On Max-Plus Linear Dynamical System Theory: The Regulation Problem ⋆

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Abstract

A class of timed discrete event systems may be modeled by using Timed-Event Graphs, a class of timed Petri nets that can have its firing dynamic described by using an algebra called "Max-plus algebra". For this kind of systems it may be desirable to enforce some timing constraints in steady state. This is what in this paper we call a "max-plus regulation problem". In this context we show a necessary condition for solving these regulations problems and in addition that this condition is sufficient for a large class of problems. The obtained controller is a simple linear static state feedback and can be computed using efficient pseudo-polynomial algorithms. Simulation results will illustrate the applicability of the proposed methodology.

Key words: Max-Plus algebra; Tropical algebra; Timed-Event Graphs; Geometric Control;

1 Introduction

Timed Event Graphs is an appropriate formalism for modeling some timed discrete event systems, see for instance Heidergott et al. (2006); Atto et al. (2011); Attia et al. (2010); Brackley et al. (2011); Amari et al. (2004); Kim and Lee (2015); Majdzik et al. (2014). These kinds of systems have their dynamics described by linear state-space models in Max-plus Algebra Baccelli et al. (1992). In some situations it may be desirable that a certain set of constraints in the state space holds. This could be done by using the state variables to design a control law, in analogy with classical control theory.

In the past decade, several papers were published in the problem of synthesizing controllers for this problem when the constraints can be written as max-plus linear equations in the space state (Amari et al. (2005, 2012); Atto et al. (2011); Katz (2007); Maia et al. (2011b,a); Gonçalves et al. (2012, 2015); Brunsch et al. (2012, 2010)). See the introduction in Gonçalves et al. (2015) for an in-depth review. In this sense, we highlight the work of Katz (2007) that treated the problem under the light of geometrical control theory, providing sufficient condition to solve a class of problems. This work was a major inspiration for the developments in our previous work, Gonçalves et al. (2015), and by consequence this one.

Although there is growing interest in the subject, an important feature of controllers was not discussed explicitly until recently: robustness. Indeed, many previous works require that the initial condition lies in a special set inside the desired specification in order to guarantee that the state will remain on it. But they did not addressed, at least not explicitly, if it is possible to drive the system from an arbitrary initial condition to the desired specification and then keep it inside this set. This important because it is closely related to another problem: what would happen if a perturbation - say a machine delays its production - inflicts the system? Would the controller be able to reject this perturbation and return to the desired specification? In other words, we ask for results for the steady-state version of the control problem. As far as the authors knowledge go, the two only papers that made this discussion explicitly was Kim and Lee (2015) and our previous work, Gonçalves et al. (2015). However, the former only deals with a specific kind of system and specification. Our previous work deals with a general system and a general specification, and we believe that in the mentioned paper we were the first to define

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and give sufficient conditions to solve this steady-state version of the control problem.

2 Contributions

This paper builds on our previous work Gonçalves et al. (2015). In that work, it was presented a steady-state version of a max-plus control problem, and two algorithms were derived to solve it in an open-loop strategy. In this paper, we improve the results on one of these algorithms, the *periodic synchronizer*, although the problem that we deal with here is not exactly the same as the one in that paper. In that paper, a larger class of constraints is considered, while here we consider a special class of these constraints - semimodules - which are very common in practice. Indeed, all constraints found in the related papers (Amari et al. (2005, 2012); Atto et al. (2011); Katz (2007); Maia et al. (2011b,a); Gonçalves et al. (2012, 2015)) can be rewritten in order to fall into this category. These problems will be denoted hereafter by *max-plus regulation problems.*

The major contribution is that we show that sufficient conditions derived in Gonçalves et al. (2015) are also *necessary* for solving all max-plus regulation problems under some weak technical assumptions. Additionally, we shown that the condition also provides a way to solve the problem in *closed-loop* for a wide class of problems. In our previous work the controller acts in open loop and depends on a scalar parameter h. This scalar parameter must be chosen in function of the initial condition and has influence in the upper bound of the number of steps to achieve convergence. Moreover, when the controller firing rate equals to the spectral radius, the open loop controller may fail unless the parameter h changes dynamically. The closed-loop approach eliminates all these problems: no longer the upper bound in the number of events to converge depends on the initial condition, only in the number of states, and the (closed-loop) approach should work even when the closed-loop radius is equal to the open-loop one, without the necessity of changing any parameter. Indeed, the only parameter is a matrix F, since the control law is a simple static max-plus linear state feedback of the form $u[k] = Fx[k]$ for a constant matrix F.

In order to characterize the class of problems for which the derived condition is necessary and sufficient, the concept of *criticality* is also introduced in this paper. This is related to the *spectrum* of the problem, another concept introduced in this paper. The spectrum is the set of all steady-state possible firing rates under control. In a nutshell, the problem is noncritical - and thus *easy* to solve- if the closed-loop controller that solves the problem is able to delay, even if a little bit, the system in comparison with its open loop behavior. On the other hand, if the problem is critical it may or may not be solved by our methodology. We discuss this topic as well in this paper. Thus, we believe that this paper presents a contribution towards a "final solution" for the regulation problem, that is, a necessary and sufficient condition *for all* problems.

3 Basic Definitions

A *Timed-Event Graph* is a subclass of timed Petri nets in which all places have one input and one output transitions. *Max-Plus algebra* is the dioid (idempotent semiring)

$$\mathbb{Z}_{max} = (\mathbb{Z} \cup \{-\infty\}, \oplus, \otimes)$$

in which $\oplus$ is the maximum and $\otimes$ is the traditional sum. More recently, it has been also called *Tropical Algebra.* The symbol $\otimes$ will be frequently omitted and so it will be interpreted by juxtaposition, just like the traditional product in the traditional algebra. So ab reads as $a \otimes b = a + b$. We denote the element $-\infty$ by the symbol $\perp$, and it will also be occasionally denoted by the "null" element. There is also a matricial analogue of this algebra, and so for two matrices A, B of appropriate dimension $A \oplus B$ and $A \otimes B$ will be interpreted as the matricial sum and product with $+$ being replaced by $\oplus$ and $\times$ by $\otimes$. An element in this algebra that has n rows and m columns will be denoted by $\mathbb{Z}^{n\times m}_{max}$, while an element with one row and m columns $\mathbb{Z}^{m}_{max}$. All vectors are column vectors. The symbol A^T denotes the transpose of the matrix A. A vector or matrix of appropriate dimension composed only of $\perp$ will also be denoted by $\perp$. The symbol I will denote the max-plus identity matrix of an appropriate order, that is, a matrix in which the diagonal element is 0 and $\perp$ otherwise. For a natural number k, the k^{th} matrix power A^k will be defined recursively as $A^{k+1} = A^k A$ and $A^0 = I$. If λ is a scalar not $\perp$ then $\lambda^{-1} = -\lambda$.

The *Kleene closure* of a square matrix A is equal to $\bigoplus_{i=0}^{\infty} A^i$. The *spectral radius* of this matrix, $\rho(A)$, is the greatest scalar λ for which there exists a vector $v \neq \perp$ in which $Av = \lambda v$. Generally, even though the entries of the matrix A lie in $\mathbb{Z}$ or are $\perp$, the spectral radius can be a rational number. However, since the units of the problem can be redefined, the entries of the matrix - and thus the spectral radius- can be re-scaled so the spectral radius is either an integer or $\perp$. Thus hereafter we can assume without loss of generality that $\rho(A) \in \mathbb{Z}_{max}$.

A *semimodule*, over a given dioid, is an analogous of vector spaces over semirings, that is, a set of elements x together with a scaling $(\lambda, x) \mapsto \lambda x$ and sum $(x, y) \mapsto x \oplus y$ operations which preserve some properties in the context of this given dioid. See Katz (2007) for the formal definition. Finally, *Im* M, the *image* of M, is the semimodule generated by the max-plus column span of the matrix M, that is, if $M \in \mathbb{Z}^{n\times m}_{max}$ then *Im* $M = \{Mv \mid v \in \mathbb{Z}^{m}_{max}\}$.

4 Regulation Problem

4.1 Problem statement

Consider a max-plus linear event-invariant dynamical system

$$x[k+1] = Ax[k] \oplus Bu[k]\ ,\ k \in \mathbb{N};$$
$$x[0] = x_0 \tag{1}$$

for $x[k] \in \mathbb{Z}^n_{max}$, $u[k] \in \mathbb{Z}^m_{max}$, $A \in \mathbb{Z}^{n\times n}_{max}$ and $B \in \mathbb{Z}^{n\times m}_{max}$. It is *max-plus* linear because its equations can be written in a linear way using the max-plus operators $\oplus$ and $\otimes$, the latter omitted by juxtaposition. It is *event-invariant* because the matrices A and B do not depend on the event k.

It will be assumed without loss of generality that B has no null column (a column full of $\perp$ entries). Otherwise, the corresponding control actions would play no role in the system and can be removed.

The *regulation problem*, henceforth denoted by $\mathcal{R}(A,B,E,D)$ (or $\mathcal{R}$ when it is not convenient to explicit the matrices), can be defined as follows: find a map $f : \mathbb{Z}^n_{max} \times \mathbb{N} \mapsto \mathbb{Z}^m_{max}$ such that if $u[k] = f(x[k],k)$ is taken in (1), then there exists a $p \in \mathbb{N}$ such that *for all* initial conditions $x[0]$ we have that *for all* $k \geq p$

$$Ex[k] = Dx[k].$$

The set of $x \in \mathbb{Z}^n_{max}$ such that $Ex = Dx$ will be denoted by $\mathcal{S}_{ref}(\mathcal{R})$, the *specification (reference) set*, and is clearly a semimodule. In other words, it is desired to design a (possibly event-varying) state feedback law that leads the dynamical system to a specification set in a finite number of events and keeps it there thereafter, in steady-state, whichever is the initial condition.

Note that, according to the formulation of the problem, it is possible to impose constraints only in *steady state.* If it is strictly necessary that the constraints must hold for *all* $k \geq 0$ then this technique cannot be employed. Refer to Katz (2007) for techniques in this case. Moreover, in order to the constraints to hold for all $k \geq 0$, it is necessary to limit the set of possible initial conditions x_0, since $x[0] = x_0$ must satisfy the constraint.

Note also that the fact that the specification must be reached *for any* initial condition x_0 together with the fact that the system in (1) is event-invariant implies a very interesting *robustness* property. If the system converges to the desired specification $\mathcal{S}_{ref}(\mathcal{R})$ and is driven out of it due to a perturbation, then, after a finite number of steps, it will return to it, thus effectively “rejecting” the perturbation. This happens because we can consider a new system in which the initial condition is the perturbed state, the dynamics are the same as the old one and the event k resets to 0. Since convergence happens for every initial condition and the actual state k is immaterial in the parameters of the dynamical system - it is event invariant after all- convergence is also guaranteed for this new system and then eventually the rejection of the perturbation is achieved. This characteristic is highly desirable in practice.

A problem of this form will be denoted by a *max-plus regulation problem*, since it is desirable to make the state converge to a specific set $\mathcal{S}_{ref}(\mathcal{R})$ and keep it “regulated” in it, that is, rejecting eventual perturbations.

A brief overview of the main result of this paper is given in the following:

Main result overview: For a very wide class of regulation problems $\mathcal{R}(A,B,E,D)$, it can be solved *if and only if* a specific equation, generated with the parameters A,B,E,D of the problem, has a solution. Furthermore, the control law is a simple static feedback of the form $u[k] = Fx[k]$, in which the matrix F is event and state independent. Finally, if there is n states, convergence happens in at most $n+1$ events. $\square$

The formal statement of this necessary and sufficient condition will be given in Theorem 1.

4.2 Geometrical invariance

A key concept for deriving the main result of this paper is the one of (A,B) max-plus geometrical invariance (henceforth denoted by (A,B)-MPGI), see Katz (2007). A set $\mathcal{X} \subseteq \mathbb{Z}^n_{max}$ is said to be (A,B) - MPGI if for any $x \in \mathcal{X}$ there exists an $u \in \mathbb{Z}^m_{max}$ such that $Ax \oplus Bu \in \mathcal{X}$. In words, a semimodule $\mathcal{X}$ is (A,B) - MPGI if it is possible to evolve the system according to the dynamics in (1) such that the system is always inside $\mathcal{X}$.

The first result of this paper can then be stated.

Lemma 1 *: If $\mathcal{R}(A,B,E,D)$ has a solution, then there exists an (A,B)-MPGI set $\mathcal{X}$. Furthermore, this set is inside the specification set $\mathcal{S}_{ref}(\mathcal{R})$.*

Proof: If $\mathcal{R}(A,B,E,D)$ has a solution, then there must exists a natural p such that for all $k \geq p$ it holds that $x[k] \in \mathcal{S}_{ref}(\mathcal{R})$, in which the $x[k]$ are generated according to the dynamics in (1). Thus the set

$$\mathcal{X} = \{x[k] \mid k \geq p\}$$

is an (A,B)-MPGI set inside $\mathcal{S}_{ref}(\mathcal{R})$. Indeed, for any member x' of this set, which has the form $x' = x[k]$, there exists an u', namely $u[k]$, such that $Ax' \oplus Bu' \in \mathcal{X}$, since $Ax[k] \oplus Bu[k] = x[k+1] \in \mathcal{X}$. $\square$

In this paper we are particularly interested in the (A, B)-MPGI sets that are inside the desired specification $\mathcal{S}_{ref}$. Using the definition of (A, B)-MPGI, if two sets $\mathcal{X}_1$ and $\mathcal{X}_2$ are (A, B)-MPGI, so is their union. Furthermore, if $\mathcal{X}_1 \subseteq \mathcal{S}_{ref}$ and $\mathcal{X}_2 \subseteq \mathcal{S}_{ref}$, then $\mathcal{X}_1 \cup \mathcal{X}_2 \subseteq \mathcal{S}_{ref}$. Consequently, the following result holds:

Result 1 *: (see Katz (2007)) Given a problem $\mathcal{R}$, if it has a solution then there exists the maximal (A, B)-MPGI set inside $\mathcal{S}_{ref}(\mathcal{R})$. It will be denoted henceforth by $\mathcal{K}^{\top}_{geo}(\mathcal{R})$.* $\square$

Obviously, the singleton [1] $\{\perp\}$ is an $(A, B) - MPGI$ set because we can take $u = \perp$ and then $Ax \oplus Bu = \perp \in \{\perp\}$. Therefore, this singleton is always a subset of $\mathcal{K}^{\top}_{geo}(\mathcal{R})$. The following concept will also play a crucial role in this text.

Definition 1 *: (Controllable coupled property) A problem $\mathcal{R}$ is said to be* controllable coupled *if there exists a finite natural number M such that for any vector $x \neq \perp$ inside $\mathcal{S}_{ref}(\mathcal{R})$ it holds that $|x_i - x_j|$ is bounded by W.* $\square$

This key concept was introduced in a previous work of the authors, Gonçalves et al. (2015), with the name of "coupled property" and with a slightly different, but equivalent, mathematical formulation. In this work and henceforth, it will be denoted by "controllable coupled property", since in future works other concepts of "coupled", related to the dual concept of "observability", will be introduced.

As argued in Gonçalves et al. (2015), the assumption that a problem is coupled can be taken without loss of generality. Practically, we are only interested in coupled problems. If the problem is not coupled, this implies that there is a trajectory $x[k]$ of the system such that for a particular k the difference $|x_i[k] - x_j[k]|$ is not bounded. Since the vector x represents timings, this only makes sense if $k \to \infty$, that is, $\lim_{k\to\infty} |x_i[k] - x_j[k]|$ is unbounded for two indexes i and j. This implies that the two transitions, i and j, operate in different rates in steady state. This means that no interesting synchronization in steady state, induced by the specification set, was imposed between these transitions. If this is the case, either the problem is ill-posed or can be separated in two or more independent subproblems in which then the property holds.

Checking whether the problem is coupled or not can be very onerous, because computing $\mathcal{S}_{ref}(\mathcal{R})$ can be very difficult (see Katz (2007)). An easy-to-verify sufficient condition can be derived, though. If the matrices E and D take the special form in which $E = I$, then the problem is controllable coupled if D has no $\perp$ entries. Indeed, it is very often the case that the constraints can be written as $x \geq Qx$, which can be written equivalently as $x = Q^{*}x$. In this case, it suffices that Q^{*} has no $\perp$ entries. As argued in Katz (2007), this can be achieved without loss of generality by setting loose constraints $x_i - x_j \leq W$ for a number W sufficiently large.

The following lemma can then be established.

Lemma 2 *: If $\mathcal{R}$ is controllable coupled, then $\mathcal{K}^{\top}_{geo}(\mathcal{R})$ is a finitely generated semimodule, that is, there exists a matrix K with a finite number of columns $\mathcal{K}^{\top}_{geo}(\mathcal{R}) = Im\ K$.*

Proof: Since $\mathcal{R}$ is coupled for any non-null vector x inside it we have $|x_i - x_j| \leq W$. As argued in Katz (2007), since the vectors x have integer entries there exists at most $(2W+1)^n$ - the operators being interpreted in the traditional algebra - different vectors which are not max-plus scalar multiples of each other. Therefore, if K is the matrix in which the columns are all these vectors then this matrix has a finite number of columns and generate $\mathcal{K}^{\top}_{geo}(\mathcal{R})$, that is, $\mathcal{K}^{\top}_{geo}(\mathcal{R}) \subseteq Im\ K$.

Now, any set formed by max-plus linear combinations of vectors in an (A, B)-MPGI set is also (A, B)-MPGI set. Furthemore, this set formed by linear combinations is also inside $\mathcal{S}_{ref}(\mathcal{R})$, because the specification is a semimodule. Therefore, $Im\ K$ is also (A, B)-MPGI set inside $\mathcal{S}_{ref}(\mathcal{R})$. Finally, since $\mathcal{K}^{\top}_{geo}(\mathcal{R})$ is, by definition, the maximal (A, B)-MPGI set inside $\mathcal{S}_{ref}(\mathcal{R})$ we have $Im\ K \subseteq \mathcal{K}^{\top}_{geo}(\mathcal{R})$. And the lemma is established. $\square$.

Note that the number $(2B+1)^n$ can be *very* large, which can be critical for the methodology proposed in Katz (2007) since the number of steps taken for its algorithm to converge is bounded by the so-called *volume*, which in turn is bounded by this value. However, in our case, it is necessary to guarantee that this number is finite only for theoretical reasons, since the magnitude of this number has no impact in the proposed methodology.

Finally, we recall the following result:

Result 2 *: (see Katz (2007)) If $\mathcal{K} = Im\ K$ is a finitely generated (A, B)-MPGI semimodule then there exist matrices U, V such that $AK \oplus BU = KV$.* $\square$

4.3 The control characteristic equation

The *control characteristic equation* is an equation associated to a regulation problem $\mathcal{R}(A, B, E, D)$. It was introduced in a previous work of the authors, Gonçalves et al. (2015), although it was not named at all. In the mentioned work, it was shown that this equation can provide a sufficient condition for solving regulation problems in open loop. One of the major contributions of the present work is that this equation has a much deeper

[1] Set with only one member

importance to the problem: not only solving it is *sufficient* for solving the problem in open loop for a class of problems, as presented in Gonçalves et al. (2015), but it also induces a closed-loop solution, which has a series of benefits, and its solvability is also a *necessary* condition for solving *all* regulation problems under very mild assumptions.

Definition 2 *: (Control characteristic equation) The control characteristic equation, $\mathcal{C}(\mathcal{R})$, associated a problem $\mathcal{R}(A,B,E,D)$ is the following equation for the unknowns $\{\lambda,\mu,\chi\}$:*

$$\mathcal{C}(\mathcal{R}): \begin{array}{l} (i)\ :\ \lambda\chi = A\chi \oplus B\mu; \\ (ii)\ :\ E\chi = D\chi \end{array}$$

in which $\lambda \in \mathbb{Z}$, $\chi \in \mathbb{Z}^n_{max}$ and $\mu \in \mathbb{Z}^m_{max}$. A solution $\{\lambda,\mu,\chi\}$ is said to be proper *if χ has no $\perp$ entries.* □

As it will be clear later, the unknowns $\{\lambda,\mu,\chi\}$ will induce a solution for the problem. Then, each one of these will be related to a specific behaviour in steady state. The scalar λ will command the system rate in steady state, so the greater is this number the slower the system in closed loop will be. The vector χ is related to the *state* $x[k]$ in steady state: it will be a scalar multiple of it, that is, in steady state $x[k] = h\lambda^k\chi$ for a scalar $h(x_0)$ that depends on the initial condition. Finally, the vector μ is related to the *input* in steady state: $u[k] = h\lambda^k\mu$ for the same scalar $h(x_0)$.

Definition 3 *: (Control characteristic spectrum) The control characteristic spectrum $\Lambda(\mathcal{R})$ is the set of $\lambda \in \mathbb{Z}$ such that there exists a proper solution $\{\lambda,\mu,\chi\}$ to $\mathcal{C}(\mathcal{R})$.* □

There is a series of interesting facts regarding the control characteristic equation. Firstly, it can be written as a *two-sided eigenproblem* (see Gaubert and Sergeev (2013)), as shown in our previous work (see Section V-B in Gonçalves et al. (2015)). This kind of equation has been studied recently under the light of the also recently-developed theory of *mean payoff games* (see Akian et al. (2012)), for which there is currently a pseudopolynomial algorithm to solve it (see Gaubert and Sergeev (2013)). Since we are interested in proper solutions, our previous work established that for controllable coupled problems all solutions will be proper (see Gonçalves et al. (2015), in special Proposition 5). Furthermore, it is possible to find solutions in which $\lambda \in \mathbb{Q}$, that is, a rational number. But, as argued in Gonçalves et al. (2015), we can assume without loss of generality that $\lambda \in \mathbb{Z}$ because the units of the problem can be redefined so, in these new units, λ is integer. For instance, if $\lambda = 10/3\ min$ is found, we can redefine the units of the problem from minutes to seconds, and hence $10/3\ min = 200s$. Finally, clearly due to $\mathcal{C}(\mathcal{R}) - (i)$ we have that $\lambda\chi \geq A\chi$, and a standard result in max-plus algebra implies that if χ has no $\perp$ entries - the solution is proper - it holds that $\lambda \geq \rho(A)$. Thus all members of Λ are greater than or equal to $\rho(A)$.

The following definition is also important.

Definition 4 *: (Strong control characteristic equation) The strong control characteristic equation, $\mathcal{C}_{st}(\mathcal{R})$, associated with a problem $\mathcal{R}(A,B,E,D)$ is the following equation for the unknowns $\{\lambda,\mu,\chi\}$:*

$$\mathcal{C}_{st}(\mathcal{R}): \begin{array}{l} (i)\ :\ \lambda\chi = (\lambda^{-1}A)^*B\mu; \\ (ii)\ :\ E\chi = D\chi \end{array}$$

in which $\lambda \in \mathbb{Z}$, $\chi \in \mathbb{Z}^n_{max}$ and $\mu \in \mathbb{Z}^m_{max}$. A solution $\{\lambda,\mu,\chi\}$ is said to be proper *if χ has no $\perp$ entries.* □

Indeed, any solution to $\mathcal{C}_{st}(\mathcal{R})$ is a solution to $\mathcal{C}(\mathcal{R})$, but the converse is not true unless $\lambda > \rho(A)$. This is due to a standard result in max-plus algebra (see Baccelli et al. (1992)): if we have an equation $g = M^*h$ then necessarily $g = Mg \oplus h$, but the converse is only true if $\rho(M) < 0$. The strong control characteristic equation can be solved by exploiting this property. One can solve $\mathcal{C}(\mathcal{R})$ and if $\lambda > \rho(A)$ we can claim it as a solution to $\mathcal{C}_{st}(\mathcal{R})$. If not, we can fix $\lambda = \rho(A)$ and try to solve $\mathcal{C}_{st}(\mathcal{R})$. With λ fixed, the resulting equation is max-plus linear, for which several algorithms exist (see Cuninghame-Green and Butkovic (2003); Dhingra and Gaubert (2006); Gaubert and Sergeev (2013); Lorenzo and de la Puente (2011); Allamigeon et al. (2010); Butkovic and Zimmermann (2006); Butkovic and Hegedus (1984); Truffet (2010); Gonçalves et al. (2013)). Note that the control characteristic spectra is associated with the proper solutions of $\mathcal{C}(\mathcal{R})$, not to the ones of $\mathcal{C}_{st}(\mathcal{R})$.

With the definitions given so far, a very important definition can be introduced in this text.

Definition 5 *: (Controllable critical and controllable non-critical problems) A problem $\mathcal{R}$ is said to be* controllable critical *if $\Lambda(\mathcal{R}) = \{\rho(A)\}$, that is, only $\lambda = \rho(A)$ is admissible to proper solutions. If there is other elements other than $\rho(A)$ in Λ, then the problem is said to be* controllable non-critical. □

4.4 The necessary condition

What if $\Lambda(\mathcal{R}) = \emptyset$, that is, $\mathcal{C}(\mathcal{R})$ has no proper solution? One of the main results of this paper is that if $\mathcal{R}$ is controllable coupled, then this implies that $\mathcal{R}$ has no solution at all.

Proposition 1 *: If the problem $\mathcal{R}$ is controllable coupled, then it is solvable* only if *$\mathcal{C}(\mathcal{R})$ has a proper solution.*

Proof: Suppose $\mathcal{R}$ has a solution. Invoke Result 1, then there exists a maximal (A,B)-MPGI set inside $\mathcal{S}_{ref}(\mathcal{R})$, namely $\mathcal{K}^{\top}_{geo}(\mathcal{R})$. Invoke then Lemma 2 to conclude that $\mathcal{K}^{\top}_{geo}(\mathcal{R}) = Im\ K$ for a matrix $K \in \mathbb{Z}^{n\times s}_{max}$ for a finite s. Finally, use Result 2 to conclude that the equation

$$AK \oplus BU = KV$$

has a solution. Now, $V \in \mathbb{Z}^{s\times s}_{max}$ is a finite square matrix, and hence has an eigenvector v associated with an eigenvalue λ. Post-multiply the latter equation by v and use the fact that $Vv = \lambda v$ to conclude that

$$A(Kv) \oplus B(Uv) = \lambda(Kv).$$

Furthermore, since $\mathcal{K}^{\top}_{geo}(\mathcal{R}) \subseteq \mathcal{S}_{ref}(\mathcal{R})$, it holds that $EK = DK$. Post multiply this equation by v to conclude that

$$E(Kv) = D(Kv).$$

Take $\chi = Kv$, $\mu = Uv$. Then the two previous equations show that $\mathcal{C}(\mathcal{R})$ must have a solution. It remains to establish that this solution must be proper.

Assume without loss of generality that K has no $\perp$ columns. Since the constraint $|x_i - x_j| \leq W$ must hold due to the fact that the problem is controllable coupled, K is the generator of the maximal (A, B)-MPGI set inside the constraints and no column of K is $\perp$, then all the columns of K are free of $\perp$ entries. Finally, since at least one element of v is non-null, it is an eigenvector, then Kv does not have any $\perp$ entry. This implies the desired result. □

This result shows that the control characteristic equation provides a necessary condition for all controllable coupled problems. Since, as argued previously, the controllable coupled property can be assumed without loss of generality, the equation effectively provides a necessary condition for all practical problems. It was shown in our previous work that the *strong* control characteristic equation provides a sufficient condition for solving the problem in open loop. However, the approach had issues when $\lambda = \rho(A)$, since convergence is not guaranteed when there are perturbations. The next developments will establish that this problem can be solved in a simply way using a static feedback $u[k] = Fx[k]$, which works even if $\lambda = \rho(A)$.

4.5 The sufficient condition

In this section, we will establish that the *strong* control characteristic equation provides, under some conditions, a sufficient condition for solving the regulation problem. Concerning this, the following result can be derived

Lemma 3 *: The problem $\mathcal{R}$ has a solution with a static feedback* $u[k] = Fx[k]$ if and only if *there exists a matrix F and a natural number p such that*

$$E(A \oplus BF)^p = D(A \oplus BF)^p \qquad (2)$$

Proof: For the *only if* part, note that if $u[k] = Fx[k]$ is taken, according to the dynamics in (1) we have $x[k] = (A \oplus BF)^k x_0$. Since it is necessary to have a $p \in \mathbb{N}$ such that $x[k] \in \mathcal{S}_{ref}(\mathcal{R})$ for all $k \geq p$ we must have in special that $Ex[p] = Dx[p]$, that is, $E(A \oplus BF)^p x_0 = D(A \oplus BF)^p x_0$. Since this must hold true for any x_0, we conclude the necessity of (2).

For the *if* part, note that if $E(A \oplus BF)^p = D(A \oplus BF)^p$ we can, assuming that $k \geq p$, post-multiply this equation by $(A \oplus BF)^{k-p} x_0$ to conclude that $E(A \oplus BF)^k x_0 = D(A \oplus BF)^k x_0$, that is, $Ex[k] = Dx[k]$ for $k \geq p$. □

Finding a solution to (2) is a difficult task, since it is max-plus non-linear. Even if we fix p in values and try to search for solutions, if $p \neq 1$ the resulting equation is max-plus non-linear and hard to solve. It is possible to use the method in Schutter and Moor (1996), but it can be time and space consuming. This equation will be solved indirectly by means of $\mathcal{C}_{st}(R)$. To this end, we will borrow another definition from our previous work.

Definition 6 *: (Convergence number, see Gonçalves et al. (2015)) Let $M \in \mathbb{Z}^{n\times n}_{max}$ be a matrix with $\rho(M) \leq 0$. The* convergence number*, $\kappa(M)$, is the smallest k such that*

$$\bigoplus_{i=0}^{k} M^i = M^*.$$

□

It is guaranteed that $\kappa(M)$ is finite if $\rho(M) \leq 0$. In special, since $M \in \mathbb{Z}^{n\times n}_{max}$ then $\kappa(M) \leq n$.

Before our main result, the following lemmas needs to be derived.

Lemma 4 *Let $\{\lambda, \mu, \chi\}$ be a proper solution to $\mathcal{C}_{st}(\mathcal{R})$. Let $\zeta = -\chi$. Then ζ satisfies the following equations:*

$$\begin{aligned} &(i) \ : \ \zeta^T A \preceq \lambda \zeta^T; \\ &(ii) \ : \ \zeta^T(\lambda^{-1}A)^* = \zeta^T; \\ &(iii) \ : \ \zeta^T B\mu = \lambda; \\ &(iv) \ : \ \chi\zeta^T \succeq (\lambda^{-1}A)^*. \end{aligned} \qquad (3)$$

Proof: *(i)*: Note that, since the solution is proper, χ has no $\perp$ entries and hence $\zeta = -\chi$ is well-defined. Since it is a proper solution to $\mathcal{C}_{st}(\mathcal{R})$, it is also a proper solution to $\mathcal{C}(\mathcal{R})$. Therefore, according to $\mathcal{C}(\mathcal{R}) - (i)$, we have that $\lambda\chi \succeq A\chi$. This can also be rewritten as $A \preceq \lambda\chi(-\chi)^T = \lambda\chi\zeta^T$. Pre multiplying by ζ^T the latter inequation and using the fact that $\zeta^T\chi = 0$ we can conclude that $\lambda\zeta^T \succeq \zeta^T A$.
(ii): Direct consequence of (i).
(iii): Since χ is a solution to $\mathcal{C}_{st}(\mathcal{R})$, we have that $\lambda\chi = (\lambda^{-1}A)^* B\mu$ (see $\mathcal{C}_{st}(\mathcal{R}) - (i)$). Pre multiply both

members by $\zeta^T = (-\chi)^T$ and use the fact $\zeta^T\chi = 0$ to conclude that $\lambda = \zeta^T(\lambda^{-1}A)^*B\mu$. Then use the result *(ii)* to conclude the desired result: $\lambda = \zeta^T B\mu$.
(iv): Using *(ii)*, we have that $\chi\zeta^T = \chi\zeta^T(\lambda^{-1}A)^*$. Clearly the matrix $\chi\zeta^T$ is greater or equal than the identity matrix I, since its diagonal components are $\chi_i + \zeta_i = \chi_i + (-\chi_i) = 0$. Therefore $\chi\zeta^T \succeq I$ and hence $\chi\zeta^T = \chi\zeta^T(\lambda^{-1}A)^* \succeq (\lambda^{-1}A)^*$. □

Lemma 5 *: Let $\{\lambda, \mu, \chi\}$ be a proper solution to $\mathcal{C}_{st}(\mathcal{R})$. Let $\zeta = -\chi$. Then $\zeta^T(A \oplus B\mu\zeta^T) = \lambda\zeta^T$.*

Proof: We have that $\zeta^T(A \oplus B\mu\zeta^T) = \zeta^T A \oplus (\zeta^T B\mu)\zeta^T$. Using (3)-(iii) and then (3)-(i) it is possible to conclude the desired result. □

Let $\{\lambda, \mu, \chi\}$ be a proper solution to $\mathcal{C}_{st}(\mathcal{R})$ and $\zeta = -\chi$. The feedback law $u[k] = Fx[k]$ with $F = \mu\zeta^T$ will be denoted by *spectral regulator*. The following proposition can then be established.

Proposition 2 *Let $\{\lambda, \mu, \chi\}$ be a proper solution to $\mathcal{C}_{st}(\mathcal{R}(A, B, E, D))$ and $\zeta = -\chi$. So $p = \kappa(\lambda^{-1}A) + 1$ and $F = \mu\zeta^T$ solve (2) and thus, according to Lemma 3, $u[k] = Fx[k]$ solves the regulation problem $\mathcal{R}(A, B, E, D)$.*

Proof: Let $M[k] = (A \oplus BF)^k$. By repeated applications of Lemma 5 we conclude that $\zeta^T M[k] = \lambda^k\zeta^T$. So

$$M[k+1] = (A \oplus BF)M[k] =$$
$$AM[k] \oplus B\mu(\zeta^T M[k]) = AM[k] \oplus \lambda^k B\mu\zeta^T.$$

Multiply the latter equation by $\lambda^{-(k+1)}$ and use the change of variables $\hat{M}[k] = \lambda^{-k}M[k]$ to conclude that

$$\hat{M}[k+1] = \lambda^{-1}A\hat{M}[k] \oplus \lambda^{-1}B\mu\zeta^T.$$

Iterating the latter equation and noting that $\hat{M}[0] = M[0] = I$

$$\hat{M}[k+1] = (\lambda^{-1}A)^{k+1} \oplus \lambda^{-1}\left(\bigoplus_{i=0}^{k}(\lambda^{-1}A)^i\right)B\mu\zeta^T.$$

Choose $k = \kappa(\lambda^{-1}A)$. Then, according to the definition of the convergence number

$$\hat{M}[k+1] = (\lambda^{-1}A)^{k+1} \oplus \lambda^{-1}(\lambda^{-1}A)^*B\mu\zeta^T.$$

Since $\{\lambda, \mu, \chi\}$ is a proper solution to $\mathcal{C}_{st}(\mathcal{R})$, we have from $\mathcal{C}_{st}(\mathcal{R})$-(i) that $\lambda^{-1}(\lambda^{-1}A)^*B\mu = \chi$. Therefore

$$\hat{M}[k+1] = (\lambda^{-1}A)^{k+1} \oplus \chi\zeta^T.$$

Using (3)-(iv) and noting that $(\lambda^{-1}A)^* \geq (\lambda^{-1}A)^{k+1}$ for all k.

$$\hat{M}[k+1] = \chi\zeta^T.$$

Thus, reverting to the original variable $M[k]$.

$$M[k+1] = \lambda^{k+1}\chi\zeta^T. \tag{4}$$

Now, since $\{\lambda, \mu, \chi\}$ is a proper solution to $\mathcal{C}_{st}(\mathcal{R})$ we have from $\mathcal{C}_{st}(\mathcal{R}) - (ii)$ that $E\chi = D\chi$. Post multiplying by $\lambda^{k+1}\mu$ we conclude that $E(\lambda^{k+1}\chi\mu) = D(\lambda^{k+1}\chi\mu)$. In light of (4), we can see that $EM[k+1] = DM[k+1]$ or $E(A \oplus BF)^{k+1} = D(A \oplus BF)^{k+1}$, in which $k+1 = \kappa(\lambda^{-1}A) + 1$. Therefore, $p = \kappa(\lambda^{-1}A) + 1$ can be taken and indeed (2) is solved. And the result is established. □

The previous theorem suggests that, if the spectral regulator $u[k] = Fx[k]$ is employed, in steady state ($k \geq \kappa(\lambda^{-1}A) + 1$), we will have $x[k] = M[k]x_0 = \lambda^{k+1}\chi(\zeta^T x_0)$. Since $\zeta^T x_0$ is a scalar, this implies that in steady state $x[k]$ is a scalar multiple of χ, and at each step it will increase of λ times units. Thus, χ is the "template" for the steady-state behaviour while λ is the steady-state growth rate of the firings. Furthermore, since $u[k] = Fx[k] = \mu\zeta^T x[k]$, we will have $u[k] = \lambda^{k+1}\mu(\zeta^T x_0)$ (since $\zeta^T\chi = 0$). Thus, μ is the "template" of the control input and λ is also the growth rate of the input firings.

In this sense, one can interpret that in steady state the spectral regulator is a closed loop form of the periodic synchronizer presented in Gonçalves et al. (2015), which has the form $u[k] = h\lambda^{k+1}\mu$. When $\lambda = \rho(A)$, it could be necessary to eventually adjust this parameter h to reject some perturbations. The major benefit of the closed loop is that this parameter h is nonexistent and no adjust in the matrix F is necessary. Furthermore, in the periodic synchronizer, the number of steps taken to convergence is given by $\max(r, \kappa(\lambda^{-1}A)) + 1$, where r is the smallest number such that $h(\lambda^{-1}A)^*B\mu \geq (\lambda^{-1}A)^{r+1}x_0$. This bound depends of many parameters as A, B, μ, λ, x_0 and h, whereas in the spectral regulator the bound is much simpler, $\kappa(\lambda^{-1}A) + 1$, and also clearly smaller than or equal to the former bound. All-in-all, the closed loop approach is more simple, elegant and practical.

4.6 Main result

On one hand, Proposition 1 establishes a *necessary* condition for solving all controllable coupled problems in terms of $\mathcal{C}(\mathcal{R})$. On the other hand, Proposition 2 establishes a *sufficient* condition for solving problems in terms of $\mathcal{C}_{st}(\mathcal{R})$. The following theorem agglutinates all these facts using the concept of controllable non-critical problems.

Theorem 1 *: If $\mathcal{R}$ is controllable coupled and controllable non-critical, it has a solution if and only if $\mathcal{C}(R)$ has a proper solution $\{\lambda, \chi, \mu\}$. Furthermore, it is of the form $u[k] = Fx[k]$ with $F = \mu(-\chi)^T$ and if the system has n states convergence occurs to the desired set in at most $n+1$ steps.*

Proof:

Only if: comes directly from Proposition 1.

If: If $\mathcal{R}$ is controllable non-critical, there exists a $\lambda > \rho(A)$ such that $\{\lambda, \chi, \mu\}$ is a proper solution to $\mathcal{C}(R)$. In this case, a solution to $\mathcal{C}(\mathcal{R})$ is also a solution to $\mathcal{C}_{st}(\mathcal{R})$. With such solution, Proposition 2 ensures that a feedback of the form $u[k] = Fx[k]$ solves the problem in at most $\kappa(\lambda^{-1}A) + 1$ steps. If $A \in \mathbb{Z}_{max}^{n\times n}$, it holds that $\kappa(\lambda^{-1}A) \leq n$. And the theorem is established. □

Note that the major burden in obtaining the controller is solving the associated control characteristic equation. As mentioned in Subsection 4.3, this can be done by pseudopolynomial algorithms. See Gonçalves et al. (2015) for the detailed complexity analysis. Once the triple $\{\lambda, \chi, \mu\}$ is obtained, it is very easy to compute the control law: compute $F = \mu(-\chi)^T$ then use $u[k] = Fx[k]$. Thus, the overall complexity of the control synthesis is pseudopolynomial.

4.7 False criticality

The previous theorem established results for non-critical problems. Non-critical problems are those in which the control characteristic spectra has something other than the open loop spectral radius. Considering that λ is the growth rate in steady state in closed loop, this implies that non-critical problems are those in which the feedback can delay, even if infinitesimally, the open-loop behavior. Remember that since our problem deals only with integers then the smallest non-zero possible delay is one time unit. But since this time unit can be redefined arbitrarily - as, for instance, one nanosecond - we could say that the effective delay can be “infinitesimal”.

It is frequently the case that we want that the feedbacks does not delay the system, because this could imply that, for instance, the production rate is reduced. Thus, it is interesting to consider the case in which $\lambda = \rho(A)$. Indeed, note that the definition of controllable non-critical implies that it can be *any* number different (and thus greater, since all members of Λ are greater than or equal to $\rho(A)$) than $\rho(A)$, even infinitesimally. Thus, intuitively, any critical problem could be made non-critical by changing slightly - infinitesimally - their parameters A, B, E, D. In practice this would not produce any change in the system since the parameters would not change noticeably, while the acquisition of the non-criticality would allow us to use the developed methodology.

While some critical problems can be transformed in non-critical problems using this trick, unfortunately this is not the case with all of them. Some problems do not have this property: while infinitesimal changes can make changes in their control characteristic spectra, it also make a change in the system spectral radius and it will be the case that the (perturbed) spectrum will continue to be equal to the singleton composed of the (perturbed) spectral radius, therefore preserving criticality. It can be the case that in order to obtain non-criticality a *relevant* change must be made in their parameters.

Since the important part is to have a solution to both $\mathcal{C}_{st}(\mathcal{R})$ and $\mathcal{C}(\mathcal{R})$, something which is possible with the trick, the following definition will be made.

Definition 7 *: (False and true criticality) A problem $\mathcal{R}$ is said to be* controllable false critical *if it is controllable critical and there exists a proper solution from $\mathcal{C}_{st}(\mathcal{R})$ which is also a proper solution to $\mathcal{C}(\mathcal{R})$. A problem is said to be* controllable true critical *if it is controllable critical and no proper solution from $\mathcal{C}_{st}(\mathcal{R})$ is also a proper solution to $\mathcal{C}(\mathcal{R})$.* □

Example: the following problem is controllable false critical. Consider the system:

$$x[k+1] = \begin{pmatrix} 1 & \perp \\ 0 & \perp \end{pmatrix} x[k] \oplus \begin{pmatrix} 0 \\ \perp \end{pmatrix} u[k];$$

with the constraint $x_1[k] - x_2[k] = 1$. Note that, according to the dynamical equations, $x_2[k] = x_1[k-1]$, and thus the constraint can be rewritten as $x_1[k] - x_1[k-1] = 1$, that is, we are specifying a rate of 1 time units. This problem is controllable critical, $\Lambda(\mathcal{R}) = \{\rho(A)\} = \{1\}$, but there exists an infinitesimal change in its parameters that cause the problem to be controllable non-critical. Indeed, consider the perturbed problem $\mathcal{R}_\delta$ with the same dynamical system but constraint $x_1[k] - x_2[k] = 1 + \delta$ for $\delta \geq 0$. It is possible to see that $\Lambda(\mathcal{R}_\delta) = \{1 + \delta\}$, and thus for any small δ the control characteristic spectra will have something different than the perturbed problem spectral radius, which is the same as the unperturbed one: $\rho(A) = 1$. Thus, there is a solution to $\mathcal{C}(\mathcal{R})$ which also is a solution to $\mathcal{C}_{st}(\mathcal{R})$. This implies that the problem is controllable false critical. □

Example: the following problem is controllable true critical. Consider the system:

$$x[k+1] = \begin{pmatrix} 1 & \perp \\ \perp & 2 \end{pmatrix} x[k] \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} u[k];$$

with the constraint $x_2[k] - x_1[k] = 1$. This problem is controllable critical, $\Lambda(\mathcal{R}) = \{\rho(A)\} = \{2\}$, but no infinitesimal change in its parameters will make the control characteristic spectrum differs from the singleton composed of the spectral radius. Indeed, no solution from $\mathcal{C}(\mathcal{R})$ is a solution to $\mathcal{C}_{st}(\mathcal{R})$. Thus, it is controllable true critical. □

Controllable false critical problems can be solved by finding this common solution $\{\lambda, \chi, \mu\}$ and then computing $F = \mu\zeta^T$. In practice, they are controllable non-critical. Controllable true critical problems, on the other hand, cannot be solved with the proposed methodology. They demand a considerably more complex theory that will be object of future papers.

It is perhaps reassuring to know that the great majority of problems seems to be either controllable non-critical or controllable false critical. Indeed, using arguments of cardinality, the singleton $\{\rho(A)\}$ is irrelevant in comparison to the other infinitely many possibilities of non-critical spectra, that is, all the other members in $2^{[\rho(A),\infty)\cap\mathbb{Z}}$. Thus, at least in a first glance, it seems that the matrices A, B, E, D must have a very special structure so it holds. Another, more practical argument, is that all other regulation-like problems that the authors solved when developing their research are either controllable non-critical or controllable false critical. This includes the problems found in Katz (2007); Gonçalves et al. (2015); Brunsch (2014); Attia et al. (2010); Maia et al. (2011a); Atto et al. (2011); Amari et al. (2012); Kim and Lee (2015) to cite some.

5 Illustrative example

The following system comes from a simplified (but equivalent) system presented in Kim and Lee (2015). It models a single-armed cluster tool robot that processes semiconductor wafers.

$$x[k+1] = \begin{pmatrix} \perp & \perp & 120 & 5 & \perp \\ \perp & \perp & 140 & 100 & \perp \\ \perp & \perp & 155 & 115 & \perp \\ \perp & \perp & 175 & 135 & \perp \\ \perp & \perp & \perp & 0 & \perp \end{pmatrix} x[k] \oplus \begin{pmatrix} \perp \\ \perp \\ \perp \\ 0 \\ \perp \end{pmatrix} u[k].$$

It is desirable that $x_2[k] - x_5[k]$ should be no more than 110 time units. This implies that the time between the finishing of the processing of a raw wafer in a module and the time that the wafer begins its processing in the next module is no more than 10 time units. If this time is too large, residual heat in the piece, generated after the first processing, may damage it. In order to turn the problem into controllable coupled, innocuous constraints $x_i[k] - x_j[k] \leq 250$ will also be posed.

Without any control, $u[k] = \perp$ for all k, the specification may not be achieved. Take $x[0] = [155\ 175\ 190\ 210\ 55]^T$, for which the constraint $x_2 - x_5 \leq 110$ fails. Furthermore, with $x[1] = Ax[0] = [310\ 330\ 345\ 365\ 210]$ and $x[2] = Ax[1] = [465\ 485\ 500\ 520\ 365]$ the same problem holds. This behaviour persists for all $k \geq 0$. The open loop system is unable to control to the desired specification. Therefore, we will use the proposed methodology to derive a controller.

Keeping in mind that $\rho(A) = 155$ in this case, it is possible to find that the associated control characteristic spectrum is $\Lambda(\mathcal{R}) = [155, 250] \cap \mathbb{Z}$, that is, the problem is controllable non-critical. Furthermore, $\lambda = \rho(A) = 155$ is also associated with a proper solution to the strong control characteristic equation, so $\lambda = 155$ can be taken. The associated μ is 0 and $\chi = [-230\ \ -210\ \ -195\ \ -155\ \ -310]^T$. The associated feedback matrix is thus $F = \mu(-\chi)^T = [230\ 210\ 195\ 155\ 310]$, and hence the derived control law is $u[k] = Fx[k] = 230x_1[k] \oplus 210x_2[k] \oplus 195x_3[k] \oplus 155x_4[k] \oplus 310x_5[k]$.

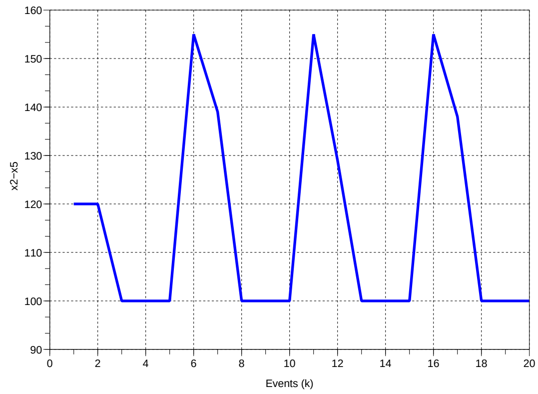

Fig. 1. Behavior of the value $x_2 - x_5$. Random delays - ranging from 0 to 20 time units- were inflicted in all the states at $k = 5$, $k = 10$ and $k = 15$. We can see that in at most 2 steps the rejection of the perturbation happens because $x_2 - x_5$ returns to the value 100, thus respecting the constraint $x_2 - x_5 \leq 100$.

The convergence number $\kappa(\lambda^{-1}A)$ equals to 2, which means that at most 3 steps convergence is achieved. Take the same initial condition as before, $x[0] = [155\ \ 175\ \ 190\ \ 210\ \ 55]^T$, for which the constraint $x_2 - x_5 \leq 110$ fails. Then:

$$x[1] = (A \oplus BF)x[0] = [310\ 330\ 345\ 385\ 210]^T$$
$$x[2] = (A \oplus BF)x[1] = [465\ 485\ 500\ 540\ 385]^T$$
$$x[3] = (A \oplus BF)x[2] = [620\ 640\ 655\ 695\ 540]^T \ldots$$

and for $k \geq 2$ all constraints hold. Thus, in this case convergence was achieved before the upper bound of 3 steps. Figure 1 also shows the behavior of the value $x_2 - x_5$ when random perturbations are inflicted in the system, thus showing that indeed the controller is able to reject perturbations.

6 Conclusion

Building on our previous work, Gonçalves et al. (2015), this paper proposes a necessary and sufficient condition for a wide class of max-plus control problems, the so-called *controllable non-critical* regulation problems. The resulting control law is a simple static feedback that can be computed efficiently even for large problems, since the major computational burden rely on solving what we define as *control characteristic equation*, which can be done with pseudo-polynomial algorithms.

In the next paper a (almost) dual theory for max-plus observers will be shown. In this sense, dual concepts to the ones introduced in this paper - as observation characteristic equation, observable critical, observable non-critical, etc- will be presented.

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