP(v \oplus Kx)$, which yields, by applying Theorem 1, the following transfer relations:

$$x = (A^*BPK)^*A^*BPv, \quad (25)$$

$$y = CA^*BP(KA^*BP)^*v. \quad (26)$$

Proposition 5: Considering a state-feedback control law $u_K = P_{opt}(v \oplus Kx)$, the greatest state-feedback matrix $K \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{p \times n}$, denoted by K_{opt} , ensuring $y = CA^*BP_{opt}(KA^*BP_{opt})^*v \preceq G_{ref}v, \forall v$, is given by:

$$K_{opt} = P_{opt} \setminus P_{opt} / (A^*BP_{opt}). \quad (27)$$

Proof: The proofs take the same steps as the ones in Proposition 4 by replacing CA^*BP_{opt} by A^*BP_{opt} . ■

VII. OBSERVER-BASED CONTROLLERS

As in the classical theory, sometimes the state is not measurable or it is too expensive to measure all the states. Hence, in this section, we propose to use the estimated state \hat{x} , obtained thanks to the observer proposed in section V, to compute the state-feedback control law as given in Proposition 5. Then this control strategy is compared with the output feedback control as given in Proposition 4. Formally, the observer-based control law $u_M = P(v \oplus M\hat{x})$ is considered, where $\hat{x} = A\hat{x} \oplus Bu \oplus L_{opt}y$ (See Eq.(14)), where the optimal observer matrix L_{opt} as given in Proposition 2 is clearly independent of the control law u_M .

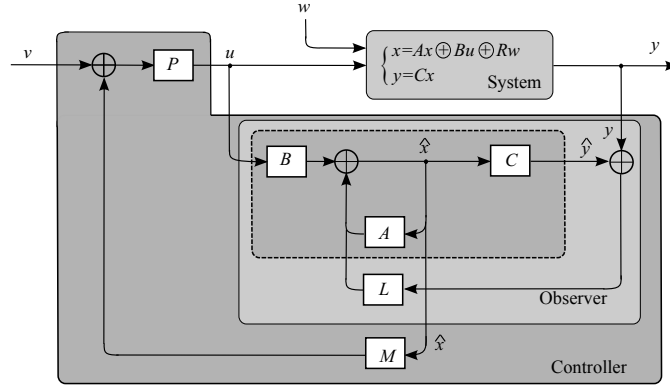


Fig. 2. The observer-based controller for max-plus linear systems.

This observer-based control law is described in Fig. 2 can be written as

$$\begin{aligned} u_M &= P(v \oplus M\hat{x}) = Pv \oplus PM(A \oplus L_{opt}C)^*Bu \\ &= (PM(A \oplus L_{opt}C)^*B)^*Pv = P(M(A \oplus L_{opt}C)^*BP)^*v. \end{aligned} \quad (28)$$

The state and the output can be written, respectively, as

$$x = A^*Bu = A^*BP(M(A \oplus L_{opt}C)^*BP)^*v, \quad (29)$$

$$y = Cx = CA^*BP(M(A \oplus L_{opt}C)^*BP)^*v, \quad (30)$$

The synthesis aim is to get the greatest control law such that the output y with this control strategy is smaller than or equal to the desired output $G_{ref}v$, *i.e.*, $y = CA^*BP(M(A \oplus L_{opt}C)^*BP)^*v \preceq G_{ref}v$, $\forall v$, which is equivalent to obtain the optimal prefilter matrix P and the optimal observer-based controller matrix M satisfying $CA^*BP(M(A \oplus L_{opt}C)^*BP)^* \preceq G_{ref}$.

Proposition 6: If the specification $CA^*BP(M(A \oplus L_{opt}C)^*BP)^*v \preceq G_{ref}v$, $\forall v$, is satisfied, then the matrix $P \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{p \times p}$ satisfies the following equality:

$$P \preceq (CA^*B) \setminus G_{ref} = P_{opt}. \quad (31)$$

Proof: According to the Kleene star operator definition ($a^* = e \oplus a \oplus a^2 \oplus \dots$), we have $CA^*BP(M(A \oplus L_{opt}C)^*BP)^* \preceq G_{ref}$. According to the residuation definition of the left product (see Example 2), the right hand side of this inequality is equivalent to $P \preceq (CA^*B) \setminus G_{ref}$. ■

Proposition 7: Considering the observer-based controller is $u_M = P_{opt}(v \oplus M\hat{x})$, the greatest feedback matrix $M \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{p \times n}$, denoted by M_{opt} , ensuring $y = CA^*BP_{opt}(M(A \oplus L_{opt}C)^*BP_{opt})^*v \preceq G_{ref}v$, $\forall v$, is given by:

$$M_{opt} = P_{opt} \setminus P_{opt} \setminus ((A \oplus L_{opt}C)^*BP_{opt}), \quad (32)$$

$$M_{opt} = P_{opt} \setminus P_{opt} \setminus (A^*BP_{opt}) = K_{opt}. \quad (33)$$

Proof:

$$\begin{aligned} & CA^*BP_{opt}(M(A \oplus L_{opt}C)^*BP_{opt})^*v \preceq G_{ref}v, & \forall v \\ \Leftrightarrow & CA^*BP_{opt}(M(A \oplus L_{opt}C)^*BP_{opt})^* \preceq G_{ref}, \\ \Leftrightarrow & P_{opt}(M(A \oplus L_{opt}C)^*BP_{opt})^* \preceq (CA^*B) \setminus G_{ref} = P_{opt}, & \text{see Example 2,} \\ \Leftrightarrow & (M(A \oplus L_{opt}C)^*BP_{opt})^* \preceq P_{opt} \setminus P_{opt}, & \text{see Example 2,} \\ \Leftrightarrow & (M(A \oplus L_{opt}C)^*BP_{opt}) \preceq P_{opt} \setminus P_{opt}, & \text{see Example 3,} \\ \Leftrightarrow & M \preceq P_{opt} \setminus P_{opt} \setminus ((A \oplus L_{opt}C)^*BP_{opt}), & \text{see Example 2.} \end{aligned}$$

By considering Lemma 4, and Proposition 2, the following equality holds $(A \oplus L_{opt}C)^*B = A^*B$. According to Proposition 5, Eq. (33) holds too. ■

Thanks to the Separation Principle, Proposition 7 shows that the controller synthesis and the observer synthesis can be done independently. In another words, first, we can find the greatest observer matrix L_{opt} to ensure the estimated output same as the original output. Second, we can find the greatest state

feedback matrix K_{opt} to ensure the greatest closed-loop transfer relation smaller than the desired transfer matrix G_{ref} . After combining the greatest observer matrix L_{opt} and the state feedback matrix K_{opt} , the observer-based controller is constructed and denoted as $u_{K_{opt}} = P_{opt}(v \oplus K_{opt}\hat{x})$, where $\hat{x} = A\hat{x} \oplus Bu \oplus L_{opt}y$. Next, Proposition 8 compares the performances between the observer-based control law, $u_{K_{opt}} = P_{opt}(v \oplus K_{opt}\hat{x})$, and the output feedback control, $u_{F_{opt}} = P_{opt}(v \oplus F_{opt}y)$.

Proposition 8: The observer-based control law $u_{K_{opt}}$ is greater than the the output feedback control law $u_{F_{opt}}$.

Proof: According to Eq. (24) and Eq. (27), the following equality holds :

$$F_{opt} = K_{opt} \not\leq C.$$

Hence, $F_{opt}C \preceq K_{opt}$ and $F_{opt}C\hat{x} \preceq K_{opt}\hat{x}$. According to Eq. (17), this inequality can be written

$$F_{opt}C((A \oplus L_{opt}C)^*Bu \oplus (A \oplus L_{opt}C)^*L_{opt}CA^*Rw) \preceq K_{opt}((A \oplus L_{opt}C)^*Bu \oplus (A \oplus L_{opt}C)^*L_{opt}CA^*Rw),$$

then, according to Corollary 2, $y = \hat{y}$, so this inequality is equivalent to

$$\begin{aligned} F_{opt}(CA^*Bu \oplus CA^*Rw) &\preceq K_{opt}((A \oplus L_{opt}C)^*Bu \oplus (A \oplus L_{opt}C)^*L_{opt}CA^*Rw), \\ \Leftrightarrow F_{opt}y &\preceq K_{opt}((A \oplus L_{opt}C)^*Bu \oplus (A \oplus L_{opt}C)^*L_{opt}CA^*Rw) = K_{opt}\hat{x}, \end{aligned}$$

and by recalling that the addition and product laws are order preserving, it appears that:

$$u_{F_{opt}} = P_{opt}(v \oplus F_{opt}y) \preceq u_{K_{opt}} = P_{opt}(v \oplus K_{opt}\hat{x}).$$

■

Proposition 8 means that the observer-based controller yields a better performance according to the just-in-time criterion, since the observer-based state-feedback control law $u_{K_{opt}}$ is greater than the output feedback law $u_{F_{opt}}$ while keeping the outputs be the same. For instance, in a manufacturing setting, the observer-based controller would provide a better scheduling by starting the process later than the output feedback control, while ensuring the same output parts finishing time. This scheduling would allow users to load the raw parts later rather than earlier to avoid unnecessary congestions in the manufacturing line.

VIII. APPLICATIONS TO A HIGH THROUGHPUT SCREENING SYSTEM

High throughput screening (HTS) is a standard technology in drug discovery. In HTS systems, the optimal scheduling is required to finish the drug screening in the shortest time, as well as to preserve the consistent time spending on each activity in the screening. If we are interested in the release event time of each activity, then we can model the HTS system as a TEG model (see [4], [5]). The HTS system in this section consists of four activities: activity 1, executed on the resource *Pipettor*, is filling the chemical

compound A into the wells of a microplate, which lasts for 2 time units. Activity 2, executed on the resource *Pipettor* as well, is filling the chemical compound B into the wells of another microplate, which lasts for 3 time units. After 1 unit waiting time for the compound A and 6 units waiting time for the compound B, activity 3 is mixing the compound B into the microplate containing the compound A for 4 time units. The mixed compound AB will be released after activity 3 right away. In activity 4, the remaining compound B will be released after 3 time units.

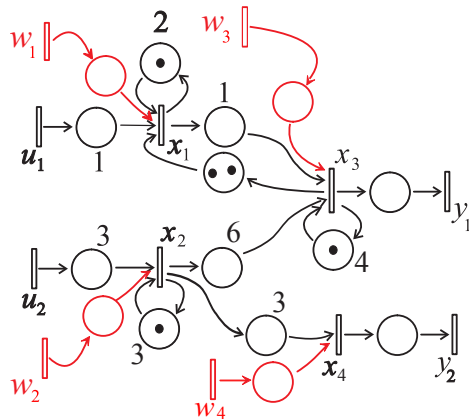


Fig. 3. The TEG model of the HTS system where the uncontrollable inputs are marked in red.

This system can be represented by the TEG model given in Fig. 3, in which x_1 denotes the release time of activity 1 on the *Pipettor* for the compound A, x_2 denotes the release time of activity 2 on the *Pipettor* for the compound B, x_3 denotes the release time of activity 3 after mixing the compounds A and B, and x_4 denotes the release time of activity 4 for the remaining compound B. The inputs u_1 and u_2 are the controls for the loading times of activity 1 and 2, respectively, so that the users can decide when to load the chemical compounds A and B. The compound A is loaded 1 time unit when it is ready. The compound B is loaded after 3 time units when the compound B is ready. The disturbance w_1 delays the release time of the compound A after activity 1, the disturbance w_2 delays the release time of the compound B after activity 2, the disturbance w_3 delays the release time of the mixed compound AB after activity 3, and the disturbance w_4 delays the release time of the remaining compound B after activity 4. The output y_1 is the release time of the mixed compound AB. The output y_2 is the release time of the unused compound B. In Fig. 3, the circles represent places and the bars represent the transitions x_i . Each black token in the places represents that the corresponding resource is available, *i.e.* the activity is ready to start. For the TEG model of a HTS system shown in Fig. 3, the system over the (max-plus)-algebra

$\overline{\mathbb{Z}}_{\max}[\gamma]$ is described as the following:

$$\begin{aligned} x &= Ax \oplus Bu \oplus Rw, \\ y &= Cx, \end{aligned}$$

where

$$A = \begin{bmatrix} 2\gamma & \varepsilon & \gamma^2 & \varepsilon \\ \varepsilon & 3\gamma & \varepsilon & \varepsilon \\ 1 & 6 & 4\gamma & \varepsilon \\ \varepsilon & 3 & \varepsilon & \varepsilon \end{bmatrix}, B = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 3 \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix}, C = \begin{bmatrix} \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix}, R = \begin{bmatrix} e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix}.$$

The example has been computed by using the toolbox `MinMaxGD`, a C++ library allowing to handle periodic series as introduced in ([12]), and it can be noted that this library is also interfaced with `Scicoslab`. In this paper the control objective is to keep the system performance, *i.e.* to obtain a just-in time control while preserving the system's full speed. The reference model transfer function series are

$$G_{ref} = CA^*B = \begin{bmatrix} 2(4\gamma)^* & 9(4\gamma)^* \\ \varepsilon & 6(3\gamma)^* \end{bmatrix}.$$

By Proposition 3 and Proposition 4, we can obtain the optimal output feedback controller as $u_{F_{opt}} = P_{opt}(v \oplus F_{opt}y)$, where

$$\begin{aligned} P_{opt} &= (CA^*B) \setminus G_{ref} = \begin{bmatrix} (4\gamma)^* & 7(4\gamma)^* \\ \varepsilon & (3\gamma)^* \end{bmatrix}, \\ F_{opt} &= P_{opt} \setminus P_{opt} \setminus (CA^*BP_{opt}) = \begin{bmatrix} -2(4\gamma)^* & 1(4\gamma)^* \\ \varepsilon & -6(3\gamma)^* \end{bmatrix}, \end{aligned}$$

which solves the model matching problem. This feedback F_{opt} is not causal because there are negative coefficients in the matrix. The canonical injection from the causal elements of $\overline{\mathbb{Z}}_{\max}[\gamma]$ (denoted $\overline{\mathbb{Z}}_{\max}[\gamma]^+$) in $\overline{\mathbb{Z}}_{\max}[\gamma]$ is also residuated (see [11] for details). Its residual is given by $\Pr(\bigoplus_{k \in \mathbb{Z}} s(k)\gamma^k) = \bigoplus_{k \in \mathbb{Z}} s_+(k)\gamma^k$ where

$$s_+(k) = \begin{cases} s(k) & \text{if } (k, s(k)) \geq (0, 0), \\ \varepsilon & \text{otherwise.} \end{cases}$$

The greatest causal feedback matrix less than or equal to F_{opt} is

$$F_{opt+} = \Pr(F_{opt}) = \begin{bmatrix} 2\gamma(4\gamma)^* & 1(4\gamma)^* \\ \varepsilon & \gamma^2(3\gamma)^* \end{bmatrix}.$$

The output feedback controller $u_{F_{opt+}} = P_{opt}(v \oplus F_{opt+}y)$ can be realized using a TEG model shown in Fig. 4. The pre-filter P_{opt} and the output-feedback control F_{opt+} are marked in gray areas. For instance,

$F_{opt+}(1, 1) = 2\gamma(4\gamma)^*$ implies that, in the TEG shown in Fig. 4, there is a cyclic component with one token and 4 time delays for a new transition ξ_3 and the output y_1 is delayed for 2 time units and one token before going through the transition ξ_3 . $F_{opt+}(1, 2) = 1(4\gamma)^*$ implies that the output y_2 is delayed for 1 time unit before going through the transition ξ_3 . $F_{opt+}(2, 2) = \gamma^2(3\gamma)^*$ implies that there is a cyclic component with one token and 3 time delays for a new transition ξ_4 and the output y_2 has two tokens before going through the transition ξ_4 . The prefilter P_{opt} can be constructed similarly in the TEG model. Then, the output feedback control law $u_{F_{opt+}} = P_{opt}(v \oplus F_{opt+}y)$ can be given in the event domain by considering the (max-plus)-algebra as follows:

$$F_{opt+y} : \begin{cases} \xi_3(k) = 4\xi_3(k-1) \oplus 2y_1(k-1) \oplus 1y_2(k), \\ \xi_4(k) = 3\xi_4(k-1) \oplus y_2(k-2), \end{cases}$$

$$P_{opt}(v \oplus F_{opt+y}) : \begin{cases} \xi_1(k) = 4\xi_1(k-1) \oplus \xi_3(k) \oplus 7\xi_4(k) \oplus v_1(k) \oplus 7v_2(k), \\ \xi_2(k) = 3\xi_2(k-1) \oplus \xi_4(k) \oplus v_2(k), \end{cases}$$

and the controls $u_1(k) = \xi_1(k)$ and $u_2(k) = \xi_2(k)$.

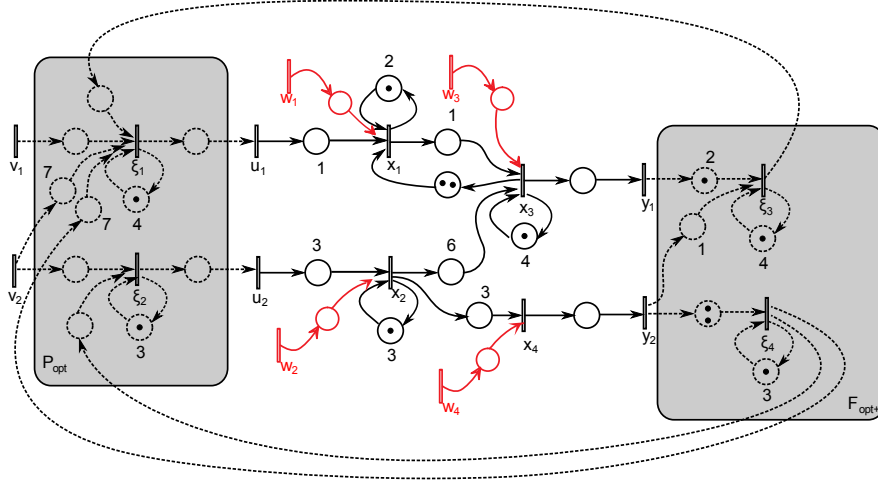


Fig. 4. The TEG realization of the causal output feedback controller $u_{F_{opt+}} = P_{opt}(v \oplus F_{opt+}y)$ for the HTS system.

The output feedback control law $u_{F_{opt+}} = P_{opt}(v \oplus F_{opt+}y)$ can also be given in the time domain by

considering the (min-plus)-algebra as follows:

$$F_{opt+y} : \begin{cases} \xi_3(t) = 1\xi_3(t-4) \oplus 1y_1(t-2) \oplus y_2(t-1), \\ \xi_4(t) = 1\xi_4(t-3) \oplus 2y_2(t), \end{cases}$$

$$P_{opt}(v \oplus F_{opt+y}) : \begin{cases} \xi_1(t) = 1\xi_1(t-4) \oplus \xi_3(t) \oplus \xi_4(t-7) \oplus v_1(t) \oplus v_2(t-7), \\ \xi_2(t) = 1\xi_2(t-3) \oplus \xi_4(t) \oplus v_2(t), \end{cases}$$

and the controls $u_1(t) = \xi_1(t)$ and $u_2(t) = \xi_2(t)$.

Now we construct the observer-based controller $u_{K_{opt}} = P_{opt}(v \oplus K_{opt}\hat{x})$ with $\hat{x} = A\hat{x} \oplus Bu \oplus L_{opt}y$, where P_{opt} is the same as above. According to Lemma 4, Lemma 5 and Eq. (27), L_{opt} and K_{opt} are computed as follows :

$$L_{opt} = L_1 \wedge L_2 = (A^*B)\phi(CA^*B) \wedge (A^*R)\phi(CA^*R) = \begin{bmatrix} \gamma^2(4\gamma)^* & 3\gamma^2(4\gamma)^* \\ \varepsilon & -3(3\gamma)^* \\ (4\gamma)^* & 3(4\gamma)^* \\ \varepsilon & (3\gamma)^* \end{bmatrix},$$

$$K_{opt} = P_{opt} \setminus P_{opt}\phi(A^*BP_{opt}) = \begin{bmatrix} -1(4\gamma)^* & 4(4\gamma)^* & -2(4\gamma)^* & 1(4\gamma)^* \\ \varepsilon & -3(3\gamma)^* & \varepsilon & -6(3\gamma)^* \end{bmatrix}.$$

Then, the causal observer matrix L_{opt+} is

$$L_{opt+} = \text{Pr}(L_{opt}) = \begin{bmatrix} \gamma^2(4\gamma)^* & 3\gamma^2(4\gamma)^* \\ \varepsilon & \gamma(3\gamma)^* \\ (4\gamma)^* & 3(4\gamma)^* \\ \varepsilon & (3\gamma)^* \end{bmatrix},$$

and the causal state-feedback matrix is

$$K_{opt+} = \text{Pr}(K_{opt}) = \begin{bmatrix} 3\gamma(4\gamma)^* & 4(4\gamma)^* & 2\gamma(4\gamma)^* & 1(4\gamma)^* \\ \varepsilon & \gamma(3\gamma)^* & \varepsilon & \gamma^2(3\gamma)^* \end{bmatrix}.$$

The observer-based controller $u_{K_{opt+}} = P_{opt}(v \oplus K_{opt+}\hat{x})$ with $\hat{x} = A\hat{x} \oplus Bu \oplus L_{opt+}y$ can be realized using a TEG model shown in Fig. 5. The pre-filter P_{opt} , the observer mapping L_{opt+} , and the state-feedback control K_{opt+} are marked in gray areas. For instance, $L_{opt+}(1,1) = \gamma^2(4\gamma)^*$ implies that, in the TEG model shown in Fig. 5, there is a cyclic component with one token and 4 time delays for a new transition ξ_3 and the output y_1 has two tokens before going through the transition ξ_3 . $L_{opt+}(1,2) = 3\gamma^2(4\gamma)^*$ implies that the output y_2 is delayed for 3 time units and has two tokens before going through the transition ξ_3 . $L_{opt+}(2,2) = \gamma(3\gamma)^*$ implies that there is a cyclic component with one token and 3 time delays for a new transition ξ_4 and the output y_2 has one token before going through the transition

ξ_4 . $L_{opt+}(3,1) = (4\gamma)^*$ implies that there is a cyclic component with one token and 4 time delays for a new transition ξ_5 . $L_{opt+}(3,2) = 3(4\gamma)^*$ implies that y_2 is delayed for 3 time units before going through the transition ξ_5 . $L_{opt+}(4,2) = (4\gamma)^*$ implies that y_2 implies that there is a cyclic component with one token and 3 time delays for a new transition ξ_6 . The observer-based state feedback matrix K_{opt+} and the prefilter P_{opt} can be explained similarly as above.

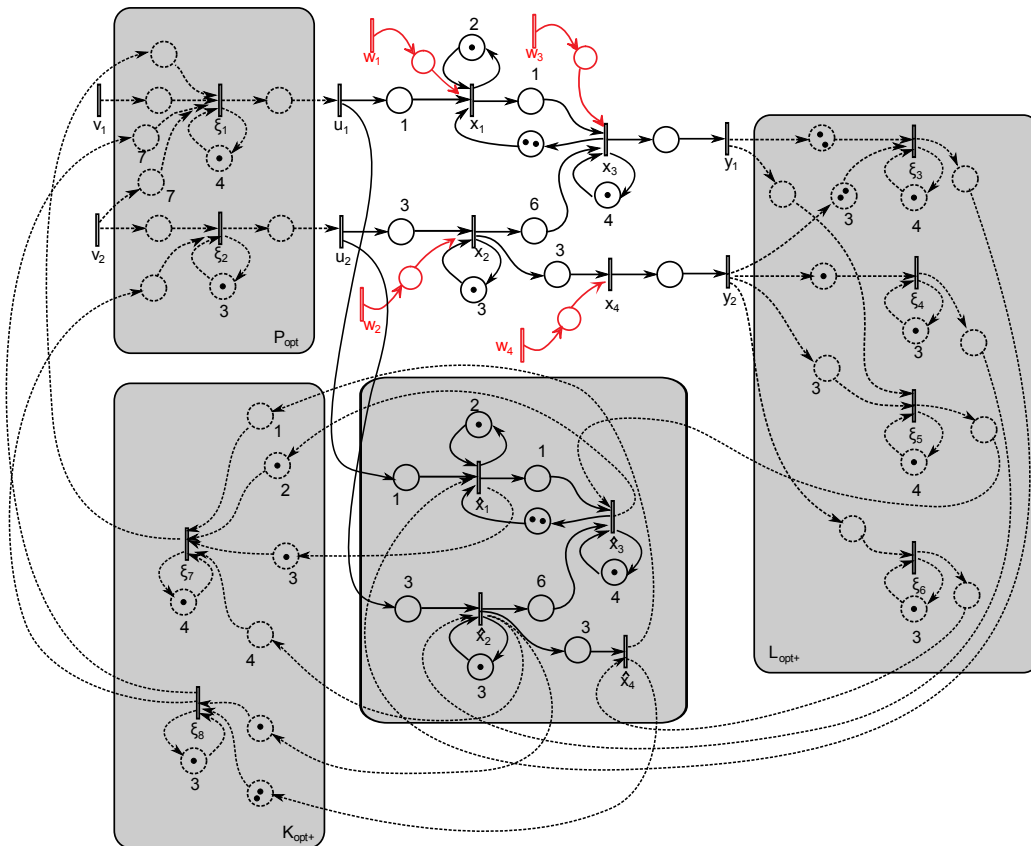


Fig. 5. The TEG realization of the observer-based controller $u_{K_{opt+}} = P_{opt}(v \oplus K_{opt+}\hat{x})$ for the HTS system.

The estimated states $\hat{x} = A\hat{x} \oplus Bu \oplus L_{opt+}y$ can be written in the event domain by considering the

(max-plus)-algebra as follows:

$$L_{opt+y} : \begin{cases} \xi_3(k) = 4\xi_3(k-1) \oplus y_1(k-2) \oplus 3y_2(k-2), \\ \xi_4(k) = 3\xi_4(k-1) \oplus y_2(k-1), \\ \xi_5(k) = 4\xi_5(k-1) \oplus y_1(k) \oplus 3y_2(k), \\ \xi_6(k) = 3\xi_6(k-1) \oplus y_2(k), \end{cases}$$

$$\hat{x} : \begin{cases} \hat{x}_1(k) = 2\hat{x}_1(k-1) \oplus \hat{x}_3(k-2) \oplus 1u_1(k) \oplus \xi_3(k), \\ \hat{x}_2(k) = 3\hat{x}_2(k-1) \oplus 3u_2(k) \oplus \xi_4(k), \\ \hat{x}_3(k) = 1\hat{x}_1(k) \oplus 6\hat{x}_2(k) \oplus 4\hat{x}_3(k-1) \oplus \xi_5(k), \\ \hat{x}_4(k) = 3\hat{x}_2(k) \oplus \xi_6(k). \end{cases}$$

where ξ_i , $i = 3, \dots, 6$, are the intermediate transitions in the TEG shown in Fig. 5. Then the event domain representation for the observer-based control law $u_{K_{opt+\hat{x}}} = P_{opt}(v \oplus K_{opt+\hat{x}})$ is obtained as follows:

$$K_{opt+\hat{x}} : \begin{cases} \xi_7(k) = 4\xi_7(k-1) \oplus 3\hat{x}_1(k-1) \oplus 4\hat{x}_2(k) \oplus 3\hat{x}_3(k-1) \oplus 1\hat{x}_4(k), \\ \xi_8(k) = 3\xi_8(k-1) \oplus \hat{x}_2(k-1) \oplus \hat{x}_4(k-2), \end{cases}$$

$$P_{opt}(v \oplus K_{opt+\hat{x}}) : \begin{cases} \xi_1(k) = 4\xi_1(k-1) \oplus \xi_7(k) \oplus 7\xi_8(k) \oplus v_1(k) \oplus 7v_2(k), \\ \xi_2(k) = 3\xi_2(k-1) \oplus \xi_8(k) \oplus v_2(k), \end{cases}$$

and $u_1(k) = \xi_1(k)$ and $u_2(k) = \xi_2(k)$, where ξ_i , $i = 1, 2, 7, 8$, are the intermediate transitions in the TEG shown in Fig. 5. Similarly, the estimated state \hat{x} can be written in time-domain equations by considering the (min-plus)-algebra.

$$L_{opt+y} : \begin{cases} \xi_3(t) = 1\xi_3(t-4) \oplus 2y_1(t) \oplus 2y_2(t-3), \\ \xi_4(t) = 1\xi_4(t-3) \oplus 1y_2(t), \\ \xi_5(t) = 1\xi_5(t-4) \oplus y_1(t) \oplus y_2(t-3), \\ \xi_6(t) = 1\xi_6(t-3) \oplus y_2(t), \end{cases}$$

$$\hat{x} : \begin{cases} \hat{x}_1(t) = 1\hat{x}_1(t-2) \oplus 2\hat{x}_3(t) \oplus u_1(t-1) \oplus \xi_3(t), \\ \hat{x}_2(t) = 1\hat{x}_2(t-3) \oplus u_2(t-3) \oplus \xi_4(t), \\ \hat{x}_3(t) = 1\hat{x}_3(t-4) \oplus \hat{x}_1(t-1) \oplus \hat{x}_2(t-6) \oplus \xi_5(t), \\ \hat{x}_4(t) = \hat{x}_2(t-3) \oplus \xi_6(t). \end{cases}$$

Then, the event domain representation for the observer-based control law $u_{K_{opt+}} = P_{opt}(v \oplus K_{opt+}\hat{x})$ is obtained as follows:

$$K_{opt+}\hat{x} : \begin{cases} \xi_7(t) = 1\xi_7(t-4) \oplus 1\hat{x}_1(t-3) \oplus \hat{x}_2(t-4) \oplus 1\hat{x}_3(t-2) \oplus \hat{x}_4(t-1), \\ \xi_8(t) = 1\xi_8(t-3) \oplus 1\hat{x}_2(t) \oplus 2\hat{x}_4(t) \end{cases}$$

$$P_{opt}(v \oplus K_{opt+}\hat{x}) : \begin{cases} \xi_1(t) = 1\xi_1(t-4) \oplus \xi_7(t) \oplus \xi_8(t-7) \oplus v_1(t) \oplus v_2(t-7), \\ \xi_2(t) = 1\xi_2(t-3) \oplus \xi_8(t) \oplus v_2(t), \end{cases}$$

and $u_1(t) = \xi_1(t)$, and $u_2(t) = \xi_2(t)$. By Proposition 8, the observer-based control law $u_{K_{opt+}} = P_{opt}(v \oplus K_{opt+}\hat{x})$ is greater than the output feedback control law $u_{F_{opt+}} = P_{opt}(v \oplus F_{opt+}y)$, for any external input v . This result can be verified in this example, because

$$F_{opt+}CA^*B \preceq K_{opt+}(A \oplus L_{opt+}C)^*B \text{ and } F_{opt+}CA^*R = K_{opt+}(A \oplus L_{opt+}C)^*L_{opt+}CA^*R$$

hold, where

$$F_{opt+}CA^*B = \begin{bmatrix} 4\gamma(4\gamma)^* & 7(4\gamma)^* \\ \varepsilon & 6\gamma^2(3\gamma)^* \end{bmatrix}, \quad K_{opt+}(A \oplus L_{opt+}C)^*B = \begin{bmatrix} (4\gamma)^* & 7(4\gamma)^* \\ \varepsilon & 3\gamma(3\gamma)^* \end{bmatrix}$$

$$F_{opt+}CA^*R = K_{opt+}(A \oplus L_{opt+}C)^*L_{opt+}CA^*R = \begin{bmatrix} 3\gamma(4\gamma)^* & 4(4\gamma)^* & 2\gamma(4\gamma)^* & \varepsilon \\ \varepsilon & 3\gamma^2(3\gamma)^* & \varepsilon & \varepsilon \end{bmatrix}.$$

IX. CONCLUSIONS

The main contribution of this paper is the design of an observer-based controller for max-plus linear systems, where only a subset of the states obtained from measurement is available for the controller. These results can be applied to fault detection, model matching, and diagnosis for max-plus linear systems. This paper first constructs the observer structure for max-plus linear systems, and then finds the greatest observer matrix such that the estimated output preserves the original output behaviors. Second, this paper calculates the greatest output feedback and state-feedback control laws such that the closed-loop transfer relation is smaller than the reference transfer relation in a model matching problem. Then, an observer-based controller is constructed using the estimated state in the TEG model of max-plus linear systems. Moreover, it is proved that the observer-based controller provides a greater control than the output feedback control, i.e. a better performance in terms of just-in-time control criterion. At last, this paper applies the observer-based controller and the output feedback synthesis to a practical application of a HTS system in drug discovery. Both of the observer-based controller and the output feedback controller are constructed in TEG models, and the observer-based controller yields a better scheduling strategy. The scheduling obtained from the observer-based controller would allow users to load the chemical compounds at late as possible to avoid unnecessary congestions according to the just-in-time criterion.

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X. APPENDIX

A. Formulas of Star Operations

$a^*(ba^*)^* = (a \oplus b)^* = (a^*b)^*a^*$	(f.1)
$(a^*)^* = a^*$	(f.2)
$(ab)^*a = a(ba)^*$	(f.3)
$a^*a^* = a^*$	(f.4)
$(a^*)^+ = (a^+)^* = a^*$	(f.5)

B. Formulas of Left Residuations

$a(a \wp x) \preceq x$	(f.6)
$a \wp (ax) \succeq x$	(f.7)
$a(a \wp (ax)) = ax$	(f.8)
$a \wp (x \wedge y) = a \wp x \wedge a \wp y$	(f.9)
$(a \oplus b) \wp x = a \wp x \wedge b \wp x$	(f.10)
$(ab) \wp x = b \wp (a \wp x)$	(f.11)
$b(a \wp x) \preceq (a \wp b) \wp x$	(f.12)
$(a \wp x)b \preceq a \wp (xb)$	(f.13)

C. Formulas of Right Residuations

$(x \wp a)a \preceq x$	(f.14)
$(xa) \wp a \succeq x$	(f.15)
$((xa) \wp a)a = xa$	(f.16)
$(x \wedge y) \wp a = x \wp a \wedge y \wp a$	(f.17)
$x \wp (a \oplus b) = x \wp a \wedge x \wp b$	(f.18)
$x \wp (ba) = (x \wp a) \wp b$	(f.19)
$(x \wp a)b \preceq x \wp (b \wp a)$	(f.20)
$b(x \wp a) \preceq (bx) \wp a$	(f.21)

XI. SOURCE CODE

```
#ifndef _WIN32
#include "../include/lminmaxgd.h"
#else
```

```

#include "..\include\lminmaxgd.h"
using namespace std;
#endif
/////////////////////////////////////////////////////////////////
/* Example : Observer based controller and comparison with output feedback
Example taken from paper
[2] "Observer-based Controllers for (Max-Plus)-Linear Systems"
Laurent Hardouin, Ying Shang, Carlos Andrey Maia, Bertrand Cottenceau
(IEEE TAC, 2010)
// see also animation on http://perso-laris.univ-angers.fr/~hardouin/Observer.html
// you must link with a library which contains all source files given in folder src
// gd.cpp poly.cpp serie.cpp smatrix.cpp tools.cpp interf.cpp
// or add these files in your project
// or alternatively you can include the following files, it is the most friendly to
*/
#include "..\src\gd.cpp"
#include "..\src\poly.cpp"
#include "..\src\serie.cpp"
#include "..\src\smatrix.cpp"
#include "..\src\tools.cpp"

/////////////////////////////////////////////////////////////////

int main()
{
try
{

    smatrix A(4,4); // matrice d'tat

    smatrix Lopt,L1,L2,Loptcausal;

    int i,j;

```

```
smatrix CA,AB,CAB,AR,CAR;
```

```
smatrix Gref;
```

```
smatrix Popt, Fopt,Kopt;
```

```
A(2,2)=gd(1,4); // (g^1,d^4)
```

```
A(2,1)=gd(0,6); //
```

```
A(2,0)=gd(0,1);
```

```
A(0,0)=gd(1,2);
```

```
A(0,2)=gd(2,0);
```

```
A(1,1)=gd(1,3);
```

```
A(3,1)=gd(0,3);
```

```
smatrix C(2,4); // output matrix only coefficient on diagonal equal to e or ep
```

```
C(0,2)=gd(0,0);
```

```
C(1,3)=gd(0,0);
```

```
smatrix B(4,2); // matrix of inputs, epsilon everywhere
```

```
B(0,0)=gd(0,1);
```

```
B(1,1)=gd(0,3);
```

```
smatrix R(4,4); // matrix connecting the uncontrollable input and the state
```

```
R(0,0)=gd(0,0);
```

```
R(1,1)=gd(0,0);
```

```
R(2,2)=gd(0,0);
```

```
smatrix As=star(A);
```

```
CA=otimes(C,As); // CA*
```

```
AB=otimes(As,B);
```

```
CAB=otimes(C,AB); // CA*B, the input output transfer matrix
```

```

cout<<" CAB"<<CAB<<endl;
AR=otimes (As, R);
CAR=otimes (C, AR); // CA*R, the disturbance output transfer matrix

cout<<" CAR"<<CAR<<endl;

L1=rfrac (AB, CAB);
L2=rfrac (AR, CAR);
Lopt=inf (L1, L2); // the uncausal observer

cout<<"Lopt noncausal : " <<Lopt<<endl;
Loptcausal=prcaus (Lopt); // the causal projection of the oberver matrix
Lopt=Loptcausal;

cout<<"Lopt causal : " <<Loptcausal<<endl;

// Below we check the Corollary 2

smatrix Yu;
Yu=otimes (Loptcausal, C);
Yu=oplus (A, Yu);
Yu=star (Yu);
Yu=otimes (C, Yu);
Yu=otimes (Yu, B);

smatrix Yw;
Yw=otimes (Loptcausal, C);
Yw=oplus (A, Yw);
Yw=star (Yw);

```

```

Yw=otimes(C, Yw);
Yw=otimes(Yw, Loptcausal);
Yw=otimes(Yw, CAR);

if (Yu==Yw)
{

cout<<" Corollary 2 is satisfied, the estimated output and the real output are equal
cout<<"Yu : " <<Yu<<endl;
cout<<"Yw : " <<Yw<<endl;
}

Gref=CAB; // reference model is chosen equal to the system, we want to keep the in
cout<<" Gref " <<Gref<<endl;

Popt=lfrac(Gref, CAB); // the prefilter

cout<<" Popt non causal " <<Popt<<endl;

smatrix Popt_causal=prcaus(Popt);

cout<<" Popt causal " <<Popt_causal<<endl;

Kopt=otimes(AB, Popt);
Kopt=rfrac(Popt, Kopt);
Kopt=lfrac(Kopt, Popt);

cout<<" Kopt non causal" <<Kopt<<endl; // the state feedback controller

Fopt=rfrac(Kopt, C); // the output feedback controller

cout<<" Fopt non causal " <<Fopt<<endl;

```

```

// causal projection
Kopt=prcaus(Kopt);

Fopt=prcaus(Fopt);

cout<<" Fopt " <<Fopt<<endl;

cout<<" Kopt " <<Kopt<<endl;

// From now I try to enlighten that the strategy with oberverbasedcontrol is better
// the feedback, I need your help Ying... :)
smatrix FoptC;
FoptC=otimes(Fopt,C);

// Below it appears that the observer based controller yields a greater control
// See Proposition 8
smatrix Xu,Xw;

Xu=otimes(Lopt,C);
Xu=oplus(A,Xu);
Xu=star(Xu);
Xu=otimes(Xu,B);
Xu=otimes(Kopt,Xu); // the transfer Kopt(A+LoptC)*B

Xw=otimes(Lopt,C);
Xw=oplus(A,Xw);
Xw=star(Xw);
Xw=otimes(Xw,Lopt);
Xw=otimes(Xw,CAR);
Xw=otimes(Kopt,Xw); // the transfer Kopt((A+LoptC)*(LoptCA)*R

```



```

smatrix FCAB, FCAR;

FCAB=otimes(Fopt,CAB);
FCAR=otimes(Fopt,CAR);

if(!(FCAB==Xu) || !(FCAR==Xw))
{
cout<<" FCAB " <<FCAB<<endl;
cout<<" Xu " <<Xu<<endl;
cout<<" FCAR " <<FCAR<<endl;
cout<<" Xw " <<Xw<<endl;
smatrix test1=oplus(FCAB,Xu);
smatrix test2=oplus(FCAR,Xw);
if ((test1==Xu) && (test2==Xw))
{
    cout<<" the oberver-based controller yields a greater control than the out.
}

}

return(0);
}

catch(mem_limite l)
{
    cout<<"Exception : too many coefficents in polynom " <<l.memoire<<endl;
    return(1);
}

catch(taille_incorrecte obj)
{ // 0 : r non causal
    // 1 : tentative d'accs un element d'une matrice avec un indice incorrect

```

```
// 2 : matrice de taille incompatible pour oplus, inf, otimes, rfrac, lfrac
// 3 : etoile de matrice carre uniquement
    cout<<"Exception  "<<obj.erreur<<endl;
    return(1);
}
}
```