A super-eigenvector approach to control constrained max-plus linear systems

C. A. Maia, L. Hardouin, R. Santos-Mendes and J. J. Loiseau

Abstract— **The control of timed Petri nets subject to synchronization and time delay phenomena is addressed in this paper. This class of timed Petri nets can be described by using the max-plus algebra. The objective is to design a feedback controller for a max-plus linear system to ensure that the system evolution respects time constraints imposed to the state that can be expressed by a semimodule. In order to achieve this goal, an approach based on the definition of the super-eigenvector of a matrix is proposed. Under some conditions, it ensures the existence of a feedback and allows us to compute it. The contribution is illustrated by a transportation control problem taken from literature.**

I. Introduction

Many Engineered systems, such as manufacturing, transportation and communication networks, can be modeled by using Discrete Event Systems (DES) theory [7]. Timed Event Graphs (TEG) are a class of timed Petri net in which all places have one input and one output transition [26], [4]. TEG can be used to model DES that are subject to synchronization and delay phenomena. The dynamic behavior of TEG can be described by linear equations by using suitable idempotent semirings or dioids [4].

Many results have been achieved concerning TEG description in idempotent semirings [4], [15]. Among them it can be noticed results about performance analysis and controller synthesis. About the control setting many problems have been addressed. In [25], [19] it is proposed a control strategy when some system inputs are unknown; In [9] the authors have proposed a feedback approach for reference control. In [21] it is proposed a multivariable control and [20] have considered parameter uncertainties by using interval analysis. The model-reference control based on pre-compensation and feedback is presented in [23] and [24]. A Finite-horizon control problem was addressed in [27]. An observer design is presented in [14].

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Regarding constraint control problems, several results were obtained for some classes of problems. For instance see the papers [3], [28], [18], [16], [12], [22]. Actually there are two main approaches to deal with the problem: one based on the dioid of series $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ and another one based on the dioid $\mathbb{Z}_{\max}$. The main characteristic of the first one is fact that the approach is based on transfer techniques, which allows to deal with a given class of problems (see for instance [28], [16]). On the other hand, the approaches based dioid $\mathbb{Z}_{\max}$ enable us to deal directly with the system realization (see for instance [18]).

In this paper the control problem for max-plus linear system initially developed by [18] is considered. The objective is to find a control law to ensure the state of the system will remain in a given semimodule. Unlike that approach the proposed one is based on the super-eigenvector of a matrix [1]. Results concerning the existence and computation of a feedback are presented. The contribution of this paper is illustrated by a transportation control problem, in which a feedback control is designed in order to guarantee that the state respect some synchronization constraints, which can be represented by a semimodule.

The paper organization is as follows. Section II introduces algebraic framework considered in the sequel, especially the idempotent semiring and Residuation theories. In Section III the control problem and some theoretical results are introduced. Numerical results for a transportation control problem are presented in section IV. Finally remarks and a conclusion are given in section V.

II. Preliminaries

Discrete Event Systems subject to synchronization and delay phenomena can be described by using dioids (actually an idempotent semiring), defined by a set $\mathbb{D}$ in which the elements can be manipulated by using the operations $\oplus$ and $\otimes$. The operation $\oplus$ is associative, commutative and idempotent, that is, $a \oplus a = a, \forall a \in \mathbb{D}$. The operation $\otimes$ is associative and distributive at left and at right with respect to $\oplus$. Moreover, $\forall a$, $a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$, that is, ε is absorbing with respect to $\otimes$. In a dioid, a partial order relation is defined by $b \preceq a$ iff $a = a \oplus b$ and $x \wedge y$ denotes the greatest lower bound between x and y. A dioid $\mathbb{D}$ is said to be complete if it is closed for infinite $\oplus$-sums and if $\otimes$ distributes over infinite $\oplus$-sums. Most of the time the symbol $\otimes$ will be omitted as in conventional algebra, moreover $a^i = a \otimes a^{i-1}$ and $a^0 = e$. It is straightforward to check that $(\mathbb{Z} \cup \{-\infty\}, max, +)$ is a dioid, hereafter called $\mathbb{Z}_{\max}$.

The state evolution of a max-plus linear system can be described by the following equation:

$$x(k) \quad = \quad Ax(k-1) \oplus Bu(k) \qquad (1)$$

in which vectors $x(k) \in (\mathbb{Z}_{\max})^n$, $u(k) \in (\mathbb{Z}_{\max})^p$ represent the vector state and the input of the system. The state of the system gives the date of event occurrence. More precisely, $x_i(k)$ is a dater function associated to an event x_i, hence it represents the date of its k^{th} occurrence. Similarly, for the input u_j, the date of its k^{th} occurrence is denoted $u_j(k)$. Moreover A and B are matrices of appropriate dimension, representing the system parameters. According to the dater function definition $x(k) \preceq x(k+1)$, that is the event numbered k must occur before the $(k+1)^{th}$ one. Hence $x_i(k)$ is a non decreasing function and therefore matrix A is such that $\forall i \; A_{ii} \succeq e$ or in matrix form $A \succeq I$ in which I is the identity matrix (a matrix with entries equal to ε except for the diagonal, in which entries are equal to e).

It is important to remark that max-plus linear systems can be handled by using Scilab toolboxes [17], [10], which can be downloaded from Scilab web site or authors ones.

The solutions of Inequality $ax \preceq x$ is relevant in many max-plus linear systems problems. An important result is presented in Lem. 1 (for more details, please see [4], Lem. 4.77.)

Lemma 1: In a complete dioid $\mathcal{D}$, we have the following equivalence $ax \preceq x \Leftrightarrow x = a^*x$, in which $a^* = \bigoplus_{i \in \mathbb{N}} a^i$ ($(.)^*$ is called "Kleene star operator").

Inversion of mappings is an important issue in many control applications. Unfortunately, in a general manner, mappings defined over idempotent semiring do not admit inverse. However the residuation theory allows to characterize the solution set of an inequality such as $f(x) \preceq y$, which is useful in control problems. The reader may consult [5] to obtain a complete presentation of this theory.

Definition 1 (Isotone mapping): f is an isotone mapping if it preserves order, that is, $a \preceq b \Longrightarrow f(a) \preceq f(b)$.

Definition 2 (Residuated mapping): An isotone mapping $f : \mathcal{D} \to \mathcal{E}$, where $\mathcal{D}$ and $\mathcal{E}$ are partially ordered sets, is a *residuated mapping* if for all $y \in \mathcal{E}$ there exists a greatest element x that satisfies the inequality $f(x) \preceq y$. This greatest element is denoted by $f^\sharp(y)$ and mapping $f^\sharp$ is called the *residual* of f.

Mappings $L_a : x \mapsto a \otimes x$ and $R_a : x \mapsto x \otimes a$ defined over a complete idempotent semiring $\mathbb{D}$ are both residuated [4]. Their residuals are isotone mappings denoted respectively by $L_a^\sharp(x) = a \backslash\!\!\circ\, x$ and $R_a^\sharp(x) = x \,\circ\!\!\!/\, a$.

Dually, if there exists a least element x for the inequality $y \preceq f(x)$ it is denoted by $f^\flat(y)$. Mapping $f^\flat$ is called the *dual residual* of f. Function $T(x) = x \oplus a$, defined over a complete idempotent semiring $\mathbb{D}$, is dually residuated, and its residual is denoted by $T^\flat(x) = x \ominus a$.

As a direct consequence of the definitions presented so far, we can derive some useful relations involving complete idempotent semiring. These relations are given below (see [4] for more details).

$$\begin{array}{rcll} axb \preceq y & \Leftrightarrow & x \preceq a \backslash\!\!\circ\, y \,\circ\!\!\!/\, b & (2) \\ a(a \backslash\!\!\circ\, y \,\circ\!\!\!/\, b)b & \preceq & y & (3) \\ a \oplus x \succeq y & \Leftrightarrow & x \succeq y \ominus a & (4) \\ a \oplus (y \ominus a) & \succeq & y & (5) \\ a^* a^* & = & a^* & (6) \end{array}$$

A. Semimodules

A semimodule is equivalent to the notion of linear vector space in a semiring setting. A semimodule defined from a dioid $(\mathcal{D}, \oplus, \otimes, \varepsilon_s, e)$ is a comutative monoid $(\mathcal{M}, \widehat{\oplus})$ with neutral element $\varepsilon_{\mathcal{M}}$, equipped with a map $(\mathcal{D} \times \mathcal{M}) \mapsto \mathcal{M}$, that is $(\lambda, v) \mapsto \lambda.v$ (left action), for which:

$$\begin{array}{rcl} (\lambda \otimes \mu).v & = & \lambda.(\mu.v), \\ \lambda.(u \widehat{\oplus} v) & = & \lambda.u \widehat{\oplus} \lambda.v, \\ (\lambda \oplus \mu).v & = & \lambda.v \widehat{\oplus} \mu.v, \\ \varepsilon_s.v & = & v, \\ \lambda.\varepsilon_{\mathcal{M}} & = & \varepsilon_{\mathcal{M}}, \\ e.v & = & v, \end{array}$$

for all $u, v \in \mathcal{M}$ and $\lambda, \mu \in \mathcal{D}$. For more details see [13], [8]. A subsemimodule is a subset $\mathcal{S} \subset \mathcal{M}$ for which if $u, v \in \mathcal{S}$ and $\lambda, \mu \in \mathcal{D}$ then $\lambda.v \widehat{\oplus} \mu.v \in \mathcal{S}$. In this paper we will consider the subsemimodule of semimodule of the n-dimensional vectors with entries in $\mathcal{D}$ equipped with the operations $(u \widehat{\oplus} v)_i = u_i \oplus v_i$ and $\lambda.v = \lambda \otimes v$. In this context the set of all solutions of the system $Ax = Bx$, for which A, B, x have entries in $\mathbb{Z}_{\max}$, can be characterized a finitely generated semimodule [6], [13], that is, it can be expressed as an image of a matrix with entries in $\mathbb{Z}_{\max}$. A discussion on this fact and an algorithm to compute the set of solutions is presented in the appendix.

The eigenvalue and eigenvector of a given square matrix $A \in \overline{\mathbb{Z}}_{max}^{n \times n}$ is defined as in classical system theory, i.e., an eigenvector associated to a given eigenvalue λ is all v such that $Av = \lambda v$. In this context, it is important to remark that the eigenvalues can be expressed as a countable number.

Definition 3: We define by $\mathcal{V}(A, \lambda)$ the semimodule of the eigenvectors of the matrix $A \in \overline{\mathbb{Z}}_{max}^{n \times n}$ associated to the eigenvalue λ.

Remark 1: We must observe that the equation $Av = \lambda v$ can be easily put in the form $Ax = Bx$. Therefore $\mathcal{V}(A, \lambda)$ can be expressed as a finitely generated semimodule. To this end write λv as $(\lambda I)v$, in which I is the identity matrix.

In order to generalize this concept, the following definition will be considered.

Definition 4 (λ-super-eigenvector): A super-eigenvector associated a given value λ (or λ-super-eigenvector) is defined as all v such that $Av \preceq \lambda v$ (see [13], [1]). Moreover the set of all vectors $v \in \overline{\mathbb{Z}}_{max}^{n}$ satisfying $Av = \lambda v$ (respectively $Av \preceq \lambda v$) is called the λ-eigenspace (respectively λ-super-eigenspace) of A.

We say that v has a full support if its entries are different from ε (see [1]). It can be shown that if there exists a full support λ-super-eigenvector for a matrix A then $\lambda \geq \rho(A)$, in which $\rho(A)$ is the maximum circuit mean of A. For more details see [1].

Remark 2: We can see that the set of all λ-super-eigenvector of the matrix $A \in \overline{\mathbb{Z}}_{max}^{n \times n}$ can be characterized by a semimodule. This semimodule is equal to $\mathcal{V}(A \oplus \lambda I, \lambda)$, since $Av \preceq \lambda v$ is equivalent to $(A \oplus \lambda I)v = \lambda v$. In addition this semimodule is a finitely generated one, and can be explicitly computed by using Property 1.

Property 1: The semimodule $\mathcal{V}(A \oplus \lambda I, \lambda)$ is generated by $\mathsf{Im}(\lambda^{-1}A)^*$.

Proof:
Indeed $Av \preceq \lambda v \Leftrightarrow (\lambda^{-1}A)v \preceq v \Leftrightarrow (\lambda^{-1}A)^* v = v \Leftrightarrow v \in \mathsf{Im}(\lambda^{-1}A)^*$. These equivalences come from Lem. 1 and Eq. 6 . ■

B. Dealing with Max-plus Equations

As discussed previously, in order to manipulate semimodules defined from the dioid $\mathbb{Z}_{\max}$, it is important to compute the set of all solutions of the system $Ax = Bx$, for which A, B, x have entries in $\mathbb{Z}_{\max}$. It was proven by [6], [13], [2] that this set can be expressed as an image of a matrix with entries in $\mathbb{Z}_{\max}$. In the appendix a relatively simple proof of this fact and an algorithm to compute the solutions are given.

Another important equation for this paper is the one of the form $C \oplus EXG = D$, in which X is a matrix to be found. In this sense, the following lemma is useful to check if this equation has a solution.

Lemma 2: The equation $C \oplus EXG = D$, in which C, D, E, X, G are matrices of appropriated dimension, with entries in a given complete dioid, has a solution for X if and only if $D \succeq C$ and:

$$D \ominus C \preceq E(E \backslash D \phi G)G. \tag{7}$$

Proof:
If there exists a solution X for the equation, obviously $D \succeq C$ and the definition of the dual residuation (Rel. 4) ensures that $EXG \succeq D \ominus C$. On the other hand $EXG \preceq D$, so $EXG \preceq E(E \backslash D \phi G)G$ (by using Rel. 2 and isotony of $\otimes$). As a result $D \ominus C \preceq E(E \backslash D \phi G)G$.

Conversely if $D \succeq C$ and $D \ominus C \preceq E(E \backslash D \phi G)G$ then:

$$\begin{aligned} (D \ominus C) \oplus C &\preceq E(E \backslash D \phi G)G \oplus C \ (isotony\ of\ \oplus), \\ D &\preceq E(E \backslash D \phi G)G \oplus C \ (By\ Rel.5). \end{aligned} \tag{8}$$

By the residuation definition (Rel. 3), $E(E \backslash D \phi G)G \preceq D$. Since $D \succeq C$, then:

$$C \oplus E(E \backslash D \phi G)G \preceq D. \tag{9}$$

Therefore Ineq's. 8 and 9 ensure that $X = E \backslash D \phi G$ is a solution for the equation $C \oplus EXG = D$. ■

Property 2: If X_1 is a solution for the equation $C \oplus EXG = D$, all elements of the set

$$\{X \quad | \quad X_1 \preceq X \preceq E \backslash D \phi G\}, \tag{10}$$

are solutions as well. Moreover:

$$D \ominus C \preceq EX_1G. \tag{11}$$

Proof:
If X_1 is a solution for the above equation, we can easily check, by using residuation definition and the isotony of $\oplus$, that $E \backslash D \phi G$ is the greatest solution. Therefore the isotony of $\oplus$ and $\otimes$ ensures that all elements of the set $\{X \quad | \quad X_1 \preceq X \preceq E \backslash D \phi G\ \}$ are solutions for the considered equation. Ineq. 11 is straightforward from dual residuation definition. ■

Remark 3: Provided that Ineq. 7 is ensured, the greatest of equation $C \oplus EXG = D$ solution is obviously given by $E \backslash D \phi G$. The smallest solution does not exist in general, since multiplication is not a dually residuable operation. However, Ineq. 11 can be useful to find smaller solutions than $E \backslash D \phi G$.

III. Control Problem

Definition 5 (Control problem): The aim is to find a feedback control law $(u(k) = Fx(k-1))$ for the following system:

$$x(k) = Ax(k-1) \oplus Bu(k), \tag{12}$$

to ensure that the state evolution will respect the following constraint:

$$x(k) \in \mathcal{M}, \tag{13}$$

for which $\mathcal{M}$ is a finitely generated semimodule.
In other words, the control problem can be solved by finding a feedback control law $u(k) = Fx(k-1)$, in which $F \in \overline{\mathbb{Z}}_{max}^{p \times n}$, to ensure that $x(k) \in \mathcal{M} (\forall k \geq 0)$. In this sense, by using Eq. 12 the state evolution can be computed by:

$$x(k) = (A \oplus BF)x(k-1), \quad \forall k \geq 1. \tag{14}$$

We remark that restrictions in the form of equalities and inequalities can be easily put in the form of the constraint 13. This kind of problem appears, for instance, in manufacturing systems and transportation networks for which the aim is to ensure that the dates of event occurrence (i.e. $x(k)$) are constrained to a given semimodule. Obviously, depending on the characteristics of the semimodule $\mathcal{M}$ the control problem cannot be solved. This is illustrated by Example 1.

Example 1: It is easy to see that the following system:

$$\begin{aligned} x_1(k) &= x_1(k-1) \oplus u(k), &(15)\\ x_2(k) &= 2x_1(k-1) \oplus x_2(k-1) \oplus 2u(k), &(16) \end{aligned}$$

can not ensure the restriction $x(k) \in \mathsf{Im}\begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}$, since it is equivalent to $x_2(k) \preceq x_1(k)$ and the system enforces $x_2(k) \succeq 2x_1(k-1)$.

By using the results obtained previously, we can state the Proposition 1.

Proposition 1: Assume that $\exists \lambda \succeq \rho(A)$ such that $\mathcal{V}(A \oplus \lambda I, \lambda) \cap \mathcal{M} \neq \emptyset$. If there exists a λ-super-eigenvector, v, in the semimodule $\mathcal{V}(A \oplus \lambda I, \lambda) \cap \mathcal{M}$ such that

$$(\lambda v \ominus Av) \preceq (B(B \backslash (\lambda v) \phi v)v, \tag{17}$$

then the control problem proposed in Def.5 has a solution.

Proof:
If $v \in \mathcal{V}(A \oplus \lambda I, \lambda) \cap \mathcal{M}$, then $Av \preceq \lambda v$. As $(\lambda v \circ\!\!\!- Av) \preceq (B(B \backslash\!\!\!\circ (\lambda v) \circ\!\!\!/ v)v$, take $C = Av$, $D = \lambda v$, $E = B$ and $G = v$, then lemma 2 ensures that there exists a solution F for the equation:

$$(A \oplus BF)v = \lambda v. \tag{18}$$

Since $v \in \mathcal{V}(A \oplus \lambda I, \lambda) \cap \mathcal{M}$, take $x(0) = v$ then $x(0) \in \mathcal{M}$. As a result, by Eq.18, $x(1) = \lambda x(0) \in \mathcal{M}$. The proof is completed by induction, resulting in $x(k) = \lambda x(k-1) \in \mathcal{M}$, $\forall k > 0$. ■

Remark 4: We remark that if all conditions of Proposition 1 are fulfilled, then Property 2 ensures that the greatest solution is given by $F = B \backslash\!\!\!\circ (\lambda v) \circ\!\!\!/ v$.

IV. AN ILLUSTRATIVE EXAMPLE: A SMALL TRANSPORTATION NETWORK CONTROL

The following figure shows a TEG model for a small train network, which was built based on the description is given in [11], [18]. In this network, there is a train service from

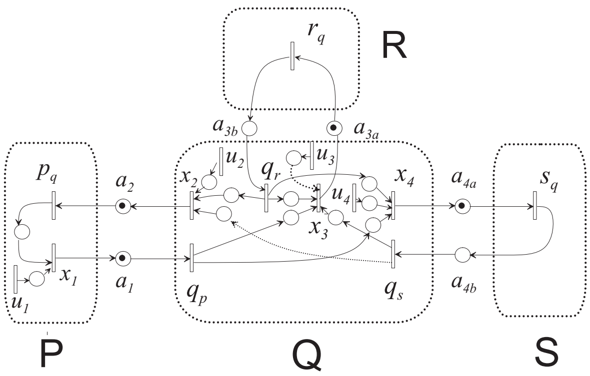

Fig. 1. TEG for the Transportation Network

P via Q to S and vice-versa and there is a train service from Q to R and vice-versa. At station Q trains from P and S have to give connection to the train with destination R and vice-versa. In the TEG presented in Fig. 1, transitions x_i, $i = 1, \ldots 4$, denote the departure events of the trains leaving the station in the directions PQ, QP, QR and QS respectively.These transitions are connected to control inputs u_1 to u_4 allowing to delay the corresponding firing dates. The other transitions denoted p_q, r_q, s_q, q_p, q_s and q_r represent the arrival events of trains to the stations in the six possible directions, that is directions QP, QR and so on. To respect some connection conditions, we must ensure that:

1) the departure of the train from P must occur after the arrival of train coming from Q to P;
2) the departure of the train going from Q to P must occur after the arrivals of the trains coming from S to Q and from R to Q;
3) the departure of the train going from Q to R must occur after the arrivals of the trains coming from P to Q and from S to Q;
4) the departure of the train going from Q to S must occur after the arrivals of the trains coming from P to Q and from R to Q.

Moreover, in the presented TEG, a_1, a_2, a_{3a}, a_{3b}, a_{4a} and a_{4b} denote the travelling times from the indicated stations. Following [11], the travelling times are chosen as: $a_1 = 14$, $a_2 = 17$, $a_{3a} \otimes a_{3b} = 11$ and $a_{4a} \otimes a_{4b} = 9$.

$$x(k) = Ax(k-1) \oplus Bu(k), \tag{19}$$

in which $x_i(k)$, $i = 1, \ldots 4$, denote the data of the k^{th} departure of trains leaving stations and are assumed to be controllable, that is, their departures can be delayed. The matrices are such that:

$$A = \begin{bmatrix} \varepsilon & 17 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 11 & 9 \\ 14 & \varepsilon & 11 & 9 \\ 14 & \varepsilon & 11 & \varepsilon \end{bmatrix},$$

and B is an identity matrix. In [18] the objective of the feedback control is to guarantee that the closed-loop system respects the set of constraints presented below.

- The time between two consecutive trains departures must not exceed a given limit, that is $x_i(k) - L_i \leq x_i(k-1)$.
- The waiting time of the passengers coming from a given station to go to another station must not exceed a given limit, that is $x_j(k) - A_{ji} - M_{ij} \leq x_i(k-1)$.

These constraints can be written as $E_r x(k) \preceq x(k-1)$. Since $x_i(k)$ is a non decreasing sequence and it must respect Eq.19, the complete set of constraint is given by:

$$\begin{aligned} (A \oplus I)x(k-1) &\preceq x(k) & (20)\\ E_r x(k) &\preceq x(k-1) & (21) \end{aligned}$$

We need to write these constraints as a semimodule, as requested in the control problem definition (Def.5). To this end we rewrite the equation of the system as:

$$\widehat{x}(k) = \widehat{A}\widehat{x}(k-1) \oplus \widehat{B}u(k), \tag{22}$$

in which $\widehat{x}(k) = [x(k)^T \ x(k-1)^T]^T$,

$$\widehat{A} = \begin{bmatrix} A & \varepsilon \\ I & \varepsilon \end{bmatrix},$$

and

$$\widehat{B} = \begin{bmatrix} I \\ \varepsilon \end{bmatrix}.$$

As a consequence the constraints can be rewritten as $E\widehat{x}(k) \preceq \widehat{x}(k)$ (see [18] for details), with matrix E given by:

$$E = \begin{bmatrix} \varepsilon & A \oplus I \\ E_r & \varepsilon \end{bmatrix},$$

in which

$$E_r = \begin{bmatrix} -15 & \varepsilon & -18 & -18 \\ -21 & -15 & \varepsilon & \varepsilon \\ \varepsilon & -15 & -15 & -15 \\ \varepsilon & -13 & -13 & -15 \end{bmatrix}.$$

Therefore the constraints can be expressed as $\widehat{x}(k) \in \mathsf{Im}E^*$ according to Lemma 1. In other words, $\widehat{x}(k)$ must belong to the semimodule $\mathcal{M}$ characterized by $\mathsf{Im}E^*$.

Following Proposition 1, we must deal with the semimodule $\mathcal{V}(A \oplus \lambda I, \lambda) \cap \mathcal{M}$. This semimodule can be completely characterized by solving an equation of the form $G \otimes v = H \otimes v$, in which G and H are matrices and v is a vector (see [8] for more details). To solve this equation, we can use, for instance, the algorithm presented in [2], which is an improvement of the ideas presented by [6].

As a result, by taking $\lambda = 14$, which is equal to $\rho(A)$, the computation of the semimodule $\mathcal{V}(A \oplus \lambda I, \lambda) \cap \mathcal{M}$ leads to:

$$\mathsf{Im}\,[17 \quad 14 \quad 17 \quad 18 \quad 3 \quad 0 \quad 3 \quad 4]^T.$$

As a consequence, we can easily verify that:

$$v = [17 \quad 14 \quad 17 \quad 18 \quad 3 \quad 0 \quad 3 \quad 4]^T$$

belongs to

$$\mathcal{V}(A \oplus \lambda I, \lambda) \cap \mathcal{M},$$

and it is such that:

$$(\lambda v \ominus Av) \preceq (B(B \backslash (\lambda v) \not/ v)v. \tag{23}$$

Therefore Proposition 1 ensures that there exists a solution for the proposed control problem. After some numerical manipulations (see Remark 3), we can show that one solution is given by a matrix denoted by F_1, which has null entries except for $F_1(4,4) = 14$. This solution is smaller than the one obtained by [18] and it was computed in a simpler way. Moreover, by Remark 4, the greatest feedback is given by:

$$F_2 = \begin{bmatrix} 14 & 17 & 14 & 13 & 28 & 31 & 28 & 27 \\ 11 & 14 & 11 & 10 & 25 & 28 & 25 & 24 \\ 14 & 17 & 14 & 13 & 28 & 31 & 28 & 27 \\ 15 & 18 & 15 & 14 & 29 & 32 & 29 & 28 \end{bmatrix}$$

As a consequence of Property 2, by taking $x(0) = v$, we can ensure that all F such that:

$$F_1 \preceq F \preceq F_2, \tag{24}$$

is a solution for the control problem.

V. Conclusion

The objective of this paper was to design a controller to guarantee that a max-plus system evolves without violating time constraints characterized by a semimodule. The presented approach is based on the algebraic property of the system and on the notion of super-eigenvector of a matrix. If some conditions are fulfilled, the controller matrix can be obtained by solving max-plus linear equations. To illustrate the contribution of this paper a transportation network problem, taken from the literature, was solved.

VI. Appendix: generation of all solution for the equation $Ax = Bx$

We are interested in equations based on the dioid $\mathbb{R}_{\max} = (\mathbb{R} \cup \{\varepsilon\}, max, +)$, as proposed[1] in [6]. We know that all solution of the equation $Ax = Bx$, for which A and $B \in (\mathbb{R}_{\max})^{m \times n}$ and $x \in (\mathbb{R}_{\max})^n$ can be expressed as finitely generated semimodule and it can be computed by the the algorithm presented in [6]. This algorithm was recently improved in [2]. The objective of this appendix is to discuss this problem and give a simple proof of this fact.

Lets us start by considering the case in which the matrices A and B are line vectors, that is $A = [a_1 \dots a_n]$ and $B = [b_1 \dots b_n]$. The solution for problem, for which A and B are matrices of general dimensions, can be obtained in a straightforward manner by solving the problem line by line, i.e, we solve for the first line of matrix A and B and use the result to solve the problem the second line of the matrices and so on. Explicitly the equation to be solved is:

$$a_1 \otimes x_1 \oplus \dots \oplus a_n \otimes x_n = b_1 \otimes x_1 \oplus \dots \oplus b_n \otimes x_n. \tag{25}$$

Without loss of generality, we also assume that these vectors are such that[2] $a_k \oplus b_k \neq \varepsilon$ for all $k \in \{1, \dots, n\}$. In this sense, if there exists a non null solution for the problem, then:

$$\exists (i,j) \quad | \quad a_i \otimes x_i = b_j \otimes x_j, \tag{26}$$

for which

$$(a_k \otimes x_k \preceq a_i \otimes x_i) \quad \& \quad (b_k \otimes x_k \preceq b_j \otimes x_j), \quad \forall k. \tag{27}$$

Since the solution is non null, $\exists k$ such that $x_k \neq \varepsilon$. Since $a_k \oplus b_k \neq \varepsilon$, by assumption, then $a_k \otimes x_k \neq \varepsilon$ or $b_k \otimes x_k \neq \varepsilon$. As a result by Ineq. 27, we ensure that $a_i \otimes x_i = b_j \otimes x_j \neq \varepsilon$. Therefore:

$$a_i \otimes b_j \neq \varepsilon. \tag{28}$$

In this case, we can see that $[x_i \quad x_j]^t \in \mathsf{Im}[b_j \quad a_i]^t$ and $a_i \succeq b_i$ and $b_j \succeq a_j$. Furthermore, we can observe that all vectors $v^{(l,p)} \in (\mathbb{R}_{\max})^n$, such that $v^{(l,p)}(l) = b_p$, $v^{(l,p)}(p) = a_l$ and $v^{(l,p)}(k) = \varepsilon$ for $k \notin \{l,p\}$, for which $a_l \succeq b_l$ and $b_p \succeq a_p$, generate a solution for the Eq. 25. This fact motivates the definition of the following set:

$$\Upsilon = \{(l,p) \quad | \quad (a_l \succeq b_l) \quad \& \quad (b_p \succeq a_p)\}. \tag{29}$$

As a consequence, all vectors in the image of the matrix, in which columns are the vectors $v^{(l,p)}$, $(l,p) \in \Upsilon$ are a solution for the Eq. 25. Hereafter we denote this matrix by M. In the following, we show that all solution for the Eq.25 belongs to the image of M, that is the semimodule is finitely generated.

If there exists a non null solution $x = [x_1 \dots x_n]^t$ for the problem, then:

[1]The development for the case in which the equations are base one dioid $\mathbb{Z}_{\max}$ is the same.

[2]If $a_k \oplus b_k = \varepsilon$, there is no restriction for the values of x_k, that is, the value of x_k does not interfere on the other entries of x, and so the the study of Eq. 25 can be simplified for the case in which at least one coefficient is non null.

$$\exists (i,j) \in \Upsilon \quad | \quad a_i \otimes x_i = b_j \otimes x_j, \tag{30}$$

such that Ineq.'s 27 and 28 hold true. Therefore, as discussed previously, x_i and x_j is generated by the vector $\beta v^{(i,j)}$ by taking β such that $x_i = \beta b_j$. As a consequence it remains to show that all other non-null entry x_k such that $k \notin \{i,j\}$ can be generated by a linear combination of the columns of M. In this sense, we can only have the two possibilities presented below.

1) ($a_k \succeq b_k$): since $b_j \succeq a_j$ then $(k,j) \in \Upsilon$. We will show that x_k can be generated by the image of $v^{(k,j)}$. In this sense if we chose α_k such that $x_k = \alpha_k \otimes b_j$. It remains to show that $\alpha_k \otimes a_k \preceq x_j$, since x_j is already generated by the image of $v^{(i,j)}$. We know by Ineq. 27 that $a_k \otimes x_k \preceq a_i x_i$, since $a_i x_i = b_j x_j$ then $a_k \otimes \alpha_k \otimes b_j \preceq b_j x_j$. By Ineq. 28, b_j is a non null scalar number, then x_i, x_j and x_k are generated by $\alpha_k v^{c(i,j,k)} \oplus \beta v^{(i,j)}$, in which $c(i,j,k) = (k,j)$.
2) ($b_k \succeq a_k$): since $a_i \succeq b_i$, then $(i,k) \in \Upsilon$. The proof follows the same reasoning of the item (1), that is, we will show that x_k can be generated by the image of $v^{(i,k)}$. To this end we chose α_k such that $x_k = \alpha_k \otimes a_i$ and we must ensure that $\alpha_k \otimes b_k \preceq x_i$, since x_i is already generated by the image of $v^{(i,j)}$. By Ineq. 27, we know that $b_k \otimes x_k \preceq b_j x_j$, since $b_j x_j = a_i x_i$ then $b_k \otimes \alpha_k \otimes a_i \preceq a_i x_i$. By Ineq. 28, a_i is a non null scalar number, then x_i, x_j and x_k are generated by $\alpha_k v^{c(i,j,k)} \oplus \beta v^{(i,j)}$, in which $c(i,j,k) = (i,k)$.

As a result, thanks to the idempotency of the dioid, the non null solution x is described as linear combination of the columns of M, that is $x \in \mathsf{Im}M$. Explicitly:

$$x = \bigoplus_{\forall k \notin \{i,j\} \& (x_k \neq \varepsilon)} (\alpha_k v^{c(i,j,k)}) \quad \oplus \beta v^{(i,j)} \quad \in \mathsf{Im}M, \tag{31}$$

in which $c(i,j,k)$ is taken as (k,j) if $(a_k \succeq b_k)$ or (i,k) otherwise.

Finally, it is important to remark that the case for which $a_k \oplus b_k = \varepsilon$ is taken into account by adding a column in the matrix M in which the k^{th} entry is equal to e and all others are null.

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