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Modélisation et contrôle d'un système de convoyage automatisé.

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Resumo

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Abstract

In this document will be presented a control strategy for Max-Plus-Linear (MPL) systems, a class of discrete-event systems widely used in synchronization and scheduling applications, such as: production systems (flexible workshops, assembly lines), communication networks (computer networks) and for transport systems (road, rail and air traffic). This document presents an observer-based controller with a real system application, located at ISTIA -École d'Ingénieurs de l'Université d'Angers.

Keywords: Discrete Event Systems, (Max,+) Linear Systems, Control Theory.

Resumo

Neste documento será apresentada uma estratégia de controle para sistemas Max-Plus lineares, uma classe de sistemas à eventos discretos largamente utilizada em sincronização e aplicações de escalonamento, tais como: sistemas de produção (oficinas flexíveis, linhas de montagem), redes de comunicação (redes de computadores) e sistemas de transporte (rodoviário, ferroviário e aéreo). Este documento apresenta um controlador baseado em observador, com um sistema real de aplicação, localizado no ISTIA - École d'Ingénieurs de l'Université d'Angers.

Palavras-chave: Sistemas à eventos discretos, Sistemas (Max,+) lineares, Teoria de controle.

Résumé

Dans ce document sera présenté une stratégie de contrôle pour les systèmes linéaires Max-Plus, une classe de systèmes à événements discrets largement utilisés pour des applications de synchronisation et d'ordonnancement, tels que les systèmes de production (ateliers flexibles, lignes d'assemblage), les réseaux de communication (réseaux informatiques) et des systèmes de transport (routier, ferroviaire et aérien). Ce document présente un contrôleur basé en observateur avec un système d'application réelle, située dans ISTIA - Ecole d'Ingénieurs de l'Université d'Angers.

Mots-clés: Systèmes à evénements discrets, Systèmes (Max,+) linéaires, Théorie de contrôle.

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1.1 Motivation

1.2 LARIS

The Angevin Laboratory for Research in Systems Engineering is a host team of the University of Angers, consisting of 3 interconnected teams:

- Dynamical Systems and Optimisation (SDP);
- Information, Signal, Image and Life Sciences (ISISV);
- Operational Safety and assistance in Decision (SFD)

1.3 Objectives

The specific objectives of this work are:

- To represent the automated conveyor system, located at ISTIA, using (max,+ algebra);
- To use and test the performance of an Observer-based Controller for (Max,+) Linear Systems.

1.4 Organization

This document is organized as follows:

- Chapter 1 - Introduction;
- Chapters 2 and 3 - Algebraic Preliminaries/System Modelling and Control Theory: Chapters 3 and 4 were granted by Prof. Dr. Laurent Hardouin, the following chapters are present in his handout [1]. These chapters present the main algebraic tools useful in the sequence. In the last section of chapter 3 I will introduce the Modified Observer-based Controller, my first participation during the work in LARIS.
- Chapter 4 - Automated conveyor system: A TEG model for the automated conveyor system is proposed in Dr. Vinícius Mariano's thesis [21] and used to construct a model with the Two-dimentinal description, the semiring $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$.

- Chapter 5 - Implementation;
- Chapter 6 - Simulation results;
- Chapter 7 - Conclusion.

2 Algebraic Preliminaries

This chapter presents the main algebraic tools useful in the sequence. It is not an exhaustive presentation but a survey of definitions, theorems necessary to prove the results introduced in the next chapters. In a first reading, the engineer could skip this chapter and pick inside the useful results during the reading of the chapters dedicated to the applications. The theory of (max,+) linear systems is partially based on the lattice theory and the manner to invert mappings defined over ordered sets, the following references were also source of inspiration [2], [3], [4].
The chapter is organized as follows :

- Elementary definitions about lattice theory basic facts are recalled in a first part, they will be useful to understand the proof of the results given in the sequel. There are usual notions for computer scientists but not necessary for engineers involved in the automatic control theory.
- The algebraic structure considered is the idempotent semiring.

2.1 Lattices and order sets

For very detailed presentation about Lattices, the reader is invited to consult the following references [2–4]. Some recalls are also available in chapter 4 of [5].

Definition 1 (Order relation, ordered set)**.** *An order relation is a binary relation which is reflexive, transitive and anti-symmetric : Let E be a set and a binary relation on this set denoted $\preceq$, this relation is an order relation if and only if for all x, y, and z elements of E :*

- $x \preceq x$ *(reflexivity)*
- $(x \preceq y$ *et* $y \preceq x) \Rightarrow x = y$ *(anti-symmetry)*
- $(x \preceq y$ *et* $y \preceq z) \Rightarrow x \preceq z$ *(transitivity)*

An ordered set is a set $(E, \preceq)$ endowed with an order relation.

Let $x, y \in (E, \preceq)$, x and y are said comparable (according to the order relation $\preceq$) if

$$x \preceq y \text{ or } y \preceq x.$$

Conversely, two elements $x, y \in (E, \preceq)$ such that $x \npreceq y$ and $y \npreceq x$, are said to be not comparable.

If $\forall x, y \in (E, \preceq)$, x and y are comparable then the order is said to be a total order, and $(E, \preceq)$ is said to be totally ordered. Conversely, if it exists a couple $x, y \in (E, \preceq)$, such that $x \neq y$ and x and y are not comparable, then the order is said to be partial and $(E, \preceq)$ is said to be partially ordered.

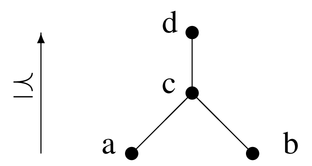
Remark 2. *In ambiguous situation, the order relation of set E will be denoted $\preceq_E$.*

All subset F of an ordered set $(E, \preceq)$ is an ordered set with an order relation restricted to the elements of F, and denoted $\preceq_F$. This order is simply defined as

$$x, y \in F \subset E, \quad x \preceq y \iff x \preceq_F y.$$

Remark 3. *If $(E, \preceq)$ is partially ordered, a subset $F \subset E$ can be such that all the elements of F be not comparable.*

A finite ordered set $(E, \preceq)$ can be represented by a graph called *Hasse diagram*. Each element of E is depicted by a vertex(•). An edge lying two vertices of the diagram means that the elements depicted by these vertices are comparable. By convention, the order is chosen to increase from the bottom to the top of the diagram.



$a \preceq c, \ b \preceq c, \ c \preceq d$

$a \preceq d, \ b \preceq d$ (by transitivity of the order $\preceq$)

Figure 1 – Hasse diagram of an ordered set $(\{a, b, c, d\}, \preceq)$

On figure 1, set $E = \{a, b, c, d\}$ is partially ordered according to order relation $\preceq$ despited by the diagram. Subset $F = \{a, b\} \subset E$ is an ordered set by considering the restriction of $\preceq$ to F, and all its elements are not comparable.

Remark 4. *A totally ordered set is also called a chain according to its Hasse diagram which looks like a chain.*

Example 5 (Ordered sets)**.**

- $(\mathbb{R}, \leq)$, $(\mathbb{Z}, \leq)$, $(\mathbb{N}, \leq)$, $(\mathbb{Q}, \leq)$ *where $\leq$ is the natural order, are totally ordered sets.*
- *Let E be a set, and $\mathcal{P}(E)$ be the set of all the subset of E. The latest is an ordered set by the inclusion relation. This ordered set is denoted $(\mathcal{P}(E), \subset)$ and it is a partially ordered set. Indeed, two disjoints subsets of E are not comparable according to the order $\subset$.*

- *Let* $(E, \preceq_E)$ *be an ordered set. The set of matrices with entries in* E*, namely* $E^{n\times m}$ *is an ordered set. Even if* $\preceq_E$ *is total on* E*, the order obtained on* $E^{n\times m}$ *is only partial. Furthermore,*

$$A \preceq_{E^{n\times m}} B \Leftrightarrow a_{ij} \preceq_E b_{ij} \forall (i,j),$$

where A *and* B *are matrices of* $E^{n\times m}$*.*

- *Let* $(E, \preceq_E)$ *and* $(F, \preceq_)$ F *be ordered sets. The set* $Map(E,F)$ *of all mappings from* E *to* F *can be ordered by defining the following order relation*

$$f \preceq_{Map(E,F)} g \Leftrightarrow f(x) \preceq_F g(x) \forall x \in E.$$

Definition 6 (Upper bound, lower bound)**.** *Let* $(E, \preceq)$ *be an ordered set and* $F \subset E$ *a non empty subset of* E*. An element* $z \in E$ *satisfying* $\forall x \in F, z \succeq x$ *(resp.* $z \preceq x$*) is called upper bound (resp. lower bound) of set* F*.*

Remark 7 (Bounds of a set)**.** *When it exists, the least upper bound (lub) of set* $F \subset E$ *is denoted* $\bigvee F$*. As well, when it exists, the greatest lower bound (glb) of* F *is denoted* $\bigwedge F$*. When these bounds are defined, all the upper bounds of* F *are greater than or equal to* $\bigvee F$ *and all lower bounds of* F *are lower than or equal to* $\bigwedge F$*.*

Definition 8. *Let* E *be an ordered set.*

- *if* $x \vee y$ *exists* $\forall x, y \in E$ *then* E *is a sup-semi-lattice.*
- *if* $\bigvee F$ *exists* $\forall F \subset E$ *then* F *is a complete sup-semi-lattice.*

Definition 9. *Let* E *be an ordered set.*

- *if* $x \wedge y$ *exists* $\forall x, y \in E$ *then* E *is an inf-semi-lattice.*
- *if* $\bigwedge F$ *exists* $\forall F \subset E$ *then* F *is a complete inf-semi-lattice.*

Definition 10. *Let* E *be an ordered set.*

- *if* $x \vee y$ *and* $x \wedge y$ *exist* $\forall x, y \in E$ *then* E *is a lattice.*
- *if* $\bigvee F$ *and* $\bigwedge F$ *exist* $\forall F \subset E$ *then* F *is a complete lattice.*

Lemma 11. *Let* E *be a lattice and* $a, b \in E$*, then the following equivalences hold :*

$$a \preceq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a.$$

Theorem 12. *Let* E *be a lattice,* $\forall a, b, c \in E$*,* $\vee$ *and* $\wedge$ *the following properties hold:*

• *associativity :*	$a \vee (b \vee c) = (a \vee b) \vee c$	*and*	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$,
• *commutativity :*	$a \vee b = b \vee a$	*and*	$a \wedge b = b \wedge a$,
• *idempotency :*	$a \vee a = a$	*and*	$a \wedge a = a$,
• *absorption :*	$a \vee (a \wedge b) = a$	*and*	$a \wedge (a \vee b) = a$.

Remark 13 (Duality principle)**.** *The inverse of an order relation $\preceq$ is an order relation denoted $\preceq^*$. Consequently, if $(E, \preceq)$ is a sup-semi-lattice, $(E, \preceq^*)$ is an inf-semi-lattice, and vice versa. Furthermore, a relation involving $\preceq$, $\vee$ and $\wedge$ is still valid by replacing $\preceq$ by $\preceq^*$ and by permuting $\vee$ and $\wedge$. This is called the duality principle.*

Example 14. *The ordered set depicted by the Hasse diagram of figure 1 is a sup-semi-lattice. But it is not an inf-semi-lattice since $a \wedge b$ does not exist.*

Example 15. *Let $E = \{a, b, c\}$ and $\mathcal{P}(E)$ the set of subset of E. Then $(\mathcal{P}(E), \cup, \cap)$ has a lattice structure if the empty set is considered as the lowest subset of E. The Hasse diagram of this lattice is given in figure 2.*

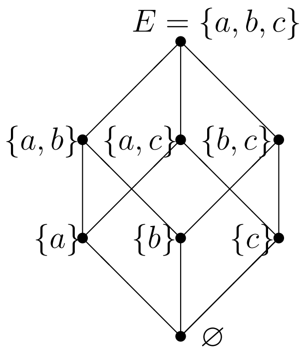


Figure 2 – Hasse diagram of the lattice $(\mathcal{P}(E), \cup, \cap)$ with $E = \{a, b, c\}$.

Example 16. *Let $(\mathbb{N}^*, \preceq_{div})$ be the set of positive integers, where the order on $\mathbb{N}^*$ is defined as*

$$a \preceq_{div} b \iff a \text{ divides } b, \tag{2.1}$$

is a lattice. The lattice laws of $(\mathbb{N}^*, \preceq_{div})$ are defined as $a \vee b = ppcm(a, b)$ and $a \wedge b = pgcd(a, b)$. In figure 3 the Hasse diagram corresponding to the set $E = \{1, 2, 3, 4, 5, 6\}$ according to the order relation $\preceq_{div}$ is given. Set $(E, \preceq_{div}) \subset \mathbb{N}^*$ is an inf-semi-lattice.

Example 17. *By adding element $+\infty$ to $\mathbb{Z}$, set $(\mathbb{Z} \cup \{+\infty\}, \leq)$ is $\vee$-complete totally ordered set. On the other hand, $(\mathbb{Q} \cup \{+\infty\}, \leq)$ is a totally ordered set which is not $\vee$-complete and not $\wedge$-complete. For instance, subset $\{x \in \mathbb{Q} | x \leq \sqrt{2}\}$ of $\mathbb{Q}$ has no least upper bound (lub) in $\mathbb{Q}$.*

Definition 18 (Distributive lattice)**.** *Lattice $(E, \vee, \wedge)$ is distributive if laws $\vee$ and $\wedge$ distribute on each other, i.e.*

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \tag{2.2}$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c). \tag{2.3}$$

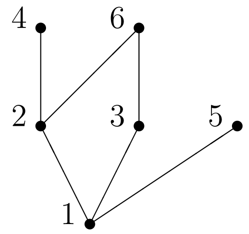


Figure 3 – Hasse diagram of semi-lattice $(\{1, 2, 3, 4, 5, 6\}, \preceq_{div})$.

Definition 19 (sub-semi-lattice, semi-lattice)**.** *Let $(E, \vee, \wedge)$ a lattice. F is said sub-$\vee$-semi-lattice of $(E, \vee)$ if $F \subset E$ and if F is closed for the law $\vee$. Furthermore, F is said to be a sub-lattice of E if $F \subset E$ and if F is closed for the laws $\vee$ and $\wedge$.*

Remark 20. *A subset F of lattice $(E, \preceq)$ can be a lattice (for the restricted order $\preceq$ to F), even if the law of lattice $(F, \preceq)$ is not the one of lattice $(E, \preceq)$, $i.e.$, F is not a sub-lattice of E. Let $(F, \cup, \cap)$ and $(E, \vee, \wedge)$ be lattices such that $F \subset E$ and such that the order of F is the restricted order of E to F, $i.e.$*

$$\forall x, y \in F \subset E,\ x \preceq y \iff y = y \vee x \iff y = y \cup x.$$

Then the following inequalities hold

$$\forall x, y \in F,\quad x \cap y \preceq x \wedge y \preceq x \vee y \preceq x \cup y.$$

To illustrate this point, let us consider the Hasse diagram of a finite lattice $(E, \preceq)$ with 6 elements (fig. 4). Elements a and b are not comparable according to the order $\preceq$ but have $a \vee b$ as least upper bound and $a \wedge b$ as greatest lower bound (in E). Set E has a minimal element 0_E (with $a \wedge b \succ 0_E$) and a maximal element π_E (with $\pi_E \succ a \vee b$).

If subset F of E is assumed equal to $F = E \setminus \{a \vee b, a \wedge b\}$ ($\setminus$ corresponds to the ensemblist substraction), then, set F is still a lattice (the dotted edge have to be considered). Nevertheless, it can be remarked that neither $a \vee b$, nor $a \wedge b$ are defined in F. Conversely, set $\{a, b\} \subset F$ admits element 0_E as unique lower bound and element π_E as unique upper bound (in F). Hence, set F has a structure of lattice, according to the restricted order of E to F, but is not a sub-lattice of E since it is not closed for $\vee$ and $\wedge$.

2.2 Idempotent semiring

Idempotent semiring, also called dioid (see [5]), is an algebraic structure. Boolean algebra and max-plus algebra are certainly the most popular. Below the definitions and properties of these algebraic structures are recalled.

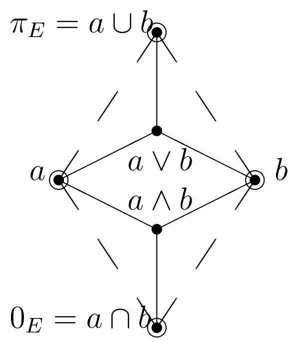


Figure 4 – F: ∘ and E: • with $F = E \backslash \{a \vee b, a \wedge b\}$

Definition 21 (Monoid)**.** *$(M, \oplus, \varepsilon)$ is a monoid if $\oplus$ is a closed law, associative, and having a neutral element denoted ε ($\forall m \in M, m \oplus \varepsilon = \varepsilon \oplus m = m$). If law $\oplus$ is commutative, the monoid is said to be commutative.*

Definition 22 (Semiring, dioid)**.** *$(\mathcal{D}, \oplus, \otimes)$ is an idempotent semiring, also called dioid in literature ([?], [5]), if the following axioms hold*

- *$(\mathcal{D}, \oplus, \varepsilon)$ is an idempotent commutative monoid , $\forall a \in \mathcal{D}, a \oplus a = a$,*
- *$(\mathcal{D}, \otimes, e)$ is a monoid,*
- *law $\otimes$ distributes over law $\oplus$,*
- *ε is absorbing for law $\otimes$, $\forall a \in \mathcal{D}, a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$.*

Furthermore, if $(\mathcal{D}, \otimes, e)$ is a commutative monoid, the idempotent semiring $(\mathcal{D}, \oplus, \otimes)$ is said to be commutative.

Definition 23 (Order relation)**.** *Since an idempotent semiring $\mathcal{D}$ has a structure of commutative idempotent monoid $(\mathcal{D}, \oplus)$, $\mathcal{D}$ has a structure of sup-semi-lattice for the order defined as follows :*

$$a \succeq b \iff a = a \oplus b.$$

Definition 24 (Complete idempotent semiring)**.** *An idempotent semiring is complete if it is closed for infinite sum and if the law $\otimes$ distributes over infinite sum, i.e. if $\forall c \in \mathcal{D}$ and $\forall \mathcal{X} \subset \mathcal{D}$*

$$c \otimes (\bigoplus_{x \in \mathcal{X}} x) = \bigoplus_{x \in \mathcal{X}} c \otimes x.$$

Remark 25. *Since a semiring has a structure of sup-semi-lattice $(\mathcal{D}, \preceq)$, if it is complete, it admits a greatest element. This greatest element will be denoted $\top$. Element $\top$ corresponds to the sum of all the elements of $\mathcal{D}$, i.e. : $\top = \bigoplus_{x \in \mathcal{D}} x$. Furthermore, an idempotent semiring admits always ε as bottom element. A complete idempotent semiring is then a complete sup-semi-lattice with a minimal element. According to definition 10, a complete idempotent semiring has then a structure of complete lattice for the order $\preceq$.*

Definition 26. *If $\mathcal{D}$ is a complete idempotent semiring, then the law $\wedge$ defined as*

$$a \wedge b = \bigoplus_{x \preceq a, x \preceq b} x,$$

is associative, commutative and idempotent, furthermore $(\mathcal{D}, \oplus, \wedge)$ is a complete lattice and the following equivalences hold

$$a = a \oplus b \Leftrightarrow a \succeq b \Leftrightarrow b = a \wedge b. \tag{2.4}$$

Remark 27. *According to definition 24, laws $\oplus$, $\otimes$, $\wedge$ are order preserving, $i.e$, $\forall a, b, c \in \mathcal{D}$ the following implication hold :*

$$\begin{aligned} a \preceq b &\Rightarrow a \otimes c \preceq b \otimes c, \\ a \preceq b &\Rightarrow a \oplus c \preceq b \oplus c, \\ a \preceq b &\Rightarrow a \wedge c \preceq b \wedge c. \end{aligned}$$

Remark 28. *Relation (2.4) looks like operators $\oplus$ and $\wedge$ play a symmetric role. That is true from the lattice point of view since operator $\oplus$ corresponds to $\vee$ and thanks to the duality principle. But that is false if we consider the second operator of the semiring $\otimes$, since there is no distributivity of $\wedge$ over $\otimes$ in a general manner. Nevertheless, the product being order preserving (see Remark 27, the following sub distributivity holds :*

$$c \otimes (a \wedge b) \preceq (c \otimes a) \wedge (c \otimes b) \ \ \forall a, b, c \in \mathcal{D}.$$

Example 29 ((max,plus) algebra)**.** $\overline{\mathbb{Z}}_{\max} = (\mathbb{Z} \cup \{-\infty, +\infty\}, \max, +)$ *is a complete idempotent semiring such that $a \oplus b = max(a, b)$, $a \otimes b = a + b$, $a \wedge b = min(a, b)$ with $\varepsilon = -\infty$, $e = 0$, and $\top = +\infty$. The order $\preceq$ is total and corresponds to the natural order $\leq$. By extension $\overline{\mathbb{Z}}_{\max}^{n\times m}$ is a semiring of matrices with entries in $\overline{\mathbb{Z}}_{\max}$. Matrix $\varepsilon \in \overline{\mathbb{Z}}_{\max}^{n\times m}$ will be such that all its entries are equal to $\varepsilon \in \overline{\mathbb{Z}}_{\max}$, matrix $\varepsilon \in \overline{\mathbb{Z}}_{\max}^{n\times n}$ will be such that all the entries are equal to $\varepsilon \in \overline{\mathbb{Z}}_{\max}$ except the diagonal entries which are equal to $e \in \overline{\mathbb{Z}}_{\max}$. This semiring will be of main interest in the sequel and section 2.6 will be devoted to present some of its properties.*

Example 30 ((min,plus) algebra)**.** $\overline{\mathbb{Z}}_{\min} = (\mathbb{Z} \cup \{-\infty, +\infty\}, \min, +)$ *is a complete idempotent semiring such that $a \oplus b = min(a, b)$, $a \otimes b = a + b$, $a \wedge b = max(a, b)$ with $\varepsilon = +\infty$, $e = 0$, and $\top = -\infty$. The order $\preceq$ is total and corresponds to the inverse of the natural order ($i.e.$, $2 \preceq 1$). Semiring of matrices $\overline{\mathbb{Z}}_{\min}^{n\times m}$ is a semiring of matrices with the entries in $\overline{\mathbb{Z}}_{\min}$.*

Example 31 ((max,min) algebra)**.** *The set $(\mathbb{Z} \cup \{-\infty, +\infty\}, max, min)$ is a complete idempotent semiring such that $a \oplus b = max(a, b)$, $a \otimes b = min(a, b)$ with $\varepsilon = -\infty$, $e = +\infty$, and $\top = +\infty$, in this semiring $a \wedge b = min(a, b)$.*

Definition 32 (Distributive semiring)**.** *A semiring $\mathcal{D}$ is distributive if it is complete and if the lattice $(\mathcal{D}, \oplus, \wedge)$ is distributive (cf. Definition 18).*

Remark 33. *Laws $\oplus$ and $\wedge$ are order preserving (see property 27), hence if the semiring is not distributive, the following inequalities hold :*

$$a \oplus (b \wedge c) \preceq (a \oplus b) \wedge (a \oplus c),$$
$$a \wedge (b \oplus c) \succeq (a \wedge b) \oplus (a \wedge c).$$

Definition 34 (Subsemiring)**.** *Let $(\mathcal{D}, \oplus, \otimes)$ be a semiring and $\mathcal{C} \subset \mathcal{D}$. $(\mathcal{C}, \oplus, \otimes)$ is a subsemiring of $\mathcal{D}$ if $\varepsilon, e \in \mathcal{C}$ and if $\mathcal{C}$ is closed for laws $\oplus$ and $\otimes$. A subsemiring is complete if it is closed for infinite sums too.*

2.3 Mappings defined over idempotent semirings

Definition 35 (Continuity)**.** *A mapping Π defined from a complete idempotent semiring $\mathcal{D}$ to a complete idempotent semiring $\mathcal{C}$ is lower semi-continuous (denoted $l.s.c.$), respectively upper semi-continuous (denoted $u.s.c$), if for all finite or infinite set $\mathcal{X}$ of $\mathcal{D}$,*

$$\Pi(\bigoplus_{x \in \mathcal{X}} x) = \bigoplus_{x \in \mathcal{X}} \Pi(x),$$

respectively,

$$\Pi(\bigwedge_{x \in \mathcal{X}} x) = \bigwedge_{x \in \mathcal{X}} \Pi(x).$$

Mapping Π is continuous if it is both $l.s.c.$ and $u.s.c.$.

Definition 36 (Isotone, antitone, monotone)**.** *Let $\Pi : \mathcal{D} \to \mathcal{C}$ be a mapping, with $\mathcal{D}$ and $\mathcal{C}$ two idempotent semirings :*

- *mapping Π is* isotone *if it is order preserving, $i.e.$, $\forall x, x' \in \mathcal{D}\ x \preceq x' \Rightarrow \Pi(x) \preceq \Pi(x')$,*
- *mapping Π is* antitone *if it inverts the order, $i.e.$, $\forall x, x' \in \mathcal{D}\ x \preceq x' \Rightarrow \Pi(x) \succeq \Pi(x')$,*
- *mapping Π is* monotone, *$i.e.$,* Π isotone or Π antitone.

Remark 37. *The composition of monotone functions is a monotone function and it can easily be checked that :*

- *The composition of isotone functions is an isotone function.*
- *The composition of two antitone functions is isotone.*
- *The composition of an isotone function with an antitone function is antitone.*

Theorem 38. *Let $\Pi : \mathcal{D} \to \mathcal{C}$ be a mapping with $\mathcal{D}$ and $\mathcal{C}$ two semirings :*

1. *if* Π *is* $l.s.c$ *then it is isotone.*

2. *if* Π *is* $u.s.c$ *then it is isotone.*

Remark 39. *If* $\Pi : \mathcal{D} \to \mathcal{C}$ *is an isotone mapping, the following inequality holds*

$$\Pi(x \oplus x') \succeq \Pi(x) \oplus \Pi(x') \qquad \forall x, x' \in \mathcal{D},$$

since

$$\left.\begin{array}{lll} x \oplus x' \succeq x & \Rightarrow & \Pi(x \oplus x') \succeq \Pi(x) \\ x \oplus x' \succeq x' & \Rightarrow & \Pi(x \oplus x') \succeq \Pi(x') \end{array}\right\} \Rightarrow \Pi(x \oplus x') \succeq \Pi(x) \oplus \Pi(x').$$

and dually :

$$\Pi(x \wedge x') \preceq \Pi(x) \wedge \Pi(x') \quad , \quad \forall x, x' \in \mathcal{D}.$$

Definition 40 (Homomorphism)**.** *Mapping* $\Pi : \mathcal{D} \to \mathcal{C}$ *defined over idempotent semiring is an homomorphism if*

$$\forall a, b \in \mathcal{D} \qquad \Pi(a \oplus b) = \Pi(a) \oplus \Pi(b) \text{ and } \Pi(\varepsilon) = \varepsilon \tag{2.5}$$

$$\Pi(a \otimes b) = \Pi(a) \otimes \Pi(b) \text{ and } \Pi(e) = e \tag{2.6}$$

A mapping satisfying only (2.5) is said to be $\oplus$*-morphism,* $i.e.$*, the image of the sum of elements in* $\mathcal{D}$ *is the sum, in* $\mathcal{C}$*, of their image. A mapping satisfying only (2.6) is said to be* $\otimes$*-morphism,* $i.e.$*, the image of the product of two elements of* $\mathcal{D}$ *is the product, in* $\mathcal{C}$*, of their image.*

Definition 41 (Isomorphism)**.** *Mapping* Π *defined from an idempotent semiring* $\mathcal{D}$ *to another one* $\mathcal{C}$ *is an isomorphism if the inverse of* Π *is defined and if* Π *and its inverse mapping are homomorphisms.*

Definition 42 (Equivalence relation)**.** *An equivalence relation* $\mathcal{R}$*, in set* E*, is a binary relation which is:*

- *reflexive :* $\forall\, x \in E, x\mathcal{R}x$*,*
- *symmetric :* $\forall\, x, y \in E, x\mathcal{R}y \Rightarrow y\mathcal{R}x$*,*
- *transitive :* $\forall\, x, y, z \in E, (x\mathcal{R}y \text{ and } y\mathcal{R}z) \Rightarrow x\mathcal{R}z$*.*

Definition 43 (Congruence)**.** *A congruence in a semiring* $\mathcal{D}$ *is an equivalence relation (denoted* $\mathcal{R}$*) compatible with the semiring laws ,* $i.e.$

$$a\,\mathcal{R}\,b \Rightarrow (a \oplus c)\,\mathcal{R}\,(b \oplus c), \quad (a \otimes c)\,\mathcal{R}\,(b \otimes c), \quad \forall a, b, c \in \mathcal{D}.$$

Theorem 44 (Semiring quotient, [5])**.** *Let* $\mathcal{D}$ *be an idempotent semiring and* $\mathcal{R}$ *a congruence over* $\mathcal{D}$*. By denoting* $[a] = \{x \in \mathcal{D} \,|\, x\,\mathcal{R}\,a\}$ *the equivalence class of* $a \in \mathcal{D}$*, the semiring quotient of* $\mathcal{D}$ *by this congruence is a semiring denoted* $\mathcal{D}_{/\mathcal{R}}$ *with the following sum and product*

$$\begin{array}{rcl} [a] \oplus [b] & \triangleq & [a \oplus b] \\ [a] \otimes [b] & \triangleq & [a \otimes b]. \end{array} \tag{2.7}$$

Theorem 45 ([5]). *Let Π be an homomorphism from $\mathcal{D}$ in $\mathcal{C}$. Relation $\mathcal{R}_\Pi$ defined by*

$$a\,\mathcal{R}_\Pi\,b \iff \Pi(a) = \Pi(b), \quad \forall a, b \in \mathcal{D},$$

is a congruence.

Definition 46 (Image). *The image of mapping $\Pi : \mathcal{D} \to \mathcal{C}$ is denoted $\mathsf{Im}\Pi$ and is defined as follows:*

$$\mathsf{Im}\Pi = \{y \in \mathcal{C} | y = \Pi(x) \textit{ for some } x \in \mathcal{D}\}.$$

Definition 47 (Kernel). *Let $\Pi : \mathcal{D} \to \mathcal{C}$ be an homomorphism. The kernel of this mapping, denoted $ker\Pi$, is defined as follows :*

$$\ker\Pi = \{(x, x')|\Pi(x) = \Pi(x')\}$$

the following equivalence relation:

$$x \overset{\ker\Pi}{\equiv} x' \iff \Pi(x) = \Pi(x')$$

is a congruence, according to Theorem 45.

Definition 48 (Coimage). *Quotient $\mathcal{D}_{/\ker\Pi}$ is the quotient of $\mathcal{D}$ by this congruence, it is called the coimage of Π. Hence, mapping $\mathcal{D}_{/\ker\Pi} \to \mathsf{Im}\Pi$, $[x]_\Pi \mapsto \Pi(x)$ is an isomorphism.*

Definition 49 (Closure mapping). *Let $\mathsf{Id}_\mathcal{D} : \mathcal{D} \to \mathcal{D}, x \mapsto x$ be the identity mapping. A closure mapping $\Pi : \mathcal{D} \to \mathcal{D}$ is such that:*

- *it is extensive : $\Pi \succeq \mathsf{Id}_\mathcal{D}$,*
- *it is idempotent : $\Pi \circ \Pi = \Pi$,*
- *it is isotone : $\forall x, x' \in \mathcal{D}, \;\; x \preceq x' \Rightarrow \Pi(x) \preceq \Pi(x')$.*

Conversely, if Π is isotone and $\Pi = \Pi \circ \Pi \preceq \mathsf{Id}_\mathcal{D}$ then Π is called a dual closure mapping.

Definition 50 (Canonical injection of a subset). *Let $\mathcal{U}$ a subset of $\mathcal{D}$. The canonical injection of $\mathcal{U}$ in $\mathcal{D}$ is denoted $\mathsf{Id}_{|\mathcal{U}} : \mathcal{U} \to \mathcal{D}$, and is defined as $\mathsf{Id}_{|\mathcal{U}}(u) = u$ for all $u \in \mathcal{U}$.*

Definition 51 (Mapping restriction to a domain). *Let $\Pi : \mathcal{D} \to \mathcal{C}$ and $\mathcal{U} \subset \mathcal{D}$. Mapping $\Pi_{|\mathcal{U}} : \mathcal{U} \to \mathcal{C}$ is such that*

$$\Pi_{|\mathcal{U}} \;\;=\;\; \Pi \circ \mathsf{Id}_{|\mathcal{U}}$$

where $\mathsf{Id}_{|\mathcal{U}} : \mathcal{U} \to \mathcal{D}$ is the canonical injection of $\mathcal{U}$ in $\mathcal{D}$. Mapping $\Pi_{|\mathcal{U}}$ will be called the restriction of Π to domain $\mathcal{U}$.

Remark 52. *According to this definition, it can be noticed that the canonical injection is the restriction of the identity mapping. Formally* $\mathsf{Id}_{|\mathcal{U}} : \mathcal{U} \to \mathcal{D}$ *with* $\mathcal{U} \subset \mathcal{D}$ *is the restriction of the identity mapping* $\mathsf{Id}_{\mathcal{D}}$ *to the domain* $\mathcal{U}$*, formally* $\mathsf{Id}_{\mathcal{D}|\mathcal{U}}$ *and is denoted* $\mathsf{Id}_{|\mathcal{U}}$ *in order to simplify notation since there is no ambiguity.*

Definition 53 (Image mapping)**.** *The image of* $\Pi : \mathcal{D} \to \mathcal{C}$ *is the canonical injection of* $\Pi(\mathcal{D})$ *in* $\mathcal{C}$ *; This mapping will be denoted* $I_{\mathsf{Im}\Pi}$.

Definition 54 (Mapping Restriction to a co-domain)**.** *Let* $\Pi : \mathcal{D} \to \mathcal{C}$ *and* $\mathsf{Im}\Pi \subset \mathcal{V} \subset \mathcal{C}$. *Mapping* ${}_{\mathcal{V}|}\Pi : \mathcal{D} \to \mathcal{V}$ *is defined as follows*

$$\Pi = \left(\mathsf{Id}_{|\mathcal{V}}\right) \circ \left({}_{\mathcal{V}|}\Pi\right)$$

where $\mathsf{Id}_{|\mathcal{V}} : \mathcal{V} \to \mathcal{C}$ *represents the canonical injection of* $\mathcal{V}$ *in* $\mathcal{C}$. *Mapping* ${}_{\mathcal{V}|}\Pi$ *will be called the restriction of* Π *to the co-domain* $\mathcal{V}$.

2.4 Fixed points of monotone mappings

This section presents some useful results in order to deal with fixed point equations. Actually, it will be recall that iterative algorithms can be used to compute fixed points of equations involving monotone mappings. The results are based on Knaster-Tarski theorem which states that the set of fixed points of an order preserving mapping Π, defined over a complete lattice, is also a complete lattice. This theorem guarantees the existence of at least one fixed point of Π, and even the existence of a least (or greatest) fixed point. Assuming some properties of continuity the Kleene fixed-point theorem states that the least fixed point is the supremum of the increasing Kleene chain of Π.

Below the results are adapted to the setting of semirings by recalling that a complete semiring is a complete lattice (see Remark 25).

Theorem 55. *Let* $\Pi : \mathcal{D} \to \mathcal{D}$ *be an isotone mapping with* $\mathcal{D}$ *a complete semiring. Let* $\mathcal{Y} = \{x \in \mathcal{D} | \Pi(x) = x\}$ *be the set of fixed points of* Π.

1. $\bigwedge_{y \in \mathcal{Y}} y \in \mathcal{Y}$, *and* $\bigwedge_{y \in \mathcal{Y}} y = \bigwedge\{x \in \mathcal{D} | \Pi(x) \preceq x\}$ *is the least fixed point of* Π.

2. $\bigvee_{y \in \mathcal{Y}} y \in \mathcal{Y}$, *and* $\bigvee_{y \in \mathcal{Y}} y = \bigvee\{x \in \mathcal{D} | x \preceq \Pi(x)\}$ *is the greatest fixed point of* Π.

Since $\mathcal{D}$ is complete, Theorem 55 ensures existence of both least and greatest fixed point of monotone mapping defined over an idempotent semiring. Below a constructive algorithm yielding the greatest fixed point is given.

Remark 56. *By considering the mapping $\Pi : x \mapsto \Psi(x) \wedge x_i$, with $\Psi : \mathcal{D} \to \mathcal{D}$ an isotone mapping and $x_i \in \mathcal{D}$, the algorithm will give the greatest fixed point of Π which corresponds to the greatest fixed point of Ψ lower than or equal to x_i.*

Remark 57. *A dual algorithm can be used to find a least fixed point, it is sufficient to start the algorithm with $x_0 = \bigwedge \mathcal{D} = \varepsilon_{\mathcal{D}}$. In this case by considering the mapping $\Pi : x \mapsto \Psi(x) \oplus x_i$, with $\Psi : \mathcal{D} \to \mathcal{D}$ an isotone mapping and $x_i \in \mathcal{D}$, the algorithm will give the least fixed point of Π which corresponds to the least fixed point of Ψ greater than or equal to x_i.*

When the mapping Π is semi-continuous the following theorem can be considered (see [5], section 4.5).

Theorem 58. *Let $\mathcal{D}$ be a complete semiring and $\Pi : \mathcal{D} \to \mathcal{D}$ be a mapping and $\mathcal{Y} = \{x \in \mathcal{D} | \Pi(x) = x\}$ be the set of fixed points of Π. The two following statements hold :*

1. *if Π is lower semi-continuous $(l.s.c.)$ then $\bigwedge_{y \in \mathcal{Y}} y = \Pi^*(\bigwedge_{x \in \mathcal{D}} x)$,*

2. *if Π is upper semi-continuous $(u.s.c)$ then $\bigvee_{y \in \mathcal{Y}} y = \Pi_*(\bigvee_{x \in \mathcal{D}} x)$,*

where Π^* and Π_* are defined as follows :

$$\begin{aligned} \Pi^*(x) &= \bigoplus_{i \succeq 0} \Pi^i(x), \\ \Pi_*(x) &= \bigwedge_{i \succeq 0} \Pi^i(x), \end{aligned}$$

with Π^0 is the identity mapping, and for each $i \succeq 0$, $\Pi^{i+1} = \Pi \circ \Pi^i$.

2.4.1 Properties of the Kleene star operator

The mapping $\mathcal{S} : \mathcal{D} \to \mathcal{D}, \mapsto x^* = \bigoplus_{i \in \mathbb{N}} x^i$ is frequently involved. Hence this section is dedicated to recall specific properties of this mapping and more specifically to the Kleene star operator denoted x^*. The following mapping $P : x \mapsto x^+ = \bigoplus_{k \geq 1} x^k$ is also of interest and is considered. Of course $x^* = e \oplus x^*$. According to Definition 49 the both mappings are closure mappings.

Theorem 59. *The implicit equation*

$$x = a \otimes x \oplus b \tag{2.8}$$

has $x = a^* b = (\bigoplus_{k \geq 0} a^k) b$ as smallest solution.

Remark 60. *The proof could be done by using Theorem 58 and the* $l.s.c.$ *mapping* $\Pi : \mathcal{D} \to \mathcal{D}, x \mapsto a \otimes x \oplus b$.

$$\begin{array}{rcl} \Pi^*(x) = \bigoplus_{i\in\mathbb{N}} \Pi^i(x) & = & \Pi^0(x) \oplus \Pi(x) \oplus \Pi^2(x) \oplus ... \\ & = & x \oplus (ax \oplus b) \oplus (a(ax \oplus b) \oplus b) \oplus (a(a(ax \oplus b) \oplus b) \oplus b) \oplus ... \\ & = & a^*x \oplus a^*b = a^*(x \oplus b). \end{array}$$

Hence the smallest fixed point is $\Pi^*(\bigwedge_{x\in\mathcal{D}} x) = \Pi^*(\varepsilon) = a^*b$.

It must be noted that this fixed point is also the smallest solution of the inequality $a \otimes x \oplus b \preceq x$.

Property 61. *Let* $\mathcal{D}$ *be a complete semiring.* $\forall a, b \in \mathcal{D}$

$$a^+ \preceq a^* \tag{2.9}$$
$$a^*a^* = a^* \tag{2.10}$$
$$(a^*)^* = a^* \tag{2.11}$$
$$(a^+)^* = a^* \tag{2.12}$$
$$a(ba)^* = (ab)^*a \tag{2.13}$$
$$(a \oplus b)^* = (a^*b)^*a^* = b^*(ab^*)^* = (a \oplus b)^*a^* = b^*(a \oplus b)^* \tag{2.14}$$
$$(a^*)^+ = a^* \tag{2.15}$$
$$(a^+)^+ = a^+ \tag{2.16}$$
$$(ab^*)^+ = a(a \oplus b)^* \tag{2.17}$$
$$(ab^*)^* = e \oplus a(a \oplus b)^* \tag{2.18}$$

Furthermore, if $\mathcal{D}$ *is commutative* ($i.e.$, $a \otimes b = b \otimes a$) *then*

$$(a \oplus b)^* = a^*b^*. \tag{2.19}$$

2.5 Residuation theory

In general the mappings defined on ordered sets have not inverse mappings. Nevertheless, by considering some assumption about continuity, the residuation theory yields an answer to some problems like : what is the greatest solution of inequality $\Pi(x) \preceq b$? Or dually what is the least solution of inequality $\Pi(x) \succeq b$? In particular, it is possible to characterize some residuated mappings which are a kind of pseudo-inverse mappings.

This theory is very close, in consideration of the inversion of the order relation, of the Galois theory. Indeed, from a mapping and its residual it is possible to obtain a Galois connection. For these points the readers are invited to consult the following references : [3].

About residuation theory and for historical references, the readers can consult [4]. In this chapter this theory is considered in the semiring framework, according to the chapter 4 of [5], and to the following references [8], [2], the following PhD can also be useful to get some refinements [7,9,10]. Even if majority of the results are from these references (especially [8], [2]), the proofs are recalled for their pedagogical aspect and they constitute interesting exercises.

Definition 62 (Lower set, closed lower set, upper set, closed upper set)**.** *A lower set is a nonempty subset L of $\mathcal{D}$ such that*

$$(x \in L \text{ and } y \preceq x) \Rightarrow y \in L.$$

A closed lower set (generated by k) is a lower set denoted $_{\downarrow}\mathcal{K} = \{x | x \preceq k\}$. An upper set is a subset U of $\mathcal{D}$ such that

$$(u \in U \text{ and } y \preceq u) \Rightarrow y \in U.$$

A closed upper set (generated by k) is an upper set denoted $_{\uparrow}\mathcal{K} = \{x | x \succeq k\}$.

Definition 63 (Residuated mapping, dually residuated mapping)**.** *An isotone mapping $\Pi : \mathcal{D} \to \mathcal{B}$ is said to be residuated, if equation $\Pi(x) \preceq b$ has a greatest solution in $\mathcal{D}$ for all $b \in \mathcal{B}$.*

It is said dually residuated, if equation $\Pi(x) \succeq b$ has a least solution in $\mathcal{D}$ for all $b \in \mathcal{B}$.

The following theorems yield some necessary and sufficient conditions to characterize theses mappings.

Theorem 64 ([5],Th. 4.50)**.** *Let $\Pi : \mathcal{D} \to \mathcal{B}$ an isotone mapping. The following statements are equivalent*

1. *Π is residuated.*
2. *It exists an unique isotone and u.s.c. mapping denoted $\Pi^\sharp : \mathcal{B} \to \mathcal{D}$ such that $\Pi \circ \Pi^\sharp \preceq \mathsf{Id}_{\mathcal{B}}$ and $\Pi^\sharp \circ \Pi \succeq \mathsf{Id}_{\mathcal{D}}$.*
3. *$\Pi(\varepsilon_{\mathcal{D}}) = \varepsilon_{\mathcal{B}}$ and Π is l.s.c.*

when it exists the mapping $H^\sharp$ is called the residual of the residuated mapping H.

Theorem 65 ([5],Th. 4.52)**.** *Let $\Phi : \mathcal{D} \to \mathcal{B}$ be an isotone mapping. The following statements are equivalent :*

1. *Φ is dually residuated,*
2. *it exists an unique isotone and l.s.c. mapping $\Phi^\flat : \mathcal{B} \to \mathcal{D}$ such that $\Phi \circ \Phi^\flat \succeq \mathsf{Id}_{\mathcal{B}}$ and $\Phi^\flat \circ \Phi \preceq \mathsf{Id}_{\mathcal{D}}$.*
3. *$\Phi(\top_{\mathcal{D}}) = \top_{\mathcal{B}}$ and Φ is u.s.c.*

Remark 66. *According to Theorems 64 and 65, it is clear that $\Pi^\sharp$ is dually residuated and that $\Phi^\flat$ is residuated, furthermore $(\Pi^\sharp)^\flat = \Pi$ and $(\Phi^\flat)^\sharp = \Phi$.*

Theorem 67 ([5],Th. 4.56). *Let $\Pi : \mathcal{D} \to \mathcal{B}$ be a residuated mapping, the following statements hold :*

$$\Pi \circ \Pi^{\sharp} \circ \Pi = \Pi, \tag{2.20}$$

$$\Pi^{\sharp} \circ \Pi \circ \Pi^{\sharp} = \Pi^{\sharp}, \tag{2.21}$$

$$\Phi \circ \Phi^{\flat} \circ \Phi = \Psi, \tag{2.22}$$

$$\Phi^{\flat} \circ \Phi \circ \Phi^{\flat} = \Psi^{\flat}, \tag{2.23}$$

$$\Pi^{\sharp} \text{ is dually residuated and } (\Pi^{\sharp})^{\flat} = \Pi, \tag{2.24}$$

$$\Psi^{\flat} \text{ is residuated and } (\Psi^{\flat})^{\sharp} = \Psi, \tag{2.25}$$

$$\Pi^{\sharp} \circ \Pi = \mathsf{Id}_{\mathcal{D}} \Leftrightarrow \Pi \text{ is injective } \Leftrightarrow \Pi^{\sharp} \text{ is surjective}, \tag{2.26}$$

$$\Pi \circ \Pi^{\sharp} = \mathsf{Id}_{\mathcal{B}} \Leftrightarrow \Pi^{\sharp} \text{ is injective } \Leftrightarrow \Pi \text{ is surjective}. \tag{2.27}$$

the same statements holds true for dually residuated mapping by replacing $\sharp$ by $\flat$.

Theorem 68. *Let $\Pi : \mathcal{D} \to \mathcal{B}$ and $\Psi : \mathcal{B} \to \mathcal{C}$ two residuated mappings and $\Phi : \mathcal{D} \to \mathcal{B}$ and $\Theta : \mathcal{B} \to \mathcal{C}$ two dually residuated mappings, the following statement holds:*

1. $(\Psi \circ \Pi)^{\sharp} = \Pi^{\sharp} \circ \Psi^{\sharp}$,
2. $(\Theta \circ \Phi)^{\flat} = \Phi^{\flat} \circ \Theta^{\flat}$.

Theorem 69 ([5], Th. 4.56). *Let $\Pi : \mathcal{D} \to \mathcal{B}$ and $\Psi : \mathcal{D} \to \mathcal{B}$ be residuated mappings. The following properties hold :*

1. $\Pi \preceq \Psi \Leftrightarrow \Psi^{\sharp} \preceq \Pi^{\sharp}$,
2. $(\Pi \oplus \Psi)^{\sharp} = \Pi^{\sharp} \wedge \Psi^{\sharp}$.

Let $\Theta : \mathcal{D} \to \mathcal{B}$ and $\Phi : \mathcal{D} \to \mathcal{B}$ be dually residuated mappings. The following properties hold :

3. $\Phi \preceq \Theta \Leftrightarrow \Theta^{\flat} \preceq \Phi^{\flat}$,

4. $(\Phi \wedge \Theta)^\flat = \Phi^\flat \oplus \Theta^\flat$.

Theorem 70 ([8]). *Let $\Pi : \mathcal{D} \to \mathcal{B}$ and $\Psi : \mathcal{D} \to \mathcal{B}$ be two residuated mappings, then the following equivalence holds :*

$$\mathsf{Im}\,\Pi \subset \mathsf{Im}\Psi \Leftrightarrow \Psi \circ \Psi^\sharp \circ \Pi = \Pi.$$

Theorem 71 ([8, 11], Projection on the image of a mapping). *Let $\Pi : \mathcal{D} \to \mathcal{B}$ be a residuated mapping, mapping $P_\Pi = \Pi \circ \Pi^\sharp$ is a dual closure mapping. Furthermore $P_\Pi(b)$, with $b \in \mathcal{B}$, is the greatest element in $\mathsf{Im}\Pi$ less than or equal to b.*
Let $\Psi : \mathcal{D} \to \mathcal{C}$ be a dually residuated mapping, mapping $P_\Psi = \Psi \circ \Psi^\flat$ is a closure mapping. Furthermore $P_\Psi(c)$, with $c \in \mathcal{C}$, is the lowest element in $\mathsf{Im}\Psi$ greater than or equal to c.

Theorem 72 ([4]). *Let $\mathcal{C}$ be a complete subsemiring of $\mathcal{D}$. Let $\mathsf{Id}_{|\mathcal{C}} : \mathcal{C} \to \mathcal{D}$, $x \mapsto x$ be the canonical injection. The injection $\mathsf{Id}_{|\mathcal{C}}$ is both residuated and dually residuated and their residuals are projectors.*

Theorem 73. *Let $\Pi : \mathcal{D} \to \mathcal{B}$ be a mapping. The following statements hold :*

1. *if Π is residuated then ${}_{\mathcal{E}|}\Pi$ is residuated, with $\mathcal{E}$ such that $\mathsf{Im}\Pi \subset \mathcal{E} \subset \mathcal{B}$ and*
$$({}_{\mathcal{E}|}\Pi)^\sharp = \Pi^\sharp \circ \mathsf{Id}_{|\mathcal{E}} = \Pi^\sharp_{|\mathcal{E}};$$
2. *if Π is residuated then $\Pi_{|\mathcal{C}}$ is residuated, with $\mathcal{C}$ such that $\mathsf{Im}\Pi^\sharp \subset \mathcal{C} \subset \mathcal{D}$ and*
$$(\Pi_{|\mathcal{C}})^\sharp = {}_{\mathcal{C}|}\Pi^\sharp;$$
3. *if Π is dually residuated then ${}_{\mathcal{E}|}\Pi$ is dually residuated, with $\mathcal{E}$ such that $\mathsf{Im}\Pi \subset \mathcal{E} \subset \mathcal{B}$ and*
$$({}_{\mathcal{E}|}\Pi)^\flat = \Pi^\flat \circ \mathsf{Id}_{|\mathcal{E}} = \Pi^\flat_{|\mathcal{E}};$$
4. *if Π is dually residuated then $\Pi_{|\mathcal{C}}$ is dually residuated, with $\mathcal{C}$ such that $\mathsf{Im}\Pi^\flat \subset \mathcal{C} \subset \mathcal{D}$ and*
$$(\Pi_{|\mathcal{C}})^\flat = {}_{\mathcal{C}|}\Pi^\flat.$$

Theorem 74 ([4]). *Let $\Pi : \mathcal{D} \to \mathcal{D}$ a closure mapping. The restriction ${}_{\mathsf{Im}\Pi|}\Pi$ is residuated and its residual is*

$$\left({}_{\mathsf{Im}\Pi|}\Pi\right)^\sharp = \mathsf{Id}_{|\mathsf{Im}\Pi}$$

with $\mathsf{Id}_{|\mathsf{Im}\Pi}$ the canonical injection of $\mathsf{Im}\Pi$ in $\mathcal{D}$.

Example 75. *The following mappings are considered :*

$$\begin{array}{rcccl} L_a & : & \mathcal{D} & \to & \mathcal{D} \\ & : & x & \mapsto & a \otimes x \qquad \text{(left product by } a), \\ R_a & : & \mathcal{D} & \to & \mathcal{D} \\ & : & x & \mapsto & x \otimes a \qquad \text{(right product by } a). \end{array} \tag{2.28}$$

According to semiring definition (see Definition 22) these mappings are $l.s.c$ ($L_a(x_1 \oplus x_2) = L_a(x_1) \oplus L_a(x_2)$) and such that $L_a(\varepsilon) = \varepsilon$), hence according to Definition 64 these mappings are residuated. The residual mappings are denoted :

$$\begin{array}{rcll} L_a^\sharp(x) & = & a \backslash\!\!\circ\, x & \text{(left division by } a), \\ R_a^\sharp(x) & = & x \,\circ\!\!/\, a & \text{(right division by } a). \end{array} \tag{2.29}$$

Example 76 ([4], [12], [8], [9]). *it is possible to show that the following mappings are residuated, with $\mathcal{D}$ and $\mathcal{C}$ some ordered sets and $\mathcal{C}^{\text{op}}$ the dual set $\mathcal{C}$, $i.e.$ the same set endowed with the opposite order ($i.e.$, $a \preceq_{\mathcal{C}^{\text{op}}} b \Leftrightarrow a \succeq_{\mathcal{C}} b$).*

$$\begin{array}{rcccl} \Lambda_a & : & \mathcal{D} & \to & \mathcal{C}^{\text{op}} \\ & & x & \mapsto & x \backslash\!\!\circ\, a, \\ \Psi_a & : & \mathcal{C}^{\text{op}} & \to & \mathcal{D} \\ & & x & \mapsto & a \,\circ\!\!/\, x. \end{array} \tag{2.30}$$

The residual mappings are given below :

$$\begin{array}{rcccl} \Lambda_a^\sharp = \Psi_a & : & \mathcal{C}^{\text{op}} & \to & \mathcal{D} \\ & & x & \mapsto & a \,\circ\!\!/\, x, \\ \Psi_a^\sharp = \Lambda_a & : & \mathcal{D} & \to & \mathcal{C}^{\text{op}} \\ & & x & \mapsto & x \backslash\!\!\circ\, a. \end{array} \tag{2.31}$$

Remark 77. *This result shows that the greatest solution in $\mathcal{D}$ of inequality $x \backslash\!\!\circ\, a \succeq b$ is $a \,\circ\!\!/\, b$, and $b \backslash\!\!\circ\, a$ is the greatest solution of the inequality $a \,\circ\!\!/\, x \succeq b$.*

Theorem 78 ([5]). *The implicit equation*

$$x = a \backslash\!\!\circ\, x \wedge b$$

admits $x = a^* \backslash\!\!\circ\, b$ as greatest solution.

Proof : : First note that mapping $\Pi : \mathcal{D} \to \mathcal{D}, x \mapsto a \backslash\!\!\circ\, x \wedge b$ is $u.s.c.$, $i.e.$, $\Pi(x_1 \wedge x_2) = \Pi(x_1) \wedge \Pi(x_2)$. Hence according to Theorem 58,

$$\begin{array}{rcl} \Pi^*(x) = \bigwedge_{i \in \mathbb{N}} \Pi^i(x) & = & \Pi^0(x) \wedge \Pi(x) \wedge \Pi^2(x) \wedge \ldots \\ & = & x \wedge (a \backslash\!\!\circ\, x \wedge b) \wedge (a \backslash\!\!\circ\,(a \backslash\!\!\circ\, x \wedge b) \wedge b) \wedge (a \backslash\!\!\circ\,(a \backslash\!\!\circ\,(a \backslash\!\!\circ\, x \wedge b) \wedge b) \wedge b) \wedge \ldots \\ & = & x \wedge (a \backslash\!\!\circ\, x \wedge b) \wedge (a^2 \backslash\!\!\circ\, x \wedge a \backslash\!\!\circ\, b) \wedge b) \wedge (a^3 \backslash\!\!\circ\, x \wedge a^2 \backslash\!\!\circ\, b) \wedge a \backslash\!\!\circ\, b) \wedge b) \wedge \ldots \\ & = & a^* \backslash\!\!\circ\, x \wedge a^* \backslash\!\!\circ\, b = a^* \backslash\!\!\circ\,(x \wedge b). \end{array}$$

These developments are mainly based on equations **??** and **??**. Hence the greatest fixed point is obtained by considering $\Pi(\top) = a^* \backslash\!\!\circ\, b$. □

Below the closure mappings are considered. It is recalled that a closure mapping is residuated if its co-domain is restricted to its image.

Example 79 ([7]). *Mapping $S : \mathcal{D} \to \mathcal{D}, x \mapsto x^*$ is a closure mapping. Hence $({}_{\mathsf{Im}S|}S)$ is residuated and its residual is $({}_{\mathsf{Im}S|}S)^\sharp = I_{\mathsf{Im}S}$. In other words, inequality $x^* \preceq a^*$ admits $x = a^*$ as greatest solution.*

Example 80 ([7]). *Mapping $P : \mathcal{D} \to \mathcal{D}, x \mapsto \bigoplus_{i \geq 1} x^i = x^+$ is a closure mapping. Hence $({}_{\mathsf{Im}P|}P)$ is residuated and its residual is $({}_{\mathsf{Im}P|}P)^\sharp = I_{\mathsf{Im}P}$. In other words, inequality $x^+ \preceq a^+$ admits $x = a^+$ as greatest solution.*

Example 81 ([13]). *Isotone mapping $Q_a : \mathcal{D} \to \mathcal{D}, x \mapsto (xa)^*x$ is a closure mapping. Hence ${}_{\mathsf{Im}Q_a|}Q_a$ is residuated and its residual is $({}_{\mathsf{Im}Q_a|}Q_a)^\sharp = I_{\mathsf{Im}Q_a}$. In otehr words, inequality $(xa)^*x \preceq b$, with $b \in \mathsf{Im}Q_a$, admits $x = b$ as greatest solution, furthermore the following equlity holds $(ba)^*b = b$.*

Example 82 ([7], [14]). *Let $M_h : \mathcal{D} \to \mathcal{D}, x \mapsto h(xh)^*$ an isotone mapping defined over complete idempotent semiring and sets*

$$\begin{aligned} \mathcal{G}_1 &= \{g \in \mathcal{D} \mid \exists a \in \mathcal{D}, g = a^*h\}, \\ \mathcal{G}_2 &= \{g \in \mathcal{D} \mid \exists b \in \mathcal{D}, g = hb^*\}. \end{aligned}$$

It can be shown that ${}_{\mathcal{G}_1|}M_h$ et ${}_{\mathcal{G}_2|}M_h$ are residuated with:

$$({}_{\mathcal{G}_1|}M_h)^\sharp(x) = h \circ\!\!\!\backslash\, x \circ\!\!\!/\, h,$$

$$({}_{\mathcal{G}_2|}M_h)^\sharp(x) = h \circ\!\!\!\backslash\, x \circ\!\!\!/\, h.$$

Below, it is recalled that the canonical injection from a complete subsemiring into a complete semiring is residuated.

Theorem 83 (Projection Lemma [4]). *Let $\mathcal{D}$ be a complete semiring and $\mathcal{D}_{sub}$ a complete subsemiring of $\mathcal{D}$. The canonical injection $I_{\mathcal{D}_{sub}} : \mathcal{D}_{sub} \to \mathcal{D}, x \mapsto x$ is residuated. The residual is denoted as $\mathsf{Pr}_{\mathcal{D}_{sub}} = I^\sharp_{\mathcal{D}_{sub}}$ and is such that :*

(i) $\mathsf{Pr}_{\mathcal{D}_{sub}} \circ \mathsf{Pr}_{\mathcal{D}_{sub}} = \mathsf{Pr}_{\mathcal{D}_{sub}}$,

(ii) $\mathsf{Pr}_{\mathcal{D}_{sub}} \preceq \mathsf{Id}_{\mathcal{D}}$,

(iii) $x \in \mathcal{D}_{sub} \iff \mathsf{Pr}_{\mathcal{D}_{sub}}(x) = x$.

Proposition 84 ([7]). *Let $\Pi : \mathcal{C} \to \mathcal{D}$ a residuated mapping defined over complete idempotent semiring and $I_{\mathcal{C}_{sub}}$ the canonical injection of the subsemiring $\mathcal{C}_{sub}$ into $\mathcal{C}$. Mapping $\Pi \circ I_{\mathcal{C}_{sub}}(x) \preceq b$ is residuated and its residual is given by*

$$\left(\Pi_{|\mathcal{C}_{sub}}\right)^\sharp(b) = \left(\Pi \circ I_{\mathcal{C}_{sub}}\right)^\sharp(b) = \mathsf{Pr}_{\mathcal{C}_{sub}} \circ \Pi^\sharp(b). \tag{2.32}$$

Proposition 85. *Let $\Pi : \mathcal{C} \to \mathcal{D}$ be a residuated mapping defined over complete idempotent semiring and $I_{\mathcal{D}_{sub}}$ the canonical injection of the complete subsemiring $\mathcal{D}_{sub}$ (with $\mathsf{Im}\Pi \subset \mathcal{D}_{sub} \subset \mathcal{D}$) into $\mathcal{D}$. Mapping ${}_{\mathcal{D}_{sub}|}\Pi$ is residuated and*

$$\left({}_{\mathcal{D}_{sub}|}\Pi\right)^{\sharp} \quad = \quad \Pi^{\sharp} \circ I_{\mathcal{D}_{sub}} = \left(\Pi^{\sharp}\right)_{|\mathcal{D}_{sub}}.$$

The classical kernel definition of a mapping ($i.e.$, set $\{x|\Pi(x) = \varepsilon\}$) has a weak sense when mappings are defined over lattices. Hence the following definition is classically considered for these mappings.

Definition 86 (Kernel)**.** *The kernel of mapping $C : \mathcal{X} \to \mathcal{Y}$, denoted $\ker C$, is defined by the following equivalence relation*

$$x \overset{\ker C}{\equiv} x' \quad \Longleftrightarrow \quad C(x) = C(x'). \tag{2.33}$$

This relation defines a congruence. The quotient set $\mathcal{X}_{/\ker C}$ is then the set of equivalence classes modulo $\ker C$.

Notation 87. *An equivalence class of $\mathcal{X}_{/\ker C}$ will be denoted $[x]_C$.*

Proposition 88 ([15])**.** *If $C : \mathcal{X} \to \mathcal{Y}$ ia a residuated mapping then each equivalence class $[x]_C$ has one and only one element of $\mathsf{Im}C^{\sharp}$, furthermore it is the greatest element of this class.*

2.6 Semiring $\overline{\mathbb{Z}}_{\max}$

This section aims to present the semiring (max,+), denoted $\overline{\mathbb{Z}}_{\max}$ and already introduced in example 29.

2.6.1 Matrices sum

Let A and B be two matrices $\in \overline{\mathbb{Z}}_{\max}^{n\times p}$, the matrices sum is a matrix defined as follows:

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$$

Example 89. *Let $A = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} e & 8 \\ 1 & 3 \end{pmatrix}$ the sum is equal to :*

$$A \oplus B = \begin{pmatrix} 2 & 8 \\ 3 & 7 \end{pmatrix}$$

2.6.2 Matrices product

Let $A \in \overline{\mathbb{Z}}_{\max}^{m\times p}$, $B \in \overline{\mathbb{Z}}_{\max}^{p\times n}$ and $C \in \overline{\mathbb{Z}}_{\max}^{m\times n}$ be three matirces, the matrices product is defined as follow :

$$C_{ij} = (A \otimes B)_{ij} = \bigoplus_{k=1}^{p} A_{ik} \otimes B_{kj}$$

In the sequel the null matrix will be denoted ε, $i.e.$ the matrix whose all entries are equal to ε. In the same manner the identity matirx will be denoted e, $i.e.$ the matrix whose all entries are equal to ε excepted diagonal entries which are equal to e.
By extension for $n \in \mathbb{N}$, $A^n = \underbrace{A \otimes A \otimes ... \otimes A}_{nfois}$ with $A^0 = e$ the identity matrix.

Example 90. *Let* $A = \begin{pmatrix} 2 & 5 \\ \varepsilon & 3 \\ 1 & 8 \end{pmatrix}$ *and* $B = \begin{pmatrix} e \\ 1 \end{pmatrix}$ *be matrices :*

$$C = A \otimes B = \begin{pmatrix} 6 \\ 4 \\ 9 \end{pmatrix}$$

. Let us recall that $e = 0$ *and* $\varepsilon = -\infty$.

2.7 Equation $x = ax \oplus b$

This equation can admit an infinity of solution but according to corrollary **??**, it admits a least solution, denoted a^*b with $a^* = \bigoplus_{i \in \mathbb{N}} a^i$. Classicaly, the star computation of matrix can be done by considering the star of scalar, by considering a king of Gauss elimintation. Below it is the Jordan algorithm which is proposed to compute the star of matrix $A \in \overline{\mathbb{Z}}_{\max}^{n \times n}$.

$$
\begin{array}{l}
A^{(0)} = A; \\
for(k = 1; k == n; k++) \\
\{ \\
for(i = 1; i == n; i++) \\
\{ \\
for(j = 1; j == n; j++) \\
\{ \\
A_{ij}^{(k)} = A_{ij}^{(k-1)} \oplus A_{ik}^{(k-1)} (A_{kk}^{(k-1)})^* A_{kj}^{(k-1)} \\
\} \\
\} \\
\} \\
A^* = e \oplus A^{(n)}
\end{array}
$$

Remark 91. *It is possible to associate a graph to these matrices. Let* A *be a matrix such that* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$*, it corresponds to the graph of figure 5. Entry* a_{ij} *characterizes the weight associated to the edge lying node* j *to* i*. Then* A^k *yields the weight of path of length* k *lying each node of the graph, i.e.* $(A^k)_{ij}$ *represents the greatest weight of the path between* j *and* i *whose length is* k*.*

Example 92. *Let*

$$x = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & 3 \\ 4 & \varepsilon & \varepsilon \end{pmatrix} x \oplus \begin{pmatrix} 2 \\ \varepsilon \\ 5 \end{pmatrix}$$

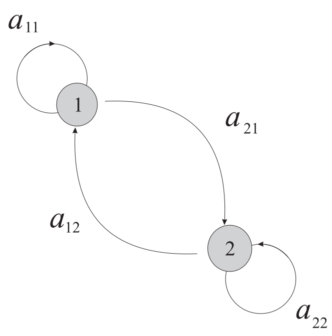


Figure 5 – The graph corresponding to matrix of size 2×2

be an implicit equation, with $x \in \overline{\mathbb{Z}}^3_{\max}$, the computation of a^* yields :

$$a^* = \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \end{pmatrix} \oplus \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & 3 \\ 4 & \varepsilon & \varepsilon \end{pmatrix} \oplus \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ 7 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \oplus \varepsilon = \begin{pmatrix} e & \varepsilon & \varepsilon \\ 7 & e & 3 \\ 4 & \varepsilon & e \end{pmatrix}$$

and the least solution is then

$$x = a^* b = \begin{pmatrix} 2 \\ 9 \\ 6 \end{pmatrix}$$

2.7.1 Equation $ax \preceq b$

In a complete idempotent semiring equation $ax \preceq b$ has a greatest solution denoted $a \backslash b$ (see example 75), it corresponds to the residuation of mapping $L_a : x \mapsto ax$. In the same manner $xa \preceq b$ admits b / a as greatest solution. By considering $A, D \in \overline{\mathbb{Z}}^{m\times n}_{\max}$, $B \in \overline{\mathbb{Z}}^{m\times p}_{\max}$, $C \in \overline{\mathbb{Z}}^{n\times p}_{\max}$, entry of matrix C:

$$\begin{aligned} C_{ij} &= \bigwedge_{k=1}^{m} (A_{ki} \backslash B_{kj}) \\ D_{ij} &= \bigwedge_{k=1}^{p} (B_{ik} / C_{jk}) \end{aligned}$$

Example 93. *Let $A = \begin{pmatrix} 2 & 5 \\ \varepsilon & 3 \\ 1 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 6 \\ 4 \\ 9 \end{pmatrix}$ be matrices then :*

$$C = A \backslash B = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

In the present case $A \otimes (A \backslash B) = B$ and it is to compare with example 90.

2.7.2 Spectral theory of matrices

Definition 94 (Graphe fortement connexe)**.** *A graph is said to be strongly connected if it exists a path between i and j two nodes, $\forall i, j$.*

Definition 95 (Irreducible Matrix)**.** *Matrix $A \in \overline{\mathbb{Z}}_{\max}^{n \times n}$ is said to be irreductible if the corresponding graph is strongly connected, conversely, A is said to be reducible.*

Definition 96 (Trace of a matrix)**.** *The trace of a matrix is classically defined:*

$$trace(A) = \bigoplus_{i=1}^{n} (A)_{ii}.$$

Remark 97. *Remark 91 said that $(A^j)_{ii}$ represents the maximal weight of all the circuit of length j crossing i. From defintion 96, it comes that $\bigoplus_{i=1}^{n} (A^j)_{ii} = trace(A^j)$ is the greatest among all the weight associated to each node $i \in [1, n]$.*

Definition 98. *$(trace(A^j))^{(\frac{1}{j})}$ corresponds to the mean weight associated to the circuits of length j.*

Definition 99.

$$\lambda = \bigoplus_{j=1}^{n} (trace(A^j))^{(\frac{1}{j})}$$

is called the maximal mean cycle of matrix A. It is the greatest mean weight for all paths whose length belongs to $[1, n]$ (n being the maximal length of an elementary circuit).

Definition 100 (Eigen vector, eigen value)**.** *Let $A \in \overline{\mathbb{Z}}_{\max}^{n \times n}$ be a matrix, λ a scalar, and $x \in \overline{\mathbb{Z}}_{\max}^{n}$ a vector. λ is an eigen value if :*

$$Ax = \lambda x$$

and x is an eigen vector.

Theorem 101. *If A is irreducible, it exists an unique eigen value. It is equal to the greates mean cycle of matrix A.*

Example 102. *Let $A = \begin{pmatrix} 2 & 5 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 3 & 3 \\ e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon \end{pmatrix}$ be a matrix, the greatest mean cycle of this irreducible matrix is equal to $\lambda = 8/3$.*

3 System Modelling and Control Theory

3.1 Model for systems subject to synchronization and delay

3.1.1 Daters equations, the event point of view

In the beginning of the $60th$ Cuninghame-Green started to represent manufacturing system in $(max, plus)$ algebra. In the very beginning of the $80th$, a team of INRIA Rocquencourt was interested by the description of Timed Event Graphs (TEGs) in $(max, plus)$ algebra (see e.g. [16]. TEGs constitute a subclass of timed Petri nets whose each place admits one and only one upstream transition and one and only one downstream transition.

Figure 6 represents a TEG. It is constituted of places (represented by circle) and transitions (represented by bars or rectangles) which are connected by directed arcs. Places contain an integer number of tokens (small black circles), and delays is associated to each place which represents the minimal time a token has to spend in the corresponding place (the lack of delay means the delay is equal to 0). Tokens move from place to place and represent the dynamical behavior of the graph, to cross a transition this one has to be fired. A transition is fired if each upstream place (connected to the transition thanks to a directed arc) has a valid token, $i.e.$ a token having spent the minimal time specified by the delay of the corresponding place, then the valid token is removed from the upstream places and added in all the downstream places. In Fig. 6 the input transition are labeled u_1,u_2, the internal transitions are labeled $x_i, i \in [1, 6]$ and y is the output transition. The TEGs model delay and synchronization phenomena, indeed the tokens have to spend a minimal duration in places and the firing of transition occurs when all places are with a valid token, it can be seen as a "rendez-vous" between tokens. This TEG can represent the behavior of an assembly line, constituted of 3 machines M_1, M_2 and M_3. The firing of the input transition u_1 characterizes the input of raw materials in the system, then a token is put in the place located between u_1 and x_1 and it must stay at least one time unit before to be considered as valid to contribute to the firing of the downstream transition x_1. The firing of transition x_1 represents the input of raw material in machine M_1. This firing will occur when a token is available in the place located between transitions x_2 et x_1 (this token means that a machine is available), and when a valid token will be in the place between transitions u_1 and x_1. The delay of 2 time units associated to the place between transitions x_1 and x_2 represents the processing time of the machine M_1, hence firing of transition x_2 represents the output of the machine M_1. The functioning rule of machine M_2 is the same, only the delays are different. Transition x_5 represents the input of machine M_3 which ensures the assembly of products coming from machines M_1 and M_2, it will be fired when a valid token will be available in the two upstream transitions. This machine is able to process three parts simultaneously.

The idea of [16, 17] was to show that these dynamical systems which are, $a\ priori$ non linear, can be represented by a linear system of equations in a specific algebraic structure. In particular they showed a TEG admits a canonical linear model in $(max, plus)$ algebra. The dynamical model considered is given below and is very reminiscent to the one of classical dynamical linear system :

$$x(k) = Ax(k-1) \oplus Bu(k) \tag{3.1a}$$

$$y(k) = Cx(k) \tag{3.1b}$$

To obtain this model a dater function is associated to each transition, this function aims to date the occurrence of the firing of the corresponding transitions (it means that the event point of view is considered). It will represent the life of the transition. Formally for transition x_j, the following function is considered : $\mathbb{Z} \to \mathbb{Z}, k \mapsto x_j(k)$ where $x_j(k)$ is the date of the firing of the token numbered k.

For the TEG of Fig. 6, this yields :

$$\begin{array}{rcl} x_1(k) & = & max(1 + u_1(k), x_2(k-1)) \\ x_2(k) & = & 2 + x_1(k) \\ x_3(k) & = & max(2 + u_2(k), x_4(k-1)) \\ x_4(k) & = & 5 + x_3(k) \\ x_5(k) & = & max(3 + x_4(k), 1 + x_2(k), x_6(k-3)) \\ x_6(k) & = & 2 + x_5(k) \\ y(k) & = & x_6(k) \end{array} \tag{3.2}$$

The operator $plus$ models the delay and the operator max models the synchronization between two dater functions. These non-linear equations become linear equations in the idempotent semiring $\mathbb{Z}_{\max}$ (cf. definition 29), then :

$$\begin{array}{rcl} x_1(k) & = & 1 \otimes u_1(k) \oplus x_2(k-1) \\ x_2(k) & = & 2 \otimes x_1(k) \\ x_3(k) & = & 2 \otimes u_2(k) \oplus x_4(k-1) \\ x_4(k) & = & 5 \otimes x_3(k) \\ x_5(k) & = & 3 \otimes x_4(k) \oplus 1 \otimes x_2(k) \oplus x_6(k-3) \\ x_6(k) & = & 2 \otimes x_5(k) \\ y(k) & = & x_6(k) \end{array} \tag{3.3}$$

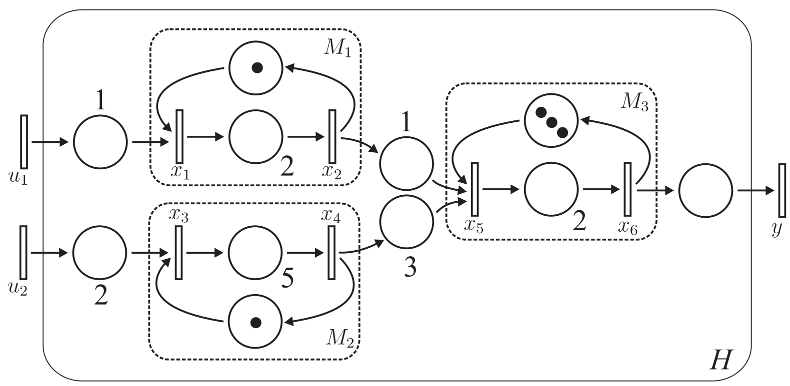


Figure 6 – A Timed Event Graph which can represent a manufacturing system with three machines labeled M_1 to M_3.

or, by considering vector notation, and vector state : $x(k) = \begin{pmatrix} x_1(k) & x_2(k) & x_3(k) & x_4(k) & x_5(k) & x_6(k) \end{pmatrix}^t$, input vector $u(k) = \begin{pmatrix} u_1(k) & u_2(k) \end{pmatrix}^t$ and output vector $y(k) = (y(k))$:

$$x(k) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon & 3 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon \end{pmatrix} x(k) \oplus \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} x(k-1) \oplus \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} x(k-2)$$

$$\oplus \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} x(k-3) \oplus \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & 2 \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix} x(k)$$

In a general manner, the model is obtained under the following formalism over $\overline{\mathbb{Z}}_{max}$:

$$\begin{aligned} x(k) &= \bigoplus_{i=0}^{a} A_i x(k-i) \oplus \bigoplus_{j=0}^{b} B_j u(k-j), \\ y(k) &= \bigoplus_{l=0}^{c} C_l x(k-l). \end{aligned}$$

After some modifications, it is possible to obtain an explicit form with a recurrence of 1 on the vector state. The system admits then the following formalism :

$$\begin{aligned} x(k) &= A_0 x(k) \oplus A_1 x(k-1) \oplus B_0 u(k), \\ y(k) &= C_0 x(k). \end{aligned}$$

Practically, this is obtained by enlarging the graph in order to guarantee that each place be initially with at the most one token, figure 7 yields an extension of the TEG of figure 6, and leads to the model :

$$x(k) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} x(k) \oplus \begin{pmatrix} \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \end{pmatrix} x(k-1) \oplus \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & 2 \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \end{pmatrix} x(k)$$

Equation (3.1.3.1) is an implicit equation with $x(k)$.

$$x(k) = Ax(k-1) \oplus Bu(k) \tag{3.4}$$

$$y(k) = Cx(k) \tag{3.5}$$

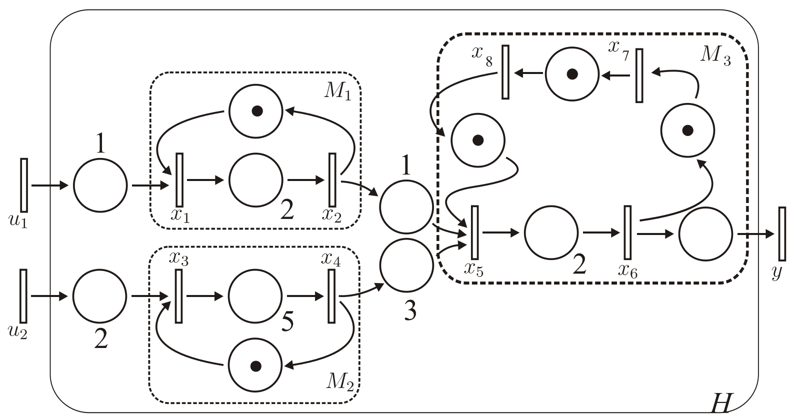


Figure 7 – Example an assembly line (event extension).

with $A = A_0^* A_1$ and $B = A_0^* B_0$. For this example this leads to :

$$A_0^* = \begin{pmatrix} e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 2 & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 5 & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & 1 & 8 & 3 & 0 & \varepsilon & \varepsilon & \varepsilon \\ 5 & 3 & 10 & 5 & 2 & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix} \quad A = A_0^* A_1 = \begin{pmatrix} \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 3 & \varepsilon & 8 & \varepsilon & \varepsilon & \varepsilon & e \\ \varepsilon & 5 & \varepsilon & 10 & \varepsilon & \varepsilon & \varepsilon & 2 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \end{pmatrix} \quad B = A_0^* B_0 = \begin{pmatrix} 1 & \varepsilon \\ 3 & \varepsilon \\ \varepsilon & 2 \\ \varepsilon & 7 \\ 4 & 10 \\ 6 & 12 \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}$$

The $(max, +)$ toolbox of Scilab, is very efficient to handle this kind of model (see <http://scilab.org/contib> and also <http://maxplus.org>).

3.1.2 Counters Equations, the time point of view

From a dual point of view, the behavior of the previous system can be described by considering a dynamic system in the time domain. A counter function is associated to each transition, it aims to count the number of firing at a time t for the corresponding transition, formally for transition $x_j : \mathbb{Z} \to \mathbb{Z}, t \mapsto x_j(t)$ with $x_j(t)$ the number of firing for transition x_j.

For TEG of figure 6, the following system is obtained :

$$\begin{array}{rcl} x_1(t) & = & min(u_1(t-1), 1 + x_2(t)) \\ x_2(t) & = & x_1(t-2) \\ x_3(t) & = & min(u_2(t-2), 1 + x_4(t)) \\ x_4(t) & = & x_3(t-5) \\ x_5(t) & = & min(x_4(t-3), x_2(t-1), 3 + x_6(t)) \\ x_6(t) & = & x_5(t-2) \\ y(t) & = & x_6(t) \end{array} \tag{3.6}$$

These dynamic equations are actually linear in the idempotent semiring $\overline{\mathbb{Z}}_{\min}$ (cf. example 30), therefore :

$$
\begin{array}{rcl}
x_1(t) & = & u_1(t-1) \oplus 1 \otimes x_2(t) \\
x_2(t) & = & x_1(t-2) \\
x_3(t) & = & u_2(t-2) \oplus 1 \otimes x_4(t) \\
x_4(t) & = & x_3(t-5) \\
x_5(t) & = & x_4(t-3) \oplus x_2(t-1) \oplus 3 \otimes x_6(t) \\
x_6(t) & = & x_5(t-2) \\
y(t) & = & x_6(t)
\end{array}
\tag{3.7}
$$

or by considering the following state vector $x(t) = \begin{pmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) & x_5(t) & x_6(t) \end{pmatrix}^t$, input vector $u(t) = \begin{pmatrix} u_1(t) & u_2(t) \end{pmatrix}^t$ and output vector $y(t) = (y(t))$:

$$
x(t) = \begin{pmatrix}
\varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix} x(t) \oplus
\begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix} x(t-1) \oplus
\begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon
\end{pmatrix} x(t-2)
$$

$$
\oplus \begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix} x(t-3) \oplus
\begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix} x(t-5) \oplus
\begin{pmatrix}
e & \varepsilon \\
\varepsilon & \varepsilon \\
\varepsilon & \varepsilon \\
\varepsilon & \varepsilon \\
\varepsilon & \varepsilon \\
\varepsilon & \varepsilon
\end{pmatrix} u(t-1) \oplus
\begin{pmatrix}
\varepsilon & \varepsilon \\
\varepsilon & e \\
\varepsilon & \varepsilon \\
\varepsilon & \varepsilon \\
\varepsilon & \varepsilon \\
\varepsilon & \varepsilon
\end{pmatrix} u(t-2)
$$

$$
y(t) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix} x(t)
$$

In a general manner the system can be represented in the following manner over $\overline{\mathbb{Z}}_{\min}$:

$$
x(t) = \bigoplus_{i=0}^{a} A_i x(t-i) \oplus \bigoplus_{j=0}^{b} B_j u(t-j), \tag{3.8}
$$

$$
y(t) = \bigoplus_{l=0}^{c} C_l x(t-l). \tag{3.9}
$$

After some extension of the state (see for $e.g.$ [18]), it is possible to get a recursive formulation with a delay of one time unit, it consists in increasing the state in order to have only temporization of one time unit on each place. Therefore, the system obtained is as follows :

$$
x(t) = A_0 x(t) \oplus A_1 x(t-1) \oplus B_0 u(t), \tag{3.10}
$$

$$
y(t) = C_0 x(t). \tag{3.11}
$$

Figure 8 yields the corresponding extension of the TEG of figure 6, it corresponds to the following matrices A_0, A_1, B_0 et C_0:

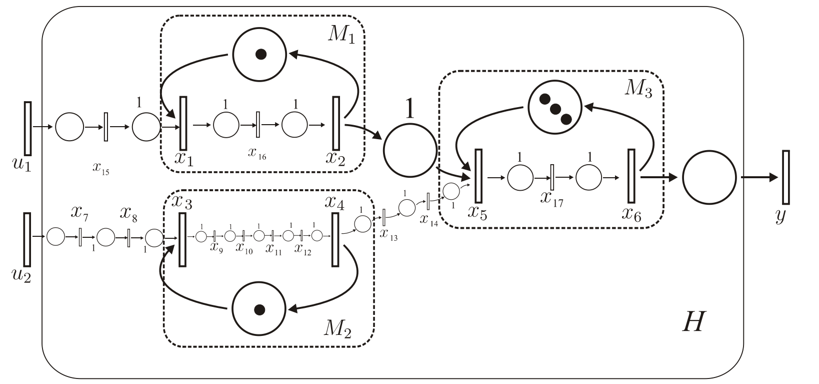


Figure 8 – Assembly line with temporization lower than or equal to 1 (temporal extension.)

$$A_0 = \begin{pmatrix} \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon \end{pmatrix}$$

$$A_1 = \begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix} \quad B = \begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \\
\varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix}$$

$$C = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

It is possible to obtain the following explicit formulation :

$$x(t) = Ax(t-1) \oplus Bu(t) \tag{3.12}$$

$$y(t) = Cx(t) \tag{3.13}$$

with $A = A_0^* A_1$ and $B = A_0^* B_0$, given below :

$$A = \begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & 1 & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & 3 \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix} \quad B = \begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \\
\varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix}$$

3.1.3 Semiring of formal series

It is also possible to represent the TEG behavior in a synthetic manner in semiring of some formal series, among them, the semiring of formal series in two commutative variables, γ and δ, with exponents in $\mathbb{Z}$ and boolean coefficient (e, ε), it is denoted $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ and was introduced by the (max,+) team of INRIA Rocquencourt (see cohen89a, [19], [5]). The aim of this section is to recall how this semiring is build.

3.1.3.1 Semiring $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$

Definition 103. *The γ transform of a signal is defined as follow :*

$$d(\gamma) = \bigoplus_{i\in\mathbb{Z}} d(k) \otimes \gamma^k$$

Remark 104. *This transform is analogous to the z transform of the classical system theory which allows to code a discrete trajectory by a formal series.*

Remark 105. *Since $\gamma \otimes d(\gamma) = \bigoplus_{i\in\mathbb{Z}} d(k) \otimes \gamma^{k+1} = \bigoplus_{i\in\mathbb{Z}} d(k-1) \otimes \gamma^k$, the γ operator can be seen as a backward shift operator, i.e., $x(k-1) = \gamma x(k)$.*

Definition 106 (Semiring $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$)**.** *The set of formal series in γ with exponents in $\mathbb{Z}$ and coefficients in $\overline{\mathbb{Z}}_{\max}$ has an idempotent semiring structure. The neutral element of the addition is the series defined as : $\varepsilon = \bigoplus_{i\in\mathbb{Z}} \varepsilon\gamma^k$ (where $\varepsilon = -\infty$ is the neutral element of the sum in $\overline{\mathbb{Z}}_{\max}$). The neutral element of the product law is the formal series $e(\gamma) = e\gamma^0$ (where $e = 0$ is the neutral element of the product law in $\overline{\mathbb{Z}}_{\max}$). The sum and the product (actually a Cauchy product) are defined as follows :*

$$\begin{aligned} d_1(\gamma) \oplus d_2(\gamma) &= \bigoplus_{k\in\mathbb{Z}} (d_1(k) \oplus d_2(k))\gamma^k \\ d_1(\gamma) \otimes d_2(\gamma) &= \bigoplus_{j\in\mathbb{Z}} (d_1(j) \oplus d_2(k-j))\gamma^k \end{aligned}$$

System 3.4 in $\overline{\mathbb{Z}}_{\max}$, can be easily translated in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$:

$$\begin{aligned} x(\gamma) &= \gamma A x(\gamma) \oplus B u(\gamma) \\ y &= C x(\gamma) \end{aligned}$$

A transfer relation can be computed it represents the input/output behavior of the system :

$$y(\gamma) = C(\gamma A)^* B u(\gamma) = H u(\gamma)$$

It is also possible to consider model 3.1.3.1 in idempotent semiring $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$:

$$\begin{aligned} x(\gamma) &= \bigoplus_{i=0}^{a} \gamma^i A_i x(\gamma) \oplus \bigoplus_{j=0}^{b} \gamma^j B_j u(\gamma), \\ y(\gamma) &= \bigoplus_{l=0}^{c} \gamma^l C_l x(\gamma). \end{aligned}$$

leading to the following model :

$$\begin{aligned} x(\gamma) &= A x(\gamma) \oplus B u(\gamma) \\ y &= C x(\gamma) \end{aligned}$$

with $A = \bigoplus_{i=0}^{a} \gamma^i A_i$, $B = \bigoplus_{j=0}^{b} \gamma^j B_j$ et $C = \bigoplus_{l=0}^{c} \gamma^l C_l$. Entries of matrices A,B and C are then polynomials. This formulation is not in a standard way, but it is then not necessary to increase the state vector size. In the following this kind of model will be consider, but the reader should have in mind that it can get easily a standard form.

3.1.3.2 Monotonic Trajectories

The trajectories which are solutions of the previous systems are not necessarily monotone, nevertheless the firing sequence associated to a transition of a TEG is not decreasing (the occurrence date $d(k)$ of the firing k is necessarily greater than or equal to $d(k-1)$). Formally,

$$\forall k \in \mathbb{Z} \quad d(k) \geq d(k-1) \Leftrightarrow d(k) = d(k) \oplus d(k-1).$$

is equivalent to

$$d(\gamma) = d(\gamma) \oplus \gamma d(\gamma) \Leftrightarrow d(\gamma) = \gamma^* d(\gamma).$$

This means that the γ transform of a monotonic trajectory can be written as $\gamma^* d(\gamma)$ and that the multiplication by γ^* of a non monotonic trajectory yields a monotonic non decreasing trajectory. It is a kind of filter. The set of monotonic trajectories ($i.e.$, which can be written $\gamma^* d(\gamma)$) is an idempotent sub semiring $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$, it is denoted $\gamma^*\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$. Indeed it easy to check that the sum and the product of non decreasing element is non decreasing, then they are closed operations, and obviously the series e and ε are non decreasing. In the sequel the product by γ^* will be omitted but as we will handle non decreasing trajectories the equality will have to be understood "modulo γ^*". For instance :

$$3\gamma \oplus 1\gamma^7 \oplus 5\gamma^9 = 3\gamma \oplus 5\gamma^9$$

In general manner the following simplifiaction rules will be considered :

$$\gamma^n \oplus \gamma^{n'} \quad = \quad \gamma^{min(n,n')}$$

The neutral element for the multiplication of the semiring $\gamma^*\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ is then the series $e(\gamma) = e \oplus e\gamma \oplus e\gamma^2 \oplus ... \oplus e\gamma^{+\infty}$. The neutral element for the addition of $\gamma^*\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ is then $\varepsilon(\gamma) = \varepsilon \oplus \varepsilon\gamma \oplus \varepsilon\gamma^2 \oplus ... \oplus \varepsilon\gamma^{+\infty}$, with $\varepsilon = -\infty$ the neutral element of the addition of $\overline{\mathbb{Z}}_{\max}$.

Remark 107. *In the sequel only monotonic series will be handle. Hence, in order to lighten the notations, without ambiguity , $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ must be understood as $\gamma^*\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$.*

3.1.3.3 Semiring $\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$

In a dual manner it is possible to define a transform for the trajectories considered in the temporal domain.

Definition 108. *The δ transform od a signal is defined as follows :*

$$d(\delta) = \bigoplus_{t \in \mathbb{Z}} c(t) \otimes \delta^t$$

Definition 109 (Semiring $\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$)**.** *The set of formal series in δ with exponents in $\mathbb{Z}$ ans coefficients in $\overline{\mathbb{Z}}_{\min}$ has an idempotent semiring. The neutral element of the addition is the series*

$\varepsilon = \bigoplus_{t\in\mathbb{Z}} \varepsilon\delta^t$ *(where* $\varepsilon = +\infty$ *is the neutral element of the addition in* $\overline{\mathbb{Z}}_{\min}$*). The neutral element of the multiplication is series* $e(\delta) = e\delta^0$ *(where* $e = 0$ *is the neutral element of the multiplication of* $\overline{\mathbb{Z}}_{\min}$*). The sum and the product (Cauchy product) of formal series are defined as follows :*

$$\begin{aligned} c_1(\delta) \oplus c_2(\delta) &= \bigoplus_{t\in\mathbb{Z}}(c_1(t) \oplus c_2(t))\delta^t \\ c_1(\delta) \otimes c_2(\delta) &= \bigoplus_{j\in\mathbb{Z}}(c_1(j) \oplus c_2(t-j))\gamma^t \end{aligned}$$

The system of equations 3.12 corresponds to the standardized model in the semiring $\overline{\mathbb{Z}}_{\min}$ and can be transposed in the semiring $\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$:

$$\begin{aligned} x(\delta) &= \delta A x(\delta) \oplus Bu(\delta) \\ y &= Cx(\delta) \end{aligned}$$

The input/output transfer relation can be computed :

$$y(\delta) = C(\delta A)^* Bu(\delta) = Hu(\delta)$$

In an equivalent manner, it is possible to consider the system under the form given by equation 3.1.3.1 in $\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$:

$$\begin{aligned} x(\delta) &= \bigoplus_{i=0}^{a} \delta^i A_i x(\delta) \oplus \bigoplus_{j=0}^{b} \delta^j B_j u(\delta), \\ y(\delta) &= \bigoplus_{l=0}^{c} \delta^l C_l x(\delta). \end{aligned}$$

leading to the following model:

$$\begin{aligned} x(\delta) &= Ax(\delta) \oplus Bu(\delta) \\ y &= Cx(\delta) \end{aligned}$$

with $A = \bigoplus_{i=0}^{a} \delta^i A_i$, $B = \bigoplus_{j=0}^{b} \delta^j B_j$ et $C = \bigoplus_{l=0}^{c} \delta^l C_l$. The elements of matrices A,B and C are then polynomials.

3.1.3.4 Monotonic trajectories

As in the event domain the trajectories corresponding to the firing of transitions ot TEG are monotonics (the number of events occurring at time $(t+1)$, $c(t+1)$, is necessarily greater than or equal to the number of firing being occurred at time t, $c(t)$). Formally,

$$\forall t \in \mathbb{Z} \quad c(t) \succeq c(t+1) \Leftrightarrow c(t) = c(t+1) \oplus c(t).$$

this is equivalent to

$$c(\delta) = \delta^{-1} c(\delta) \oplus c(\delta) \Leftrightarrow c(\delta) = (\delta^{-1})^* c(\delta).$$

This means that the δ transform of a monotonic trajectory belongs to semiring $(\delta^{-1})^*\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$. The following simplification rules are then considered:

$$\delta^t \oplus \delta^{t'} = \delta^{max(t,t')}$$

The neutral element of the multiplication law $(\delta^{-1})^*\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$ is series $e(\delta) = e \oplus e\delta \oplus e\delta^2 \oplus ... \oplus e\delta^{+\infty}$. The neutral element for the addition $(\delta^{-1})^*\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$ is series $\varepsilon(\delta) = \varepsilon \oplus \varepsilon\delta^1 \oplus \varepsilon\delta^2 \oplus ... \oplus \varepsilon\delta^{+\infty}$, with $\varepsilon = +\infty$ the neutral element of the addition of $\overline{\mathbb{Z}}_{\min}$.

3.1.4 Two-dimensionnal description, semiring $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$

The choose between the temporal and the event domain will be driven by the applications. Then it can be useful to keep the ability to switch between the two point of view by considering a two-dimensional description of trajectories. Below the construction of a semiring of formal series with two commutative variables, γ and δ, with exponents in $\mathbb{Z}$ and with boolean coefficient. For the example of figure 6 the following equation is obtained :

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} &= \begin{pmatrix} \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \gamma & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \delta^5 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \delta & \varepsilon & \delta^3 & \varepsilon & \gamma^3 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \oplus \begin{pmatrix} \delta & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \delta^2 \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
y &= \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}
\end{aligned} \tag{3.14}$$

corresponding to the following standard formalism :

$$\begin{aligned} x &= Ax \oplus Bu \\ y &= Cx. \end{aligned} \tag{3.15}$$

To each transition of the TEG is associated an entry of the vector $x \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]^n$ (vector corresponding to the internal transitions, in this example $n = 6$), $u \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]^m$ (vector corresponding to the inputs transitions, inthis example $m = 2$) and $y \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]^l$ (vector associated to output transitions, in this example $l = 1$). Matrices $A \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]^{n\times n}$, $B \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]^{n\times m}$, $C \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]^{l\times n}$, represents the interaction between transitions. The matrices entries are polynomials of $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. Entry $A_{5,6} = \gamma^3$ means that 3 tokens are in the place between x_6 and x_5, in a dually manner, entry $A_{4,3} = \delta^5$ means that a temporization of 5 times

units is associated to the place between transitions x_3 and x_4.
By solving the implicit equation, by considering the result on fixed point equations, an input/ouput transfer relation is obtained. In this example, the transfer matrix denoted $H \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{l\times m}$ is given by :

$$H = CA^*B = \begin{pmatrix} \delta^6(\gamma\delta^2)^* & \delta^{12}(\gamma\delta^5)^* \end{pmatrix} . \tag{3.16}$$

In a genaral manner, the entries of the transfer matrix representing a TEG are periodic and causal series, which can be written in the following standardized form : $p \oplus qr^*$, with p and q polynomials with exponents in $\mathbb{N}$ and r a monomial with exponents in $\mathbb{N}$. Polynomial p characterizes the transient behavior of the series, polynomial q represents a pattern which is repeated periodically, the periodicity is given by $r = \gamma^\nu\delta^\tau$ where ν/τ corresponds to the production rate of the series, conversely τ/ν is the cycle time of the series. An example of periodic seriesis given on figure 9.
To summarize a TEG admits a model in the subsemiring of periodic and causal series.

3.2 Control Theory

3.2.1 State feedback controller synthesis

State feedback controller synthesis is now considered, the reader interested should consult [14]. The architecture considered is given in figure 10, its state representation is given by :

$$\begin{cases} x &= Ax \oplus Bu \\ y &= Cx \end{cases} \tag{3.17}$$

with $A \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times n}$, $B \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times m}$ and $C \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{l\times n}$. The control is given by the following expression :

$$u = Kx \oplus v$$

with $K \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{m\times n}$. The system becomes

$$\begin{cases} x &= Ax \oplus B(Kx \oplus v) \\ y &= Cx \end{cases} \Rightarrow \begin{cases} x &= (A \oplus BK)x \oplus Bv \\ y &= Cx \end{cases}$$

and the input/output transfer relation with controller K is :

$$\begin{aligned} y &= C(A \oplus BK)^*Bv = G_K v. \\ G_K &= CA^*(A^*BK)^*A^*B \\ &= CA^*B(KA^*B)^*. \end{aligned}$$

The objective is to synthesize the greatest controller K, such that $G_K \preceq G_{ref}$. The optimal controller exists if $G_{ref} \in \mathcal{G}_1$, it is given by (see [14] for the proof) : :

$$K_{opt} = (H \backslash\!\!\circ G_{ref} /\!\!\circ A^*B). \tag{3.18}$$

Remark 110. *It must be noted that the output feedback controller and the state feedback controller are linked as follows :*

$$F_{opt} = K_{opt} \phi C$$

c.-à-d. $F_{opt}C = (K_{opt} \phi C)C \preceq K_{opt}$. *It comes* $(F_{opt}H)^* = (F_{opt}CA^*B)^* \preceq (K_{opt}A^*B)^* \Rightarrow CA^*B(F_{opt}CA^*B)^* = G_{F_{opt}} \preceq CA^*B(K_{opt}A^*B)^* = G_{K_{opt}}$. *In other words, the measurement of all the state improves the efficiency of the control, c.-à-d. leads to a greater control and a corrected system closer to the reference model.*

3.2.2 Observer synthesis

In this section the synthesis of an observer allowing to get an estimation of unmeasured state is proposed. It is done in an analoguous manner to the one of the classical linear systems ($i.e.$ observer in the Luenberger sense (see [20]). The architecture considered is given in Figure 11.

Matrices A, B, C and R are assumed to beknown. Without lost of generality, the rows of matrix C are such that only one entry is different of ε and that this entry is equal to e. Practically each row of matric C allows to connect one state to the output, it then represents the location of sensors yielding information of some state. The system is characterized by the following equations :

$$\begin{aligned} x &= Ax \oplus Bu \oplus Rw = A^*Bu \oplus A^*Rw \\ y &= Cx = CA^*Bu \oplus CA^*Rw \end{aligned} \tag{3.19}$$

and the observer can be written as follows :

$$\begin{aligned} \hat{x} &= A\hat{x} \oplus Bu \oplus L(\hat{y} \oplus y) \\ &= A\hat{x} \oplus Bu \oplus LC\hat{x} \oplus LCx \\ \hat{y} &= C\hat{x}. \end{aligned} \tag{3.20}$$

By introducing equation (3.19) and by solving the implicit equation, these equations (3.20) become :

$$\hat{x} = (A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCA^*Bu \oplus (A \oplus LC)^*LCA^*Rw. \tag{3.21}$$

By considering (2.14), the following equality is obtained :

$$(A \oplus LC)^* = A^*(LCA^*)^*, \tag{3.22}$$

by introducing it in equation (3.21), the following state estimation appears :

$$\hat{x} = A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^*LCA^*Bu \oplus A^*(LCA^*)^*LCA^*Rw,$$

by recalling that $(LCA^*)^*LCA^* = (LCA^*)^+$, this equation becomes:

$$\hat{x} = A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^+Bu \oplus A^*(LCA^*)^+Rw.$$

From equation (2.9), it comes $(LCA^*)^* \succeq (LCA^*)^+$, hence the observer equation can be written :

$$\begin{aligned} \hat{x} &= A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^+Rw \\ &= (A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCA^*Rw. \end{aligned} \quad (3.23)$$

(3.24)

The goal of the observer synthesis is to compute the greatest observer matrix L such that the estimated vector

$\hat{x}$ be as close as possible to state x, by respecting the following constraint $\hat{x} \preceq x$, formally this can be written as follows :

$$(A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCA^*Rw \preceq A^*Bu \oplus A^*Rw \quad \forall(u, w)$$

or in an equivalent manner :

$$(A \oplus LC)^*B \preceq A^*B \quad (3.25)$$

$$(A \oplus LC)^*LCA^*R \preceq A^*R. \quad (3.26)$$

Lemma 111. *The greatest matrix L such that $(A \oplus LC)^*B = A^*B$ is given by :*

$$L_1 = (A^*B) \phi (CA^*B). \quad (3.27)$$

Lemma 112. *The greatest matrix L satisfying $(A \oplus LC)^*LCA^*R \preceq A^*R$ is given by :*

$$L_2 = (A^*R) \phi (CA^*R). \quad (3.28)$$

Proposition 113. *$L_x = L_1 \wedge L_2$ is the greatest matrix such that :*

$$\hat{x} = A\hat{x} \oplus Bu \oplus L_x(\hat{y} \oplus y) \preceq x = Ax \oplus Bu \oplus Rw \quad \forall(u, w).$$

Corollary 114. *Matrix L_x ensures the equality between the estimated output $\hat{y}$ and output y, $i.e.$*

$$\begin{aligned} C(A \oplus L_xC)^*B &= CA^*B, \\ C(A \oplus L_xC)^*L_xCA^*R &= CA^*R. \end{aligned}$$

3.2.2.1 Application : Control with observer

Classically the observer can be used to estimate the state used to compute a closed loop control (see section 3.2.1). Even if only outputs are measured, it will be shown this strategy is more efficient than the output feedback.

The control strategy is despite in figure 12. According to equations 3.19 and 3.23, it leads to estimated state expression $\hat{x}$ and to state of the system x as follows :

$$\hat{x} = (A \oplus LC)^* Bu \oplus (A \oplus LC)^* LCA^* Rw \tag{3.29}$$

$$x = Ax \oplus Bu \oplus Rw = A^* Bu \oplus A^* Rw. \tag{3.30}$$

The control law is given by :

$$u = K\hat{x} \oplus v. \tag{3.31}$$

To compute the observer, it is assumed that the exogenous uncontrollable inputs are null, $i.e.$ $w = \varepsilon$. The transfer relation between input v and control u is then given by :

$$\begin{aligned} u &= K(A \oplus LC)^* Bu \oplus v \\ &= (K(A \oplus LC)^* B)^* v. \end{aligned} \tag{3.32}$$

The estimated state (3.29) becomes :

$$\hat{x} = (A \oplus LC)^* B(K(A \oplus LC)^* B)^* v.$$

The state of the system becomes :

$$\begin{aligned} x &= A^* Bu \\ &= A^* B(K(A \oplus LC)^* B)^* v. \end{aligned} \tag{3.33}$$

And the system output is given by :

$$y = CA^* B(K(A \oplus LC)^* B)^* v. \tag{3.34}$$

Proposition 115. *If the reference model $G_{ref} \in \mathcal{G}_1$ ($i.e.$ it can be written as $G_{ref} = M^* H$ with $H = CA^* B$ the transfer relation of the system), then it exists a greatest controller K such that the transfer relation in closed loop be lower than or equal to G_{ref}. This controller is given by :*

$$K_{opt} = H \backslash G_{ref} / ((A \oplus LC)^* B). \tag{3.35}$$

To summarize, the following expression of control u is obtained :

$$\hat{x} = (A \oplus LC \oplus BK_{opt})\hat{x} \oplus (B \;\; L)\begin{pmatrix} v \\ y \end{pmatrix} \tag{3.36}$$

$$u = K_{opt}\hat{x} \oplus v. \tag{3.37}$$

Hereafter, the performances of this control strategy is analyzed (according to the just in time criterion). Observer matrix L is supposed to respect the following inequality : $L \preceq L_x$, with L_x the optimal observer matrix of the proposition 113. It will be shown this strategy is more efficient then the one using the output feedback F_{opt}. In order to compare the both strategies, the reference model is assuemd such that $G_{ref} \in \mathcal{G}_1$. Control $u^{\hat{x}}_{opt}$ is the control computed with K_{opt} and control u^{y}_{opt} is the one obtained with the controller F_{opt}.

$$u^{\hat{x}}_{opt} = K_{opt}\hat{x} \oplus v \tag{3.38}$$

$$u^{y}_{opt} = F_{opt}y \oplus v. \tag{3.39}$$

The obective is to show that the transfer function between $u^{\hat{x}}_{opt}$ and v is greater than the one between u^{y}_{opt} and v. By considering equations (3.32), the controls become :

$$u^{\hat{x}}_{opt} = (K_{opt}(A \oplus L_{opt}C)^{*}B)^{*}v \tag{3.40}$$

$$u^{y}_{opt} = (F_{opt}CA^{*}B)^{*}v. \tag{3.41}$$

According to corollary 114, matrix L_{opt} is such that $C(A \oplus L_{opt}C)^{*}B = CA^{*}B$. Hence equation (3.41) can be written :

$$u^{y}_{opt} = (F_{opt}C(A \oplus L_{opt}C)^{*}B)^{*}v. \tag{3.42}$$

Proposition 116. *If the reference model is such that $G_{ref} \in \mathcal{G}_1$, the control using oberver L_{opt} and state feedback controller K_{opt} is greater than the control using a single output feedback F_{opt}, c.-à-d. :*

$$u^{y}_{opt} \preceq u^{\hat{x}}_{opt}.$$

3.2.2.2 Illustration

Example of figure (6) is considered. The input/ouptut model of the system is recalled :

$$H = \begin{pmatrix} \delta^{6}(\gamma\delta^{2})^{*} & \delta^{12}(\gamma\delta^{5})^{*} \end{pmatrix}$$

and the reference model to achieve : $G_{ref} = (\gamma\delta^5)^* H \in \mathcal{G}_1$:

$$G_{ref} = (\gamma\delta^5)^* H = \begin{pmatrix} \delta^6(\gamma\delta^5)^* & \delta^{12}(\gamma\delta^5)^* \end{pmatrix}.$$

By applying the results of proposition (113), the optimal observer is :

$$\begin{aligned} L_{opt} &= CA^* \backslash\!\!\circ CA^*B \phi\!\!\!/ CA^*B \\ &= \begin{pmatrix} \gamma^3\delta(\gamma\delta^2)^* \\ \gamma^2\delta(\gamma\delta^2)^* \\ \varepsilon \\ \varepsilon \\ \gamma(\gamma\delta^2)^* \\ (\gamma\delta^2)^* \end{pmatrix} \end{aligned}$$

and by considering the results of proposition 115, the state feedback controller is :

$$\begin{aligned} K_{opt} &= H \backslash\!\!\circ G_{ref} \phi\!\!\!/ ((A \oplus L_{opt}C)^* B) \\ &= \begin{pmatrix} \gamma\delta^4(\gamma\delta^5)^* & \gamma\delta^2(\gamma\delta^5)^* & \delta^4(\gamma\delta^5)^* & \gamma\delta^4(\gamma\delta^5)^* & \gamma\delta(\gamma\delta^5)^* & \gamma^2\delta^4(\gamma\delta^5)^* \\ \gamma^2\delta^3(\gamma\delta^5)^* & \gamma^2\delta(\gamma\delta^5)^* & \gamma\delta^3(\gamma\delta^5)^* & \gamma^2\delta^3(\gamma\delta^5)^* & \gamma^2(\gamma\delta^5)^* & \gamma^3\delta^3(\gamma\delta^5)^* \end{pmatrix}, \end{aligned}$$

which leads to the following transfer relations :

$$\begin{aligned} u &= (K_{opt}(A \oplus L_{opt}C)^* B)^* v \\ &= \begin{pmatrix} (\gamma\delta^5)^* & \delta^6(\gamma\delta^5)^* \\ \gamma^2\delta^4(\gamma\delta^5)^* & (\gamma\delta^5)^* \end{pmatrix} v \\ y &= CA^*B(K_{opt}(A \oplus L_{opt}C)^* B)^* v \\ &= \begin{pmatrix} \delta^6(\gamma\delta^5)^* & \delta^{12}(\gamma\delta^5)^* \end{pmatrix} v, \end{aligned}$$

3.2.3 Modified Observer-based controller

In this section will be proposed the using of an observer-based controller to compute the state-feedback control law, which was introduced by L. Hardouin, Y. Shang, C. A. Maya and B. Cottenceau in [22].In order to reduce the number of calculations, some algebraic modifications were proposed in my work.
Formally, the following control law is considered $u = P(v \oplus M\hat{x})$ where $\hat{x} = A\hat{x} \oplus Bu \oplus L_{opt}(\hat{y} \oplus y)$.

The synthesis objetive is to get the greatest control law such that the output y be smaller than or equal to the desired output $G_{ref}v$. G_{ref}, the reference model, which can be seen as a specification to achieve, it is denoted $G_{ref} \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{l\times m}$, its entries are given in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ but it can be equivalently given in an other semiring.

From a practical point of view an interesting reference model is $G_{ref} = H$, it means that the controller aims to preserve the input/output performances of the system while increasing as much as possible the controlled input.

The controllers are given by:

$$Popt = CA^{*}B \backslash G_{ref} \tag{3.43}$$

$$Mopt = Popt \backslash Popt \phi (A^{*}BPopt) = Kopt \tag{3.44}$$

And the observer Lopt:

$$L1 = (A^{*}B) \backslash (CA^{*}B) \tag{3.45}$$

$$L2 = (A^{*}R) \backslash (CA^{*}R) \tag{3.46}$$

Lopt is the greatest observer matrix such that:

$$Lopt = inf(L1, L2) \tag{3.47}$$

The state output can be written, respectively, as:

$$x = A^{*}Bu = A^{*}BP(M(A \oplus L_{opt}C)^{*}BP)^{*}v \tag{3.48}$$

$$y = Cx = CA^{*}BP(M(A \oplus L_{opt}C)^{*}BP)^{*}v \tag{3.49}$$

Defining two auxiliary matrices Q and k, its possible to simplify the control law u described in [22] :

$$Q = (MA^{*}B)^{*}P \tag{3.50}$$

$$K = QMA^{*}L \tag{3.51}$$

The control law u can be described as follows:

$$
\begin{aligned}
u &= P(v \oplus M(A^*Bu \oplus A^*Ly)) && (3.52)\\
u &= (MA^*B)^*P(v \oplus MA^*Ly) && (3.53)\\
u &= Qv \oplus QMA^*Ly && (3.54)\\
u &= Qv \oplus Ky. && (3.55)
\end{aligned}
$$

So, using the matrices Q and K previously defined, we can simplify and easily calculate the control law u in time-domain, using the reference v and the output y. Now, its not necessary to calculate the estimated state $\hat{x}$ to obtain the control law u.

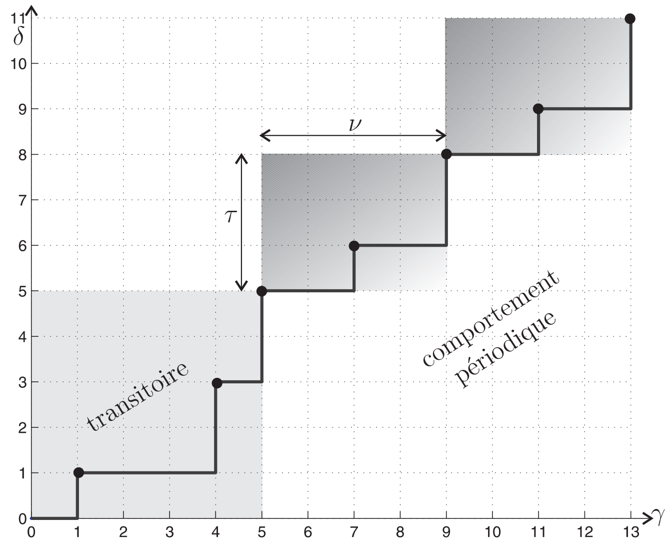


Figure 9 – Graphical representation of series $s = p \oplus qr^* = e \oplus \gamma\delta \oplus \gamma^4\delta^3 \oplus (\gamma^5\delta^5 \oplus \gamma^7\delta^6)(\gamma^4\delta^3)^*$.

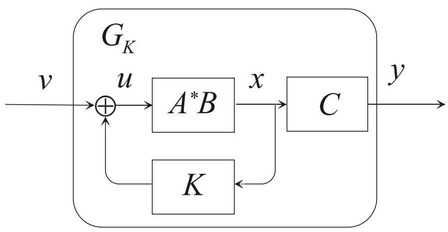


Figure 10 – Model matching : state feedback architecture.

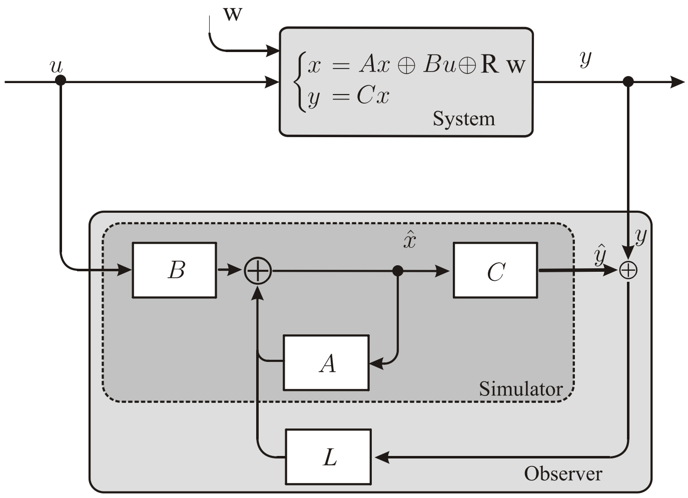


Figure 11 – Observer architecture

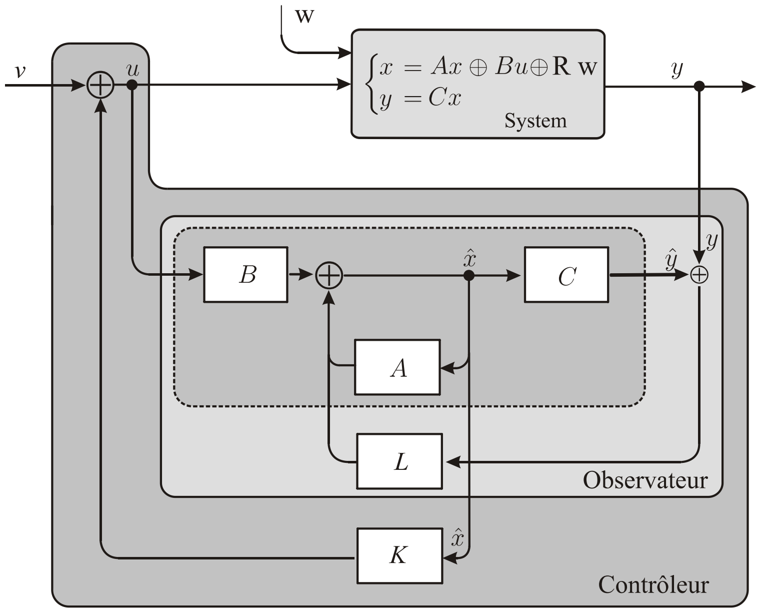


Figure 12 – Control architecture using the estimated state

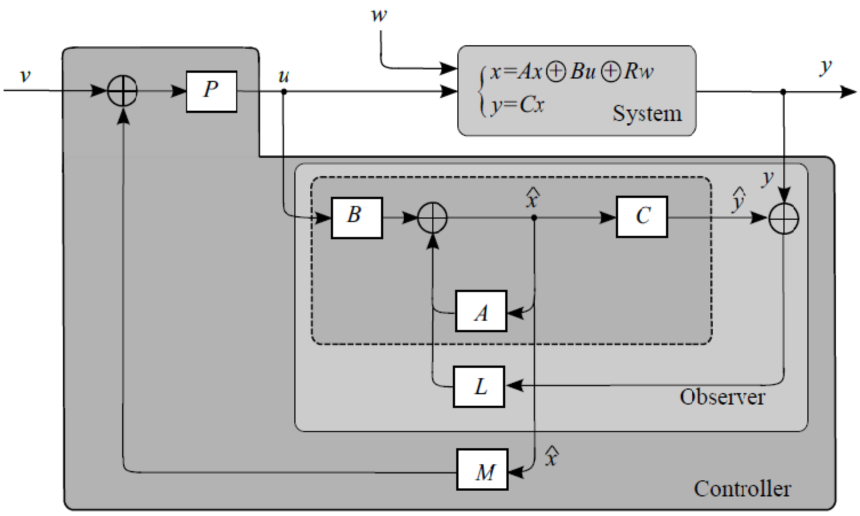


Figure 13 – The Observer-based Controller

4 Automated Conveyor System

The system located at the Laboratoire Angevin de Recherche en Ingénierie des Systèmes (LARIS) at the Université d'Angers is a conveyor belt system that moves pallets through circuits (see Figure 13). In its more general configuration, it allows by external signals the blocking of pallets using buttons located in different parts of the system and also the dynamic change of the paths that the pallets can follow through. The time in which is desirable to turn on or off the buttons or to make the path modifications are the inputs of the system. Further, there are many proximity sensors along the circuits which can detect the presence of the pallet. With this information, it is possible to have as outputs the times in which a pallet (any pallet, there is not, in principle, any distinction between them) passed through a given point. The entire system receives commands from a programable logic controller, which in turn receives actions either directly from its user interface or through a C++ program at a computer.

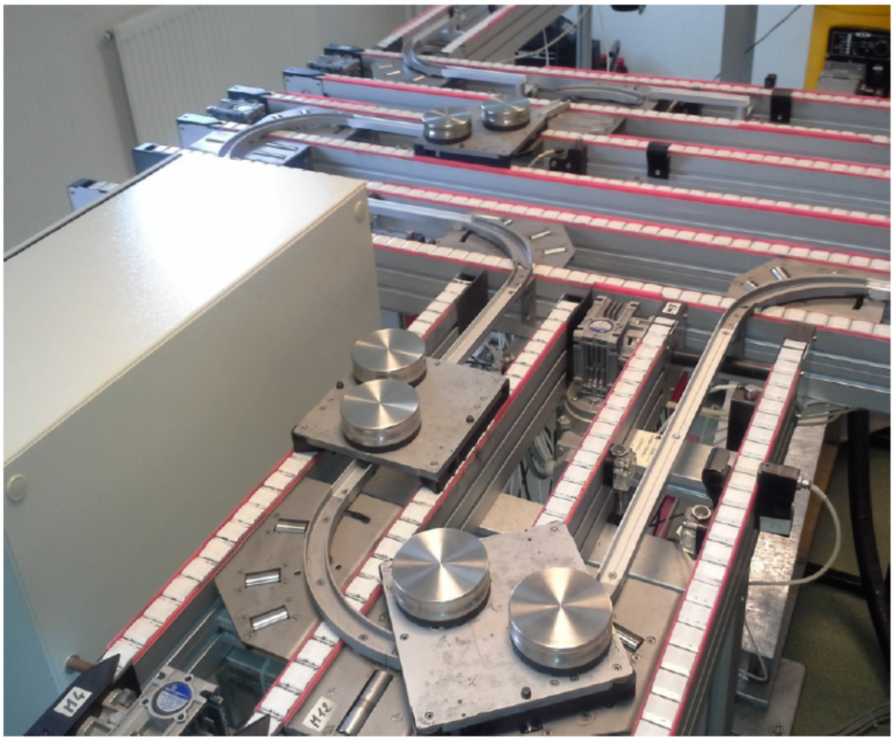


Figure 14 – Photo of the system in the Laboratoire Angevin de Recherche en Ingénierie des Systèmes (LARIS) at the Université d'Angers.

In order for the system to be modeled by a TEG, the paths need to be static through all the experiment (so, no changing of paths are allowed). Further, the buttons are programmed so, when they are turned off (that is, the button is down and the pallet is allowed to move), they automatically go on (up) again in 2 seconds. This guarantees that if more than one pallet is waiting in line, when the button fires one time, only one pallet continues.

There are two independent circuits, ten buttons (B1 to B10) and ten proximity sensors (one sensor just before each button). The upper circuit has three pallets, all of them located initially just before B1. The lower circuit has also three pallets, but two of them located just before B5 and another just before B6. The pallets in the upper circuit move clockwise while the ones in the lower circuit move counter-clockwise. For each stretch between two successive buttons (for instance, the stretch B1 - B2), there is an associated timing and also an associated capacity of pallets. The timing gives the time that a pallet needs to go from just before the initial button to the successive one when there is nothing in the path (thus, it is the minimal time). These timings were obtained through multiple experiments.

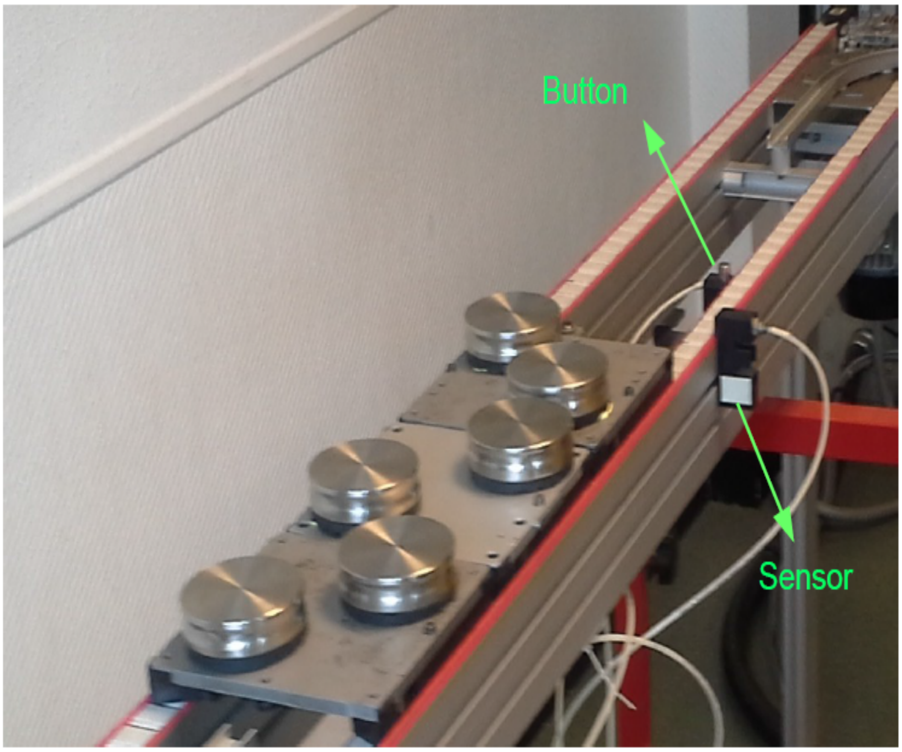


Figure 15 – Three pallets in a line, approaching a button

When the button "fires" once, only one pallet continues moving and the other two continue waiting for another firing of the button. of the results was taken as the value. Thus, from Figure 6.3 it is possible to conclude that, without anything in the path, a pallet takes in average 8 seconds 1 to go from just before B1 to just before B2. The capacity tells the maximum number of tokens that this stretch can hold. Thus, the stretch B1 - B2 can hold at most 3 pallets. The system is programmed so, if the following path is full, the button will not fire (go down). This capacity constraint is inconsequential for the upper loop, since all the capacities are three and there is only three pallets in the upper loop, but it is important for the lower one.

As programmed, in order for a button to fire it is necessary that three conditions hold: (i) there must exist one presence token for that button, that is, there must be a pallet waiting just before a button, (ii) there must exist at least one capacity token for that button, that is, there

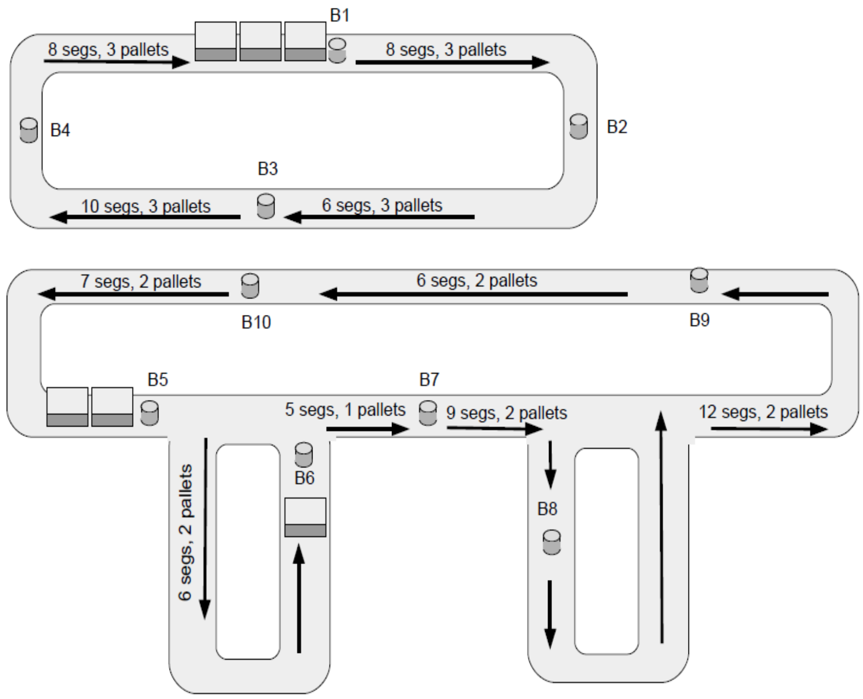


Figure 16 – Schematic picture of the transportation system

must exist at least one free space for a pallet in the following stretch (so, for example, B5 is only allowed to fire if there is at most 1 pallet in the stretch B5 - B6) and finally (iii) there must exists at least one control token, which represents an external action in the system. The first two kind of tokens obviously represent a “physical” constraint of the system, while the third one represents a logical constraint in which the engineer can act to obtain a desired behaviour. When the button fires, one of each token is consumed. Of course, the average of the values was not exactly an integer amount of seconds. Rounded numbers were used.

The system is also programmed to make a forced synchronization between the upper and lower circuits using the buttons B3 and B10. This means that, for these two buttons in particular, there is an additional fourth token necessary for firing. B3 fires only if (but not if) there is a presence token in B10 and B10 fires only if (but not if) there is a presence token in B3. This way, their firings are always synchronized.

4.1 System Description

A TEG model for this system is proposed in [21], by Dr. Vinícius Mariano Gonçalves. Before continuing, it is important to define the inputs u[k] and outputs y[k] of the system. Hence, it will be defined as ui[k], i = 1, 2, ..., 10 the time in which the kth control token is available for the ith button. Further, yi[k], i = 1, 2, ...,10 the time in which the kth palled arrived just before the ith button. Note that these definitions comply with what one can act and observe in the system. The modelling begins by analyzing each stretch. Suppose, to begin with, stretch B1 - B2. For this stretch, which holds three pallets at most, one can think in three possible status for a pallet (respective of one place in a TEG):
(i) - P1: Stopped, in the first place in line before B2;
(ii) - P2: Stopped, in the second place in line before B2;
(iii) - P3: Stopped, in the third place in line before B2;
Also, one needs four actions (respective of one transition in a TEG), labeled as x1, x2, x3 and x4:
(i) - x1: Button B1 fired, began moving to B2;
(ii) - x2: Began moving from the third place to the second place in line;
(iii) - x3: Began moving from the second place to the first place in line;
(iv) - x4: Button B2 fired, began moving to B3.

Note that:
(i) - x1 can only fire if there is at least one capacity token (that is, a free space for a pallet) in the stretch B1 - B2, there is a presence token in the previous place (there is one, initially, see Figure 6.3) and at least one control token is available. Further, at every firing of x4, one capacity token is restored to the stretch because one pallet is leaving. Since the stretch begins free of pallets (see Figure 2.3), initially there are three capacity tokens;
(ii) - A pallet can only begin to move from the third place in line to the second one if there are no pallets in the second place. Thus, there must not exist a token/pallet in P2. Further, only one token/pallet can be at P2 at a given time (because only one pallet can be at the second place) and every time x3 fires the space becomes free to a new pallet to go to the second position;
(iii) - A pallet can only begin to move from the second place in line to the first one if there are no pallets in the first place. Thus, there must not exist a token/pallet in P1. Further, only one token/pallet can be at P1 at a given time (because only one pallet can be at the first place) and every time x4 fires the space becomes free to a new pallet to go to the first position;
(iv) - Of course, x4 can only fire if there is at least one capacity token in the stretch B2! B3, which initially is devoid of pallets and thus B2 has initially three capacity tokens. Further, there must exist a presence token in P1 and also at least one control token for B2.

A final concern is that P3 can only have one token/pallet at a given time. The above constraints naturally ensure this, and thus there is no need to force it artificially. Indeed, suppose

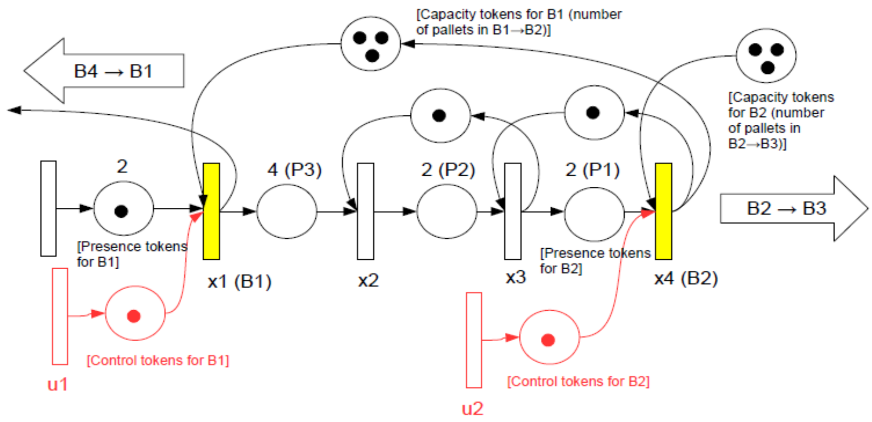


Figure 17 – TEG for B1-B2

there are two token/pallets at P3 at a given time. Then, there must be one token/pallet in P2 because, otherwise, x2 would have fired and there would be just one token/pallet in P1. This, in turn, implies that there must be at least one token/pallet in P1 because, otherwise, x3 would have fired and there would be no token/pallet in P2. Hence, there would be four tokens in P1, P2 and P3. This is impossible, because one of the constraints above ensures at most three tokens in the stretch. Then, it remains to discuss the timings of P1, P2, P3. Clearly, the sum of them needs to be 8 seconds (see Figure 16). One then needs to discover how much time it takes to a pallet to move from the second to the first position and from the third to the second position. Experiments show that this timing is of 2 seconds for both movements, and indeed this is true for all other stretches (it is simply the time the belt takes to move a pallet a distance of one length of a pallet, and thus the length of the pallet divided by the belt speed). Hence, the timing of P1 and P2 are 2 seconds and the timing of P3 is 8 - 2 - 2 = 4 seconds.

In an analogous way, models for all the stretches can be derived. It is necessary, though, to be careful about initial conditions (number of pallets initially in the stretch) and maximum number of pallets, according to Figure 15. After that, all these models can be connected (connecting the model of the stretch B1 - B2 with the one of the stretch B2 - B3 and so on). The resulting TEG can be seen in Figure 17. Note that the output transitions, drawn in green, are constructed in a way that they represents the arrival time of a pallet in each button, which is what is measured by the proximity sensor. For instance, $y2[k] = 2x3[k]$ which is the time that the kth pallet arrives at B2. Some of these output transitions have an associated place with tokens (B1, B5 and B6) because, initially, there is already a pallet close to these buttons . Note, also, the aforementioned synchronization between B3 and B10 in transition x7.

4.2 System Modelling

For the TEG model presented in Figure 18, the system over the two-dimentional description $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, presented in last chapter 3.1.4, is described as the following:

State transition matrix A:

$$A = \begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \gamma^3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma\delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\delta^4 & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \delta^2 & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon & \gamma^3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon & \gamma^3 & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^6 & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma\delta^4 & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma\delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma\delta^5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma\delta^2 & \varepsilon & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^4 & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma\delta^2 & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^5 & \varepsilon & \varepsilon & \gamma^2 & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^7 & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \gamma^2 & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^{10} & \varepsilon & \gamma & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^4 & \varepsilon
\end{pmatrix}$$

Input matrix B:

$$B = \begin{matrix}
\gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{matrix}$$

Output matrix C:

$$
C = \begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma\delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma\delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma\delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2
\end{pmatrix}
$$

The transfer function $H = CA^{\star}B$ can be seen in Appendix A.

4.3 Control System

In this section the modified observer-based controller will be used to compute the state-feedback control law, which was described in the last chapter, section 3.2.3 .

For this problem the reference model G_{ref} will be defined as $G_{ref} = H$, it means that the controller aims to preserve the input/output performances of the system while increasing as much as possible the controlled input. The controlled system will be able to retard all control entries, without losing performance.

All calculated controllers (M, P, Q and K) can be seen in Appendix A.

4.3.1 Time domain representation of u

As seen in section 3.2.3, equation 3.55, the control law u depends of the output y and the reference v. The automated conveyor system has 10 control inputs, in order to demonstrate the procedure of convertion to time domain, the control input u_0 was chosen.

$$
\begin{aligned}
u_0 &= u_y^0 + u_v^0 \\
u_0 &= Qv \oplus Ky
\end{aligned}
$$

The control law, given in the time domain for $u_y^0(t)$, can be represented using the (min-plus)-algebra as follows:

$\xi_{00}(t)$, $\xi_{01}(t)$, ..., $\xi_{19}(t)$ are intermediated transactions introduced by our controller.

$$
\begin{aligned}
\xi_{00}(t) &= 3\xi_{00}(t-45) \oplus 5y_0(t-75) \oplus 6y_0(t-80) \oplus 7y_0(t-87) \\
\xi_{01}(t) &= 3\xi_{01}(t-45) \oplus 5y_1(t-67) \oplus 6y_1(t-72) \oplus 7y_1(t-79) \\
\xi_{02}(t) &= 3\xi_{02}(t-45) \oplus 5y_2(t-63) \oplus 6y_2(t-68) \oplus 7y_2(t-75) \\
\xi_{03}(t) &= 3\xi_{03}(t-45) \oplus 8y_3(t-83) \oplus 9y_3(t-88) \oplus 10y_3(t-95) \\
\xi_{04}(t) &= 3\xi_{04}(t-45) \oplus 3y_4(t-56) \oplus 4y_4(t-61) \oplus 5y_4(t-68) \\
\xi_{05}(t) &= 3\xi_{05}(t-45) \oplus 2y_5(t-50) \oplus 3y_5(t-55) \oplus 4y_5(t-62) \\
\xi_{06}(t) &= 3\xi_{06}(t-45) \oplus 2y_6(t-45) \oplus 3y_6(t-50) \oplus 4y_6(t-57) \\
\xi_{07}(t) &= 3\xi_{07}(t-45) \oplus 4y_7(t-48) \oplus 5y_7(t-81) \oplus 6y_7(t-86) \\
\xi_{08}(t) &= 3\xi_{08}(t-45) \oplus 4y_8(t-36) \oplus 5y_8(t-69) \oplus 6y_8(t-74) \\
\xi_{09}(t) &= 3\xi_{09}(t-45) \oplus 5y_9(t-68) \oplus 6y_9(t-68) \oplus 7y_9(t-75)
\end{aligned}
$$

$$
\xi_y^0(t) = \xi_{00}(t) \oplus \xi_{01}(t) \oplus ... \oplus \xi_{08}(t) \oplus \xi_{09}(t)
$$

$$
u_y^0(t) = y_0(t-2) \oplus 1y_0(t-4) \oplus 2y_0(t-30) \oplus 3y_0(t-34) \oplus 4y_0(t-2) \oplus 2y_1(t-22) \oplus 3y_1(t-26) \oplus 4y_1(t-30) \oplus 2y_2(t-18) \oplus 3y_2(t-222) \oplus 4y_2(t-26) \oplus 3y_3(t-8) \oplus 3y_3(t-10) \oplus 4y_3(t-12) \oplus 5y_3(t-38) \oplus 6y_3(t-42) \oplus 7y_3(t-46) \oplus 2y_4(t-18) \oplus 2y_7(t-36) \oplus 3y_7(t-40) \oplus 2y_8(t-24) \oplus 3y_8(t-28) \oplus 2y_9(t-18) \oplus 3y_9(t-22) \oplus 4y_9(t-26) \oplus \xi_y^0(t)
$$

The control law, given in the time domain for $u_v^0(t)$, can be represented using the (min-plus)-algebra as follows:

$$
\begin{aligned}
\xi_{10}(t) &= 3\xi_{10}(t-45) \oplus 6v_0(t-75) \oplus 7v_0(t-80) \oplus 5v_0(t-87) \\
\xi_{11}(t) &= 3\xi_{11}(t-45) \oplus 7v_1(t-69) \oplus 8v_1(t-74) \oplus 9v_1(t-81) \\
\xi_{12}(t) &= 3\xi_{12}(t-45) \oplus 6v_2(t-63) \oplus 7v_2(t-68) \oplus 8v_2(t-75) \\
\xi_{13}(t) &= 3\xi_{13}(t-45) \oplus 9v_3(t-83) \oplus 10v_3(t-88) \oplus 11v_3(t-95) \\
\xi_{14}(t) &= 3\xi_{14}(t-45) \oplus 4v_4(t-56) \oplus 5v_4(t-61) \oplus 6v_4(t-95) \\
\xi_{15}(t) &= 3\xi_{15}(t-45) \oplus 3v_5(t-50) \oplus 4v_5(t-55) \oplus 5v_5(t-62) \\
\xi_{16}(t) &= 3\xi_{16}(t-45) \oplus 3v_6(t-45) \oplus 4v_6(t-50) \oplus 5v_6(t-57) \\
\xi_{17}(t) &= 3\xi_{17}(t-45) \oplus 5v_7(t-48) \oplus 6v_7(t-81) \oplus 7v_7(t-86) \\
\xi_{18}(t) &= 3\xi_{18}(t-45) \oplus 5v_8(t-36) \oplus 6v_8(t-69) \oplus 7v_8(t-74) \\
\xi_{19}(t) &= 3\xi_{19}(t-45) \oplus 6v_9(t-63) \oplus 7v_9(t-68) \oplus 8v_9(t-75)
\end{aligned}
$$

$$
\xi_v^0(t) = \xi_{10}(t) \oplus \xi_{11}(t) \oplus ... \oplus \xi_{18}(t) \oplus \xi_{19}(t)
$$

$$
u_v^0(t) = v_0(t) \oplus v_0(t-2) \oplus 2v_0(t-4) \oplus 3v_0(t-30) \oplus 4v_0(t-34) \oplus 5v_0(t-38) \oplus 3v_1(t) \oplus
$$

$4v_1(t-24) \oplus 5v_1(t-28) \oplus 6v_1(t-32) \oplus 3v_2(t-18) \oplus 3v_2(t-22) \oplus 5v_2(t-26) \oplus 3v_3(t-8) \oplus 4v_3(t-10) \oplus 5v_3(t-12) \oplus 6v_3(t-38) \oplus 7v_3(t-42) \oplus 8v_3(t-46) \oplus 3v_4(t-18) \oplus 3v_7(t-36) \oplus 4v_7(t-40) \oplus 3v_8(t-24) \oplus 4v_3(t-28) \oplus 3v_9(t-18) \oplus 4v_9(t-22) \oplus 5v_9(t-26) \oplus \xi_v^0(t)$

So, using the matrices Q and K previously defined, we can simplify and easily calculate the control law u in time-domain, using the reference v and the output y. Now, its not necessary to calculate the estimated state $\hat{x}$ to obtain the control law u.

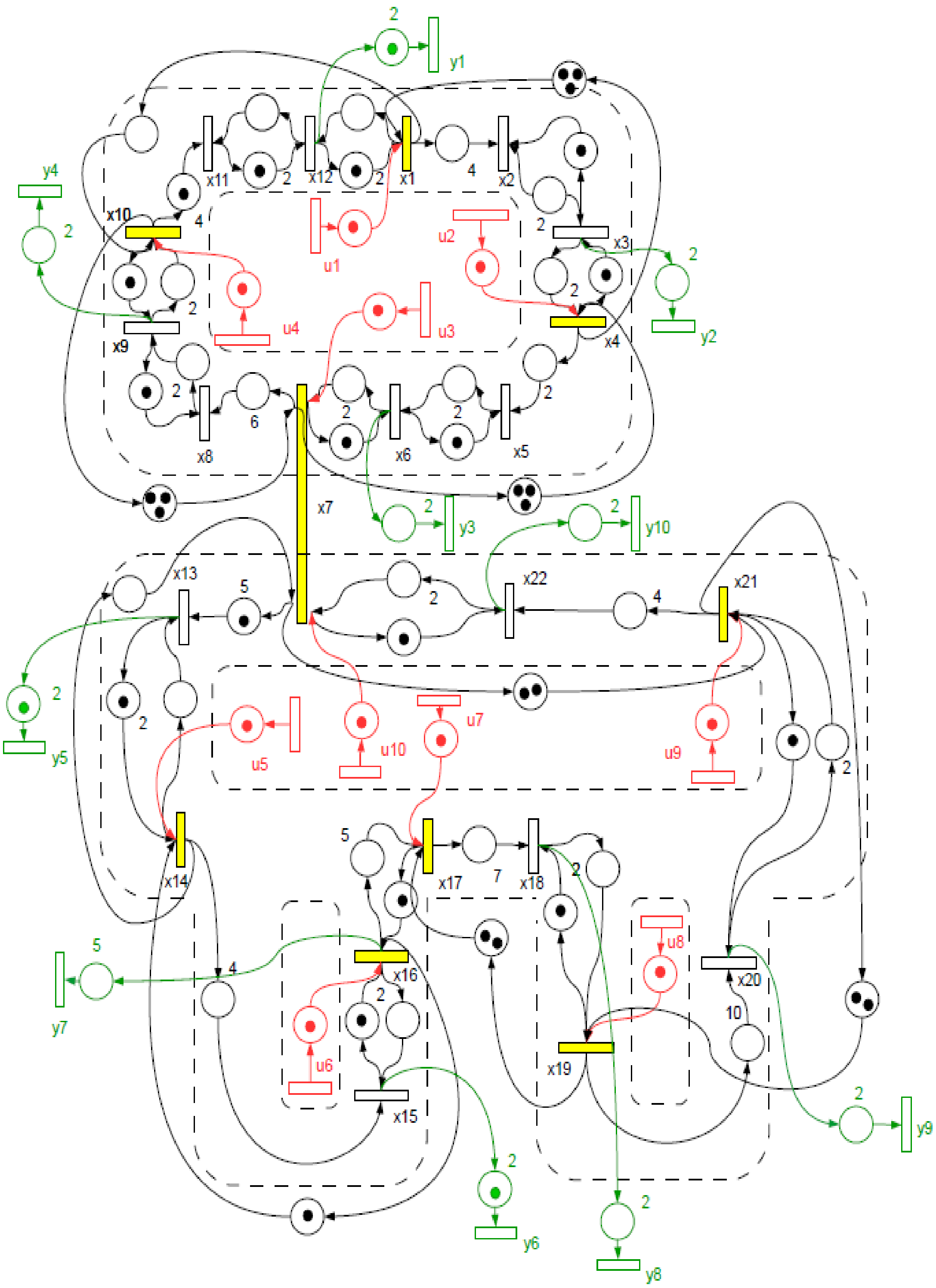


Figure 18 – Full TEG

5 Implementation

To calculate the matrices Q, K, M, P and Lopt it was necessary to use a Toolbox, developed in C++ language , to handle increasing pseudo-periodic series in semiring $\mathcal{M}_{in}^{ax}[\gamma, \delta]$, introduced by the $(max, plus)$ team of INRIA Roquencourt. The algorithms proposed in this software are initiated in 1992 in the PhD of S. Gaubert and continuated in 1994 during the master of Benoit Gruet. It is still in evolution in order to be improved until today. The C++ library can be interfaced to Scilab and more efficiently with Scicoslab. This toolbox and interfaces are downloadable in the following URL: <http://istia.univ-angers.fr/ hardouin/outils.html>. It was used the software CodeBlocks as development environment, to handle with the C++ language.

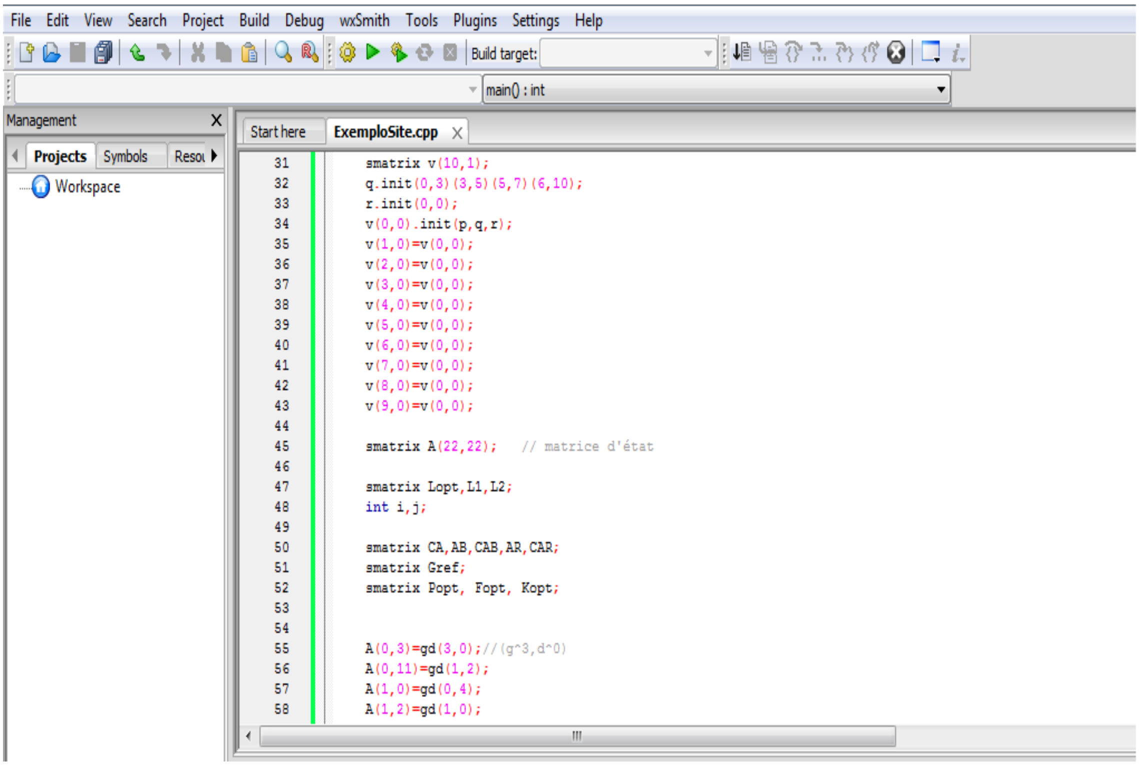


Figure 19 – Development environment - CodeBlocks

5.1 Code

The main code is presented as follows:

```
#ifndef _WIN32
#include "../include/lminmaxgd.h"
#else
#include "..\include\lminmaxgd.h"
using namespace std;
```

```
#endif

#include "..\src\gd.cpp"
#include "..\src\poly.cpp"
#include "..\src\serie.cpp"
#include "..\src\smatrix.cpp"
#include "..\src\tools.cpp"
```

At first it was necessary to include some C++ libraries in order to work with formal series, monomials, polynomials and to compute the star of a matrix. From now, will be shown how to calculate all controllers, the transfer matrix H and how to proceed a simulation using the toolbox.

```
///////

int main() // Main Function
{
try
{
poly p,q;
gd r;
smatrix v(10,1);
q.init(0,3)(3,5)(5,7)(6,10);
r.init(0,0);
v(0,0).init(p,q,r);             // Reference v
v(1,0)=v(0,0);
v(2,0)=v(0,0);
v(3,0)=v(0,0);
v(4,0)=v(0,0);
v(5,0)=v(0,0);
v(6,0)=v(0,0);
v(7,0)=v(0,0);
v(8,0)=v(0,0);
v(9,0)=v(0,0);
```

The reference v was defined using a polynomial, all entries have the same form, without losing generality.

```
smatrix A(22,22); // Matrix A

smatrix Lopt,L1,L2;
int i,j;

smatrix CA,AB,CAB,AR,CAR;
smatrix Gref;
smatrix Popt, Fopt, M;
```

The second step was the definition of all controllers and model. The state transfer matrix A:

```
A(0,3)=gd(3,0);//(g^3,d^0) // Matrix A
A(0,11)=gd(1,2);
A(1,0)=gd(0,4);
A(1,2)=gd(1,0);
A(2,1)=gd(0,2);
A(2,3)=gd(1,0);
A(3,2)=gd(0,2);
A(3,6)=gd(3,0);
A(4,2)=gd(0,2);
A(4,6)=gd(1,0);
A(5,4)=gd(0,2);
A(5,6)=gd(1,0);

A(6,5)=gd(0,2);
A(6,9)=gd(3,0);
A(6,13)=gd(0,0);
A(6,21)=gd(0,2);
A(7,6)=gd(0,6);
A(7,8)=gd(1,0);
A(8,7)=gd(0,2);
A(8,9)=gd(1,0);
A(9,0)=gd(0,0);
A(9,8)=gd(0,2);

A(10,9)=gd(1,4);
A(10,11)=gd(0,0);
A(11,0)=gd(0,0);
A(11,10)=gd(1,2);
A(12,6)=gd(1,5);
A(12,13)=gd(0,0);
A(13,12)=gd(1,2);
A(13,15)=gd(1,0);
A(14,13)=gd(0,4);
A(14,15)=gd(0,0);

A(15,14)=gd(1,2);
A(15,16)=gd(1,0);
A(16,15)=gd(0,5);
A(16,18)=gd(2,0);
A(16,15)=gd(0,5);
A(16,18)=gd(2,0);
A(17,16)=gd(0,7);
A(17,18)=gd(1,0);
A(18,17)=gd(0,2);
A(18,20)=gd(2,0);

A(19,18)=gd(0,10);
A(19,20)=gd(1,0);
A(20,6)=gd(2,0);
A(20,19)=gd(0,2);
A(21,6)=gd(1,0);
A(21,20)=gd(0,4);
```

The output matrix C and input matrix B:

```
smatrix C(10,22); // Output Matrix C
C(0,11)=gd(1,2);
C(1,2)=gd(0,2);
C(2,5)=gd(0,2);
C(3,8)=gd(0,2);
C(4,12)=gd(1,2);
C(5,14)=gd(1,2);
C(6,15)=gd(0,5);
C(7,17)=gd(0,2);
C(8,19)=gd(0,2);
C(9,21)=gd(0,2);
```

```
smatrix B(22,10); // Matrix B
B(0,0)=gd(1,0);
B(3,1)=gd(1,0);
B(6,2)=gd(1,0);
B(6,9)=gd(1,0);
B(9,3)=gd(1,0);
B(13,4)=gd(1,0);
B(15,5)=gd(1,0);
B(16,6)=gd(1,0);
B(18,7)=gd(1,0);
B(20,8)=gd(1,0);
```

Matrix R represents the connections between the uncontrollable inputs and the state matrix. In our case R is defined as identity and we will not work with system disturbances.

```
smatrix R(22,22); // Matrix R = Identity
R(0,0)=gd(0,0);
R(1,1)=gd(0,0);
R(2,2)=gd(0,0);
R(3,3)=gd(0,0);
R(4,4)=gd(0,0);
R(5,5)=gd(0,0);
R(6,6)=gd(0,0);
R(7,7)=gd(0,0);
R(8,8)=gd(0,0);
R(9,9)=gd(0,0);
R(10,10)=gd(0,0);
R(11,11)=gd(0,0);
R(12,12)=gd(0,0);
R(13,13)=gd(0,0);
R(14,14)=gd(0,0);
R(15,15)=gd(0,0);
R(16,16)=gd(0,0);
R(17,17)=gd(0,0);
```

```
R(18,18)=gd(0,0);
R(19,19)=gd(0,0);
R(20,20)=gd(0,0);
R(21,21)=gd(0,0);
```

Calculation of the Observer L_{opt}

```
smatrix As=star(A); // A*
CA=otimes(C,As);   // CA*

AB=otimes(As,B);
CAB=otimes(C,AB);
AR=otimes(As,R);
CAR=otimes(C,AR);

L1=rfrac(AB,CAB);
L2=rfrac(AR,CAR);
Lopt=inf(L1,L2);
Lopt=prcaus(Lopt);  // Observer L_opt


fstream f;
f.open("/Users/Matheus/Documents/libminmaxgd/examples/Lopt.txt",ios::out);
f<<Lopt;
f.close();
```

The desired behavior G_{ref} was defined equal to the transfer function $G_{ref} = H = CA^{\star}B$. Controllers M and P_{opt} are also calculated. It is important to mention that as we are working with a real system, the use of causal controllers is required.

```
Gref=CAB; // Desired behavior Gref = CA*B =H

Popt=lfrac(Gref,CAB);
M=otimes(AB,Popt);
M=rfrac(Popt,M);
M=lfrac(M,Popt);
M=prcaus(M); // Causal M

Fopt=rfrac(M,C);
Fopt=prcaus(Fopt); // Causal Fopt


fstream g;
g.open("/Users/Matheus/Documents/libminmaxgd/examples/Fopt.txt",ios::out);
g<<Fopt;
g.close();


fstream h;
```

```
h.open("/Users/Matheus/Documents/libminmaxgd/examples/M.txt",ios::out);
h<<M;
h.close();
```

Calculation of Q and K.

```
// - - - - - - - - -  Control - - - - - - -

smatrix Q;

Q = otimes(As,B);
Q = otimes(Kopt,Q);
Q = star(Q);
Q = otimes(Q,Popt);
Q = prcaus(Q); // Causal Matrix Q

fstream l;
l.open("/Users/Matheus/Documents/libminmaxgd/examples/Q.txt",ios::out);
l<<Q;
l.close();

smatrix K;

K = otimes(Q,Kopt);
K = otimes(K,As);
K = otimes(K,Lopt);
K = prcaus(K); // Causal Matrix K

fstream m;
m.open("/Users/Matheus/Documents/libminmaxgd/examples/K.txt",ios::out);
m<<K;
m.close();
```

To conclude, is presented the simulation, where will be compared the controlled system responce, denoted by Y and the desired output, denoted by Y2.

```
// ------------------- Simulation ---------------------

smatrix Y;

Y=otimes(Lopt,C);
Y=oplus(A,Y);
Y=star(Y);
Y=otimes(Kopt,Y);
Y=otimes(Y,B);
Y=otimes(Y,Popt);
Y=star(Y);
Y=otimes(Popt,Y);
Y=otimes(B,Y);
Y=otimes(As,Y);
```

```
Y=otimes(C,Y);
Y=otimes(Y,v);

cout<<"Y"<<Y<<endl;

smatrix Y2;

Y2=otimes(Gref,v);

cout<<"Y2"<<Y2<<endl;



return(0);
}

catch(mem_limite l)
{
cout<<"Exception : too many coefficents in polynom "<<l.memoire<<endl;
return(1);
}

catch(taille_incorrecte obj)
{ // 0 : r non causal
// 1 : tentative d'accès à un element d'une matrice avec un indice incorrect
// 2 : matrice de taille incompatible pour oplus, inf, otimes, rfrac, lfrac
// 3 : etoile de matrice carrée uniquement
cout<<"Exception  "<<obj.erreur<<endl;
return(1);
}
}
```

Using the $\mathcal{M}_{in}^{ax}[\gamma, \delta]$ C++ library we could easily work with periodic series, an effective toolbox, that doesn't need an huge computational effort to solve our problem. Simulation results will be presented in the next chapter.

6 Results

In order to test the efficacy of the observer-based controller a simulation was developed. Remembering, the objective is to get the greatest control law such that the output y be smaller than or equal to the desired output $G_{ref}v$.The desired behavior of the system G_{ref}, was defined equal to the transfer function H, which means $G_{ref} = H = CA^{\star}B$.

It was performed a simple comparison between the controlled system responce, denoted by Y and the desired output, denoted by Y2.

$$Y = Cx = CA^{*}BP(M(A \oplus L_{opt}C)^{*}BP)^{*}v \tag{6.1}$$

$$Y2 = G_{ref}v = Hv \tag{6.2}$$

The following code represents how to calculate Y and Y2 using the Toolbox, developed in C++ language , to handle increasing pseudo-periodic series in semiring $\mathcal{M}_{in}^{ax}[\gamma, \delta]$, introduced by the $(max, plus)$ team of INRIA Roquencourt.

```
----------------------------------- Simulation ----------------------------------
smatrix Y;

Y=otimes(Lopt,C);
Y=oplus(A,Y);
Y=star(Y);
Y=otimes(Kopt,Y);
Y=otimes(Y,B);
Y=otimes(Y,Popt);
Y=star(Y);
Y=otimes(Popt,Y);
Y=otimes(B,Y);
Y=otimes(As,Y);
Y=otimes(C,Y);
Y=otimes(Y,v);

cout<<"Y"<<Y<<endl;

smatrix Y2;

Y2=otimes(Gref,v);

cout<<"Y2"<<Y2<<endl;
---------------------------------------------------------------------------------
```

6.1 Simulation Results

The result obtained from the application changes some characters in the semiring $\mathcal{M}_{in}^{ax}[\gamma,\delta]$ representation ($g \Leftrightarrow \gamma$, $d \Leftrightarrow \delta$ and + $\Leftrightarrow \oplus$). The simulation result, provided by the console application is shown below:

```
Y =
[0,0] =  g^2 d^5+ g^3 d^7+( g^4 d^53+ g^5 d^59+ g^6 d^65)[ g^3 d^45]*
[1,0] =  g^1 d^11+ g^2 d^13+ g^3 d^15+( g^4 d^61+ g^5 d^67+ g^6 d^73)[ g^3 d^45]*
[2,0] =  g^1 d^15+( g^2 d^39+ g^3 d^45+ g^4 d^65)[ g^3 d^45]*
[3,0] = ( g^1 d^45+ g^2 d^51+ g^3 d^57)[ g^3 d^45]*
[4,0] =  g^2 d^5+( g^3 d^42+ g^4 d^48+ g^5 d^54)[ g^3 d^45]*
[5,0] =  g^2 d^9+ g^3 d^11+( g^4 d^48+ g^5 d^54+ g^6 d^60)[ g^3 d^45]*
[6,0] =  g^1 d^8+ g^2 d^14+ g^3 d^19+( g^4 d^53+ g^5 d^59+ g^6 d^65)[ g^3 d^45]*
[7,0] =  g^1 d^17+ g^2 d^23+ g^3 d^28+( g^4 d^62+ g^5 d^68+ g^6 d^74)[ g^3 d^45]*
[8,0] = ( g^1 d^29+ g^2 d^35+ g^3 d^41)[ g^3 d^45]*
[9,0] = ( g^1 d^35+ g^2 d^41+ g^3 d^47)[ g^3 d^45]*


Y2 =
[0,0] =  g^2 d^5+ g^3 d^7+( g^4 d^53+ g^5 d^59+ g^6 d^65)[ g^3 d^45]*
[1,0] =  g^1 d^11+ g^2 d^13+ g^3 d^15+( g^4 d^61+ g^5 d^67+ g^6 d^73)[ g^3 d^45]*
[2,0] =  g^1 d^15+( g^2 d^39+ g^3 d^45+ g^4 d^65)[ g^3 d^45]*
[3,0] = ( g^1 d^45+ g^2 d^51+ g^3 d^57)[ g^3 d^45]*
[4,0] =  g^2 d^5+( g^3 d^42+ g^4 d^48+ g^5 d^54)[ g^3 d^45]*
[5,0] =  g^2 d^9+ g^3 d^11+( g^4 d^48+ g^5 d^54+ g^6 d^60)[ g^3 d^45]*
[6,0] =  g^1 d^8+ g^2 d^14+ g^3 d^19+( g^4 d^53+ g^5 d^59+ g^6 d^65)[ g^3 d^45]*
[7,0] =  g^1 d^17+ g^2 d^23+ g^3 d^28+( g^4 d^62+ g^5 d^68+ g^6 d^74)[ g^3 d^45]*
[8,0] = ( g^1 d^29+ g^2 d^35+ g^3 d^41)[ g^3 d^45]*
[9,0] = ( g^1 d^35+ g^2 d^41+ g^3 d^47)[ g^3 d^45]*
```

Writing in standardized form, we have:

$$
\begin{aligned}
Y_0 &= \gamma^2\delta^5 \oplus \gamma^3\delta^7 \oplus (\gamma^4\delta^{53} \oplus \gamma^5\delta^{59} \oplus \gamma^6\delta^{65})[\gamma^3\delta^{45}]* \\
Y_1 &= \gamma^1\delta^{11} \oplus \gamma^2\delta^{13} \oplus \gamma^3\delta^{15} \oplus (\gamma^4\delta^{61} \oplus \gamma^5\delta^{67} \oplus \gamma^6\delta^{73})[\gamma^3\delta^{45}]* \\
Y_2 &= \gamma^1\delta^{15} \oplus (\gamma^2\delta^{39} \oplus \gamma^3\delta^{45} \oplus \gamma^4\delta^{65})[\gamma^3\delta^{45}]* \\
Y_3 &= (\gamma^1\delta^{45} \oplus \gamma^2\delta^{51} \oplus \gamma^3\delta^{57})[\gamma^3\delta^{45}]* \\
Y_4 &= \gamma^2\delta^5 \oplus (\gamma^3\delta^{42} \oplus \gamma^4\delta^{48} \oplus \gamma^5\delta^{54})[\gamma^3\delta^{45}]* \\
Y_5 &= \gamma^2\delta^9 \oplus \gamma^3\delta^{11} \oplus (\gamma^4\delta^{48} \oplus \gamma^5\delta^{54} \oplus \gamma^6\delta^{60})[\gamma^3\delta^{45}]* \\
Y_6 &= \gamma^1\delta^8 \oplus \gamma^2\delta^{14} \oplus \gamma^3\delta^{19} \oplus (\gamma^4\delta^{53} \oplus \gamma^5\delta^{59} \oplus \gamma^6\delta^{65})[\gamma^3\delta^{45}]* \\
Y_7 &= \gamma^1\delta^{17} \oplus \gamma^2\delta^{23} \oplus \gamma^3\delta^{28} \oplus (\gamma^4\delta^{62} \oplus \gamma^5\delta^{68} \oplus \gamma^6\delta^{74})[\gamma^3\delta^{45}]* \\
Y_8 &= (\gamma^1\delta^{29} \oplus \gamma^2\delta^{35} \oplus \gamma^3\delta^{41})[\gamma^3\delta^{45}]* \\
Y_9 &= (\gamma^1\delta^{35} \oplus \gamma^2\delta^{41} \oplus \gamma^3\delta^{47})[\gamma^3\delta^{45}]*
\end{aligned}
$$

$$
\begin{aligned}
Y2_0 &= \gamma^2\delta^5 \oplus \gamma^3\delta^7 \oplus (\gamma^4\delta^{53} \oplus \gamma^5\delta^{59} \oplus \gamma^6\delta^{65})[\gamma^3\delta^{45}]* \\
Y2_1 &= \gamma^1\delta^{11} \oplus \gamma^2\delta^{13} \oplus \gamma^3\delta^{15} \oplus (\gamma^4\delta^{61} \oplus \gamma^5\delta^{67} \oplus \gamma^6\delta^{73})[\gamma^3\delta^{45}]* \\
Y2_2 &= \gamma^1\delta^{15} \oplus (\gamma^2\delta^{39} \oplus \gamma^3\delta^{45} \oplus \gamma^4\delta^{65})[\gamma^3\delta^{45}]* \\
Y2_3 &= (\gamma^1\delta^{45} \oplus \gamma^2\delta^{51} \oplus \gamma^3\delta^{57})[\gamma^3\delta^{45}]* \\
Y2_4 &= \gamma^2\delta^5 \oplus (\gamma^3\delta^{42} \oplus \gamma^4\delta^{48} \oplus \gamma^5\delta^{54})[\gamma^3\delta^{45}]* \\
Y2_5 &= \gamma^2\delta^9 \oplus \gamma^3\delta^{11} \oplus (\gamma^4\delta^{48} \oplus \gamma^5\delta^{54} \oplus \gamma^6\delta^{60})[\gamma^3\delta^{45}]* \\
Y2_6 &= \gamma^1\delta^8 \oplus \gamma^2\delta^{14} \oplus \gamma^3\delta^{19} \oplus (\gamma^4\delta^{53} \oplus \gamma^5\delta^{59} \oplus \gamma^6\delta^{65})[\gamma^3\delta^{45}]* \\
Y2_7 &= \gamma^1\delta^{17} \oplus \gamma^2\delta^{23} \oplus \gamma^3\delta^{28} \oplus (\gamma^4\delta^{62} \oplus \gamma^5\delta^{68} \oplus \gamma^6\delta^{74})[\gamma^3\delta^{45}]* \\
Y2_8 &= (\gamma^1\delta^{29} \oplus \gamma^2\delta^{35} \oplus \gamma^3\delta^{41})[\gamma^3\delta^{45}]* \\
Y2_9 &= (\gamma^1\delta^{35} \oplus \gamma^2\delta^{41} \oplus \gamma^3\delta^{47})[\gamma^3\delta^{45}]*
\end{aligned}
$$

As mentioned in Chapter 3, the system responses are periodic and causal series, which can be written in the following standardized form : $p \oplus qr^*$, with p and q polynomials with exponents in $\mathbb{N}$ and r a monomial with exponents in$\mathbb{N}$. Polynomial p characterizes the transient behavior of the series, polynomial q represents a pattern which is repeated periodically, the periodicity is given by $r = \gamma^\nu\delta^\tau$ where ν/τ corresponds to the production rate of the series, conversely τ/ν is the cycle time of the series.

6.2 Simulation Analysis

Analyzing the response of the controlled system Y, some conclusions can be taken:

- Y_0, Y_1, Y_2, Y_4, Y_5, Y_6 and Y_7 have a transient behavior before reaching the steady state;
- Y_3, Y_8 and Y_9 only have the steady state;
- The periodicity is given by $r = \gamma^\nu\delta^\tau = \gamma^3\delta^{45}$;
- The production rate is $\tau/\nu = 45/3$.

At least, analysing the performance of the controlled system and the desired behavior, by comparing the polynomials Y_0 and $Y2_0$, respectively, the output of the controlled system and desired output.

$Y_0 = \gamma^2\delta^5 \oplus \gamma^3\delta^7 \oplus (\gamma^4\delta^{53} \oplus \gamma^5\delta^{59} \oplus \gamma^6\delta^{65})[\gamma^3\delta^{45}]*$

$Y2_0 = \gamma^2\delta^5 \oplus \gamma^3\delta^7 \oplus (\gamma^4\delta^{53} \oplus \gamma^5\delta^{59} \oplus \gamma^6\delta^{65})[\gamma^3\delta^{45}]*$

Can be noticed that $Y_0 = Y2_0$. Which means that the observer-based controller has reached the objective. This result is present for all trajectories of Y_k and $Y2_k$.

7 Conclusion

The first general objective of the project - to develop a model for the automated conveyor system located in LARIS using a TEG model proposed by Dr. Vinícius Mariano Gonçalves in his thesis [21] and the two-dimentional description $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ for (Max,+) linear systems, was achieved.

Using the Observer-based Controller for (Max,+) linear systems, based on the work introduced by Prof. Laurent Hardouin in [22] , the second general objective of the project was to control the automated conveyor system. Some algebraic manipulations were proposed by me, in order to simplify the way of obtaining the control law u in the time domain. Now, it is not necessary to calculate the u using the estimated state $\hat{x}$. The control law can be calculated based on the reference input v and the output of the system Y.

To calculate the observer-based controller it was necessary to use a Toolbox, developed in C++ language , to handle with increasing pseudo-periodic series in semiring $\mathcal{M}_{in}^{ax}[\gamma,\delta]$, introduced by the $(max, plus)$ team of INRIA Roquencourt.

After the calculation of the controllers, a simulation of the controlled system has been proposed. The output of the controlled system Y was compared with the desired output $Y2 = G_{ref}v$. As we can observe in Chapter 6 specification has been reached. Therefore, the controlled system has the same performance of the reference system G_{ref}, but their controlled inputs u are delayed as much as possible. For just-in-time problems this is a very important result.

As a possible continuation of this work, uncertain TEG models can be used to represent possible modelling errors.

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Appendix

APPENDIX A – Transfer Function and Controllers

The transfer function H of the system is given by:

```
H =
[0,0] =  g^2 d^2+ g^3 d^4+ g^4 d^30+ g^5 d^34+ g^6 d^38+( g^7 d^75+ g^8 d^80+ g^9 d^87)[ g^3 d^
[0,1] =  g^5 d^24+ g^6 d^28+ g^7 d^32+( g^8 d^69+ g^9 d^74+ g^10 d^81)[ g^3 d^45]*
[0,2] =  g^4 d^18+ g^5 d^22+ g^6 d^26+( g^7 d^63+ g^8 d^68+ g^9 d^75)[ g^3 d^45]*
[0,3] =  g^4 d^8+ g^5 d^10+ g^6 d^12+ g^7 d^38+ g^8 d^42+ g^9 d^46+( g^10 d^83+ g^11 d^88+ g^12
[0,4] =  g^4 d^18+( g^5 d^56+ g^6 d^61+ g^7 d^68)[ g^3 d^45]*
[0,5] = ( g^4 d^50+ g^5 d^55+ g^6 d^62)[ g^3 d^45]*
[0,6] = ( g^4 d^45+ g^5 d^50+ g^6 d^57)[ g^3 d^45]*
[0,7] =  g^4 d^36+ g^5 d^40+( g^6 d^48+ g^7 d^81+ g^8 d^86)[ g^3 d^45]*
[0,8] =  g^4 d^24+ g^5 d^28+( g^6 d^36+ g^7 d^69+ g^8 d^74)[ g^3 d^45]*
[0,9] =  g^4 d^18+ g^5 d^22+ g^6 d^26+( g^7 d^63+ g^8 d^68+ g^9 d^75)[ g^3 d^45]*
[1,0] =  g^1 d^8+ g^2 d^10+ g^3 d^12+ g^4 d^38+ g^5 d^42+ g^6 d^46+( g^7 d^83+ g^8 d^88+ g^9 d^
[1,1] =  g^2 d^2+ g^3 d^4+ g^4 d^8+ g^5 d^32+ g^6 d^36+ g^7 d^40+( g^8 d^77+ g^9 d^82+ g^10 d^8
[1,2] =  g^4 d^26+ g^5 d^30+ g^6 d^34+( g^7 d^71+ g^8 d^76+ g^9 d^83)[ g^3 d^45]*
[1,3] =  g^4 d^16+ g^5 d^18+ g^6 d^20+ g^7 d^46+ g^8 d^50+ g^9 d^54+( g^10 d^91+ g^11 d^96+ g^1
[1,4] =  g^4 d^26+( g^5 d^64+ g^6 d^69+ g^7 d^76)[ g^3 d^45]*
[1,5] = ( g^4 d^58+ g^5 d^63+ g^6 d^70)[ g^3 d^45]*
[1,6] = ( g^4 d^53+ g^5 d^58+ g^6 d^65)[ g^3 d^45]*
[1,7] =  g^4 d^44+ g^5 d^48+( g^6 d^56+ g^7 d^89+ g^8 d^94)[ g^3 d^45]*
[1,8] =  g^4 d^32+ g^5 d^36+( g^6 d^44+ g^7 d^77+ g^8 d^82)[ g^3 d^45]*
[1,9] =  g^4 d^26+ g^5 d^30+ g^6 d^34+( g^7 d^71+ g^8 d^76+ g^9 d^83)[ g^3 d^45]*
[2,0] =  g^1 d^12+ g^2 d^16+ g^3 d^20+( g^4 d^42+ g^5 d^61+ g^6 d^66)[ g^3 d^45]*
[2,1] =  g^2 d^6+ g^3 d^10+ g^4 d^14+( g^5 d^36+ g^6 d^55+ g^7 d^60)[ g^3 d^45]*
[2,2] =  g^2 d^4+ g^3 d^8+( g^4 d^30+ g^5 d^49+ g^6 d^54)[ g^3 d^45]*
[2,3] =  g^4 d^20+ g^5 d^24+ g^6 d^28+( g^7 d^50+ g^8 d^69+ g^9 d^74)[ g^3 d^45]*
[2,4] =  g^2 d^4+( g^3 d^42+ g^4 d^47+ g^5 d^68)[ g^3 d^45]*
[2,5] = ( g^2 d^36+ g^3 d^41+ g^4 d^62)[ g^3 d^45]*
[2,6] = ( g^2 d^31+ g^3 d^36+ g^4 d^57)[ g^3 d^45]*
[2,7] =  g^2 d^22+ g^3 d^26+( g^4 d^48+ g^5 d^67+ g^6 d^72)[ g^3 d^45]*
[2,8] =  g^2 d^10+ g^3 d^14+( g^4 d^36+ g^5 d^55+ g^6 d^60)[ g^3 d^45]*
[2,9] =  g^2 d^4+ g^3 d^8+( g^4 d^30+ g^5 d^49+ g^6 d^54)[ g^3 d^45]*
[3,0] =  g^1 d^22+ g^2 d^26+ g^3 d^30+( g^4 d^67+ g^5 d^72+ g^6 d^79)[ g^3 d^45]*
[3,1] =  g^2 d^16+ g^3 d^20+ g^4 d^24+( g^5 d^61+ g^6 d^66+ g^7 d^73)[ g^3 d^45]*
[3,2] =  g^1 d^10+ g^2 d^14+ g^3 d^18+( g^4 d^55+ g^5 d^60+ g^6 d^67)[ g^3 d^45]*
[3,3] =  g^2 d^2+ g^3 d^4+ g^4 d^30+ g^5 d^34+ g^6 d^38+( g^7 d^75+ g^8 d^80+ g^9 d^87)[ g^3 d^
[3,4] =  g^1 d^10+( g^2 d^48+ g^3 d^53+ g^4 d^60)[ g^3 d^45]*
[3,5] = ( g^1 d^42+ g^2 d^47+ g^3 d^54)[ g^3 d^45]*
[3,6] = ( g^1 d^37+ g^2 d^42+ g^3 d^49)[ g^3 d^45]*
[3,7] =  g^1 d^28+ g^2 d^32+( g^3 d^40+ g^4 d^73+ g^5 d^78)[ g^3 d^45]*
[3,8] =  g^1 d^16+ g^2 d^20+( g^3 d^28+ g^4 d^61+ g^5 d^66)[ g^3 d^45]*
[3,9] =  g^1 d^10+ g^2 d^14+ g^3 d^18+( g^4 d^55+ g^5 d^60+ g^6 d^67)[ g^3 d^45]*
[4,0] =  g^3 d^19+ g^4 d^23+ g^5 d^27+( g^6 d^64+ g^7 d^69+ g^8 d^76)[ g^3 d^45]*
[4,1] =  g^4 d^13+ g^5 d^17+ g^6 d^21+( g^7 d^58+ g^8 d^63+ g^9 d^70)[ g^3 d^45]*
[4,2] =  g^3 d^7+ g^4 d^11+ g^5 d^15+( g^6 d^52+ g^7 d^57+ g^8 d^64)[ g^3 d^45]*
[4,3] =  g^6 d^27+ g^7 d^31+ g^8 d^35+( g^9 d^72+ g^10 d^77+ g^11 d^84)[ g^3 d^45]*
[4,4] =  g^2 d^2+ g^3 d^7+( g^4 d^45+ g^5 d^50+ g^6 d^57)[ g^3 d^45]*
```

```
[4,5] = ( g^3 d^39+ g^4 d^44+ g^5 d^51)[ g^3 d^45]*
[4,6] = ( g^3 d^34+ g^4 d^39+ g^5 d^46)[ g^3 d^45]*
[4,7] =  g^3 d^25+ g^4 d^29+( g^5 d^37+ g^6 d^70+ g^7 d^75)[ g^3 d^45]*
[4,8] =  g^3 d^13+ g^4 d^17+( g^5 d^25+ g^6 d^58+ g^7 d^63)[ g^3 d^45]*
[4,9] =  g^3 d^7+ g^4 d^11+ g^5 d^15+( g^6 d^52+ g^7 d^57+ g^8 d^64)[ g^3 d^45]*
[5,0] =  g^4 d^25+ g^5 d^29+ g^6 d^33+( g^7 d^70+ g^8 d^75+ g^9 d^82)[ g^3 d^45]*
[5,1] =  g^5 d^19+ g^6 d^23+ g^7 d^27+( g^8 d^64+ g^9 d^69+ g^10 d^76)[ g^3 d^45]*
[5,2] =  g^4 d^13+ g^5 d^17+ g^6 d^21+( g^7 d^58+ g^8 d^63+ g^9 d^70)[ g^3 d^45]*
[5,3] =  g^7 d^33+ g^8 d^37+ g^9 d^41+( g^10 d^78+ g^11 d^83+ g^12 d^90)[ g^3 d^45]*
[5,4] =  g^2 d^6+ g^3 d^8+ g^4 d^13+( g^5 d^51+ g^6 d^56+ g^7 d^63)[ g^3 d^45]*
[5,5] =  g^2 d^2+ g^3 d^7+( g^4 d^45+ g^5 d^50+ g^6 d^57)[ g^3 d^45]*
[5,6] =  g^3 d^2+( g^4 d^40+ g^5 d^45+ g^6 d^52)[ g^3 d^45]*
[5,7] =  g^4 d^31+ g^5 d^35+( g^6 d^43+ g^7 d^76+ g^8 d^81)[ g^3 d^45]*
[5,8] =  g^4 d^19+ g^5 d^23+( g^6 d^31+ g^7 d^64+ g^8 d^69)[ g^3 d^45]*
[5,9] =  g^4 d^13+ g^5 d^17+ g^6 d^21+( g^7 d^58+ g^8 d^63+ g^9 d^70)[ g^3 d^45]*
[6,0] =  g^4 d^30+ g^5 d^35+ g^6 d^40+( g^7 d^75+ g^8 d^80+ g^9 d^87)[ g^3 d^45]*
[6,1] =  g^5 d^24+ g^6 d^29+ g^7 d^34+( g^8 d^69+ g^9 d^74+ g^10 d^81)[ g^3 d^45]*
[6,2] =  g^4 d^18+ g^5 d^23+ g^6 d^28+( g^7 d^63+ g^8 d^68+ g^9 d^75)[ g^3 d^45]*
[6,3] =  g^7 d^38+ g^8 d^43+ g^9 d^48+( g^10 d^83+ g^11 d^88+ g^12 d^95)[ g^3 d^45]*
[6,4] =  g^2 d^11+ g^3 d^16+ g^4 d^21+( g^5 d^56+ g^6 d^61+ g^7 d^68)[ g^3 d^45]*
[6,5] =  g^1 d^5+ g^2 d^10+ g^3 d^15+( g^4 d^50+ g^5 d^55+ g^6 d^62)[ g^3 d^45]*
[6,6] =  g^2 d^5+ g^3 d^10+( g^4 d^45+ g^5 d^50+ g^6 d^57)[ g^3 d^45]*
[6,7] = ( g^4 d^36+ g^5 d^41+ g^6 d^48)[ g^3 d^45]*
[6,8] = ( g^4 d^24+ g^5 d^29+ g^6 d^36)[ g^3 d^45]*
[6,9] =  g^4 d^18+ g^5 d^23+ g^6 d^28+( g^7 d^63+ g^8 d^68+ g^9 d^75)[ g^3 d^45]*
[7,0] =  g^4 d^39+ g^5 d^44+ g^6 d^49+( g^7 d^84+ g^8 d^89+ g^9 d^96)[ g^3 d^45]*
[7,1] =  g^5 d^33+ g^6 d^38+ g^7 d^43+( g^8 d^78+ g^9 d^83+ g^10 d^90)[ g^3 d^45]*
[7,2] =  g^4 d^27+ g^5 d^32+ g^6 d^37+( g^7 d^72+ g^8 d^77+ g^9 d^84)[ g^3 d^45]*
[7,3] =  g^7 d^47+ g^8 d^52+ g^9 d^57+( g^10 d^92+ g^11 d^97+ g^12 d^104)[ g^3 d^45]*
[7,4] =  g^2 d^20+ g^3 d^25+ g^4 d^30+( g^5 d^65+ g^6 d^70+ g^7 d^77)[ g^3 d^45]*
[7,5] =  g^1 d^14+ g^2 d^19+ g^3 d^24+( g^4 d^59+ g^5 d^64+ g^6 d^71)[ g^3 d^45]*
[7,6] =  g^1 d^9+ g^2 d^14+ g^3 d^19+( g^4 d^54+ g^5 d^59+ g^6 d^66)[ g^3 d^45]*
[7,7] =  g^2 d^2+ g^3 d^9+( g^4 d^45+ g^5 d^50+ g^6 d^57)[ g^3 d^45]*
[7,8] = ( g^4 d^33+ g^5 d^38+ g^6 d^45)[ g^3 d^45]*
[7,9] =  g^4 d^27+ g^5 d^32+ g^6 d^37+( g^7 d^72+ g^8 d^77+ g^9 d^84)[ g^3 d^45]*
[8,0] = ( g^4 d^51+ g^5 d^56+ g^6 d^63)[ g^3 d^45]*
[8,1] = ( g^5 d^45+ g^6 d^50+ g^7 d^57)[ g^3 d^45]*
[8,2] = ( g^4 d^39+ g^5 d^44+ g^6 d^51)[ g^3 d^45]*
[8,3] = ( g^7 d^59+ g^8 d^64+ g^9 d^71)[ g^3 d^45]*
[8,4] = ( g^2 d^32+ g^3 d^37+ g^4 d^44)[ g^3 d^45]*
[8,5] = ( g^1 d^26+ g^2 d^31+ g^3 d^38)[ g^3 d^45]*
[8,6] = ( g^1 d^21+ g^2 d^26+ g^3 d^33)[ g^3 d^45]*
[8,7] =  g^1 d^12+ g^2 d^14+( g^3 d^24+ g^4 d^57+ g^5 d^62)[ g^3 d^45]*
[8,8] =  g^2 d^2+( g^3 d^12+ g^4 d^45+ g^5 d^50)[ g^3 d^45]*
[8,9] = ( g^4 d^39+ g^5 d^44+ g^6 d^51)[ g^3 d^45]*
[9,0] =  g^2 d^14+ g^3 d^18+( g^4 d^57+ g^5 d^62+ g^6 d^69)[ g^3 d^45]*
[9,1] =  g^3 d^8+ g^4 d^12+( g^5 d^51+ g^6 d^56+ g^7 d^63)[ g^3 d^45]*
[9,2] =  g^2 d^2+ g^3 d^6+( g^4 d^45+ g^5 d^50+ g^6 d^57)[ g^3 d^45]*
[9,3] =  g^5 d^22+ g^6 d^26+( g^7 d^65+ g^8 d^70+ g^9 d^77)[ g^3 d^45]*
[9,4] = ( g^2 d^38+ g^3 d^43+ g^4 d^50)[ g^3 d^45]*
[9,5] = ( g^1 d^32+ g^2 d^37+ g^3 d^44)[ g^3 d^45]*
[9,6] = ( g^1 d^27+ g^2 d^32+ g^3 d^39)[ g^3 d^45]*
[9,7] =  g^1 d^18+ g^2 d^20+( g^3 d^30+ g^4 d^63+ g^5 d^68)[ g^3 d^45]*
[9,8] =  g^1 d^6+ g^2 d^8+( g^3 d^18+ g^4 d^51+ g^5 d^56)[ g^3 d^45]*
[9,9] =  g^2 d^2+ g^3 d^6+( g^4 d^45+ g^5 d^50+ g^6 d^57)[ g^3 d^45]*
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The observer L_{opt}:

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Lopt =

[0,0] =  g^0 d^0+ g^1 d^2+ g^2 d^4+ g^3 d^30+ g^4 d^34+ g^5 d^38+( g^6 d^75+ g^7 d^80+ g^8 d^87
[0,1] =  g^3 d^22+ g^4 d^26+ g^5 d^30+( g^6 d^67+ g^7 d^72+ g^8 d^79)[ g^3 d^45]*
[0,2] =  g^3 d^18+ g^4 d^22+ g^5 d^26+( g^6 d^63+ g^7 d^68+ g^8 d^75)[ g^3 d^45]*
[0,3] =  g^3 d^8+ g^4 d^10+ g^5 d^12+ g^6 d^38+ g^7 d^42+ g^8 d^46+( g^9 d^83+ g^10 d^88+ g^11
[0,4] =  g^3 d^18+( g^4 d^56+ g^5 d^61+ g^6 d^68)[ g^3 d^45]*
[0,5] = ( g^3 d^50+ g^4 d^55+ g^5 d^62)[ g^3 d^45]*
[0,6] = ( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[0,7] =  g^3 d^36+ g^4 d^40+( g^5 d^48+ g^6 d^81+ g^7 d^86)[ g^3 d^45]*
[0,8] =  g^3 d^24+ g^4 d^28+( g^5 d^36+ g^6 d^69+ g^7 d^74)[ g^3 d^45]*
[0,9] =  g^3 d^18+ g^4 d^22+ g^5 d^26+( g^6 d^63+ g^7 d^68+ g^8 d^75)[ g^3 d^45]*
[1,0] =  g^0 d^4+ g^1 d^6+ g^2 d^8+ g^3 d^34+ g^4 d^38+ g^5 d^42+( g^6 d^79+ g^7 d^84+ g^8 d^91
[1,1] =  g^2 d^0+ g^3 d^26+ g^4 d^30+ g^5 d^34+( g^6 d^71+ g^7 d^76+ g^8 d^83)[ g^3 d^45]*
[1,2] =  g^3 d^22+ g^4 d^26+ g^5 d^30+( g^6 d^67+ g^7 d^72+ g^8 d^79)[ g^3 d^45]*
[1,3] =  g^3 d^12+ g^4 d^14+ g^5 d^16+ g^6 d^42+ g^7 d^46+ g^8 d^50+( g^9 d^87+ g^10 d^92+ g^11
[1,4] =  g^3 d^22+( g^4 d^60+ g^5 d^65+ g^6 d^72)[ g^3 d^45]*
[1,5] = ( g^3 d^54+ g^4 d^59+ g^5 d^66)[ g^3 d^45]*
[1,6] = ( g^3 d^49+ g^4 d^54+ g^5 d^61)[ g^3 d^45]*
[1,7] =  g^3 d^40+ g^4 d^44+( g^5 d^52+ g^6 d^85+ g^7 d^90)[ g^3 d^45]*
[1,8] =  g^3 d^28+ g^4 d^32+( g^5 d^40+ g^6 d^73+ g^7 d^78)[ g^3 d^45]*
[1,9] =  g^3 d^22+ g^4 d^26+ g^5 d^30+( g^6 d^67+ g^7 d^72+ g^8 d^79)[ g^3 d^45]*
[2,0] =  g^0 d^6+ g^1 d^8+ g^2 d^10+ g^3 d^36+ g^4 d^40+ g^5 d^44+( g^6 d^81+ g^7 d^86+ g^8 d^9
[2,1] =  g^1 d^0+ g^2 d^2+ g^3 d^28+ g^4 d^32+ g^5 d^36+( g^6 d^73+ g^7 d^78+ g^8 d^85)[ g^3 d^
[2,2] =  g^3 d^24+ g^4 d^28+ g^5 d^32+( g^6 d^69+ g^7 d^74+ g^8 d^81)[ g^3 d^45]*
[2,3] =  g^3 d^14+ g^4 d^16+ g^5 d^18+ g^6 d^44+ g^7 d^48+ g^8 d^52+( g^9 d^89+ g^10 d^94+ g^11
[2,4] =  g^3 d^24+( g^4 d^62+ g^5 d^67+ g^6 d^74)[ g^3 d^45]*
[2,5] = ( g^3 d^56+ g^4 d^61+ g^5 d^68)[ g^3 d^45]*
[2,6] = ( g^3 d^51+ g^4 d^56+ g^5 d^63)[ g^3 d^45]*
[2,7] =  g^3 d^42+ g^4 d^46+( g^5 d^54+ g^6 d^87+ g^7 d^92)[ g^3 d^45]*
[2,8] =  g^3 d^30+ g^4 d^34+( g^5 d^42+ g^6 d^75+ g^7 d^80)[ g^3 d^45]*
[2,9] =  g^3 d^24+ g^4 d^28+ g^5 d^32+( g^6 d^69+ g^7 d^74+ g^8 d^81)[ g^3 d^45]*
[3,0] =  g^0 d^8+ g^1 d^10+ g^2 d^12+ g^3 d^38+ g^4 d^42+ g^5 d^46+( g^6 d^83+ g^7 d^88+ g^8 d^
[3,1] =  g^0 d^0+ g^1 d^2+ g^2 d^4+ g^3 d^30+ g^4 d^34+ g^5 d^38+( g^6 d^75+ g^7 d^80+ g^8 d^87
[3,2] =  g^3 d^26+ g^4 d^30+ g^5 d^34+( g^6 d^71+ g^7 d^76+ g^8 d^83)[ g^3 d^45]*
[3,3] =  g^3 d^16+ g^4 d^18+ g^5 d^20+ g^6 d^46+ g^7 d^50+ g^8 d^54+( g^9 d^91+ g^10 d^96+ g^11
[3,4] =  g^3 d^26+( g^4 d^64+ g^5 d^69+ g^6 d^76)[ g^3 d^45]*
[3,5] = ( g^3 d^58+ g^4 d^63+ g^5 d^70)[ g^3 d^45]*
[3,6] = ( g^3 d^53+ g^4 d^58+ g^5 d^65)[ g^3 d^45]*
[3,7] =  g^3 d^44+ g^4 d^48+( g^5 d^56+ g^6 d^89+ g^7 d^94)[ g^3 d^45]*
[3,8] =  g^3 d^32+ g^4 d^36+( g^5 d^44+ g^6 d^77+ g^7 d^82)[ g^3 d^45]*
[3,9] =  g^3 d^26+ g^4 d^30+ g^5 d^34+( g^6 d^71+ g^7 d^76+ g^8 d^83)[ g^3 d^45]*
[4,0] =  g^0 d^8+ g^1 d^12+ g^2 d^16+( g^3 d^38+ g^4 d^57+ g^5 d^62)[ g^3 d^45]*
[4,1] =  g^0 d^0+ g^1 d^4+ g^2 d^8+( g^3 d^30+ g^4 d^49+ g^5 d^54)[ g^3 d^45]*
[4,2] =  g^1 d^0+ g^2 d^4+( g^3 d^26+ g^4 d^45+ g^5 d^50)[ g^3 d^45]*
[4,3] =  g^3 d^16+ g^4 d^20+ g^5 d^24+( g^6 d^46+ g^7 d^65+ g^8 d^70)[ g^3 d^45]*
[4,4] =  g^1 d^0+( g^2 d^38+ g^3 d^43+ g^4 d^64)[ g^3 d^45]*
[4,5] = ( g^1 d^32+ g^2 d^37+ g^3 d^58)[ g^3 d^45]*
[4,6] = ( g^1 d^27+ g^2 d^32+ g^3 d^53)[ g^3 d^45]*
[4,7] =  g^1 d^18+ g^2 d^22+( g^3 d^44+ g^4 d^63+ g^5 d^68)[ g^3 d^45]*
[4,8] =  g^1 d^6+ g^2 d^10+( g^3 d^32+ g^4 d^51+ g^5 d^56)[ g^3 d^45]*
[4,9] =  g^1 d^0+ g^2 d^4+( g^3 d^26+ g^4 d^45+ g^5 d^50)[ g^3 d^45]*
[5,0] =  g^0 d^10+ g^1 d^14+ g^2 d^18+( g^3 d^40+ g^4 d^59+ g^5 d^64)[ g^3 d^45]*
[5,1] =  g^0 d^2+ g^1 d^6+ g^2 d^10+( g^3 d^32+ g^4 d^51+ g^5 d^56)[ g^3 d^45]*
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[5,2] =  g^1 d^2+ g^2 d^6+( g^3 d^28+ g^4 d^47+ g^5 d^52)[ g^3 d^45]*
[5,3] =  g^3 d^18+ g^4 d^22+ g^5 d^26+( g^6 d^48+ g^7 d^67+ g^8 d^72)[ g^3 d^45]*
[5,4] =  g^1 d^2+( g^2 d^40+ g^3 d^45+ g^4 d^66)[ g^3 d^45]*
[5,5] = ( g^1 d^34+ g^2 d^39+ g^3 d^60)[ g^3 d^45]*
[5,6] = ( g^1 d^29+ g^2 d^34+ g^3 d^55)[ g^3 d^45]*
[5,7] =  g^1 d^20+ g^2 d^24+( g^3 d^46+ g^4 d^65+ g^5 d^70)[ g^3 d^45]*
[5,8] =  g^1 d^8+ g^2 d^12+( g^3 d^34+ g^4 d^53+ g^5 d^58)[ g^3 d^45]*
[5,9] =  g^1 d^2+ g^2 d^6+( g^3 d^28+ g^4 d^47+ g^5 d^52)[ g^3 d^45]*
[6,0] =  g^0 d^12+ g^1 d^16+ g^2 d^20+( g^3 d^57+ g^4 d^62+ g^5 d^69)[ g^3 d^45]*
[6,1] =  g^0 d^4+ g^1 d^8+ g^2 d^12+( g^3 d^49+ g^4 d^54+ g^5 d^61)[ g^3 d^45]*
[6,2] =  g^0 d^0+ g^1 d^4+ g^2 d^8+( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[6,3] =  g^3 d^20+ g^4 d^24+ g^5 d^28+( g^6 d^65+ g^7 d^70+ g^8 d^77)[ g^3 d^45]*
[6,4] =  g^0 d^0+( g^1 d^38+ g^2 d^43+ g^3 d^50)[ g^3 d^45]*
[6,5] = ( g^0 d^32+ g^1 d^37+ g^2 d^44)[ g^3 d^45]*
[6,6] = ( g^0 d^27+ g^1 d^32+ g^2 d^39)[ g^3 d^45]*
[6,7] =  g^0 d^18+ g^1 d^22+( g^2 d^30+ g^3 d^63+ g^4 d^68)[ g^3 d^45]*
[6,8] =  g^0 d^6+ g^1 d^10+( g^2 d^18+ g^3 d^51+ g^4 d^56)[ g^3 d^45]*
[6,9] =  g^0 d^0+ g^1 d^4+ g^2 d^8+( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[7,0] =  g^0 d^18+ g^1 d^22+ g^2 d^26+( g^3 d^63+ g^4 d^68+ g^5 d^75)[ g^3 d^45]*
[7,1] =  g^0 d^10+ g^1 d^14+ g^2 d^18+( g^3 d^55+ g^4 d^60+ g^5 d^67)[ g^3 d^45]*
[7,2] =  g^0 d^6+ g^1 d^10+ g^2 d^14+( g^3 d^51+ g^4 d^56+ g^5 d^63)[ g^3 d^45]*
[7,3] =  g^2 d^0+ g^3 d^26+ g^4 d^30+ g^5 d^34+( g^6 d^71+ g^7 d^76+ g^8 d^83)[ g^3 d^45]*
[7,4] =  g^0 d^6+( g^1 d^44+ g^2 d^49+ g^3 d^56)[ g^3 d^45]*
[7,5] = ( g^0 d^38+ g^1 d^43+ g^2 d^50)[ g^3 d^45]*
[7,6] = ( g^0 d^33+ g^1 d^38+ g^2 d^45)[ g^3 d^45]*
[7,7] =  g^0 d^24+ g^1 d^28+( g^2 d^36+ g^3 d^69+ g^4 d^74)[ g^3 d^45]*
[7,8] =  g^0 d^12+ g^1 d^16+( g^2 d^24+ g^3 d^57+ g^4 d^62)[ g^3 d^45]*
[7,9] =  g^0 d^6+ g^1 d^10+ g^2 d^14+( g^3 d^51+ g^4 d^56+ g^5 d^63)[ g^3 d^45]*
[8,0] =  g^0 d^20+ g^1 d^24+ g^2 d^28+( g^3 d^65+ g^4 d^70+ g^5 d^77)[ g^3 d^45]*
[8,1] =  g^0 d^12+ g^1 d^16+ g^2 d^20+( g^3 d^57+ g^4 d^62+ g^5 d^69)[ g^3 d^45]*
[8,2] =  g^0 d^8+ g^1 d^12+ g^2 d^16+( g^3 d^53+ g^4 d^58+ g^5 d^65)[ g^3 d^45]*
[8,3] =  g^1 d^0+ g^2 d^2+ g^3 d^28+ g^4 d^32+ g^5 d^36+( g^6 d^73+ g^7 d^78+ g^8 d^85)[ g^3 d^45]*
[8,4] =  g^0 d^8+( g^1 d^46+ g^2 d^51+ g^3 d^58)[ g^3 d^45]*
[8,5] = ( g^0 d^40+ g^1 d^45+ g^2 d^52)[ g^3 d^45]*
[8,6] = ( g^0 d^35+ g^1 d^40+ g^2 d^47)[ g^3 d^45]*
[8,7] =  g^0 d^26+ g^1 d^30+( g^2 d^38+ g^3 d^71+ g^4 d^76)[ g^3 d^45]*
[8,8] =  g^0 d^14+ g^1 d^18+( g^2 d^26+ g^3 d^59+ g^4 d^64)[ g^3 d^45]*
[8,9] =  g^0 d^8+ g^1 d^12+ g^2 d^16+( g^3 d^53+ g^4 d^58+ g^5 d^65)[ g^3 d^45]*
[9,0] =  g^0 d^22+ g^1 d^26+ g^2 d^30+( g^3 d^67+ g^4 d^72+ g^5 d^79)[ g^3 d^45]*
[9,1] =  g^0 d^14+ g^1 d^18+ g^2 d^22+( g^3 d^59+ g^4 d^64+ g^5 d^71)[ g^3 d^45]*
[9,2] =  g^0 d^10+ g^1 d^14+ g^2 d^18+( g^3 d^55+ g^4 d^60+ g^5 d^67)[ g^3 d^45]*
[9,3] =  g^0 d^0+ g^1 d^2+ g^2 d^4+ g^3 d^30+ g^4 d^34+ g^5 d^38+( g^6 d^75+ g^7 d^80+ g^8 d^87)[ g^
[9,4] =  g^0 d^10+( g^1 d^48+ g^2 d^53+ g^3 d^60)[ g^3 d^45]*
[9,5] = ( g^0 d^42+ g^1 d^47+ g^2 d^54)[ g^3 d^45]*
[9,6] = ( g^0 d^37+ g^1 d^42+ g^2 d^49)[ g^3 d^45]*
[9,7] =  g^0 d^28+ g^1 d^32+( g^2 d^40+ g^3 d^73+ g^4 d^78)[ g^3 d^45]*
[9,8] =  g^0 d^16+ g^1 d^20+( g^2 d^28+ g^3 d^61+ g^4 d^66)[ g^3 d^45]*
[9,9] =  g^0 d^10+ g^1 d^14+ g^2 d^18+( g^3 d^55+ g^4 d^60+ g^5 d^67)[ g^3 d^45]*
[10,0] =  g^0 d^0+ g^1 d^26+ g^2 d^30+ g^3 d^34+( g^4 d^71+ g^5 d^76+ g^6 d^83)[ g^3 d^45]*
[10,1] =  g^1 d^18+ g^2 d^22+ g^3 d^26+( g^4 d^63+ g^5 d^68+ g^6 d^75)[ g^3 d^45]*
[10,2] =  g^1 d^14+ g^2 d^18+ g^3 d^22+( g^4 d^59+ g^5 d^64+ g^6 d^71)[ g^3 d^45]*
[10,3] =  g^1 d^4+ g^2 d^6+ g^3 d^8+ g^4 d^34+ g^5 d^38+ g^6 d^42+( g^7 d^79+ g^8 d^84+ g^9 d^91)[ g
[10,4] =  g^1 d^14+( g^2 d^52+ g^3 d^57+ g^4 d^64)[ g^3 d^45]*
[10,5] = ( g^1 d^46+ g^2 d^51+ g^3 d^58)[ g^3 d^45]*
[10,6] = ( g^1 d^41+ g^2 d^46+ g^3 d^53)[ g^3 d^45]*
[10,7] =  g^1 d^32+ g^2 d^36+( g^3 d^44+ g^4 d^77+ g^5 d^82)[ g^3 d^45]*
[10,8] =  g^1 d^20+ g^2 d^24+( g^3 d^32+ g^4 d^65+ g^5 d^70)[ g^3 d^45]*
```

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[10,9] =  g^1 d^14+ g^2 d^18+ g^3 d^22+( g^4 d^59+ g^5 d^64+ g^6 d^71)[ g^3 d^45]*
[11,0] =  g^0 d^0+ g^1 d^2+ g^2 d^28+ g^3 d^32+ g^4 d^36+( g^5 d^73+ g^6 d^78+ g^7 d^85)[ g^3 d
[11,1] =  g^2 d^20+ g^3 d^24+ g^4 d^28+( g^5 d^65+ g^6 d^70+ g^7 d^77)[ g^3 d^45]*
[11,2] =  g^2 d^16+ g^3 d^20+ g^4 d^24+( g^5 d^61+ g^6 d^66+ g^7 d^73)[ g^3 d^45]*
[11,3] =  g^2 d^6+ g^3 d^8+ g^4 d^10+ g^5 d^36+ g^6 d^40+ g^7 d^44+( g^8 d^81+ g^9 d^86+ g^10 d
[11,4] =  g^2 d^16+( g^3 d^54+ g^4 d^59+ g^5 d^66)[ g^3 d^45]*
[11,5] = ( g^2 d^48+ g^3 d^53+ g^4 d^60)[ g^3 d^45]*
[11,6] = ( g^2 d^43+ g^3 d^48+ g^4 d^55)[ g^3 d^45]*
[11,7] =  g^2 d^34+ g^3 d^38+( g^4 d^46+ g^5 d^79+ g^6 d^84)[ g^3 d^45]*
[11,8] =  g^2 d^22+ g^3 d^26+( g^4 d^34+ g^5 d^67+ g^6 d^72)[ g^3 d^45]*
[11,9] =  g^2 d^16+ g^3 d^20+ g^4 d^24+( g^5 d^61+ g^6 d^66+ g^7 d^73)[ g^3 d^45]*
[12,0] =  g^1 d^17+ g^2 d^21+ g^3 d^25+( g^4 d^62+ g^5 d^67+ g^6 d^74)[ g^3 d^45]*
[12,1] =  g^1 d^9+ g^2 d^13+ g^3 d^17+( g^4 d^54+ g^5 d^59+ g^6 d^66)[ g^3 d^45]*
[12,2] =  g^1 d^5+ g^2 d^9+ g^3 d^13+( g^4 d^50+ g^5 d^55+ g^6 d^62)[ g^3 d^45]*
[12,3] =  g^4 d^25+ g^5 d^29+ g^6 d^33+( g^7 d^70+ g^8 d^75+ g^9 d^82)[ g^3 d^45]*
[12,4] =  g^0 d^0+ g^1 d^5+( g^2 d^43+ g^3 d^48+ g^4 d^55)[ g^3 d^45]*
[12,5] = ( g^1 d^37+ g^2 d^42+ g^3 d^49)[ g^3 d^45]*
[12,6] = ( g^1 d^32+ g^2 d^37+ g^3 d^44)[ g^3 d^45]*
[12,7] =  g^1 d^23+ g^2 d^27+( g^3 d^35+ g^4 d^68+ g^5 d^73)[ g^3 d^45]*
[12,8] =  g^1 d^11+ g^2 d^15+( g^3 d^23+ g^4 d^56+ g^5 d^61)[ g^3 d^45]*
[12,9] =  g^1 d^5+ g^2 d^9+ g^3 d^13+( g^4 d^50+ g^5 d^55+ g^6 d^62)[ g^3 d^45]*
[13,0] =  g^2 d^19+ g^3 d^23+ g^4 d^27+( g^5 d^64+ g^6 d^69+ g^7 d^76)[ g^3 d^45]*
[13,1] =  g^2 d^11+ g^3 d^15+ g^4 d^19+( g^5 d^56+ g^6 d^61+ g^7 d^68)[ g^3 d^45]*
[13,2] =  g^2 d^7+ g^3 d^11+ g^4 d^15+( g^5 d^52+ g^6 d^57+ g^7 d^64)[ g^3 d^45]*
[13,3] =  g^5 d^27+ g^6 d^31+ g^7 d^35+( g^8 d^72+ g^9 d^77+ g^10 d^84)[ g^3 d^45]*
[13,4] =  g^0 d^0+ g^1 d^2+ g^2 d^7+( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[13,5] =  g^1 d^0+( g^2 d^39+ g^3 d^44+ g^4 d^51)[ g^3 d^45]*
[13,6] = ( g^2 d^34+ g^3 d^39+ g^4 d^46)[ g^3 d^45]*
[13,7] =  g^2 d^25+ g^3 d^29+( g^4 d^37+ g^5 d^70+ g^6 d^75)[ g^3 d^45]*
[13,8] =  g^2 d^13+ g^3 d^17+( g^4 d^25+ g^5 d^58+ g^6 d^63)[ g^3 d^45]*
[13,9] =  g^2 d^7+ g^3 d^11+ g^4 d^15+( g^5 d^52+ g^6 d^57+ g^7 d^64)[ g^3 d^45]*
[14,0] =  g^2 d^23+ g^3 d^27+ g^4 d^31+( g^5 d^68+ g^6 d^73+ g^7 d^80)[ g^3 d^45]*
[14,1] =  g^2 d^15+ g^3 d^19+ g^4 d^23+( g^5 d^60+ g^6 d^65+ g^7 d^72)[ g^3 d^45]*
[14,2] =  g^2 d^11+ g^3 d^15+ g^4 d^19+( g^5 d^56+ g^6 d^61+ g^7 d^68)[ g^3 d^45]*
[14,3] =  g^5 d^31+ g^6 d^35+ g^7 d^39+( g^8 d^76+ g^9 d^81+ g^10 d^88)[ g^3 d^45]*
[14,4] =  g^0 d^4+ g^1 d^6+ g^2 d^11+( g^3 d^49+ g^4 d^54+ g^5 d^61)[ g^3 d^45]*
[14,5] =  g^0 d^0+ g^1 d^5+( g^2 d^43+ g^3 d^48+ g^4 d^55)[ g^3 d^45]*
[14,6] =  g^1 d^0+( g^2 d^38+ g^3 d^43+ g^4 d^50)[ g^3 d^45]*
[14,7] =  g^2 d^29+ g^3 d^33+( g^4 d^41+ g^5 d^74+ g^6 d^79)[ g^3 d^45]*
[14,8] =  g^2 d^17+ g^3 d^21+( g^4 d^29+ g^5 d^62+ g^6 d^67)[ g^3 d^45]*
[14,9] =  g^2 d^11+ g^3 d^15+ g^4 d^19+( g^5 d^56+ g^6 d^61+ g^7 d^68)[ g^3 d^45]*
[15,0] =  g^3 d^25+ g^4 d^30+ g^5 d^35+( g^6 d^70+ g^7 d^75+ g^8 d^82)[ g^3 d^45]*
[15,1] =  g^3 d^17+ g^4 d^22+ g^5 d^27+( g^6 d^62+ g^7 d^67+ g^8 d^74)[ g^3 d^45]*
[15,2] =  g^3 d^13+ g^4 d^18+ g^5 d^23+( g^6 d^58+ g^7 d^63+ g^8 d^70)[ g^3 d^45]*
[15,3] =  g^6 d^33+ g^7 d^38+ g^8 d^43+( g^9 d^78+ g^10 d^83+ g^11 d^90)[ g^3 d^45]*
[15,4] =  g^1 d^6+ g^2 d^11+ g^3 d^16+( g^4 d^51+ g^5 d^56+ g^6 d^63)[ g^3 d^45]*
[15,5] =  g^0 d^0+ g^1 d^5+ g^2 d^10+( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[15,6] =  g^1 d^0+ g^2 d^5+( g^3 d^40+ g^4 d^45+ g^5 d^52)[ g^3 d^45]*
[15,7] = ( g^3 d^31+ g^4 d^36+ g^5 d^43)[ g^3 d^45]*
[15,8] = ( g^3 d^19+ g^4 d^24+ g^5 d^31)[ g^3 d^45]*
[15,9] =  g^3 d^13+ g^4 d^18+ g^5 d^23+( g^6 d^58+ g^7 d^63+ g^8 d^70)[ g^3 d^45]*
[16,0] =  g^3 d^30+ g^4 d^35+ g^5 d^40+( g^6 d^75+ g^7 d^80+ g^8 d^87)[ g^3 d^45]*
[16,1] =  g^3 d^22+ g^4 d^27+ g^5 d^32+( g^6 d^67+ g^7 d^72+ g^8 d^79)[ g^3 d^45]*
[16,2] =  g^3 d^18+ g^4 d^23+ g^5 d^28+( g^6 d^63+ g^7 d^68+ g^8 d^75)[ g^3 d^45]*
[16,3] =  g^6 d^38+ g^7 d^43+ g^8 d^48+( g^9 d^83+ g^10 d^88+ g^11 d^95)[ g^3 d^45]*
[16,4] =  g^1 d^11+ g^2 d^16+ g^3 d^21+( g^4 d^56+ g^5 d^61+ g^6 d^68)[ g^3 d^45]*
[16,5] =  g^0 d^5+ g^1 d^10+ g^2 d^15+( g^3 d^50+ g^4 d^55+ g^5 d^62)[ g^3 d^45]*
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[16,6] =  g^0 d^0+ g^1 d^5+ g^2 d^10+( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[16,7] =  g^2 d^0+( g^3 d^36+ g^4 d^41+ g^5 d^48)[ g^3 d^45]*
[16,8] = ( g^3 d^24+ g^4 d^29+ g^5 d^36)[ g^3 d^45]*
[16,9] =  g^3 d^18+ g^4 d^23+ g^5 d^28+( g^6 d^63+ g^7 d^68+ g^8 d^75)[ g^3 d^45]*
[17,0] =  g^3 d^37+ g^4 d^42+ g^5 d^47+( g^6 d^82+ g^7 d^87+ g^8 d^94)[ g^3 d^45]*
[17,1] =  g^3 d^29+ g^4 d^34+ g^5 d^39+( g^6 d^74+ g^7 d^79+ g^8 d^86)[ g^3 d^45]*
[17,2] =  g^3 d^25+ g^4 d^30+ g^5 d^35+( g^6 d^70+ g^7 d^75+ g^8 d^82)[ g^3 d^45]*
[17,3] =  g^6 d^45+ g^7 d^50+ g^8 d^55+( g^9 d^90+ g^10 d^95+ g^11 d^102)[ g^3 d^45]*
[17,4] =  g^1 d^18+ g^2 d^23+ g^3 d^28+( g^4 d^63+ g^5 d^68+ g^6 d^75)[ g^3 d^45]*
[17,5] =  g^0 d^12+ g^1 d^17+ g^2 d^22+( g^3 d^57+ g^4 d^62+ g^5 d^69)[ g^3 d^45]*
[17,6] =  g^0 d^7+ g^1 d^12+ g^2 d^17+( g^3 d^52+ g^4 d^57+ g^5 d^64)[ g^3 d^45]*
[17,7] =  g^1 d^0+ g^2 d^7+( g^3 d^43+ g^4 d^48+ g^5 d^55)[ g^3 d^45]*
[17,8] = ( g^3 d^31+ g^4 d^36+ g^5 d^43)[ g^3 d^45]*
[17,9] =  g^3 d^25+ g^4 d^30+ g^5 d^35+( g^6 d^70+ g^7 d^75+ g^8 d^82)[ g^3 d^45]*
[18,0] = ( g^3 d^39+ g^4 d^44+ g^5 d^51)[ g^3 d^45]*
[18,1] = ( g^3 d^31+ g^4 d^36+ g^5 d^43)[ g^3 d^45]*
[18,2] = ( g^3 d^27+ g^4 d^32+ g^5 d^39)[ g^3 d^45]*
[18,3] = ( g^6 d^47+ g^7 d^52+ g^8 d^59)[ g^3 d^45]*
[18,4] = ( g^1 d^20+ g^2 d^25+ g^3 d^32)[ g^3 d^45]*
[18,5] = ( g^0 d^14+ g^1 d^19+ g^2 d^26)[ g^3 d^45]*
[18,6] = ( g^0 d^9+ g^1 d^14+ g^2 d^21)[ g^3 d^45]*
[18,7] =  g^0 d^0+ g^1 d^2+( g^2 d^12+ g^3 d^45+ g^4 d^50)[ g^3 d^45]*
[18,8] = ( g^2 d^0+ g^3 d^33+ g^4 d^38)[ g^3 d^45]*
[18,9] = ( g^3 d^27+ g^4 d^32+ g^5 d^39)[ g^3 d^45]*
[19,0] = ( g^3 d^49+ g^4 d^54+ g^5 d^61)[ g^3 d^45]*
[19,1] = ( g^3 d^41+ g^4 d^46+ g^5 d^53)[ g^3 d^45]*
[19,2] = ( g^3 d^37+ g^4 d^42+ g^5 d^49)[ g^3 d^45]*
[19,3] = ( g^6 d^57+ g^7 d^62+ g^8 d^69)[ g^3 d^45]*
[19,4] = ( g^1 d^30+ g^2 d^35+ g^3 d^42)[ g^3 d^45]*
[19,5] = ( g^0 d^24+ g^1 d^29+ g^2 d^36)[ g^3 d^45]*
[19,6] = ( g^0 d^19+ g^1 d^24+ g^2 d^31)[ g^3 d^45]*
[19,7] =  g^0 d^10+ g^1 d^12+( g^2 d^22+ g^3 d^55+ g^4 d^60)[ g^3 d^45]*
[19,8] =  g^1 d^0+( g^2 d^10+ g^3 d^43+ g^4 d^48)[ g^3 d^45]*
[19,9] = ( g^3 d^37+ g^4 d^42+ g^5 d^49)[ g^3 d^45]*
[20,0] =  g^2 d^12+( g^3 d^51+ g^4 d^56+ g^5 d^63)[ g^3 d^45]*
[20,1] =  g^2 d^4+( g^3 d^43+ g^4 d^48+ g^5 d^55)[ g^3 d^45]*
[20,2] =  g^2 d^0+( g^3 d^39+ g^4 d^44+ g^5 d^51)[ g^3 d^45]*
[20,3] =  g^5 d^20+( g^6 d^59+ g^7 d^64+ g^8 d^71)[ g^3 d^45]*
[20,4] = ( g^1 d^32+ g^2 d^37+ g^3 d^44)[ g^3 d^45]*
[20,5] = ( g^0 d^26+ g^1 d^31+ g^2 d^38)[ g^3 d^45]*
[20,6] = ( g^0 d^21+ g^1 d^26+ g^2 d^33)[ g^3 d^45]*
[20,7] =  g^0 d^12+ g^1 d^14+( g^2 d^24+ g^3 d^57+ g^4 d^62)[ g^3 d^45]*
[20,8] =  g^0 d^0+ g^1 d^2+( g^2 d^12+ g^3 d^45+ g^4 d^50)[ g^3 d^45]*
[20,9] =  g^2 d^0+( g^3 d^39+ g^4 d^44+ g^5 d^51)[ g^3 d^45]*
[21,0] =  g^1 d^12+ g^2 d^16+( g^3 d^55+ g^4 d^60+ g^5 d^67)[ g^3 d^45]*
[21,1] =  g^1 d^4+ g^2 d^8+( g^3 d^47+ g^4 d^52+ g^5 d^59)[ g^3 d^45]*
[21,2] =  g^1 d^0+ g^2 d^4+( g^3 d^43+ g^4 d^48+ g^5 d^55)[ g^3 d^45]*
[21,3] =  g^4 d^20+ g^5 d^24+( g^6 d^63+ g^7 d^68+ g^8 d^75)[ g^3 d^45]*
[21,4] = ( g^1 d^36+ g^2 d^41+ g^3 d^48)[ g^3 d^45]*
[21,5] = ( g^0 d^30+ g^1 d^35+ g^2 d^42)[ g^3 d^45]*
[21,6] = ( g^0 d^25+ g^1 d^30+ g^2 d^37)[ g^3 d^45]*
[21,7] =  g^0 d^16+ g^1 d^18+( g^2 d^28+ g^3 d^61+ g^4 d^66)[ g^3 d^45]*
[21,8] =  g^0 d^4+ g^1 d^6+( g^2 d^16+ g^3 d^49+ g^4 d^54)[ g^3 d^45]*
[21,9] =  g^1 d^0+ g^2 d^4+( g^3 d^43+ g^4 d^48+ g^5 d^55)[ g^3 d^45]*
```

Feedback controller M:

```
M =

[0,0] =  g^0 d^2+ g^1 d^4+ g^2 d^30+ g^3 d^34+ g^4 d^38+( g^5 d^75+ g^6 d^80+ g^7 d^87)[ g^3 d^
[0,1] =  g^1 d^0+ g^2 d^26+ g^3 d^30+ g^4 d^34+( g^5 d^71+ g^6 d^76+ g^7 d^83)[ g^3 d^45]*
[0,2] =  g^2 d^24+ g^3 d^28+ g^4 d^32+( g^5 d^69+ g^6 d^74+ g^7 d^81)[ g^3 d^45]*
[0,3] =  g^2 d^0+ g^3 d^24+ g^4 d^28+ g^5 d^32+( g^6 d^69+ g^7 d^74+ g^8 d^81)[ g^3 d^45]*
[0,4] =  g^2 d^22+ g^3 d^26+ g^4 d^30+( g^5 d^67+ g^6 d^72+ g^7 d^79)[ g^3 d^45]*
[0,5] =  g^2 d^20+ g^3 d^24+ g^4 d^28+( g^5 d^65+ g^6 d^70+ g^7 d^77)[ g^3 d^45]*
[0,6] =  g^2 d^18+ g^3 d^22+ g^4 d^26+( g^5 d^63+ g^6 d^68+ g^7 d^75)[ g^3 d^45]*
[0,7] =  g^2 d^12+ g^3 d^16+ g^4 d^20+ g^5 d^46+( g^6 d^62+ g^7 d^69+ g^8 d^95)[ g^3 d^45]*
[0,8] =  g^2 d^10+ g^3 d^12+ g^4 d^18+ g^5 d^42+ g^6 d^46+( g^7 d^67+ g^8 d^88+ g^9 d^95)[ g^3
[0,9] =  g^2 d^8+ g^3 d^10+ g^4 d^12+ g^5 d^38+ g^6 d^42+ g^7 d^46+( g^8 d^83+ g^9 d^88+ g^10 d
[0,10] =  g^1 d^4+ g^2 d^6+ g^3 d^8+ g^4 d^34+ g^5 d^38+ g^6 d^42+( g^7 d^79+ g^8 d^84+ g^9 d^9
[0,11] =  g^0 d^2+ g^1 d^4+ g^2 d^6+ g^3 d^32+ g^4 d^36+ g^5 d^40+( g^6 d^77+ g^7 d^82+ g^8 d^8
[0,12] =  g^2 d^17+ g^3 d^21+( g^4 d^58+ g^5 d^63+ g^6 d^70)[ g^3 d^45]*
[0,13] =  g^2 d^18+( g^3 d^56+ g^4 d^61+ g^5 d^68)[ g^3 d^45]*
[0,14] =  g^2 d^14+( g^3 d^52+ g^4 d^57+ g^5 d^64)[ g^3 d^45]*
[0,15] = ( g^2 d^50+ g^3 d^55+ g^4 d^62)[ g^3 d^45]*
[0,16] = ( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[0,17] = ( g^2 d^38+ g^3 d^43+ g^4 d^50)[ g^3 d^45]*
[0,18] =  g^2 d^36+ g^3 d^40+( g^4 d^48+ g^5 d^81+ g^6 d^86)[ g^3 d^45]*
[0,19] =  g^2 d^26+ g^3 d^30+( g^4 d^38+ g^5 d^71+ g^6 d^76)[ g^3 d^45]*
[0,20] =  g^2 d^24+ g^3 d^28+( g^4 d^36+ g^5 d^69+ g^6 d^74)[ g^3 d^45]*
[0,21] =  g^2 d^20+ g^3 d^24+( g^4 d^32+ g^5 d^65+ g^6 d^70)[ g^3 d^45]*
[1,0] =  g^0 d^10+ g^1 d^36+ g^2 d^40+ g^3 d^44+( g^4 d^81+ g^5 d^86+ g^6 d^93)[ g^3 d^45]*
[1,1] =  g^0 d^6+ g^1 d^32+ g^2 d^36+ g^3 d^40+( g^4 d^77+ g^5 d^82+ g^6 d^89)[ g^3 d^45]*
[1,2] =  g^0 d^4+ g^1 d^30+ g^2 d^34+ g^3 d^38+( g^4 d^75+ g^5 d^80+ g^6 d^87)[ g^3 d^45]*
[1,3] =  g^0 d^2+ g^1 d^6+ g^2 d^30+ g^3 d^34+ g^4 d^38+( g^5 d^75+ g^6 d^80+ g^7 d^87)[ g^3 d^
[1,4] =  g^0 d^2+ g^1 d^28+ g^2 d^32+ g^3 d^36+( g^4 d^73+ g^5 d^78+ g^6 d^85)[ g^3 d^45]*
[1,5] =  g^0 d^0+ g^1 d^26+ g^2 d^30+ g^3 d^34+( g^4 d^71+ g^5 d^76+ g^6 d^83)[ g^3 d^45]*
[1,6] =  g^1 d^24+ g^2 d^28+ g^3 d^32+( g^4 d^69+ g^5 d^74+ g^6 d^81)[ g^3 d^45]*
[1,7] =  g^1 d^18+ g^2 d^22+ g^3 d^26+ g^4 d^52+( g^5 d^68+ g^6 d^75+ g^7 d^101)[ g^3 d^45]*
[1,8] =  g^1 d^16+ g^2 d^18+ g^3 d^24+ g^4 d^48+ g^5 d^52+( g^6 d^73+ g^7 d^94+ g^8 d^101)[ g^3
[1,9] =  g^1 d^12+ g^2 d^16+ g^3 d^18+ g^4 d^44+ g^5 d^48+ g^6 d^52+( g^7 d^89+ g^8 d^94+ g^9 d
[1,10] =  g^0 d^8+ g^1 d^12+ g^2 d^14+ g^3 d^40+ g^4 d^44+ g^5 d^48+( g^6 d^85+ g^7 d^90+ g^8 d
[1,11] =  g^0 d^10+ g^1 d^12+ g^2 d^38+ g^3 d^42+ g^4 d^46+( g^5 d^83+ g^6 d^88+ g^7 d^95)[ g^3
[1,12] =  g^1 d^23+ g^2 d^27+( g^3 d^64+ g^4 d^69+ g^5 d^76)[ g^3 d^45]*
[1,13] =  g^1 d^24+( g^2 d^62+ g^3 d^67+ g^4 d^74)[ g^3 d^45]*
[1,14] =  g^1 d^20+( g^2 d^58+ g^3 d^63+ g^4 d^70)[ g^3 d^45]*
[1,15] = ( g^1 d^56+ g^2 d^61+ g^3 d^68)[ g^3 d^45]*
[1,16] = ( g^1 d^51+ g^2 d^56+ g^3 d^63)[ g^3 d^45]*
[1,17] = ( g^1 d^44+ g^2 d^49+ g^3 d^56)[ g^3 d^45]*
[1,18] =  g^1 d^42+ g^2 d^46+( g^3 d^54+ g^4 d^87+ g^5 d^92)[ g^3 d^45]*
[1,19] =  g^1 d^32+ g^2 d^36+( g^3 d^44+ g^4 d^77+ g^5 d^82)[ g^3 d^45]*
[1,20] =  g^1 d^30+ g^2 d^34+( g^3 d^42+ g^4 d^75+ g^5 d^80)[ g^3 d^45]*
[1,21] =  g^1 d^26+ g^2 d^30+( g^3 d^38+ g^4 d^71+ g^5 d^76)[ g^3 d^45]*
[2,0] =  g^0 d^16+ g^1 d^20+( g^2 d^57+ g^3 d^62+ g^4 d^69)[ g^3 d^45]*
[2,1] =  g^0 d^12+ g^1 d^16+( g^2 d^53+ g^3 d^58+ g^4 d^65)[ g^3 d^45]*
[2,2] =  g^0 d^10+ g^1 d^14+( g^2 d^51+ g^3 d^56+ g^4 d^63)[ g^3 d^45]*
[2,3] =  g^0 d^6+ g^1 d^10+ g^2 d^14+( g^3 d^51+ g^4 d^56+ g^5 d^63)[ g^3 d^45]*
[2,4] =  g^0 d^8+ g^1 d^12+( g^2 d^49+ g^3 d^54+ g^4 d^61)[ g^3 d^45]*
[2,5] =  g^0 d^6+ g^1 d^10+( g^2 d^47+ g^3 d^52+ g^4 d^59)[ g^3 d^45]*
[2,6] =  g^0 d^4+ g^1 d^8+( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[2,7] =  g^1 d^2+ g^2 d^28+( g^3 d^44+ g^4 d^51+ g^5 d^77)[ g^3 d^45]*
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[2,8] =  g^1 d^0+ g^2 d^24+ g^3 d^28+( g^4 d^49+ g^5 d^70+ g^6 d^77)[ g^3 d^45]*
[2,9] =  g^2 d^20+ g^3 d^24+ g^4 d^28+( g^5 d^65+ g^6 d^70+ g^7 d^77)[ g^3 d^45]*
[2,10] =  g^1 d^16+ g^2 d^20+ g^3 d^24+( g^4 d^61+ g^5 d^66+ g^6 d^73)[ g^3 d^45]*
[2,11] =  g^0 d^14+ g^1 d^18+ g^2 d^22+( g^3 d^59+ g^4 d^64+ g^5 d^71)[ g^3 d^45]*
[2,12] =  g^0 d^3+( g^1 d^40+ g^2 d^45+ g^3 d^52)[ g^3 d^45]*
[2,13] = ( g^0 d^38+ g^1 d^43+ g^2 d^50)[ g^3 d^45]*
[2,14] = ( g^0 d^34+ g^1 d^39+ g^2 d^46)[ g^3 d^45]*
[2,15] = ( g^0 d^37+ g^1 d^44+ g^2 d^77)[ g^3 d^45]*
[2,16] = ( g^0 d^32+ g^1 d^39+ g^2 d^72)[ g^3 d^45]*
[2,17] = ( g^0 d^25+ g^1 d^32+ g^2 d^65)[ g^3 d^45]*
[2,18] =  g^0 d^22+( g^1 d^30+ g^2 d^63+ g^3 d^68)[ g^3 d^45]*
[2,19] =  g^0 d^12+( g^1 d^20+ g^2 d^53+ g^3 d^58)[ g^3 d^45]*
[2,20] =  g^0 d^10+( g^1 d^18+ g^2 d^51+ g^3 d^56)[ g^3 d^45]*
[2,21] =  g^0 d^6+( g^1 d^14+ g^2 d^47+ g^3 d^52)[ g^3 d^45]*
[3,0] =  g^0 d^26+ g^1 d^30+( g^2 d^67+ g^3 d^72+ g^4 d^79)[ g^3 d^45]*
[3,1] =  g^0 d^22+ g^1 d^26+( g^2 d^63+ g^3 d^68+ g^4 d^75)[ g^3 d^45]*
[3,2] =  g^0 d^20+ g^1 d^24+( g^2 d^61+ g^3 d^66+ g^4 d^73)[ g^3 d^45]*
[3,3] =  g^0 d^16+ g^1 d^20+ g^2 d^24+( g^3 d^61+ g^4 d^66+ g^5 d^73)[ g^3 d^45]*
[3,4] =  g^0 d^18+ g^1 d^22+( g^2 d^59+ g^3 d^64+ g^4 d^71)[ g^3 d^45]*
[3,5] =  g^0 d^16+ g^1 d^20+( g^2 d^57+ g^3 d^62+ g^4 d^69)[ g^3 d^45]*
[3,6] =  g^0 d^14+ g^1 d^18+( g^2 d^55+ g^3 d^60+ g^4 d^67)[ g^3 d^45]*
[3,7] =  g^0 d^8+ g^1 d^12+ g^2 d^38+( g^3 d^54+ g^4 d^61+ g^5 d^87)[ g^3 d^45]*
[3,8] =  g^0 d^4+ g^1 d^10+ g^2 d^34+ g^3 d^38+( g^4 d^59+ g^5 d^80+ g^6 d^87)[ g^3 d^45]*
[3,9] =  g^0 d^2+ g^1 d^4+ g^2 d^30+ g^3 d^34+ g^4 d^38+( g^5 d^75+ g^6 d^80+ g^7 d^87)[ g^3 d^45]*
[3,10] =  g^0 d^0+ g^1 d^26+ g^2 d^30+ g^3 d^34+( g^4 d^71+ g^5 d^76+ g^6 d^83)[ g^3 d^45]*
[3,11] =  g^0 d^24+ g^1 d^28+ g^2 d^32+( g^3 d^69+ g^4 d^74+ g^5 d^81)[ g^3 d^45]*
[3,12] =  g^0 d^13+( g^1 d^50+ g^2 d^55+ g^3 d^62)[ g^3 d^45]*
[3,13] = ( g^0 d^48+ g^1 d^53+ g^2 d^60)[ g^3 d^45]*
[3,14] = ( g^0 d^44+ g^1 d^49+ g^2 d^56)[ g^3 d^45]*
[3,15] = ( g^0 d^47+ g^1 d^54+ g^2 d^87)[ g^3 d^45]*
[3,16] = ( g^0 d^42+ g^1 d^49+ g^2 d^82)[ g^3 d^45]*
[3,17] = ( g^0 d^35+ g^1 d^42+ g^2 d^75)[ g^3 d^45]*
[3,18] =  g^0 d^32+( g^1 d^40+ g^2 d^73+ g^3 d^78)[ g^3 d^45]*
[3,19] =  g^0 d^22+( g^1 d^30+ g^2 d^63+ g^3 d^68)[ g^3 d^45]*
[3,20] =  g^0 d^20+( g^1 d^28+ g^2 d^61+ g^3 d^66)[ g^3 d^45]*
[3,21] =  g^0 d^16+( g^1 d^24+ g^2 d^57+ g^3 d^62)[ g^3 d^45]*
[4,0] =  g^1 d^19+ g^2 d^23+ g^3 d^27+( g^4 d^64+ g^5 d^69+ g^6 d^76)[ g^3 d^45]*
[4,1] =  g^1 d^15+ g^2 d^19+ g^3 d^23+( g^4 d^60+ g^5 d^65+ g^6 d^72)[ g^3 d^45]*
[4,2] =  g^1 d^13+ g^2 d^17+ g^3 d^21+( g^4 d^58+ g^5 d^63+ g^6 d^70)[ g^3 d^45]*
[4,3] =  g^2 d^13+ g^3 d^17+ g^4 d^21+( g^5 d^58+ g^6 d^63+ g^7 d^70)[ g^3 d^45]*
[4,4] =  g^1 d^11+ g^2 d^15+ g^3 d^19+( g^4 d^56+ g^5 d^61+ g^6 d^68)[ g^3 d^45]*
[4,5] =  g^1 d^9+ g^2 d^13+ g^3 d^17+( g^4 d^54+ g^5 d^59+ g^6 d^66)[ g^3 d^45]*
[4,6] =  g^1 d^7+ g^2 d^11+ g^3 d^15+( g^4 d^52+ g^5 d^57+ g^6 d^64)[ g^3 d^45]*
[4,7] =  g^2 d^5+ g^3 d^9+ g^4 d^35+( g^5 d^51+ g^6 d^58+ g^7 d^84)[ g^3 d^45]*
[4,8] =  g^3 d^7+ g^4 d^31+ g^5 d^35+( g^6 d^56+ g^7 d^77+ g^8 d^84)[ g^3 d^45]*
[4,9] =  g^4 d^27+ g^5 d^31+ g^6 d^35+( g^7 d^72+ g^8 d^77+ g^9 d^84)[ g^3 d^45]*
[4,10] =  g^3 d^23+ g^4 d^27+ g^5 d^31+( g^6 d^68+ g^7 d^73+ g^8 d^80)[ g^3 d^45]*
[4,11] =  g^2 d^21+ g^3 d^25+ g^4 d^29+( g^5 d^66+ g^6 d^71+ g^7 d^78)[ g^3 d^45]*
[4,12] =  g^0 d^2+ g^1 d^6+ g^2 d^10+( g^3 d^47+ g^4 d^52+ g^5 d^59)[ g^3 d^45]*
[4,13] =  g^0 d^2+ g^1 d^7+( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[4,14] =  g^1 d^3+( g^2 d^41+ g^3 d^46+ g^4 d^53)[ g^3 d^45]*
[4,15] =  g^0 d^1+( g^1 d^39+ g^2 d^44+ g^3 d^51)[ g^3 d^45]*
[4,16] = ( g^1 d^34+ g^2 d^39+ g^3 d^46)[ g^3 d^45]*
[4,17] = ( g^1 d^27+ g^2 d^32+ g^3 d^39)[ g^3 d^45]*
[4,18] =  g^1 d^25+ g^2 d^29+( g^3 d^37+ g^4 d^70+ g^5 d^75)[ g^3 d^45]*
[4,19] =  g^1 d^15+ g^2 d^19+( g^3 d^27+ g^4 d^60+ g^5 d^65)[ g^3 d^45]*
[4,20] =  g^1 d^13+ g^2 d^17+( g^3 d^25+ g^4 d^58+ g^5 d^63)[ g^3 d^45]*
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[4,21] =  g^1 d^9+ g^2 d^13+( g^3 d^21+ g^4 d^54+ g^5 d^59)[ g^3 d^45]*
[5,0] =  g^2 d^25+ g^3 d^30+ g^4 d^35+( g^5 d^70+ g^6 d^75+ g^7 d^82)[ g^3 d^45]*
[5,1] =  g^2 d^21+ g^3 d^26+ g^4 d^31+( g^5 d^66+ g^6 d^71+ g^7 d^78)[ g^3 d^45]*
[5,2] =  g^2 d^19+ g^3 d^24+ g^4 d^29+( g^5 d^64+ g^6 d^69+ g^7 d^76)[ g^3 d^45]*
[5,3] =  g^3 d^19+ g^4 d^24+ g^5 d^29+( g^6 d^64+ g^7 d^69+ g^8 d^76)[ g^3 d^45]*
[5,4] =  g^2 d^17+ g^3 d^22+ g^4 d^27+( g^5 d^62+ g^6 d^67+ g^7 d^74)[ g^3 d^45]*
[5,5] =  g^2 d^15+ g^3 d^20+ g^4 d^25+( g^5 d^60+ g^6 d^65+ g^7 d^72)[ g^3 d^45]*
[5,6] =  g^2 d^13+ g^3 d^18+ g^4 d^23+( g^5 d^58+ g^6 d^63+ g^7 d^70)[ g^3 d^45]*
[5,7] =  g^3 d^12+ g^4 d^17+ g^5 d^43+( g^6 d^57+ g^7 d^64+ g^8 d^90)[ g^3 d^45]*
[5,8] =  g^4 d^15+ g^5 d^38+ g^6 d^43+( g^7 d^62+ g^8 d^83+ g^9 d^90)[ g^3 d^45]*
[5,9] =  g^5 d^33+ g^6 d^38+ g^7 d^43+( g^8 d^78+ g^9 d^83+ g^10 d^90)[ g^3 d^45]*
[5,10] =  g^4 d^29+ g^5 d^34+ g^6 d^39+( g^7 d^74+ g^8 d^79+ g^9 d^86)[ g^3 d^45]*
[5,11] =  g^3 d^27+ g^4 d^32+ g^5 d^37+( g^6 d^72+ g^7 d^77+ g^8 d^84)[ g^3 d^45]*
[5,12] =  g^1 d^8+ g^2 d^13+ g^3 d^18+( g^4 d^53+ g^5 d^58+ g^6 d^65)[ g^3 d^45]*
[5,13] =  g^0 d^6+ g^1 d^11+ g^2 d^16+( g^3 d^51+ g^4 d^56+ g^5 d^63)[ g^3 d^45]*
[5,14] =  g^0 d^2+ g^1 d^7+ g^2 d^12+( g^3 d^47+ g^4 d^52+ g^5 d^59)[ g^3 d^45]*
[5,15] =  g^0 d^5+ g^1 d^10+( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[5,16] =  g^0 d^0+ g^1 d^5+( g^2 d^40+ g^3 d^45+ g^4 d^52)[ g^3 d^45]*
[5,17] = ( g^2 d^33+ g^3 d^38+ g^4 d^45)[ g^3 d^45]*
[5,18] = ( g^2 d^31+ g^3 d^36+ g^4 d^43)[ g^3 d^45]*
[5,19] = ( g^2 d^21+ g^3 d^26+ g^4 d^33)[ g^3 d^45]*
[5,20] = ( g^2 d^19+ g^3 d^24+ g^4 d^31)[ g^3 d^45]*
[5,21] = ( g^2 d^15+ g^3 d^20+ g^4 d^27)[ g^3 d^45]*
[6,0] =  g^2 d^30+ g^3 d^35+ g^4 d^40+( g^5 d^75+ g^6 d^80+ g^7 d^87)[ g^3 d^45]*
[6,1] =  g^2 d^26+ g^3 d^31+ g^4 d^36+( g^5 d^71+ g^6 d^76+ g^7 d^83)[ g^3 d^45]*
[6,2] =  g^2 d^24+ g^3 d^29+ g^4 d^34+( g^5 d^69+ g^6 d^74+ g^7 d^81)[ g^3 d^45]*
[6,3] =  g^3 d^24+ g^4 d^29+ g^5 d^34+( g^6 d^69+ g^7 d^74+ g^8 d^81)[ g^3 d^45]*
[6,4] =  g^2 d^22+ g^3 d^27+ g^4 d^32+( g^5 d^67+ g^6 d^72+ g^7 d^79)[ g^3 d^45]*
[6,5] =  g^2 d^20+ g^3 d^25+ g^4 d^30+( g^5 d^65+ g^6 d^70+ g^7 d^77)[ g^3 d^45]*
[6,6] =  g^2 d^18+ g^3 d^23+ g^4 d^28+( g^5 d^63+ g^6 d^68+ g^7 d^75)[ g^3 d^45]*
[6,7] =  g^3 d^17+ g^4 d^22+ g^5 d^48+( g^6 d^62+ g^7 d^69+ g^8 d^95)[ g^3 d^45]*
[6,8] =  g^4 d^20+ g^5 d^43+ g^6 d^48+( g^7 d^67+ g^8 d^88+ g^9 d^95)[ g^3 d^45]*
[6,9] =  g^5 d^38+ g^6 d^43+ g^7 d^48+( g^8 d^83+ g^9 d^88+ g^10 d^95)[ g^3 d^45]*
[6,10] =  g^4 d^34+ g^5 d^39+ g^6 d^44+( g^7 d^79+ g^8 d^84+ g^9 d^91)[ g^3 d^45]*
[6,11] =  g^3 d^32+ g^4 d^37+ g^5 d^42+( g^6 d^77+ g^7 d^82+ g^8 d^89)[ g^3 d^45]*
[6,12] =  g^1 d^13+ g^2 d^18+ g^3 d^23+( g^4 d^58+ g^5 d^63+ g^6 d^70)[ g^3 d^45]*
[6,13] =  g^0 d^11+ g^1 d^16+ g^2 d^21+( g^3 d^56+ g^4 d^61+ g^5 d^68)[ g^3 d^45]*
[6,14] =  g^0 d^7+ g^1 d^12+ g^2 d^17+( g^3 d^52+ g^4 d^57+ g^5 d^64)[ g^3 d^45]*
[6,15] =  g^0 d^10+ g^1 d^15+( g^2 d^50+ g^3 d^55+ g^4 d^62)[ g^3 d^45]*
[6,16] =  g^0 d^5+ g^1 d^10+( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[6,17] =  g^1 d^3+( g^2 d^38+ g^3 d^43+ g^4 d^50)[ g^3 d^45]*
[6,18] =  g^1 d^0+( g^2 d^36+ g^3 d^41+ g^4 d^48)[ g^3 d^45]*
[6,19] = ( g^2 d^26+ g^3 d^31+ g^4 d^38)[ g^3 d^45]*
[6,20] = ( g^2 d^24+ g^3 d^29+ g^4 d^36)[ g^3 d^45]*
[6,21] = ( g^2 d^20+ g^3 d^25+ g^4 d^32)[ g^3 d^45]*
[7,0] = ( g^2 d^39+ g^3 d^44+ g^4 d^51)[ g^3 d^45]*
[7,1] = ( g^2 d^35+ g^3 d^40+ g^4 d^47)[ g^3 d^45]*
[7,2] = ( g^2 d^33+ g^3 d^38+ g^4 d^45)[ g^3 d^45]*
[7,3] = ( g^3 d^33+ g^4 d^38+ g^5 d^45)[ g^3 d^45]*
[7,4] = ( g^2 d^31+ g^3 d^36+ g^4 d^43)[ g^3 d^45]*
[7,5] = ( g^2 d^29+ g^3 d^34+ g^4 d^41)[ g^3 d^45]*
[7,6] = ( g^2 d^27+ g^3 d^32+ g^4 d^39)[ g^3 d^45]*
[7,7] = ( g^3 d^26+ g^4 d^33+ g^5 d^59)[ g^3 d^45]*
[7,8] = ( g^4 d^31+ g^5 d^52+ g^6 d^59)[ g^3 d^45]*
[7,9] = ( g^5 d^47+ g^6 d^52+ g^7 d^59)[ g^3 d^45]*
[7,10] = ( g^4 d^43+ g^5 d^48+ g^6 d^55)[ g^3 d^45]*
[7,11] = ( g^3 d^41+ g^4 d^46+ g^5 d^53)[ g^3 d^45]*
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[7,12] = ( g^1 d^22+ g^2 d^27+ g^3 d^34)[ g^3 d^45]*
[7,13] = ( g^0 d^20+ g^1 d^25+ g^2 d^32)[ g^3 d^45]*
[7,14] = ( g^0 d^16+ g^1 d^21+ g^2 d^28)[ g^3 d^45]*
[7,15] = ( g^0 d^19+ g^1 d^26+ g^2 d^59)[ g^3 d^45]*
[7,16] = ( g^0 d^14+ g^1 d^21+ g^2 d^54)[ g^3 d^45]*
[7,17] = ( g^0 d^7+ g^1 d^14+ g^2 d^47)[ g^3 d^45]*
[7,18] =  g^0 d^2+( g^1 d^12+ g^2 d^45+ g^3 d^50)[ g^3 d^45]*
[7,19] = ( g^1 d^2+ g^2 d^35+ g^3 d^40)[ g^3 d^45]*
[7,20] = ( g^1 d^0+ g^2 d^33+ g^3 d^38)[ g^3 d^45]*
[7,21] = ( g^2 d^29+ g^3 d^34+ g^4 d^41)[ g^3 d^45]*
[8,0] =  g^1 d^12+( g^2 d^51+ g^3 d^56+ g^4 d^63)[ g^3 d^45]*
[8,1] =  g^1 d^8+( g^2 d^47+ g^3 d^52+ g^4 d^59)[ g^3 d^45]*
[8,2] =  g^1 d^6+( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[8,3] =  g^2 d^6+( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[8,4] =  g^1 d^4+( g^2 d^43+ g^3 d^48+ g^4 d^55)[ g^3 d^45]*
[8,5] =  g^1 d^2+( g^2 d^41+ g^3 d^46+ g^4 d^53)[ g^3 d^45]*
[8,6] =  g^1 d^0+( g^2 d^39+ g^3 d^44+ g^4 d^51)[ g^3 d^45]*
[8,7] =  g^2 d^20+( g^3 d^38+ g^4 d^45+ g^5 d^71)[ g^3 d^45]*
[8,8] =  g^3 d^20+( g^4 d^43+ g^5 d^64+ g^6 d^71)[ g^3 d^45]*
[8,9] =  g^4 d^20+( g^5 d^59+ g^6 d^64+ g^7 d^71)[ g^3 d^45]*
[8,10] =  g^3 d^16+( g^4 d^55+ g^5 d^60+ g^6 d^67)[ g^3 d^45]*
[8,11] =  g^2 d^14+( g^3 d^53+ g^4 d^58+ g^5 d^65)[ g^3 d^45]*
[8,12] = ( g^1 d^34+ g^2 d^39+ g^3 d^46)[ g^3 d^45]*
[8,13] = ( g^0 d^32+ g^1 d^37+ g^2 d^44)[ g^3 d^45]*
[8,14] = ( g^0 d^28+ g^1 d^33+ g^2 d^40)[ g^3 d^45]*
[8,15] = ( g^0 d^31+ g^1 d^38+ g^2 d^71)[ g^3 d^45]*
[8,16] = ( g^0 d^26+ g^1 d^33+ g^2 d^66)[ g^3 d^45]*
[8,17] = ( g^0 d^19+ g^1 d^26+ g^2 d^59)[ g^3 d^45]*
[8,18] =  g^0 d^14+( g^1 d^24+ g^2 d^57+ g^3 d^62)[ g^3 d^45]*
[8,19] =  g^0 d^4+( g^1 d^14+ g^2 d^47+ g^3 d^52)[ g^3 d^45]*
[8,20] =  g^0 d^2+( g^1 d^12+ g^2 d^45+ g^3 d^50)[ g^3 d^45]*
[8,21] = ( g^1 d^8+ g^2 d^41+ g^3 d^46)[ g^3 d^45]*
[9,0] =  g^0 d^16+ g^1 d^20+( g^2 d^57+ g^3 d^62+ g^4 d^69)[ g^3 d^45]*
[9,1] =  g^0 d^12+ g^1 d^16+( g^2 d^53+ g^3 d^58+ g^4 d^65)[ g^3 d^45]*
[9,2] =  g^0 d^10+ g^1 d^14+( g^2 d^51+ g^3 d^56+ g^4 d^63)[ g^3 d^45]*
[9,3] =  g^0 d^6+ g^1 d^10+ g^2 d^14+( g^3 d^51+ g^4 d^56+ g^5 d^63)[ g^3 d^45]*
[9,4] =  g^0 d^8+ g^1 d^12+( g^2 d^49+ g^3 d^54+ g^4 d^61)[ g^3 d^45]*
[9,5] =  g^0 d^6+ g^1 d^10+( g^2 d^47+ g^3 d^52+ g^4 d^59)[ g^3 d^45]*
[9,6] =  g^0 d^4+ g^1 d^8+( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[9,7] =  g^1 d^2+ g^2 d^28+( g^3 d^44+ g^4 d^51+ g^5 d^77)[ g^3 d^45]*
[9,8] =  g^1 d^0+ g^2 d^24+ g^3 d^28+( g^4 d^49+ g^5 d^70+ g^6 d^77)[ g^3 d^45]*
[9,9] =  g^2 d^20+ g^3 d^24+ g^4 d^28+( g^5 d^65+ g^6 d^70+ g^7 d^77)[ g^3 d^45]*
[9,10] =  g^1 d^16+ g^2 d^20+ g^3 d^24+( g^4 d^61+ g^5 d^66+ g^6 d^73)[ g^3 d^45]*
[9,11] =  g^0 d^14+ g^1 d^18+ g^2 d^22+( g^3 d^59+ g^4 d^64+ g^5 d^71)[ g^3 d^45]*
[9,12] =  g^0 d^3+( g^1 d^40+ g^2 d^45+ g^3 d^52)[ g^3 d^45]*
[9,13] = ( g^0 d^38+ g^1 d^43+ g^2 d^50)[ g^3 d^45]*
[9,14] = ( g^0 d^34+ g^1 d^39+ g^2 d^46)[ g^3 d^45]*
[9,15] = ( g^0 d^37+ g^1 d^44+ g^2 d^77)[ g^3 d^45]*
[9,16] = ( g^0 d^32+ g^1 d^39+ g^2 d^72)[ g^3 d^45]*
[9,17] = ( g^0 d^25+ g^1 d^32+ g^2 d^65)[ g^3 d^45]*
[9,18] =  g^0 d^22+( g^1 d^30+ g^2 d^63+ g^3 d^68)[ g^3 d^45]*
[9,19] =  g^0 d^12+( g^1 d^20+ g^2 d^53+ g^3 d^58)[ g^3 d^45]*
[9,20] =  g^0 d^10+( g^1 d^18+ g^2 d^51+ g^3 d^56)[ g^3 d^45]*
[9,21] =  g^0 d^6+( g^1 d^14+ g^2 d^47+ g^3 d^52)[ g^3 d^45]*
```

The matrix K is given by:

```
K =

[0,0] =  g^0 d^2+ g^1 d^4+ g^2 d^30+ g^3 d^34+ g^4 d^38+( g^5 d^75+ g^6 d^80+ g^7 d^87)[ g^3 d^
[0,1] =  g^2 d^22+ g^3 d^26+ g^4 d^30+( g^5 d^67+ g^6 d^72+ g^7 d^79)[ g^3 d^45]*
[0,2] =  g^2 d^18+ g^3 d^22+ g^4 d^26+( g^5 d^63+ g^6 d^68+ g^7 d^75)[ g^3 d^45]*
[0,3] =  g^2 d^8+ g^3 d^10+ g^4 d^12+ g^5 d^38+ g^6 d^42+ g^7 d^46+( g^8 d^83+ g^9 d^88+ g^10 d
[0,4] =  g^2 d^18+( g^3 d^56+ g^4 d^61+ g^5 d^68)[ g^3 d^45]*
[0,5] = ( g^2 d^50+ g^3 d^55+ g^4 d^62)[ g^3 d^45]*
[0,6] = ( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[0,7] =  g^2 d^36+ g^3 d^40+( g^4 d^48+ g^5 d^81+ g^6 d^86)[ g^3 d^45]*
[0,8] =  g^2 d^24+ g^3 d^28+( g^4 d^36+ g^5 d^69+ g^6 d^74)[ g^3 d^45]*
[0,9] =  g^2 d^18+ g^3 d^22+ g^4 d^26+( g^5 d^63+ g^6 d^68+ g^7 d^75)[ g^3 d^45]*
[1,0] =  g^0 d^10+ g^1 d^36+ g^2 d^40+ g^3 d^44+( g^4 d^81+ g^5 d^86+ g^6 d^93)[ g^3 d^45]*
[1,1] =  g^0 d^2+ g^1 d^28+ g^2 d^32+ g^3 d^36+( g^4 d^73+ g^5 d^78+ g^6 d^85)[ g^3 d^45]*
[1,2] =  g^1 d^24+ g^2 d^28+ g^3 d^32+( g^4 d^69+ g^5 d^74+ g^6 d^81)[ g^3 d^45]*
[1,3] =  g^1 d^12+ g^2 d^16+ g^3 d^18+ g^4 d^44+ g^5 d^48+ g^6 d^52+( g^7 d^89+ g^8 d^94+ g^9 d
[1,4] =  g^1 d^24+( g^2 d^62+ g^3 d^67+ g^4 d^74)[ g^3 d^45]*
[1,5] = ( g^1 d^56+ g^2 d^61+ g^3 d^68)[ g^3 d^45]*
[1,6] = ( g^1 d^51+ g^2 d^56+ g^3 d^63)[ g^3 d^45]*
[1,7] =  g^1 d^42+ g^2 d^46+( g^3 d^54+ g^4 d^87+ g^5 d^92)[ g^3 d^45]*
[1,8] =  g^1 d^30+ g^2 d^34+( g^3 d^42+ g^4 d^75+ g^5 d^80)[ g^3 d^45]*
[1,9] =  g^1 d^24+ g^2 d^28+ g^3 d^32+( g^4 d^69+ g^5 d^74+ g^6 d^81)[ g^3 d^45]*
[2,0] =  g^0 d^16+ g^1 d^20+( g^2 d^57+ g^3 d^62+ g^4 d^69)[ g^3 d^45]*
[2,1] =  g^0 d^8+ g^1 d^12+( g^2 d^49+ g^3 d^54+ g^4 d^61)[ g^3 d^45]*
[2,2] =  g^0 d^4+ g^1 d^8+( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[2,3] =  g^2 d^20+ g^3 d^24+ g^4 d^28+( g^5 d^65+ g^6 d^70+ g^7 d^77)[ g^3 d^45]*
[2,4] = ( g^0 d^38+ g^1 d^43+ g^2 d^50)[ g^3 d^45]*
[2,5] = ( g^0 d^37+ g^1 d^44+ g^2 d^77)[ g^3 d^45]*
[2,6] = ( g^0 d^32+ g^1 d^39+ g^2 d^72)[ g^3 d^45]*
[2,7] =  g^0 d^22+( g^1 d^30+ g^2 d^63+ g^3 d^68)[ g^3 d^45]*
[2,8] =  g^0 d^10+( g^1 d^18+ g^2 d^51+ g^3 d^56)[ g^3 d^45]*
[2,9] =  g^0 d^4+ g^1 d^8+( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[3,0] =  g^0 d^26+ g^1 d^30+( g^2 d^67+ g^3 d^72+ g^4 d^79)[ g^3 d^45]*
[3,1] =  g^0 d^18+ g^1 d^22+( g^2 d^59+ g^3 d^64+ g^4 d^71)[ g^3 d^45]*
[3,2] =  g^0 d^14+ g^1 d^18+( g^2 d^55+ g^3 d^60+ g^4 d^67)[ g^3 d^45]*
[3,3] =  g^0 d^2+ g^1 d^4+ g^2 d^30+ g^3 d^34+ g^4 d^38+( g^5 d^75+ g^6 d^80+ g^7 d^87)[ g^3 d^
[3,4] = ( g^0 d^48+ g^1 d^53+ g^2 d^60)[ g^3 d^45]*
[3,5] = ( g^0 d^47+ g^1 d^54+ g^2 d^87)[ g^3 d^45]*
[3,6] = ( g^0 d^42+ g^1 d^49+ g^2 d^82)[ g^3 d^45]*
[3,7] =  g^0 d^32+( g^1 d^40+ g^2 d^73+ g^3 d^78)[ g^3 d^45]*
[3,8] =  g^0 d^20+( g^1 d^28+ g^2 d^61+ g^3 d^66)[ g^3 d^45]*
[3,9] =  g^0 d^14+ g^1 d^18+( g^2 d^55+ g^3 d^60+ g^4 d^67)[ g^3 d^45]*
[4,0] =  g^1 d^19+ g^2 d^23+ g^3 d^27+( g^4 d^64+ g^5 d^69+ g^6 d^76)[ g^3 d^45]*
[4,1] =  g^1 d^11+ g^2 d^15+ g^3 d^19+( g^4 d^56+ g^5 d^61+ g^6 d^68)[ g^3 d^45]*
[4,2] =  g^1 d^7+ g^2 d^11+ g^3 d^15+( g^4 d^52+ g^5 d^57+ g^6 d^64)[ g^3 d^45]*
[4,3] =  g^4 d^27+ g^5 d^31+ g^6 d^35+( g^7 d^72+ g^8 d^77+ g^9 d^84)[ g^3 d^45]*
[4,4] =  g^0 d^2+ g^1 d^7+( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[4,5] =  g^0 d^1+( g^1 d^39+ g^2 d^44+ g^3 d^51)[ g^3 d^45]*
[4,6] = ( g^1 d^34+ g^2 d^39+ g^3 d^46)[ g^3 d^45]*
[4,7] =  g^1 d^25+ g^2 d^29+( g^3 d^37+ g^4 d^70+ g^5 d^75)[ g^3 d^45]*
[4,8] =  g^1 d^13+ g^2 d^17+( g^3 d^25+ g^4 d^58+ g^5 d^63)[ g^3 d^45]*
[4,9] =  g^1 d^7+ g^2 d^11+ g^3 d^15+( g^4 d^52+ g^5 d^57+ g^6 d^64)[ g^3 d^45]*
[5,0] =  g^2 d^25+ g^3 d^30+ g^4 d^35+( g^5 d^70+ g^6 d^75+ g^7 d^82)[ g^3 d^45]*
[5,1] =  g^2 d^17+ g^3 d^22+ g^4 d^27+( g^5 d^62+ g^6 d^67+ g^7 d^74)[ g^3 d^45]*
```

```
[5,2] =  g^2 d^13+ g^3 d^18+ g^4 d^23+( g^5 d^58+ g^6 d^63+ g^7 d^70)[ g^3 d^45]*
[5,3] =  g^5 d^33+ g^6 d^38+ g^7 d^43+( g^8 d^78+ g^9 d^83+ g^10 d^90)[ g^3 d^45]*
[5,4] =  g^0 d^6+ g^1 d^11+ g^2 d^16+( g^3 d^51+ g^4 d^56+ g^5 d^63)[ g^3 d^45]*
[5,5] =  g^0 d^5+ g^1 d^10+( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[5,6] =  g^0 d^0+ g^1 d^5+( g^2 d^40+ g^3 d^45+ g^4 d^52)[ g^3 d^45]*
[5,7] = ( g^2 d^31+ g^3 d^36+ g^4 d^43)[ g^3 d^45]*
[5,8] = ( g^2 d^19+ g^3 d^24+ g^4 d^31)[ g^3 d^45]*
[5,9] =  g^2 d^13+ g^3 d^18+ g^4 d^23+( g^5 d^58+ g^6 d^63+ g^7 d^70)[ g^3 d^45]*
[6,0] =  g^2 d^30+ g^3 d^35+ g^4 d^40+( g^5 d^75+ g^6 d^80+ g^7 d^87)[ g^3 d^45]*
[6,1] =  g^2 d^22+ g^3 d^27+ g^4 d^32+( g^5 d^67+ g^6 d^72+ g^7 d^79)[ g^3 d^45]*
[6,2] =  g^2 d^18+ g^3 d^23+ g^4 d^28+( g^5 d^63+ g^6 d^68+ g^7 d^75)[ g^3 d^45]*
[6,3] =  g^5 d^38+ g^6 d^43+ g^7 d^48+( g^8 d^83+ g^9 d^88+ g^10 d^95)[ g^3 d^45]*
[6,4] =  g^0 d^11+ g^1 d^16+ g^2 d^21+( g^3 d^56+ g^4 d^61+ g^5 d^68)[ g^3 d^45]*
[6,5] =  g^0 d^10+ g^1 d^15+( g^2 d^50+ g^3 d^55+ g^4 d^62)[ g^3 d^45]*
[6,6] =  g^0 d^5+ g^1 d^10+( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[6,7] =  g^1 d^0+( g^2 d^36+ g^3 d^41+ g^4 d^48)[ g^3 d^45]*
[6,8] = ( g^2 d^24+ g^3 d^29+ g^4 d^36)[ g^3 d^45]*
[6,9] =  g^2 d^18+ g^3 d^23+ g^4 d^28+( g^5 d^63+ g^6 d^68+ g^7 d^75)[ g^3 d^45]*
[7,0] = ( g^2 d^39+ g^3 d^44+ g^4 d^51)[ g^3 d^45]*
[7,1] = ( g^2 d^31+ g^3 d^36+ g^4 d^43)[ g^3 d^45]*
[7,2] = ( g^2 d^27+ g^3 d^32+ g^4 d^39)[ g^3 d^45]*
[7,3] = ( g^5 d^47+ g^6 d^52+ g^7 d^59)[ g^3 d^45]*
[7,4] = ( g^0 d^20+ g^1 d^25+ g^2 d^32)[ g^3 d^45]*
[7,5] = ( g^0 d^19+ g^1 d^26+ g^2 d^59)[ g^3 d^45]*
[7,6] = ( g^0 d^14+ g^1 d^21+ g^2 d^54)[ g^3 d^45]*
[7,7] =  g^0 d^2+( g^1 d^12+ g^2 d^45+ g^3 d^50)[ g^3 d^45]*
[7,8] = ( g^1 d^0+ g^2 d^33+ g^3 d^38)[ g^3 d^45]*
[7,9] = ( g^2 d^27+ g^3 d^32+ g^4 d^39)[ g^3 d^45]*
[8,0] =  g^1 d^12+( g^2 d^51+ g^3 d^56+ g^4 d^63)[ g^3 d^45]*
[8,1] =  g^1 d^4+( g^2 d^43+ g^3 d^48+ g^4 d^55)[ g^3 d^45]*
[8,2] =  g^1 d^0+( g^2 d^39+ g^3 d^44+ g^4 d^51)[ g^3 d^45]*
[8,3] =  g^4 d^20+( g^5 d^59+ g^6 d^64+ g^7 d^71)[ g^3 d^45]*
[8,4] = ( g^0 d^32+ g^1 d^37+ g^2 d^44)[ g^3 d^45]*
[8,5] = ( g^0 d^31+ g^1 d^38+ g^2 d^71)[ g^3 d^45]*
[8,6] = ( g^0 d^26+ g^1 d^33+ g^2 d^66)[ g^3 d^45]*
[8,7] =  g^0 d^14+( g^1 d^24+ g^2 d^57+ g^3 d^62)[ g^3 d^45]*
[8,8] =  g^0 d^2+( g^1 d^12+ g^2 d^45+ g^3 d^50)[ g^3 d^45]*
[8,9] =  g^1 d^0+( g^2 d^39+ g^3 d^44+ g^4 d^51)[ g^3 d^45]*
[9,0] =  g^0 d^16+ g^1 d^20+( g^2 d^57+ g^3 d^62+ g^4 d^69)[ g^3 d^45]*
[9,1] =  g^0 d^8+ g^1 d^12+( g^2 d^49+ g^3 d^54+ g^4 d^61)[ g^3 d^45]*
[9,2] =  g^0 d^4+ g^1 d^8+( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
[9,3] =  g^2 d^20+ g^3 d^24+ g^4 d^28+( g^5 d^65+ g^6 d^70+ g^7 d^77)[ g^3 d^45]*
[9,4] = ( g^0 d^38+ g^1 d^43+ g^2 d^50)[ g^3 d^45]*
[9,5] = ( g^0 d^37+ g^1 d^44+ g^2 d^77)[ g^3 d^45]*
[9,6] = ( g^0 d^32+ g^1 d^39+ g^2 d^72)[ g^3 d^45]*
[9,7] =  g^0 d^22+( g^1 d^30+ g^2 d^63+ g^3 d^68)[ g^3 d^45]*
[9,8] =  g^0 d^10+( g^1 d^18+ g^2 d^51+ g^3 d^56)[ g^3 d^45]*
[9,9] =  g^0 d^4+ g^1 d^8+( g^2 d^45+ g^3 d^50+ g^4 d^57)[ g^3 d^45]*
```

The matrix Q is given by:

```
Q =

[0,0] =  g^0 d^0+ g^1 d^2+ g^2 d^4+ g^3 d^30+ g^4 d^34+ g^5 d^38+( g^6 d^75+ g^7 d^80+ g^8 d^87)[ g^
[0,1] =  g^3 d^0+ g^4 d^24+ g^5 d^28+ g^6 d^32+( g^7 d^69+ g^8 d^74+ g^9 d^81)[ g^3 d^45]*
```

```
[0,2] =  g^3 d^18+ g^4 d^22+ g^5 d^26+( g^6 d^63+ g^7 d^68+ g^8 d^75)[ g^3 d^45]*
[0,3] =  g^3 d^8+ g^4 d^10+ g^5 d^12+ g^6 d^38+ g^7 d^42+ g^8 d^46+( g^9 d^83+ g^10 d^88+ g^11
[0,4] =  g^3 d^18+( g^4 d^56+ g^5 d^61+ g^6 d^68)[ g^3 d^45]*
[0,5] = ( g^3 d^50+ g^4 d^55+ g^5 d^62)[ g^3 d^45]*
[0,6] = ( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[0,7] =  g^3 d^36+ g^4 d^40+( g^5 d^48+ g^6 d^81+ g^7 d^86)[ g^3 d^45]*
[0,8] =  g^3 d^24+ g^4 d^28+( g^5 d^36+ g^6 d^69+ g^7 d^74)[ g^3 d^45]*
[0,9] =  g^3 d^18+ g^4 d^22+ g^5 d^26+( g^6 d^63+ g^7 d^68+ g^8 d^75)[ g^3 d^45]*
[1,0] =  g^0 d^8+ g^1 d^10+ g^2 d^36+ g^3 d^40+ g^4 d^44+( g^5 d^81+ g^6 d^86+ g^7 d^93)[ g^3 d
[1,1] =  g^0 d^0+ g^1 d^2+ g^2 d^6+ g^3 d^30+ g^4 d^34+ g^5 d^38+( g^6 d^75+ g^7 d^80+ g^8 d^87
[1,2] =  g^2 d^24+ g^3 d^28+ g^4 d^32+( g^5 d^69+ g^6 d^74+ g^7 d^81)[ g^3 d^45]*
[1,3] =  g^2 d^12+ g^3 d^16+ g^4 d^18+ g^5 d^44+ g^6 d^48+ g^7 d^52+( g^8 d^89+ g^9 d^94+ g^10
[1,4] =  g^2 d^24+( g^3 d^62+ g^4 d^67+ g^5 d^74)[ g^3 d^45]*
[1,5] = ( g^2 d^56+ g^3 d^61+ g^4 d^68)[ g^3 d^45]*
[1,6] = ( g^2 d^51+ g^3 d^56+ g^4 d^63)[ g^3 d^45]*
[1,7] =  g^2 d^42+ g^3 d^46+( g^4 d^54+ g^5 d^87+ g^6 d^92)[ g^3 d^45]*
[1,8] =  g^2 d^30+ g^3 d^34+( g^4 d^42+ g^5 d^75+ g^6 d^80)[ g^3 d^45]*
[1,9] =  g^2 d^24+ g^3 d^28+ g^4 d^32+( g^5 d^69+ g^6 d^74+ g^7 d^81)[ g^3 d^45]*
[2,0] =  g^0 d^12+ g^1 d^16+ g^2 d^20+( g^3 d^57+ g^4 d^62+ g^5 d^69)[ g^3 d^45]*
[2,1] =  g^1 d^6+ g^2 d^10+ g^3 d^14+( g^4 d^51+ g^5 d^56+ g^6 d^63)[ g^3 d^45]*
[2,2] =  g^0 d^0+ g^1 d^4+ g^2 d^8+( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[2,3] =  g^3 d^20+ g^4 d^24+ g^5 d^28+( g^6 d^65+ g^7 d^70+ g^8 d^77)[ g^3 d^45]*
[2,4] =  g^0 d^0+( g^1 d^38+ g^2 d^43+ g^3 d^50)[ g^3 d^45]*
[2,5] = ( g^0 d^32+ g^1 d^37+ g^2 d^44)[ g^3 d^45]*
[2,6] = ( g^0 d^27+ g^1 d^32+ g^2 d^39)[ g^3 d^45]*
[2,7] =  g^0 d^18+ g^1 d^22+( g^2 d^30+ g^3 d^63+ g^4 d^68)[ g^3 d^45]*
[2,8] =  g^0 d^6+ g^1 d^10+( g^2 d^18+ g^3 d^51+ g^4 d^56)[ g^3 d^45]*
[2,9] =  g^0 d^0+ g^1 d^4+ g^2 d^8+( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[3,0] =  g^0 d^22+ g^1 d^26+ g^2 d^30+( g^3 d^67+ g^4 d^72+ g^5 d^79)[ g^3 d^45]*
[3,1] =  g^1 d^16+ g^2 d^20+ g^3 d^24+( g^4 d^61+ g^5 d^66+ g^6 d^73)[ g^3 d^45]*
[3,2] =  g^0 d^10+ g^1 d^14+ g^2 d^18+( g^3 d^55+ g^4 d^60+ g^5 d^67)[ g^3 d^45]*
[3,3] =  g^0 d^0+ g^1 d^2+ g^2 d^4+ g^3 d^30+ g^4 d^34+ g^5 d^38+( g^6 d^75+ g^7 d^80+ g^8 d^87
[3,4] =  g^0 d^10+( g^1 d^48+ g^2 d^53+ g^3 d^60)[ g^3 d^45]*
[3,5] = ( g^0 d^42+ g^1 d^47+ g^2 d^54)[ g^3 d^45]*
[3,6] = ( g^0 d^37+ g^1 d^42+ g^2 d^49)[ g^3 d^45]*
[3,7] =  g^0 d^28+ g^1 d^32+( g^2 d^40+ g^3 d^73+ g^4 d^78)[ g^3 d^45]*
[3,8] =  g^0 d^16+ g^1 d^20+( g^2 d^28+ g^3 d^61+ g^4 d^66)[ g^3 d^45]*
[3,9] =  g^0 d^10+ g^1 d^14+ g^2 d^18+( g^3 d^55+ g^4 d^60+ g^5 d^67)[ g^3 d^45]*
[4,0] =  g^2 d^19+ g^3 d^23+ g^4 d^27+( g^5 d^64+ g^6 d^69+ g^7 d^76)[ g^3 d^45]*
[4,1] =  g^3 d^13+ g^4 d^17+ g^5 d^21+( g^6 d^58+ g^7 d^63+ g^8 d^70)[ g^3 d^45]*
[4,2] =  g^2 d^7+ g^3 d^11+ g^4 d^15+( g^5 d^52+ g^6 d^57+ g^7 d^64)[ g^3 d^45]*
[4,3] =  g^5 d^27+ g^6 d^31+ g^7 d^35+( g^8 d^72+ g^9 d^77+ g^10 d^84)[ g^3 d^45]*
[4,4] =  g^0 d^0+ g^1 d^2+ g^2 d^7+( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[4,5] =  g^1 d^1+( g^2 d^39+ g^3 d^44+ g^4 d^51)[ g^3 d^45]*
[4,6] = ( g^2 d^34+ g^3 d^39+ g^4 d^46)[ g^3 d^45]*
[4,7] =  g^2 d^25+ g^3 d^29+( g^4 d^37+ g^5 d^70+ g^6 d^75)[ g^3 d^45]*
[4,8] =  g^2 d^13+ g^3 d^17+( g^4 d^25+ g^5 d^58+ g^6 d^63)[ g^3 d^45]*
[4,9] =  g^2 d^7+ g^3 d^11+ g^4 d^15+( g^5 d^52+ g^6 d^57+ g^7 d^64)[ g^3 d^45]*
[5,0] =  g^3 d^25+ g^4 d^30+ g^5 d^35+( g^6 d^70+ g^7 d^75+ g^8 d^82)[ g^3 d^45]*
[5,1] =  g^4 d^19+ g^5 d^24+ g^6 d^29+( g^7 d^64+ g^8 d^69+ g^9 d^76)[ g^3 d^45]*
[5,2] =  g^3 d^13+ g^4 d^18+ g^5 d^23+( g^6 d^58+ g^7 d^63+ g^8 d^70)[ g^3 d^45]*
[5,3] =  g^6 d^33+ g^7 d^38+ g^8 d^43+( g^9 d^78+ g^10 d^83+ g^11 d^90)[ g^3 d^45]*
[5,4] =  g^1 d^6+ g^2 d^11+ g^3 d^16+( g^4 d^51+ g^5 d^56+ g^6 d^63)[ g^3 d^45]*
[5,5] =  g^0 d^0+ g^1 d^5+ g^2 d^10+( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[5,6] =  g^1 d^0+ g^2 d^5+( g^3 d^40+ g^4 d^45+ g^5 d^52)[ g^3 d^45]*
[5,7] = ( g^3 d^31+ g^4 d^36+ g^5 d^43)[ g^3 d^45]*
[5,8] = ( g^3 d^19+ g^4 d^24+ g^5 d^31)[ g^3 d^45]*
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[5,9] =  g^3 d^13+ g^4 d^18+ g^5 d^23+( g^6 d^58+ g^7 d^63+ g^8 d^70)[ g^3 d^45]*
[6,0] =  g^3 d^30+ g^4 d^35+ g^5 d^40+( g^6 d^75+ g^7 d^80+ g^8 d^87)[ g^3 d^45]*
[6,1] =  g^4 d^24+ g^5 d^29+ g^6 d^34+( g^7 d^69+ g^8 d^74+ g^9 d^81)[ g^3 d^45]*
[6,2] =  g^3 d^18+ g^4 d^23+ g^5 d^28+( g^6 d^63+ g^7 d^68+ g^8 d^75)[ g^3 d^45]*
[6,3] =  g^6 d^38+ g^7 d^43+ g^8 d^48+( g^9 d^83+ g^10 d^88+ g^11 d^95)[ g^3 d^45]*
[6,4] =  g^1 d^11+ g^2 d^16+ g^3 d^21+( g^4 d^56+ g^5 d^61+ g^6 d^68)[ g^3 d^45]*
[6,5] =  g^0 d^5+ g^1 d^10+ g^2 d^15+( g^3 d^50+ g^4 d^55+ g^5 d^62)[ g^3 d^45]*
[6,6] =  g^0 d^0+ g^1 d^5+ g^2 d^10+( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[6,7] =  g^2 d^0+( g^3 d^36+ g^4 d^41+ g^5 d^48)[ g^3 d^45]*
[6,8] = ( g^3 d^24+ g^4 d^29+ g^5 d^36)[ g^3 d^45]*
[6,9] =  g^3 d^18+ g^4 d^23+ g^5 d^28+( g^6 d^63+ g^7 d^68+ g^8 d^75)[ g^3 d^45]*
[7,0] = ( g^3 d^39+ g^4 d^44+ g^5 d^51)[ g^3 d^45]*
[7,1] = ( g^4 d^33+ g^5 d^38+ g^6 d^45)[ g^3 d^45]*
[7,2] = ( g^3 d^27+ g^4 d^32+ g^5 d^39)[ g^3 d^45]*
[7,3] = ( g^6 d^47+ g^7 d^52+ g^8 d^59)[ g^3 d^45]*
[7,4] = ( g^1 d^20+ g^2 d^25+ g^3 d^32)[ g^3 d^45]*
[7,5] = ( g^0 d^14+ g^1 d^19+ g^2 d^26)[ g^3 d^45]*
[7,6] = ( g^0 d^9+ g^1 d^14+ g^2 d^21)[ g^3 d^45]*
[7,7] =  g^0 d^0+ g^1 d^2+( g^2 d^12+ g^3 d^45+ g^4 d^50)[ g^3 d^45]*
[7,8] = ( g^2 d^0+ g^3 d^33+ g^4 d^38)[ g^3 d^45]*
[7,9] = ( g^3 d^27+ g^4 d^32+ g^5 d^39)[ g^3 d^45]*
[8,0] =  g^2 d^12+( g^3 d^51+ g^4 d^56+ g^5 d^63)[ g^3 d^45]*
[8,1] =  g^3 d^6+( g^4 d^45+ g^5 d^50+ g^6 d^57)[ g^3 d^45]*
[8,2] =  g^2 d^0+( g^3 d^39+ g^4 d^44+ g^5 d^51)[ g^3 d^45]*
[8,3] =  g^5 d^20+( g^6 d^59+ g^7 d^64+ g^8 d^71)[ g^3 d^45]*
[8,4] = ( g^1 d^32+ g^2 d^37+ g^3 d^44)[ g^3 d^45]*
[8,5] = ( g^0 d^26+ g^1 d^31+ g^2 d^38)[ g^3 d^45]*
[8,6] = ( g^0 d^21+ g^1 d^26+ g^2 d^33)[ g^3 d^45]*
[8,7] =  g^0 d^12+ g^1 d^14+( g^2 d^24+ g^3 d^57+ g^4 d^62)[ g^3 d^45]*
[8,8] =  g^0 d^0+ g^1 d^2+( g^2 d^12+ g^3 d^45+ g^4 d^50)[ g^3 d^45]*
[8,9] =  g^2 d^0+( g^3 d^39+ g^4 d^44+ g^5 d^51)[ g^3 d^45]*
[9,0] =  g^0 d^12+ g^1 d^16+ g^2 d^20+( g^3 d^57+ g^4 d^62+ g^5 d^69)[ g^3 d^45]*
[9,1] =  g^1 d^6+ g^2 d^10+ g^3 d^14+( g^4 d^51+ g^5 d^56+ g^6 d^63)[ g^3 d^45]*
[9,2] =  g^0 d^0+ g^1 d^4+ g^2 d^8+( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
[9,3] =  g^3 d^20+ g^4 d^24+ g^5 d^28+( g^6 d^65+ g^7 d^70+ g^8 d^77)[ g^3 d^45]*
[9,4] =  g^0 d^0+( g^1 d^38+ g^2 d^43+ g^3 d^50)[ g^3 d^45]*
[9,5] = ( g^0 d^32+ g^1 d^37+ g^2 d^44)[ g^3 d^45]*
[9,6] = ( g^0 d^27+ g^1 d^32+ g^2 d^39)[ g^3 d^45]*
[9,7] =  g^0 d^18+ g^1 d^22+( g^2 d^30+ g^3 d^63+ g^4 d^68)[ g^3 d^45]*
[9,8] =  g^0 d^6+ g^1 d^10+( g^2 d^18+ g^3 d^51+ g^4 d^56)[ g^3 d^45]*
[9,9] =  g^0 d^0+ g^1 d^4+ g^2 d^8+( g^3 d^45+ g^4 d^50+ g^5 d^57)[ g^3 d^45]*
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