SYNCHRONIZATION, LINEARITY, AND BEYOND

A TROPICAL-ALGEBRAIC APPROACH TO THE CONTROL OF DISCRETE-EVENT SYSTEMS WITH RESOURCE SHARING AND PARTIAL SYNCHRONIZATION

vorgelegt von

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ABSTRACT

This thesis is dedicated to the study of discrete-event systems (DES) exhibiting resource-sharing and partial-synchronization phenomena, the latter consisting in the presence of external signals restricting the occurrence of certain events. The results are developed within a well-established framework for the modeling and control of discrete-event systems based on tropical algebra, where the basic modeling structures form a subclass of timed Petri nets called *timed event graphs* (TEGs). A notable advantage of using TEGs is the fact that their evolution can be described by linear equations in a tropical-algebraic setting such as the max-plus or the min-plus algebra. This has given rise to a solid control theory for the class of systems that can be modeled as TEGs, including methods for feedforward, feedback, and observer-based control design. Nonetheless, this system class is not suitable for modeling practically-relevant phenomena involving concurrency. The main contribution of this thesis is to further enrich the existing TEG-based control framework by encompassing the phenomena of resource-sharing and partial synchronization, neither of which can be modeled by TEGs alone. Both these phenomena bring additional restrictions (internal in the former case, external in the latter) to the occurrence of certain events in the system. In both cases, we show that these restrictions can be expressed as inequalities in the semiring of counters by making use of an operation called Hadamard product. We then proceed to propose a formal and systematic method for optimal output-reference control of systems exhibiting either of these phenomena, where optimality is interpreted in a *just-in-time* sense: control inputs are provided as late as possible while guaranteeing that the resulting system outputs do not occur later than dictated by the reference signal. The method also extends to the cases in which the output-reference signals (in the case of resource-sharing) or the external restrictions (in the case of partial synchronization) may change unexpectedly during the operation of the system. Finally, a unified method is presented which allows to systematically obtain optimal control inputs for systems in which both treated phenomena are simultaneously present.

ZUSAMMENFASSUNG

In dieser Arbeit werden ereignisdiskrete Systeme (EDS) untersucht, die durch Ressourcen-Sharing und partielle Synchronisation gekennzeichnet sind. Unter partieller Synchronisation versteht man ein Phänomen, bei dem externe Signale das Auftreten bestimmter Ereignisse einschränken. Zur Herleitung der in dieser Arbeit erzielten Ergebnisse wird eine etablierte Vorgehensweise zur Modellierung und Regelung einer speziellen Klasse zeitbehafteter ereignisdiskreter Systeme genutzt. In dieser Vorgehensweise werden zeitbehaftete Synchronisationsgraphen (engl.: timed event graphs (TEGs)) in tropischen Algebren wie der max-plus oder der min-plus Algebra beschrieben. Dies führt zu linearen Gleichungen für die zeitliche Entwicklung der betrachteten TEGs. Auf der Grundlage dieser linearen Modelle hat sich in den letzten Jahrzehnten eine ausgereifte System- und Regelungstheorie für TEGs entwickelt. Allerdings lassen sich in dieser Systemklasse praktisch relevante Phänomene wie das Auftreten von Konflikten nicht abbilden. Der Hauptbeitrag dieser Arbeit ist eine Weiterentwicklung der für TEGs zur Verfügung stehenden System- und Regelungstheorie, so dass auch Ressourcen-Sharing und partielle Synchronisation behandelt werden können. Beide Phänome implizieren zusätzliche Restriktionen für das Auftreten bestimmter Ereignisse – interner Art im ersteren, externer Art im letzteren Fall. In beiden Fällen wird gezeigt, dass die jeweiligen Restriktionen als Ungleichungen formuliert werden können, wenn im Rahmen der benutzten tropischen Algebra ein sogenanntes Hadamard-Produkt eingeführt wird. Für Systeme mit Ressourcen-Sharing oder partieller Synchronisation wird dann eine formale Vorgehensweise zur optimalen Berechnung von Eingangsereignissen entwickelt. Optimalität ist hier im Sinne eines just-in-time Kriterums zu verstehen, d.h. alle Eingangsereignisse sollen so spät als möglich erfolgen, allerdings unter der Bedingung, dass kein Ausgangsereignis später auftritt als von einem Referenzsignal spezifiziert. Diese Vorgehensweise wird dann auf den Fall erweitert, dass Referenzsignale oder externe Restriktionen sich während des Betriebs des Systems auf unvorhergesehene Weise ändern können. Schließlich wird eine Methode zur systematischen Berechnung des optimalen Eingangs für den Fall vorgestellt, dass Ressourcen-Sharing- und partielle Synchronisationsphänome gleichzeitig auftreten.

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Finally, I acknowledge the contribution of the book *Synchronization and Linearity* [5] to the research that culminated in this thesis. To the authors, as well as to the community, I would like to point out that the title of this thesis is meant as an homage to the book, by no means having the pretension of conveying the (silly) idea that this thesis could ever have a significance even remotely comparable to that of the book.

CONTENTS

1 INTRODUCTION

In this thesis, we shall focus on the study of systems belonging to the class of *discrete-event systems* (DES), whose dynamic evolution consists in the sequential occurrence of events that cause instantaneous transitions among a discrete set of states (as a standard reference, see e. g. [9]). Two of the most commonly used formalisms to model DES are automata [24] and Petri nets [37], which have served as the basis for a rich control theory for this class of systems (see e. g. [9], [45], [50], [35]). Even though the dynamics of DES is event-driven, in some applications time plays a crucial role, for example for performance evaluation, deadline enforcement, or scheduling of time-sensitive tasks. In order to encompass such cases, timed models for DES have emerged in both the automata [4] and the Petri nets [48] fronts. Typical examples of real-world systems suitable to be modeled as DES are human-made ones found in the context of manufacturing (e. g. [10], [26], [49], [33], [40]), transportation (e. g. [23], [51], [19], [38], [29]), and computer networks (e. g. [7], [14], [46]).

The building blocks for the models considered in this thesis form a subclass of timed Petri nets called *timed event graphs* (TEGs), characterized by the fact that each place has precisely one upstream and one downstream transition and all arcs have weight one. In particular, the former restriction implies that TEGs alone are not suitable for modeling conflict or choice. They can, however, be used to model synchronization and delay phenomena, which are central in many of the application scenarios cited above. One advantage of TEGs is the well-known fact that in a suitable mathematical framework, namely a tropical semiring setting such as the max-plus or the min-plus algebra, their evolution can be described by linear equations (see [5] for a thorough coverage). Moreover, by partitioning the transitions of a TEG into input, internal, and output ones, these linear equations take the form of a linear state-space model of the system. Based thereon, an elaborate control theory has become available for this subclass of DES, carrying over some key concepts from classical control theory; these include transfer functions and transfer matrices [5, 11, 20], as well as standard control approaches like optimal output-reference feedforward control [11, 28, 34] and model-reference control with output or state feedback [13, 27, 31]. For a tutorial introduction to this control framework, the reader may refer to [22].

In the scope of this thesis, the relevant control approach is that of optimal output-reference control, with optimality being understood in a *just-in-time* sense: the goal is to fire all input transitions as late as possible while guaranteeing that the firing of output transitions is not later than specified by a reference signal. In a manufacturing context, for instance, the firing of an input transition could correspond to the provisioning of raw material, whereas the firing of an output transition signifies the completion of a workpiece. In general, a just-in-time policy aims at satisfying customer demands while minimizing internal stocks and idle waiting times.

The results presented here aim at expanding the class of systems to which the control framework discussed above can be applied. This is achieved by tackling two different phenomena that naturally arise in many applications and that cannot be dealt with by methods purely based on TEG models. The first phenomenon is *resource sharing*, and the second, consisting in the presence of external restrictions for the occurrence of certain events, has become known as *partial synchronization*. Let us now motivate the investigation of each of these phenomena and highlight the main contributions of this thesis in each context.

SYSTEMS WITH RESOURCE SHARING

Systems of practical interest often involve limited resources that are shared among different subsystems. As examples, one can think of an automated manufacturing cell where the same tool/machine may be required in several (possibly concurrent) steps of the production process, of a railway network where shared track segments are used by multiple trains, or of computational tasks competing for the use of a fixed number of processors. On the other hand, as pointed out before, TEGs cannot model concurrency or choice, which implies a TEG alone is not suitable for modeling resource-sharing phenomena. The aforementioned algebraic advantages of using TEGs have motivated considerable effort toward overcoming this limitation, leading to adaptations and enhancements of TEG-based approaches in order to encompass systems with shared resources (e. g. [36], [12], [53], [1], [8]).

In the first core part of this thesis (Part II), we consider a scenario in which a number of subsystems, each modeled as a TEG, compete for access to one or more shared resources. The objective is to determine just-in-time inputs that make sure every subsystem meets its own demand (i. e., tracks its own output-reference signal) while taking into account the limited capacity of the resources. Our approach is based on the one from [36], where the dispute for the joint resources is settled by establishing a priority policy among the subsystems.

The problem becomes more general — and significantly more challenging — if one considers that the output-reference signals may vary while the system is operating, hence requiring an on-line update of the resource-allocation schedule. Imagine a manufacturing shop floor, for instance, where an increase in the demand for high-priority products will require a readjustment of resource allocations by processing steps related to lower-priority products, or an emergency call center where the arrival of high-priority calls may render it necessary to reschedule the answer to lower-priority ones (the latter problem has been studied, e. g., in [2, 3]). As the first main contribution of the thesis, in Chapter 5 a method is proposed to optimally update the inputs of all subsystems in the case their reference signals are changed during the operation — to the best of the author's knowledge, this problem has not been dealt with before in this context. More precisely, we consider that all subsystems are initially operating under optimal schedules with respect to their individual output references and to the global priority policy. Supposing the output reference of one or more of them is updated during run-time, we show how to optimally update all their inputs so that their outputs are as close as possible to the corresponding new references and the priority policy is still observed. In the case the limited availability of the resources and the performance limitation of the subsystems make it impossible to respect some of the new references, we also provide the optimal way to relax such references, obtaining their closest possible feasible versions based on which the corresponding inputs are then optimally updated.

A more detailed comparison with related work on systems with shared resources will make more sense after the reader has become familiar with the method proposed in this thesis. Therefore, we postpone this comparison to the end of Part II (see Chapter 6).

SYSTEMS WITH PARTIAL SYNCHRONIZATION

The conditions for transition firings in TEGs are classically modeled by standard synchronization, i.e., a transition can only fire *after* the firing of certain other transitions, possibly with some delay, and the firing of one transition never disables another. In some applications, however, different forms of synchronization arise. In the second core part of this thesis (Part III), we consider *partial synchronization* (or PS, for short), a term coined in [17] where this phenomenon was originally studied in a TEG setting. It consists in the existence of external signals that limit the time instants at which certain transitions in the system are allowed to fire. This is manifested in several scenarios of practical relevance. In manufacturing, for instance, the occurrence of events corresponding to turning on different high-power demanding machines may be restricted to not occur simultaneously in order to avoid spikes in the energy consumption, or there may be time windows within which some equipment is scheduled for maintenance and, therefore, cannot operate. In transportation networks, the access to single-track segments by certain lines may be restricted according to a fixed, predetermined schedule of external lines (e. g. operated by a different company). Furthermore, we also consider the case in which the external signals restricting the occurrence of certain events may vary over time. In the manufacturing cases, the plans for utilizing heavy machinery or for performing equipment maintenance may need to be updated, whereas in transportation networks the availability of single-track segments to certain lines may be altered due, e.g., to delays or unexpected deviations from the fixed schedule of external lines.

The second main contribution of this thesis starts in Chapter 7, where an original approach to tackle the modeling and control of TEGs under PS restrictions is proposed. Given a system modeled as a TEG, a reference for its output, and predetermined external signals restricting the occurrence of one or more of its transitions, first we systematically obtain a model (overall no longer a TEG) which incorporates the given PS restrictions. Then, we obtain optimal (just-in-time) inputs which lead to tracking the output reference while making sure that the firing of those transitions under PS respect the imposed restrictions. In Chapter 8, we proceed to extend the method to the case in which PS signals may change during the operation of the system — the case of varying PS signals has not been dealt with before in this setting. With the system initially operating according to the optimal inputs computed before, suppose the PS restrictions on one or more of the affected transitions are altered at a certain time. We establish the optimal way of updating the inputs so that the new restrictions are observed and, if possible, the reference is still met. However, depending on how strict the updated PS restrictions turn out to be, tracking the original reference may become unattainable. In that case, we show how to relax the reference as little as possible to make it feasible, and this minimally-relaxed reference is then used when updating the inputs.

As before, we postpone the comparison with related work on systems with PS to the end of Part III, after our method has been presented (see Chapter 9).

OUTLINE

This thesis is divided into four parts, Part I – Part IV, of which Parts II and III are the ones containing the chief results and contributions. In more detail, the document is structured as follows.

Part I — PRELIMINARIES

CHAPTER 2 provides an overview of the mathematical concepts underlying the discussions along the thesis. Covered topics include idempotent semirings, the min-plus tropical semiring, the semiring of counters, fixed points of isotone mappings, and residuation theory.

CHAPTER 3 concerns timed event graphs (TEGs), the basic modeling elements for the systems treated in the subsequent chapters. We recall how the behavior of TEGs can be described in the semiring of counters and present some related fundamental control results.

Part II — SYSTEMS WITH SHARED RESOURCES

CHAPTER 4 consists in a method for optimal output-reference control of a collection of subsystems that compete for access to shared resources under a predetermined priority policy. Each subsystem is modeled as a TEG and is assigned its own reference signal. The global conditions imposed by the joint resources are expressed as inequalities in the semiring of counters using the Hadamard product, and the computation of the just-in-time inputs is formulated as a fixed-point problem.

CHAPTER 5 takes the scenario from Chapter 4 as a starting point and deals with the case in which the output-references may be updated during the operation. The problem of optimally updating the inputs can again be systematically solved by computing fixed points of appropriate mappings.

CHAPTER 6 is reserved for comparison of the results from Chapters 4 and 5 with selected related work from the literature.

Part III — SYSTEMS WITH PARTIAL SYNCHRONIZATION

CHAPTER 7 shows how the PS phenomenon can be captured by a Petri-net structure appended to the TEG model of a system. Analogously to Chapter 4, restrictions from PS can be expressed as inequalities in the semiring of counters using the Hadamard product, and the optimal inputs are obtained by solving fixed-point problems.

CHAPTER 8 is focused on the case of PS restrictions that can change while the system is running. As before, we present a formal and systematic method to optimally update the inputs by computing fixed points of certain mappings.

CHAPTER 9 presents a comparison of the results from Chapters 7 and 8 with selected related work from the literature.

Part IV — SYSTEMS WITH SHARED RESOURCES AND PARTIAL SYNCHRONIZATION

CHAPTER 10 merges the methods from Chapters 4 and 7 into a unified framework, capable of dealing with systems exhibiting both resource-sharing and PS phenomena. All the results from these two chapters can be adapted to this more general setting.

Part I

PRELIMINARIES

2 ALGEBRAIC SETTING

The main purpose of this chapter is to make the thesis largely self-contained. We present a summary of some basic definitions and results on idempotent semirings, with particular focus on the min-plus tropical semiring and on the semiring of counters — for a more exhaustive discussion, the reader may refer to [5] — and touch on some topics from residuation theory — see [6].

2.1 IDEMPOTENT SEMIRINGS

Definition 2.1 (IDEMPOTENT SEMIRING)**.** An *idempotent semiring* (or *dioid*) is a set $\mathcal{D}$ endowed with two binary operations, denoted $\oplus$ (*sum*) and $\otimes$ (*product*), such that the following axioms hold:

I. $\oplus$ is associative, commutative, idempotent — i. e., $(\forall a \in \mathcal{D})$ $a \oplus a = a$ — and has a neutral (*zero*) element, denoted ε;

II. $\otimes$ is associative and has a neutral (*unit*) element, denoted e;

III. $\otimes$ distributes over $\oplus$, i. e., $(\forall a, b, c \in \mathcal{D})$ $c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$ and $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$;

IV. the element ε is absorbing for $\otimes$, i. e., $(\forall a \in \mathcal{D})$ $a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$.

Remark 2.1. As in conventional algebra, the product symbol $\otimes$ is often omitted. Throughout this thesis, we assume that the product has precedence over all other binary operations in a dioid. More precisely, for any binary operator $\circledast$ on $\mathcal{D}$ and for all $a, b, c, d \in \mathcal{D}$, an expression like $ab \circledast cd$ is to be read $(a \otimes b) \circledast (c \otimes d)$. Note that this includes operators resulting from residuation (see Section 2.5). ◇

CANONICAL ORDER A canonical (partial) order relation on an idempotent semiring $\mathcal{D}$ can be defined by

$$(\forall a, b \in \mathcal{D})\ a \preceq b \Leftrightarrow a \oplus b = b\,. \tag{2.1}$$

Note that ε is the bottom element of $\mathcal{D}$, as $(\forall a \in \mathcal{D})\ \varepsilon \preceq a$.

Remark 2.2 (INFINITE SUMS). Infinite sums in an idempotent semiring are defined as the *supremum* (or *least upper bound*) with respect to the canonical order.

Definition 2.2 (COMPLETE IDEMPOTENT SEMIRING)**.** An idempotent semiring $\mathcal{D}$ is *complete* if it is closed for infinite sums — i. e., $(\forall \mathcal{X} \subseteq \mathcal{D})$

$\bigoplus_{x\in\mathcal{X}} x \in \mathcal{D}$ — and if the product distributes over infinite sums — i.e., $(\forall a \in \mathcal{D}, \forall \mathcal{X} \subseteq \mathcal{D})$

$$a \otimes \Big(\bigoplus_{x\in\mathcal{X}} x\Big) = \bigoplus_{x\in\mathcal{X}} (a \otimes x) \quad \text{and} \quad \Big(\bigoplus_{x\in\mathcal{X}} x\Big) \otimes a = \bigoplus_{x\in\mathcal{X}} (x \otimes a)\,.$$

TOP ELEMENT In a complete idempotent semiring $\mathcal{D}$, the *top element* is defined as $\top = \bigoplus_{x\in\mathcal{D}} x$. Clearly, from the definition of order (2.1) it follows that $(\forall a \in \mathcal{D})\, \top \succeq a$. Furthermore, as a consequence of axiom IV from Def. 2.1 and of the distributivity of $\otimes$ over infinite sums, one has that $\top \otimes \varepsilon = \varepsilon \otimes \top = \varepsilon$.

$\wedge$-OPERATOR In a complete idempotent semiring $\mathcal{D}$ with canonical order $\preceq$, the *greatest lower bound* (or *infimum*) operation, denoted $\wedge$, is defined by

$$(\forall a, b \in \mathcal{D}) \;\; a \wedge b = \bigoplus \{x \in \mathcal{D} \,|\, x \preceq a \text{ and } x \preceq b\}\,.$$

Operation $\wedge$ is associative, commutative, idempotent, and has $\top$ as neutral element. Moreover, for all $a, b \in \mathcal{D}$ the following equivalences hold:

$$a \oplus b = b \;\Leftrightarrow\; a \preceq b \;\Leftrightarrow\; a \wedge b = a\,. \tag{2.2}$$

Remark 2.3 (MATRIX DIOID; [5]). The set of $n \times n$-matrices with entries in a complete idempotent semiring $\mathcal{D}$, endowed with sum and product operations defined by

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}\,,$$
$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} (A_{ik} \otimes B_{kj})\,,$$

for all $i, j \in \{1, \ldots, n\}$, forms a complete idempotent semiring denoted $\mathcal{D}^{n\times n}$. Its unit element (or identity matrix) is the $n \times n$-matrix with entries equal to e on the main diagonal and ε elsewhere; the zero (resp. top) element is the $n \times n$-matrix with all entries equal to ε (resp. $\top$). The definition of order (2.1) implies, for any $A, B \in \mathcal{D}^{n\times n}$,

$$A \preceq B \;\Leftrightarrow\; (\forall i, j \in \{1, \ldots, n\})\, A_{ij} \preceq B_{ij}\,.$$

It is possible to deal with nonsquare matrices in this context by suitably padding them with ε-rows or columns; this is done only implicitly, as it does not interfere with the relevant parts of the results of operations between matrices. ◇

MATRIX NOTATION Throughout this thesis, we shall denote the i^{th} row and the j^{th} column of a matrix A by $A_{[i\cdot]}$ and $A_{[\cdot j]}$, respectively. In the case of row or column vectors, i.e., $a \in \mathcal{D}^{1\times n}$ or $a \in \mathcal{D}^{n\times 1}$ with $n \geq 2$, we denote the i^{th} entry simply by a_i.

KLEENE STAR OPERATOR In a complete idempotent semiring $\mathcal{D}$, the Kleene star operator on $a \in \mathcal{D}$ is defined as $a^* = \bigoplus_{i \geq 0} a^i$, with $a^0 = e$ and $a^i = a^{i-1} \otimes a$ for $i > 0$. Note that this unary operator has precedence over all binary operators on $\mathcal{D}$, including $\otimes$; for instance, for any $a, b \in \mathcal{D}$, the expressions ab^* and a^*b are to be read $a \otimes (b^*)$ and $(a^*) \otimes b$, respectively.

Remark 2.4 ([5]). The implicit equation $x = ax \oplus b$ over a complete idempotent semiring admits $x = a^*b$ as least solution. This applies, in particular, in the case $x, b \in \mathcal{D}^n$ and $a \in \mathcal{D}^{n \times n}$ (cf. Remark 2.3). Moreover, if x is a solution of $x = ax \oplus b$, then $x = a^*x$. ◇

2.2 THE MIN-PLUS TROPICAL SEMIRING

The set $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$, with the (standard) *minimum* operation as $\oplus$ and conventional addition as $\otimes$, forms the *min-plus tropical semiring* (or *min-plus algebra*), denoted $\overline{\mathbb{Z}}_{\min}$. Note that the canonical order $\preceq$ of $\overline{\mathbb{Z}}_{\min}$ is reversed with respect to the conventional order $\leq$ over $\mathbb{Z}$. For instance, we have $2 \oplus 5 = 2$, so $5 \preceq 2$; in general, $(\forall a, b \in \overline{\mathbb{Z}}_{\min})$ $a \preceq b \Leftrightarrow b \leq a$.

Taking $\varepsilon = +\infty$ and $e = 0$, it is straightforward to check that axioms I–IV from Def. 2.1 are obeyed, i. e., $\overline{\mathbb{Z}}_{\min}$ is an idempotent semiring. In fact, one can verify that $\overline{\mathbb{Z}}_{\min}$ is complete, as the properties from Def. 2.2 hold:

$$(\forall \mathcal{X} \subseteq \overline{\mathbb{Z}}_{\min}) \quad \bigoplus_{x \in \mathcal{X}} x \;=\; \left(\min_{x \in \mathcal{X}} x\right) \in \overline{\mathbb{Z}}_{\min}\,,$$

and also $(\forall a \in \overline{\mathbb{Z}}_{\min}, \forall \mathcal{X} \subseteq \overline{\mathbb{Z}}_{\min})$

$$a \otimes \Big(\bigoplus_{x \in \mathcal{X}} x\Big) = a + \left(\min_{x \in \mathcal{X}} x\right) = \min_{x \in \mathcal{X}} \,(a + x) = \bigoplus_{x \in \mathcal{X}} (a \otimes x)$$

and

$$\Big(\bigoplus_{x \in \mathcal{X}} x\Big) \otimes a = \left(\min_{x \in \mathcal{X}} x\right) + a = \min_{x \in \mathcal{X}} \,(x + a) = \bigoplus_{x \in \mathcal{X}} (x \otimes a)\,.$$

Recall from Remark 2.2 that, in the case $\mathcal{X}$ is an infinite subset of $\overline{\mathbb{Z}}_{\min}$, $\bigoplus_{x \in \mathcal{X}} x$ corresponds to the supremum of $\mathcal{X}$ with respect to the canonical order of $\overline{\mathbb{Z}}_{\min}$, which, in turn, amounts to the infimum with respect to the conventional order over $\mathbb{Z}$. For example, consider the set $\mathcal{X} = \{x \in \overline{\mathbb{Z}}_{\min} \setminus \{-\infty, +\infty\} \mid x \succeq 3\}$, which, in standard algebra, reads as $\mathcal{X} = \{x \in \mathbb{Z} \mid x \leq 3\}$; in $\overline{\mathbb{Z}}_{\min}$ we have $\bigoplus_{x \in \mathcal{X}} x = -\infty$, which is the infimum of $\mathcal{X}$ in standard algebra.

The top element of $\overline{\mathbb{Z}}_{\min}$ is

$$\top \;=\; \bigoplus_{x \in \overline{\mathbb{Z}}_{\min}} x \;=\; \min_{x \in \overline{\mathbb{Z}}} x \;=\; -\infty\,,$$

and $\wedge$ corresponds to the standard *maximum* operation — or, if applied over infinite sets, the *supremum* with respect to the conventional order over $\mathbb{Z}$.

Remark 2.5. As $\overline{\mathbb{Z}}_{\min}$ will be the underlying tropical semiring in the context of this thesis, the symbols ε, e, and $\top$ shall henceforth refer to the corresponding elements in $\overline{\mathbb{Z}}_{\min}$. ◇

2.3 SEMIRING OF COUNTERS

FORMAL POWER SERIES A *formal power series* in δ with coefficients in $\overline{\mathbb{Z}}_{\min}$ and exponents in $\overline{\mathbb{Z}}$ is defined by

$$s = \bigoplus_{t \in \overline{\mathbb{Z}}} s(t)\delta^t \, .$$

In this thesis, the coefficients $s(t)$ of a series will refer to the accumulated number of occurrences of certain events up to (but not including) time t. This is illustrated by the following example.

Example 2.1. Suppose the first occurrence of a given event takes place at time 3, then the next three occurrences happen at time 5, and the last two occurrences are at time 12; the corresponding series σ will be

$$\sigma = \bigoplus_{t=-\infty}^{3} e\delta^t \oplus 1\delta^4 \oplus 1\delta^5 \oplus \bigoplus_{t=6}^{12} 4\delta^t \oplus \bigoplus_{t=13}^{+\infty} 6\delta^t \, .$$

◇

The series on which we shall focus, therefore, clearly have nonincreasing coefficients (in the order of $\overline{\mathbb{Z}}_{\min}$, which, as pointed out before, is the reverse of the standard order of $\mathbb{Z}$), meaning $s(t-1) \succeq s(t)$ for all t.

Σ — THE SEMIRING OF COUNTERS The set of nonincreasing formal power series in δ with coefficients in $\overline{\mathbb{Z}}_{\min}$ and exponents in $\overline{\mathbb{Z}}$, with addition and multiplication defined by

$$s \oplus s' = \bigoplus_{t \in \overline{\mathbb{Z}}} (s(t) \oplus s'(t))\delta^t \, ,$$
$$s \otimes s' = \bigoplus_{t \in \overline{\mathbb{Z}}} \Big(\bigoplus_{\tau \in \overline{\mathbb{Z}}} (s(\tau) \otimes s'(t-\tau)) \Big) \delta^t \, ,$$

forms a complete idempotent semiring, denoted Σ. It has

- zero element s_ε given by $s_\varepsilon(t) = \varepsilon$ for all t;
- unit element s_e given by $s_e(t) = \begin{cases} e & \text{if } t \leq 0 \, , \\ \varepsilon & \text{if } t > 0 \, ; \end{cases}$
- top element $s_\top$ given by $s_\top(t) = \top$ for all t.

We refer to elements of Σ as *counters*.

It is easy to see that s_ε, s_e, respectively $s_\top$ are indeed the zero, unit, respectively top elements in Σ: $\forall s \in \Sigma$, $\forall t \in \overline{\mathbb{Z}}$,

$$\begin{aligned}(s \oplus s_\varepsilon)(t) &= s(t) \oplus s_\varepsilon(t) = s(t)\,;\\ (s \otimes s_e)(t) &= \bigoplus_{\tau \in \overline{\mathbb{Z}}} s(\tau) \otimes s_e(t-\tau)\\ &= \bigoplus_{\tau \geq t} s(\tau)\\ &= s(t) \quad \text{(as } s \text{ is nonincreasing)}\,;\\ (s \oplus s_\top)(t) &= s(t) \oplus s_\top(t) = \top\,.\end{aligned}$$

Note that the order in Σ is induced by the canonical order in $\overline{\mathbb{Z}}_{\min}$, i. e., for all $s, s' \in \Sigma$,

$$s \preceq s' \Leftrightarrow (\forall t \in \overline{\mathbb{Z}})\, s(t) \preceq s'(t)\,.$$

Remark 2.6 ([5]). The fact that multiplication $\otimes$ is (obviously) commutative in $\overline{\mathbb{Z}}_{\min}$ implies it is also commutative in Σ. ◇

COMPACT NOTATION FOR COUNTERS Counters can be represented compactly by omitting terms $s(t)\delta^t$ whenever $s(t) = s(t+1)$. For instance, counter σ from Example 2.1 can be written compactly as $\sigma = e\delta^3 \oplus 1\delta^5 \oplus 4\delta^{12} \oplus 6\delta^{+\infty}$, as illustrated in Fig. 2.1. Recall that in Example 2.1 we associated the coefficients of counter σ with the accumulated number of occurrences of a certain event, as will be done throughout this thesis. Then, one can notice that the δ-exponents of the terms appearing explicitly in the compact notation denote the time instants at which that event occurs. More precisely, in the compact notation of a counter, consecutive terms $a\delta^\tau \oplus b\delta^\lambda$ mean that $b - a$ occurrences of the associated event take place at time τ.

It is also common to omit terms with ε-coefficients. For instance, for any $\tau \in \overline{\mathbb{Z}}$, the counter with coefficients equal to e for $t \leq \tau$ and ε for $t > \tau$ is simply denoted by $e\delta^\tau$ — in particular, the unit element s_e defined above can be written $s_e = e\delta^0$.

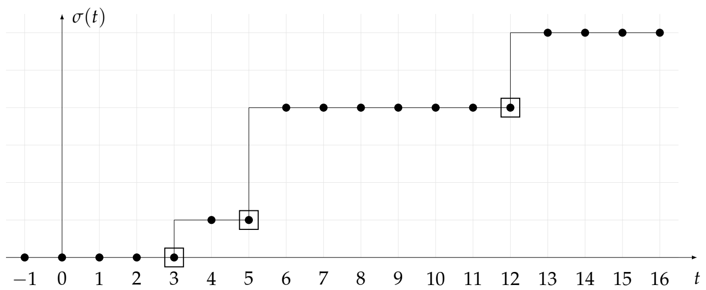

Figure 2.1: Graphical representation of counter σ from Example 2.1, which can be written compactly as $\sigma = e\delta^3 \oplus 1\delta^5 \oplus 4\delta^{12} \oplus 6\delta^{+\infty}$. The squares in the graph mark the terms appearing in the compact notation, i. e., when omitting redundant terms $\sigma(t)\delta^t$ such that $\sigma(t) = \sigma(t+1)$.

Remark 2.7. Note that, with $\tau > 0$, for any $s \in \Sigma$ we have

$$\begin{aligned}(s \otimes e\delta^{\tau})(t) &= \bigoplus_{t' \geq t-\tau} s(t') \otimes e \;\oplus\; \bigoplus_{t' < t-\tau} s(t') \otimes \varepsilon \\ &= \bigoplus_{t' \geq t-\tau} s(t') \\ &= s(t-\tau) \qquad \text{(as } s \text{ is nonincreasing)}\end{aligned}$$

for all $t \in \overline{\mathbb{Z}}$, i. e., multiplication by the counter $e\delta^{\tau}$ can be seen as a backward shift operation — a delay — by τ time units. For example, multiplying counter σ from Example 2.1 by $e\delta^3$ results in (using the compact notation introduced above) $\sigma \otimes e\delta^3 = e\delta^6 \oplus 1\delta^8 \oplus 4\delta^{15} \oplus 6\delta^{+\infty}$, which graphically has the effect of sliding all the dots (and squares, of course) in Fig. 2.1 three units to the right. ◇

2.4 FIXED POINTS OF ISOTONE MAPPINGS

ISOTONE MAPPINGS A mapping $\Pi : \mathcal{D} \to \mathcal{C}$, with $\mathcal{D}$ and $\mathcal{C}$ two idempotent semirings, is *isotone* if $(\forall a, b \in \mathcal{D})\, a \preceq b \Rightarrow \Pi(a) \preceq \Pi(b)$.

Remark 2.8. The composition of two isotone mappings is again an isotone mapping. In fact, if $\Pi_1 : \mathcal{X} \to \mathcal{C}$ and $\Pi_2 : \mathcal{D} \to \mathcal{X}$ are two isotone mappings, with $\mathcal{D}$, $\mathcal{C}$, and $\mathcal{X}$ idempotent semirings, then for any $a, b \in \mathcal{D}$ we have

$$a \preceq_{\mathcal{D}} b \;\Rightarrow\; \Pi_2(a) \preceq_{\mathcal{X}} \Pi_2(b) \;\Rightarrow\; \Pi_1\big(\Pi_2(a)\big) \preceq_{\mathcal{C}} \Pi_1\big(\Pi_2(b)\big)\,,$$

where $\preceq_{\Xi}$ denotes the canonical order on semiring Ξ, showing that the mapping $\Pi_1 \circ \Pi_2 : \mathcal{D} \to \mathcal{C}$ is isotone. ◇

Remark 2.9 ([22]). Let Π be an isotone mapping over a complete idempotent semiring $\mathcal{D}$, and let $\mathcal{Y} = \{x \in \mathcal{D} \,|\, \Pi(x) = x\}$ be the set of fixed points of Π. It follows that $\bigwedge_{y \in \mathcal{Y}} y$ (resp. $\bigoplus_{y \in \mathcal{Y}} y$) is the least (resp. greatest) fixed point of Π. ◇

Remark 2.10. Algorithms exist which allow to compute the least and greatest fixed points of isotone mappings over complete idempotent semirings. In particular, the algorithm presented in [22] is applicable to the relevant mappings considered in this thesis. ◇

2.5 RESIDUATION THEORY

Residuation theory provides, under certain conditions, greatest (resp. least) solutions to inequalities such as $f(x) \preceq b$ (resp. $f(x) \succeq b$).

Definition 2.3. An isotone mapping $f : \mathcal{D} \to \mathcal{C}$, with $\mathcal{D}$ and $\mathcal{C}$ complete idempotent semirings, is said to be residuated if for all $y \in \mathcal{C}$ there exists a greatest solution to the inequality $f(x) \preceq y$. This greatest solution is denoted $f^{\sharp}(y)$, and the mapping $f^{\sharp} : \mathcal{C} \to \mathcal{D}$, $y \mapsto \bigoplus\{x \in \mathcal{D} \,|\, f(x) \preceq y\}$, is called the *residual* of f.

Mapping f is said to be dually residuated if for all $y \in \mathcal{C}$ there exists a least solution to the inequality $f(x) \succeq y$. This least solution is denoted $f^\flat(y)$, and the mapping $f^\flat : \mathcal{C} \to \mathcal{D}$, $y \mapsto \bigwedge\{x \in \mathcal{D} \,|\, f(x) \succeq y\}$, is called the *dual residual* of f.

Note that, if equality $f(x) = y$ is solvable, $f^\sharp(y)$ yields its greatest solution (provided mapping f is residuated, understood). Similarly, as long as f is dually residuated, the least solution is given by $f^\flat(y)$.

Theorem 2.1 ([6]). *Mapping f as in Def. 2.3 is residuated if and only if there exists a unique isotone mapping $f^\sharp : \mathcal{C} \to \mathcal{D}$ such that $(\forall y \in \mathcal{C})\, f\big(f^\sharp(y)\big) \preceq y$ and $(\forall x \in \mathcal{D})\, f^\sharp\big(f(x)\big) \succeq x$.*

Remark 2.11. Let $\mathcal{D}$ be a complete idempotent semiring. For any $a \in \mathcal{D}$, mapping

$$\begin{aligned} L_a : \mathcal{D} &\to \mathcal{D} \\ x &\mapsto a \otimes x \end{aligned}$$

is residuated. Its residual is denoted by $L_a^\sharp(y) = a \backslash\!\!\circ\, y$ ($\backslash\!\!\circ$ is the "left-division" operator). This applies, in particular, to the matrix case: for $A \in \mathcal{D}^{n\times m}$, mapping

$$\begin{aligned} L_A : \mathcal{D}^{m\times p} &\to \mathcal{D}^{n\times p} \\ X &\mapsto A \otimes X \end{aligned}$$

is residuated; $L_A^\sharp(Y) = A \backslash\!\!\circ\, Y \in \mathcal{D}^{m\times p}$ can be computed as follows: for $1 \le i \le m$ and $1 \le j \le p$,

$$(A \backslash\!\!\circ\, Y)_{ij} = \bigwedge_{k=1}^{n} A_{ki} \backslash\!\!\circ\, Y_{kj}\,.$$

The reader should keep in mind that, as stated in Remark 2.1, throughout this document we assume that $\otimes$ has precedence over $\backslash\!\!\circ$, so that, for any $a, b, c, d \in \mathcal{D}$, an expression like $ab \backslash\!\!\circ\, cd$ is to be read as $(a \otimes b) \backslash\!\!\circ\, (c \otimes d)$. ◇

2.6 THE HADAMARD PRODUCT OF COUNTERS

Definition 2.4 ([21]). The Hadamard product of $s_1, s_2 \in \Sigma$, written $s_1 \odot s_2$, is the counter defined as follows:

$$(\forall t \in \overline{\mathbb{Z}}) \;\; (s_1 \odot s_2)(t) = s_1(t) \otimes s_2(t)\,.$$

Remark 2.12. The Hadamard product is associative, commutative, distributes over $\oplus$ and $\wedge$, has neutral element $e\delta^{+\infty}$, and s_ε is absorbing for it (i.e., $(\forall s \in \Sigma)\, s \odot s_\varepsilon = s_\varepsilon$). ◇

Proposition 2.2 ([21]). *For any $a \in \Sigma$, the mapping $\Pi_a : \Sigma \to \Sigma$, $x \mapsto a \odot x$, is residuated. For any $b \in \Sigma$, the counter $\Pi_a^\sharp(b)$, denoted $b \odot^\sharp a$, is the greatest $x \in \Sigma$ such that $a \odot x \preceq b$.*

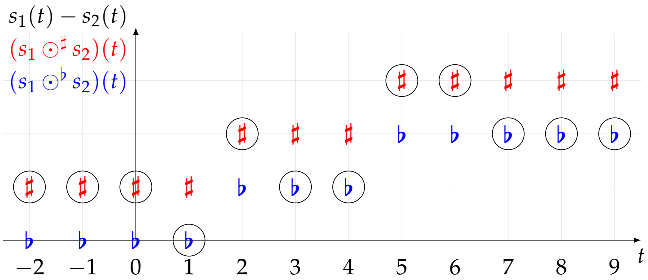

Figure 2.2: Graphical illustration of $s =$ "$s_1 - s_2$" $\notin \Sigma$ (points marked with ◯) in comparison with $s_1 \odot^\sharp s_2$ (denoted with $\sharp$) and $s_1 \odot^\flat s_2$ (denoted with $\flat$), where $s_1 = 1\delta^1 \oplus 3\delta^4 \oplus 5\delta^{+\infty}$ and $s_2 = e\delta^0 \oplus 1\delta^2 \oplus 2\delta^6 \oplus 3\delta^{+\infty}$.

Proposition 2.3 ([52]). *For $a \in \Sigma$, let $\mathcal{D}_a = \{x \in \Sigma \,|\, x = s_\varepsilon$ if $\exists t \in \mathbb{Z}$ with $a(t) = -\infty\}$ and $\mathcal{C}_a = \{y \in \Sigma \,|\, (\forall t \in \mathbb{Z})\, a(t) \in \{-\infty, +\infty\} \Rightarrow y(t) = +\infty\}$. The mapping $\Pi_a : \mathcal{D}_a \to \mathcal{C}_a$, $x \mapsto a \odot x$ is dually residuated for any $a \in \Sigma$. Its dual residual is denoted by $\Pi_a^\flat(y) = y \odot^\flat a$ and corresponds to the least $x \in \Sigma$ that satisfies $a \odot x \succeq y$.*

Remark 2.13. Given two counters $s_1, s_2 \in \Sigma$, the series s defined by $(\forall t \in \overline{\mathbb{Z}})\, s(t) = s_1(t) - s_2(t)$ is not necessarily a counter; $s_1 \odot^\sharp s_2$ is the greatest counter less than or equal to s (in the sense of a coefficient-wise order like that of Σ). Similarly, provided the conditions from Prop. 2.3 are met, $s_1 \odot^\flat s_2$ is the least counter greater than or equal to s. These ideas are illustrated in Example 2.2. ◇

Example 2.2. Consider the counters $s_1 = 1\delta^1 \oplus 3\delta^4 \oplus 5\delta^{+\infty}$ and $s_2 = e\delta^0 \oplus 1\delta^2 \oplus 2\delta^6 \oplus 3\delta^{+\infty}$. The series s with $s(t) = s_1(t) - s_2(t)$ for all t is given by

$$\Sigma \not\ni s = \bigoplus_{t=-\infty}^{0} 1\delta^t \oplus e\delta^1 \oplus 2\delta^2 \oplus 1\delta^3 \oplus 1\delta^4 \oplus 3\delta^5 \oplus 3\delta^6 \oplus \bigoplus_{t=7}^{+\infty} 2\delta^t\,,$$

whereas we have the counters $s_1 \odot^\sharp s_2 = 1\delta^1 \oplus 2\delta^4 \oplus 3\delta^{+\infty}$ and $s_1 \odot^\flat s_2 = e\delta^1 \oplus 1\delta^4 \oplus 2\delta^{+\infty}$. The comparison among series s and counters $s_1 \odot^\sharp s_2$ and $s_1 \odot^\flat s_2$ is graphically illustrated in Fig. 2.2. One can see that $s_1 \odot^\sharp s_2$ is the closest counter approximation of s from below in the sense of a coefficient-wise order like that of Σ (or from above, in the graphical sense); similarly, $s_1 \odot^\flat s_2$ is the closest counter approximation of s from above in the sense of a coefficient-wise order like that of Σ (or from below, in the graphical sense). ◇

3 TIMED EVENT GRAPHS — MODELING AND CONTROL IN THE SEMIRING OF COUNTERS

3.1 MODELING OF TEGS IN THE SEMIRING OF COUNTERS

TIMED PETRI NETS A timed Petri net is a tuple (P, T, A, w, h, v), where P is a finite set of places (graphically represented by circles), T a finite set of transitions (represented by bars), $A \subseteq (P \times T) \cup (T \times P)$ a set of arcs connecting places to transitions and transitions to places, w a weight function assigning a positive integer weight to every arc, and h a function assigning a nonnegative holding time to each place. In the following, holding times will be restricted to be integers. Furthermore, the function v assigns to each place a nonnegative integer number of tokens residing initially in this place. For any $p \in P$ and $t \in T$, if $(p, t) \in A$, we say that p is an upstream place of t, and t is a downstream transition of p; analogously, if $(t, p) \in A$, t is said to be an upstream transition of p, and p is a downstream place of t.

The dynamics of a timed Petri net is governed by the following rules: (i) a transition t can fire if all its upstream places p contain at least $w\big((p,t)\big)$ tokens that have resided there for at least $h(p)$ time units; (ii) if a transition t fires, it removes $w\big((p,t)\big)$ tokens from each of its upstream places p and deposits $w\big((t,\bar{p})\big)$ tokens in each of its downstream places $\bar{p}$. We assume that initial tokens in a place p have been residing in that place for an infinite amount of time, meaning they immediately (i. e., as soon as the system starts evolving) contribute to the firing of downstream transitions of p, regardless of the value of $h(p)$.

TIMED EVENT GRAPHS Timed event graphs (TEGs) are timed Petri nets in which each place has exactly one upstream and one downstream transition and all arcs have weight 1. In a TEG, we can distinguish input transitions (those that have no upstream place), output transitions (those that have no downstream place), and internal transitions (those that have at least one upstream and one downstream place). We typically denote input, output, and internal transitions respectively by the symbols u, y, and x, with appropriate sub- or superscripts depending on the context. Fig. 3.1 shows an example of a TEG, with input transitions u_1 and u_2, output transition y, and internal transitions x_1, x_2, and x_3.

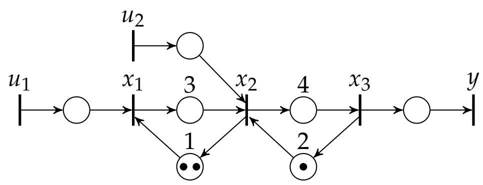

Figure 3.1: A TEG with two inputs u_1 and u_2, a single output y, and three internal transitions x_1, x_2, and x_3.

EARLIEST FIRING RULE Throughout this thesis, we shall assume that TEGs operate under the *earliest firing rule*, which states that every internal and output transition always fires as soon as it is enabled.

TEG DYNAMICS IN Σ With each transition x_i, we associate a sequence $\{x_i(t)\}_{t\in\overline{\mathbb{Z}}}$, for simplicity denoted by the same symbol, where $x_i(t)$ represents the accumulated number of firings of x_i up to time t. Similarly, we associate sequences $\{u_j(t)\}_{t\in\overline{\mathbb{Z}}}$ and $\{y(t)\}_{t\in\overline{\mathbb{Z}}}$ with transitions u_j and y, respectively. By inspection of Fig. 3.1, one can see that, at any time t, $x_1(t)$ cannot exceed the minimum between $u_1(t)$ and $x_2(t-1)+2$. This can be expressed in $\overline{\mathbb{Z}}_{\min}$ as

$$(\forall t \in \overline{\mathbb{Z}}) \;\; x_1(t) \succeq u_1(t) \oplus 2x_2(t-1)\,. \tag{3.1}$$

Under the earliest firing rule, inequality (3.1) turns into equality and, through the δ-transform, can be written in Σ as

$$x_1 = u_1 \oplus 2\delta^1 x_2\,.$$

We can obtain similar relations for x_2, x_3, and y; then, defining the vectors

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

we can write

$$\begin{aligned} x &= \begin{bmatrix} s_\varepsilon & 2\delta^1 & s_\varepsilon \\ e\delta^3 & s_\varepsilon & 1\delta^2 \\ s_\varepsilon & e\delta^4 & s_\varepsilon \end{bmatrix} x \oplus \begin{bmatrix} e\delta^0 & s_\varepsilon \\ s_\varepsilon & e\delta^0 \\ s_\varepsilon & s_\varepsilon \end{bmatrix} u\,, \\ y &= \begin{bmatrix} s_\varepsilon & s_\varepsilon & e\delta^0 \end{bmatrix} x\,. \end{aligned}$$

In general, a TEG can be described by implicit equations over Σ of the form

$$\begin{aligned} x &= Ax \oplus Bu\,, \\ y &= Cx\,. \end{aligned} \tag{3.2}$$

TRANSFER RELATIONS From Remark 2.4, the least solution of (3.2) is given by

$$x = A^*Bu \qquad \text{and} \qquad y = CA^*Bu\,. \tag{3.3}$$

We denote

$$\mathcal{F} = A^*B \qquad \text{and} \qquad \mathcal{G} = CA^*B\,, \tag{3.4}$$

where $\mathcal{G}$ is often called the (input-output) *transfer matrix* — or, in the case of a single input and a single output, *transfer function* — of the system. For instance, for the TEG from Fig. 3.1, we obtain

$$\mathcal{F} = \begin{bmatrix} e\delta^0 \oplus 2\delta^4(1\delta^6)^* & 2\delta^1(1\delta^6)^* \\ e\delta^3(1\delta^6)^* & (1\delta^6)^* \\ e\delta^7(1\delta^6)^* & e\delta^4(1\delta^6)^* \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} e\delta^7(1\delta^6)^* & e\delta^4(1\delta^6)^* \end{bmatrix}. \tag{3.5}$$

3.2 OPTIMAL CONTROL OF TEGS

Assume that a TEG to be controlled is modeled by equations (3.2) and that an output-reference $z \in \Sigma$ is given. Under the just-in-time paradigm, we aim at firing the input transitions the least possible number of times while guaranteeing that the output transition fires, by each time instant, at least as many times as specified by z. In other words, we seek the greatest (in the order of $\overline{\mathbb{Z}}_{\min}$) input (vector) u such that $y = \mathcal{G}u \preceq z$. Based on (3.3) and Remark 2.11, the solution is directly obtained by

$$u_{\text{opt}} = \mathcal{G} \between z\,. \tag{3.6}$$

Example 3.1. For the TEG from Fig. 3.1, suppose it is required that the accumulated number of firings of y be e $(= 0)$ for $t < 14$, 1 for $14 \leq t < 23$, 3 for $23 \leq t < 29$, and 4 for $t \geq 29$. In other words, one firing is required by time 14, then two more by time 23, and finally one more by time 29. This can be represented by the output-reference

$$z = e\delta^{14} \oplus 1\delta^{23} \oplus 3\delta^{29} \oplus 4\delta^{+\infty}\,.$$

Applying (3.6), we obtain the just-in-time input

$$u_{\text{opt}} = \begin{bmatrix} u_{1_{\text{opt}}} \\ u_{2_{\text{opt}}} \end{bmatrix} = \begin{bmatrix} e\delta^4 \oplus 1\delta^{10} \oplus 2\delta^{16} \oplus 3\delta^{22} \oplus 4\delta^{+\infty} \\ e\delta^7 \oplus 1\delta^{13} \oplus 2\delta^{19} \oplus 3\delta^{25} \oplus 4\delta^{+\infty} \end{bmatrix},$$

and the corresponding optimal output is

$$y_{\text{opt}} = \mathcal{G}u_{\text{opt}} = e\delta^{11} \oplus 1\delta^{17} \oplus 2\delta^{23} \oplus 3\delta^{29} \oplus 4\delta^{+\infty}\,.$$

One can easily verify that indeed $y_{\text{opt}} \preceq z$, as illustrated in Fig. 3.2. Note that, as "tracking the reference" means y approximates z from below in the sense of the canonical order in $\overline{\mathbb{Z}}_{\min}$, which is reversed with respect to the standard order, in a graphical sense (as in Fig. 3.2) this means the approximation happens from above. ◇

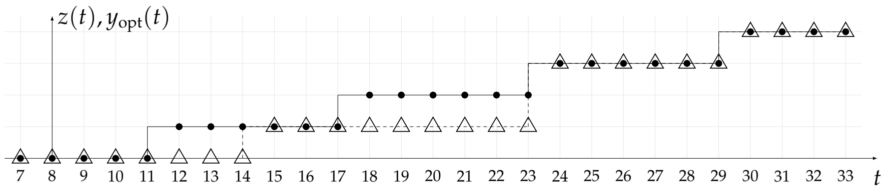

Figure 3.2: Tracking of reference z (points marked with $\triangle$) by the optimal output y_{opt} (marked with $\bullet$) obtained in Example 3.1.

3.3 CONTROL OF TEGS WITH OUTPUT-REFERENCE UPDATE

The material of this section is a dual version, adapted to the point of view of counters, of the results from [34].

In practice, it may be necessary to update the reference for the output of a system during run-time, for instance when customer demand is increased and new production objectives must be taken into account. For a system like the one from Example 3.1, let reference z be updated to a new one, z', at a certain time T. The problem at hand is to find the input u'_{opt} which optimally tracks z' without, however, changing the inputs given up to time T. Define the mapping $r_T : \Sigma \to \Sigma$ such that, for any $s \in \Sigma$, $r_T(s)$ is the counter defined by

$$[r_T(s)](t) = \begin{cases} s(t), & \text{if } t \leq T\,; \\ \varepsilon\,, & \text{if } t > T\,. \end{cases} \tag{3.7}$$

Let us extend the definition to matrices, for simplicity using the same notation: for any matrix $A \in \Sigma^{p \times q}$, r_T is applied entry-wise, i.e., $\left[r_T(A)\right]_{ij} = r_T\big([A]_{ij}\big)$ for any $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$. Our objective can then be restated as follows: find the greatest element u'_{opt} of the set

$$\mathcal{N} = \{u \in \Sigma^{m \times 1} \mid \mathcal{G}u \preceq z' \text{ and } r_T(u) = r_T(u_{\text{opt}})\}\,,$$

where m is the number of input transitions in the system and u_{opt} is the optimal input with respect to reference z, computed as in (3.6). The following theorem provides, given that certain conditions are met, a way to compute the greatest element of $\mathcal{N}$.

Theorem 3.1 ([34]). *Let $\mathcal{D}$ and $\mathcal{C}$ be complete idempotent semirings, $f_1 : \mathcal{D} \to \mathcal{C}$ and $f_2 : \mathcal{D} \to \mathcal{D}$ residuated mappings, $c_1 \in \mathcal{C}$, and $c_2 \in \mathcal{D}$. If the set*

$$\mathcal{S} = \{x \in \mathcal{D} \mid f_1(x) \preceq c_1 \text{ and } f_2(x) = c_2\}$$

is nonempty, we have $\bigoplus_{x \in \mathcal{S}} x = f_1^\sharp(c_1) \wedge f_2^\sharp(c_2)$.

An obvious correspondence between set $\mathcal{S}$ from Theorem 3.1 and set $\mathcal{N}$ can be established by taking $\mathcal{D}$ as $\Sigma^{m \times 1}$, $\mathcal{C}$ as Σ, f_1 as $L_{\mathcal{G}}$ (which is well known to be residuated — see Remark 2.11), c_1 as z', f_2 as r_T, and c_2 as $r_T(u_{\text{opt}})$.

Remark 3.1. Mapping r_T as defined in (3.7) is residuated. Its residual is the mapping $r_T^\sharp : \Sigma \to \Sigma$ such that, for any $s \in \Sigma$, $r_T^\sharp(s)$ is the counter defined by

$$[r_T^\sharp(s)](t) = \begin{cases} s(t), & \text{if } t \leq T; \\ s(T), & \text{if } t > T. \end{cases}$$

In fact, $r_T^\sharp$ is clearly isotone and we have, for any $s \in \Sigma$, $r_T\big(r_T^\sharp(s)\big) = r_T(s) \preceq s$ and $r_T^\sharp\big(r_T(s)\big) = r_T^\sharp(s) \succeq s$, so the conditions from Theorem 2.1 are fulfilled. Mapping $r_T^\sharp$ is applied to matrices entry-wise, the same way as r_T. ◇

Hence, as long as set $\mathcal{N}$ is nonempty, Theorem 3.1 provides the desired solution

$$u'_{\text{opt}} = \mathcal{G} \diagdown\!\!\!\circ z' \wedge r_T^\sharp(u_{\text{opt}}). \tag{3.8}$$

In order to check for nonemptiness of $\mathcal{N}$, let us consider the set

$$\widetilde{\mathcal{N}} = \{u \in \Sigma^{m \times 1} \,|\, r_T(u) = r_T(u_{\text{opt}})\},$$

i. e., the set of counters that up to and including time T are identical to u_{opt}. It is easy to see that the least element of $\widetilde{\mathcal{N}}$ is

$$\underline{u} = \bigwedge_{u \in \widetilde{\mathcal{N}}} u = r_T(u_{\text{opt}}). \tag{3.9}$$

In fact, since $r_T \circ r_T = r_T$ and, therefore, $r_T(\underline{u}) = r_T\big(r_T(u_{\text{opt}})\big) = r_T(u_{\text{opt}})$, we have $\underline{u} \in \widetilde{\mathcal{N}}$. As $L_{\mathcal{G}}$ is isotone, clearly if $\underline{u}$ does not lead to respecting z', then no input such that $r_T(u) = r_T(u_{\text{opt}})$ will. Formally,

$$\mathcal{N} \neq \emptyset \;\Leftrightarrow\; \mathcal{G}\underline{u} \preceq z'. \tag{3.10}$$

In the case $\mathcal{G}\underline{u} \npreceq z'$ (and hence $\mathcal{N} = \emptyset$), this means the past inputs make it impossible for the system to respect z'. Intuitively, having implemented a just-in-time policy u_{opt} for a reference z up to time T may make it impossible to satisfy a more demanding new reference z'. Since the condition $r_T(u) = r_T(u_{\text{opt}})$ cannot be relaxed, in order to have a solution we must then increase z'; more precisely, we wish to find the least counter $z'' \succeq z'$ such that the set

$$\mathcal{N}'' = \{u \in \Sigma^{m \times 1} \,|\, \mathcal{G}u \preceq z'' \text{ and } r_T(u) = r_T(u_{\text{opt}})\} \tag{3.11}$$

is not empty. The following result provides the answer.

Proposition 3.2. *Let $\mathcal{N}''$ be defined as in (3.11) and $\underline{u}$ as in (3.9). The least counter $z'' \succeq z'$ such that $\mathcal{N}'' \neq \emptyset$ is $z'' = z' \oplus \mathcal{G}\underline{u}$.*

Proof. Since $\mathcal{G}\underline{u} \preceq z' \oplus \mathcal{G}\underline{u} = z''$, we have $\underline{u} \in \mathcal{N}''$, therefore $\mathcal{N}'' \neq \emptyset$. Take now an arbitrary $\xi \succeq z'$ such that $\mathcal{N}_\xi \neq \emptyset$ (where $\mathcal{N}_\xi$ is defined like $\mathcal{N}$, only replacing z' with ξ), and take any $v \in \mathcal{N}_\xi$. Clearly $v \in \widetilde{\mathcal{N}}$ and hence $\underline{u} \preceq v$; as $L_{\mathcal{G}}$ is isotone, we have $\mathcal{G}\underline{u} \preceq \mathcal{G}v \preceq \xi$, implying $z'' = z' \oplus \mathcal{G}\underline{u} \preceq z' \oplus \xi = \xi$. □

A correspondence between sets $\mathcal{N}''$ and $\mathcal{S}$ can be established analogously to that between $\mathcal{N}$ and $\mathcal{S}$, only taking c_1 as z'' (instead of z'). Applying Theorem 3.1 and recalling that $r_T^\sharp \circ r_T = r_T^\sharp$, we obtain the optimal input

$$u'_{\text{opt}} = \mathcal{G} \diamond\!\!\!\!\; \backslash (z' \oplus \mathcal{G}\underline{u}) \;\wedge\; r_T^\sharp(u_{\text{opt}}) \,. \tag{3.12}$$

Note that, in the case $\mathcal{N} \neq \emptyset$, we have $z'' = z' \oplus \mathcal{G}\underline{u} = z'$ and therefore recover solution (3.8).

Part II

SYSTEMS WITH SHARED RESOURCES

4

MODELING AND CONTROL OF TEGS WITH SHARED RESOURCES

In this chapter, we turn our attention to systems in which a number of TEGs share one or multiple resources. We first focus on the simple case of a single shared resource (Sections 4.1 and 4.2) and then proceed to generalize the approach to the case of arbitrarily many shared resources (Section 4.3).

A preliminary version of part of the following material has appeared in [41, 44], which reflect original work from — and have as the main author and contributor — the author of this thesis.

The main ideas in Sections 4.1 and 4.2 are largely based on previous results from [36]; here, the method is extended to the case of arbitrarily many shared resources — Section 4.3 is entirely original — and to the case of TEGs with multiple inputs — Section 4.4 originates here.

4.1 MODELING OF TEGS WITH A SINGLE SHARED RESOURCE

Consider a system consisting of K subsystems — modeled as TEGs $S^1, \ldots, S^K$ — sharing a resource with finite but arbitrary capacity, as illustrated in Fig. 4.1. For the sake of clarity of exposition, for the time being we keep the discussion simpler by assuming that each subsystem S^k has only two input transitions, u^{k0} and u^{k1}; the more general case of subsystems with an arbitrary number of input transitions is covered in Section 4.4. The firing of u^{k0} can be thought of, for example, as the provisioning of raw material, whereas u^{k1} represents permission to allocate the resource; as their firing schedules can be freely assigned, these transitions play the role of *control* inputs. We assume u^{k1} to be connected to the resource-allocation transition x_A^k via a place with no holding time and no initial tokens, which translates to the ability of deciding in real time whether or not to grant a subsystem S^k access to the resource. Transitions x_R^k and y^k correspond to resource release and output, respectively.

Block H^k in Fig. 4.1 corresponds to the allocation-release dynamics of S^k. If looked at individually, it can be seen as a single-input single-output TEG, with x_A^k playing the role of the input and x_R^k that of the output transition — note that, in simple cases, H^k may consist of just a single place. Its dynamics can, therefore, be captured by a counter $H^k \in \Sigma$, for simplicity denoted by the same symbol; i. e., $x_R^k = H^k x_A^k$. We assume the initial marking inside block H^k to be such that the first firing of x_R^k cannot occur before the first firing of x_A^k; in counter terms, there exists $\tau \geq 0$ such that $H^k(t) = e$ for all $t \leq \tau$ and hence $H^k \succeq s_e$,

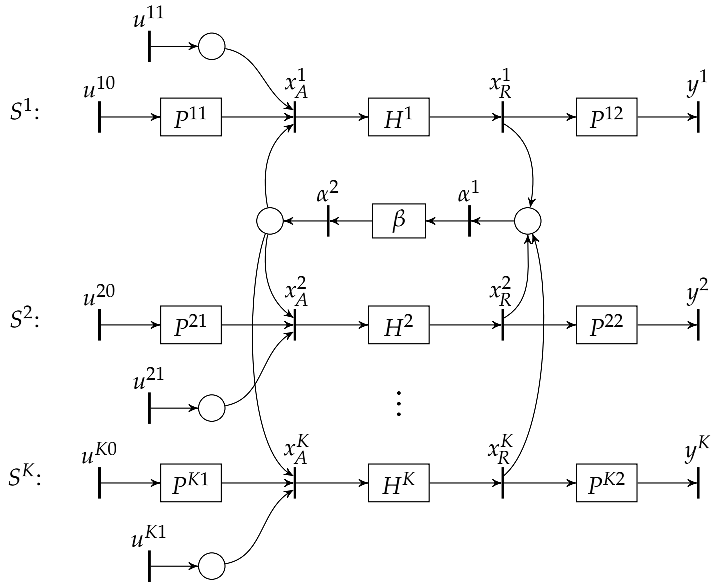

Figure 4.1: TEGs $S^1, \ldots, S^K$ with a shared resource β.

implying $x_R^k = H^k x_A^k \succeq x_A^k$. In particular, in the case block H^k is just a place, this amounts to assuming it contains no initial tokens. This assumption can be intuitively interpreted as there being no work in progress in the system before the first firing of the inputs. Similarly, blocks P^{k1} and P^{k2} correspond to the input-allocation (with respect to input u^{k0}) and release-output dynamics of S^k. Analogously to the case of H^k, we can see P^{k1} (resp. P^{k2}) as a TEG with a single input u^{k0} (resp. x_R^k) and a single output x_A^k (resp. y^k), and we make directly analogous assumptions regarding the initial marking and the firings of u^{k0} and x_A^k (resp. x_R^k and y^k).

In turn, β describes the capacity of the resource as well as the minimum delay between release and allocation events. It may, in general, be modeled as a TEG (or simply a place) with input and output transitions α^1 and α^2, respectively. These two transitions are auxiliary, used to help make some intermediate steps in the general algebraic formulation clearer (see arguments leading to inequality (4.1), below); in concrete examples, they will normally be omitted, so that all resource-release transitions x_R^k (resp. resource-allocation transitions x_A^k) will be directly connected — and hence serve as inputs (resp. outputs) — to β. We assume the resource has non-null capacity and imposes a non-null delay between release and allocation events. Note that, as a consequence of the aforementioned assumptions on the initial marking of all blocks H^k, it follows that the resource is fully available before the first firing of both inputs u^{k0} and u^{k1} of the same subsystem S^k for some $k \in \{1, \ldots, K\}$.

One can easily see that the overall system is not a TEG, as the place connecting all x_R^k to α^1 has $K > 1$ upstream transitions and the place connecting α^2 to all x_A^k has $K > 1$ downstream transitions. Consequently, it is not possible to model the behavior of the whole system by linear equations such as (3.2); in order to express the relationship among transitions (and corresponding counters) x_A^k and x_R^k, $k \in \{1, \dots, K\}$, we need the Hadamard product operation, as explained below.

Recall from Def. 2.4 that the Hadamard product amounts to the coefficient-wise standard sum of counters. From Fig. 4.1 one can see that, at any time instant t, the accumulated number of firings of α^1 cannot exceed (in the conventional sense) that of all resource-release transitions x_R^k combined. The Hadamard product allows us to express this in Σ as $x_R^1 \odot \cdots \odot x_R^K \preceq \alpha^1$. Similarly, the combined accumulated number of firings of all allocation transitions x_A^k can never exceed that of α^2, i. e., $\alpha^2 \preceq x_A^1 \odot \cdots \odot x_A^K$. Since, according to the earliest firing rule, from Fig. 4.1 we have $\alpha^2 = \beta \otimes \alpha^1$, the two inequalities above can be combined into

$$\beta \otimes \Big(\bigodot_{k=1}^{K} x_R^k \Big) \preceq \bigodot_{k=1}^{K} x_A^k \, . \tag{4.1}$$

Condition (4.1) fully captures the restrictions imposed by the dynamics and the finite capacity of the resource on the combined allocation and release schedules of all subsystems. Individually, each subsystem S^k would evolve according to equations (3.2). Collectively, they work under the extra condition that all allocation and release transitions respect (4.1).

4.2 OPTIMAL CONTROL OF TEGS WITH A SINGLE SHARED RESOURCE

For a system like the one from Fig. 4.1, let the input-output behavior of each subsystem S^k, including the resource and ignoring all other subsystems, be described by $y^k = \mathcal{G}^k u^k$, where

$$u^k = \begin{bmatrix} u^{k0} \\ u^{k1} \end{bmatrix} \qquad \text{and} \qquad \mathcal{G}^k = \begin{bmatrix} \mathcal{G}^{k0} & \mathcal{G}^{k1} \end{bmatrix} ;$$

i. e., $\mathcal{G}^k$ is the transfer matrix of S^k in the hypothetical case that no other subsystem requires the joint resource. Assume respective references z^k are given. Our control objective is to obtain just-in-time firing schedules for all inputs u^k with respect to z^k while making sure that the capacity and dynamics of the resource are observed. This means we seek, for each $k \in \{1, \dots, K\}$, the greatest u^k such that $\mathcal{G}^k u^k \preceq z^k$ and also such that the resulting resource-allocation and release schedules satisfy inequality (4.1). It should be noted that, due to the limited capacity of the resource, in general it is not possible for all subsystems to operate optimally with respect to their individual output-references, meaning they cannot achieve the same just-in-time schedule as in the case without resource sharing.

One way to settle the dispute for the resource is introducing a priority policy. We henceforth assume, without loss of generality, that the subsystems are indexed according to their priority level, meaning S^k has higher priority than S^{k+1} for all $k \in \{1, \dots, K-1\}$. We shall then adopt the following policy: for each $k \in \{2, \dots, K\}$ and for all $i \in \{1, \dots, k-1\}$, S^k cannot interfere with the performance of S^i; in other words, lower-priority subsystems cannot compromise the performance of higher-priority ones.

According to the priority policy introduced above, when computing the optimal input for a given subsystem S^k we can effectively neglect all (if any) lower-priority subsystems. In particular, the subsystem with highest priority, S^1, is not affected by this policy, and we can simply compute its optimal input u^1_{opt} by applying the method for a single TEG introduced in Section 3.2, i. e., $u^1_{\text{opt}} = \mathcal{G}^1 \mathbin{\backslash\!\!\!\!\circ}\, z^1$. The corresponding resource-allocation and release schedules are then given by $x^1_{A_{\text{opt}}} = P^{11} u^{10}_{\text{opt}} \oplus u^{11}_{\text{opt}}$ and $x^1_{R_{\text{opt}}} = H^1 x^1_{A_{\text{opt}}}$, respectively.

For S^2, we must compute the optimal input under the restriction that the optimal behavior of S^1 is unaffected; based on (4.1) and neglecting all lower-priority subsystems (i. e., all S^j with $2 < j \leq K$), this means we must respect

$$\beta \otimes (x^1_{R_{\text{opt}}} \odot x^2_R) \preceq x^1_{A_{\text{opt}}} \odot x^2_A\,. \tag{4.2}$$

In fact, we want to determine the greatest u^2 such that $\mathcal{G}^2 u^2 \preceq z^2$ and which also leads to allocation and release schedules (x^k_A and x^k_R, respectively) satisfying (4.2). Let us denote

$$\mathcal{P}^2 = \begin{bmatrix} P^{21} & s_e \end{bmatrix} \in \Sigma^{1\times 2}\,.$$

For any just-in-time input $u^2 = \begin{bmatrix} u^{20} \\ u^{21} \end{bmatrix}$ computed so that (4.2) holds, it follows that $x^2_A = \mathcal{P}^2 u^2$ and $x^2_R = H^2 x^2_A = H^2 \mathcal{P}^2 u^2$. Hence, we can rewrite (4.2) as

$$\beta \otimes (x^1_{R_{\text{opt}}} \odot H^2 \mathcal{P}^2 u^2) \preceq x^1_{A_{\text{opt}}} \odot \mathcal{P}^2 u^2\,,$$

which, in turn, thanks to the fact that left-multiplication is residuated (cf. Remark 2.11), is equivalent to

$$x^1_{R_{\text{opt}}} \odot H^2 \mathcal{P}^2 u^2 \preceq \beta \mathbin{\backslash\!\!\!\!\circ} (x^1_{A_{\text{opt}}} \odot \mathcal{P}^2 u^2)\,. \tag{4.3}$$

At this point, the fact that the Hadamard product is residuated (see Prop. 2.2) comes in handy. Applying the proposition and again Remark 2.11, inequality (4.3) leads to

$$u^2 \preceq H^2 \mathcal{P}^2 \mathbin{\backslash\!\!\!\!\circ} \left[\left(\beta \mathbin{\backslash\!\!\!\!\circ} (x^1_{A_{\text{opt}}} \odot \mathcal{P}^2 u^2) \right) \odot^\sharp x^1_{R_{\text{opt}}} \right]. \tag{4.4}$$

Combining (4.4) with $\mathcal{G}^2 u^2 \preceq z^2$ or, equivalently, with $u^2 \preceq \mathcal{G}^2 \mathbin{\backslash\!\!\!\!\circ} z^2$, we can then write

$$u^2 \preceq H^2 \mathcal{P}^2 \mathbin{\backslash\!\!\!\!\circ} \left[\left(\beta \mathbin{\backslash\!\!\!\!\circ} (x^1_{A_{\text{opt}}} \odot \mathcal{P}^2 u^2) \right) \odot^\sharp x^1_{R_{\text{opt}}} \right] \wedge \mathcal{G}^2 \mathbin{\backslash\!\!\!\!\circ} z^2\,. \tag{4.5}$$

Since for any $s_1, s_2 \in \Sigma$ it holds that $s_1 \preceq s_2 \Leftrightarrow s_1 = s_1 \wedge s_2$ (cf. (2.2)), one can see that (4.5) is equivalent to

$$u^2 \;=\; H^2\mathcal{P}^2 \diagdown\!\!\!\circ \big[\big(\beta \diagdown\!\!\!\circ (x^1_{A_\text{opt}} \odot \mathcal{P}^2 u^2)\big) \odot^\sharp x^1_{R_\text{opt}}\big] \wedge \mathcal{G}^2 \diagdown\!\!\!\circ z^2 \wedge u^2 . \qquad (4.6)$$

Our sought solution u^2_opt is, therefore, the greatest u^2 satisfying (4.6), which can be obtained by computing the greatest fixed point (provided it exists) of the mapping $\Phi^2 : \Sigma^{2\times 1} \to \Sigma^{2\times 1}$,

$$\Phi^2(u^2) \;=\; H^2\mathcal{P}^2 \diagdown\!\!\!\circ \big[\big(\beta \diagdown\!\!\!\circ (x^1_{A_\text{opt}} \odot \mathcal{P}^2 u^2)\big) \odot^\sharp x^1_{R_\text{opt}}\big] \wedge \mathcal{G}^2 \diagdown\!\!\!\circ z^2 \wedge u^2 . \qquad (4.7)$$

Notice that Φ^2 consists in a succession of order-preserving operations (product $\otimes$, Hadamard product $\odot$ and its residual $\odot^\sharp$, left-division $\diagdown\!\!\!\circ$, and infimum $\wedge$), which, in turn, can be seen as the composition of corresponding isotone mappings (for instance, following the notation of Proposition 2.2, $s_1 \odot s_2$ corresponds to $\Pi_{s_1}(s_2)$, and similarly for the other operations). Therefore, according to Remark 2.8, Φ^2 is also isotone; Remark 2.9 then ensures the existence of its greatest fixed point.

The procedure presented above applies to an arbitrary S^k, $k \in \{2, \ldots, K\}$. Again, we must compute the optimal input while guaranteeing that the optimal behavior of all higher-priority subsystems is unaffected, but we can neglect lower-priority subsystems. Based on (4.1), this means we must observe

$$\beta \otimes \Big(\bigodot_{i=1}^{k-1} x^i_{R_\text{opt}} \odot x^k_R \Big) \;\preceq\; \bigodot_{i=1}^{k-1} x^i_{A_\text{opt}} \odot x^k_A . \qquad (4.8)$$

As before, we denote $\mathcal{P}^k = [P^{k1} \;\; s_e]$ and argue that, for a just-in-time input u^k satisfying (4.8), it follows that $x^k_A = \mathcal{P}^k u^k$ and $x^k_R = H^k\mathcal{P}^k u^k$, leading to

$$\beta \otimes \Big(\bigodot_{i=1}^{k-1} x^i_{R_\text{opt}} \odot H^k\mathcal{P}^k u^k \Big) \;\preceq\; \bigodot_{i=1}^{k-1} x^i_{A_\text{opt}} \odot \mathcal{P}^k u^k . \qquad (4.9)$$

The fact that both $\odot$ and $\otimes$ are residuated allows us to manipulate (4.9) and establish an upper bound for u^k, analogous to (4.4):

$$u^k \;\preceq\; H^k\mathcal{P}^k \diagdown\!\!\!\circ \Big[\Big(\beta \diagdown\!\!\!\circ \Big(\bigodot_{i=1}^{k-1} x^i_{A_\text{opt}} \odot \mathcal{P}^k u^k \Big)\Big) \odot^\sharp \bigodot_{i=1}^{k-1} x^i_{R_\text{opt}} \Big] . \qquad (4.10)$$

We seek the greatest input u^k such that $\mathcal{G}^k u^k \preceq z^k$ (i. e., $u^k \preceq \mathcal{G}^k \diagdown\!\!\!\circ z^k$) and that (4.10) holds. Following the same reasoning as before, this optimal solution u^k_opt is the greatest fixed point of mapping $\Phi^k : \Sigma^{2\times 1} \to \Sigma^{2\times 1}$,

$$\Phi^k(u^k) \;=\; H^k\mathcal{P}^k \diagdown\!\!\!\circ \Big[\Big(\beta \diagdown\!\!\!\circ \Big(\bigodot_{i=1}^{k-1} x^i_{A_\text{opt}} \odot \mathcal{P}^k u^k \Big)\Big) \odot^\sharp \bigodot_{i=1}^{k-1} x^i_{R_\text{opt}} \Big] \wedge \mathcal{G}^k \diagdown\!\!\!\circ z^k \wedge u^k . \qquad (4.11)$$

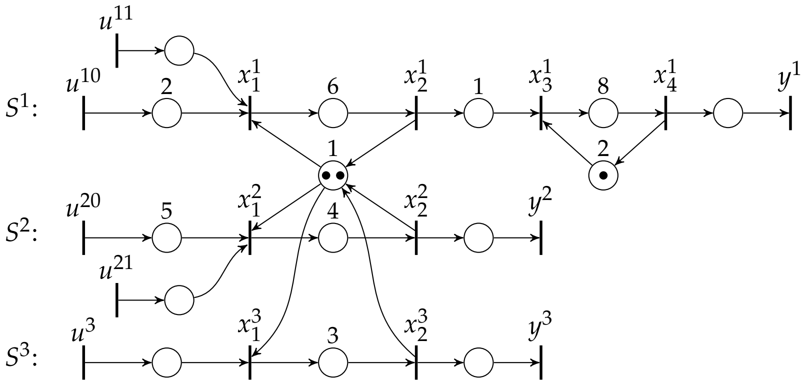

Figure 4.2: Three TEGs S^1, S^2, and S^3 with a shared resource.

Example 4.1. Consider the system shown in Figure 4.2, where three subsystems, modeled as TEGs S^1, S^2, and S^3, share a resource represented by the place with two initial tokens. Recalling the notation from Figure 4.1, in this case we have $P^{11} = e\delta^2$, $H^1 = e\delta^6$, $P^{12} = e\delta^9(1\delta^{10})^*$, $P^{21} = e\delta^5$, $H^2 = e\delta^4$, $P^{22} = e\delta^0$, $P^{31} = e\delta^0$, $H^1 = e\delta^3$, $P^{12} = e\delta^0$ — as the place connecting the input to the allocation transition of S^3 has zero holding time, the assumption of there being an input directly connected to the allocation transition is fulfilled, so transitions u^{30} and u^{31} as in Fig. 4.1 can be merged into a single input transition, here simply called u^3. The dynamics of the resource in this example is captured simply by the counter $\beta = 2\delta^1$. The transfer matrices/function of each individual subsystem (including the resource) are

$$\begin{aligned}
\mathcal{G}^1 &= \begin{bmatrix} e\delta^{17}(1\delta^{10})^* & e\delta^{15}(1\delta^{10})^* \end{bmatrix}, \\
\mathcal{G}^2 &= \begin{bmatrix} e\delta^{9}(2\delta^{5})^* & e\delta^{4}(2\delta^{5})^* \end{bmatrix}, \\
\mathcal{G}^3 &= e\delta^3(2\delta^4)^* .
\end{aligned}$$

The following references are given for the outputs of the respective subsystems: for S^1, 4 firings of the output transition are required by time 52; for S^2, 3 output firings are required by time 27, plus 2 firings at time 39; for S^3, 3 outputs are required at time 9, plus 2 at time 35. These references can be encoded in the form of counters as follows:

$$\begin{aligned}
z^1 &= e\delta^{52} \oplus 4\delta^{+\infty}, \\
z^2 &= e\delta^{27} \oplus 3\delta^{39} \oplus 5\delta^{+\infty}, \\
z^3 &= e\delta^{9} \oplus 3\delta^{35} \oplus 5\delta^{+\infty} .
\end{aligned}$$

According to the priority policy, we start by computing the optimal input for S^1 while ignoring the other subsystems. This yields

$$u^1_{\text{opt}} = \mathcal{G}^1 \lozenge z^1 = \begin{bmatrix} e\delta^5 \oplus 1\delta^{15} \oplus 2\delta^{25} \oplus 3\delta^{35} \oplus 4\delta^{+\infty} \\ e\delta^7 \oplus 1\delta^{17} \oplus 2\delta^{27} \oplus 3\delta^{37} \oplus 4\delta^{+\infty} \end{bmatrix} .$$

The resulting optimal resource-allocation and release schedules are

$$\begin{aligned} x^1_{A_{\text{opt}}} &= x^1_{1_{\text{opt}}} &= e\delta^{7} \oplus 1\delta^{17} \oplus 2\delta^{27} \oplus 3\delta^{37} \oplus 4\delta^{+\infty}, \\ x^1_{R_{\text{opt}}} &= x^1_{2_{\text{opt}}} &= e\delta^{13} \oplus 1\delta^{23} \oplus 2\delta^{33} \oplus 3\delta^{43} \oplus 4\delta^{+\infty}. \end{aligned}$$

Next, we compute the optimal input for S^2, ignoring S^3 but taking the optimal schedules of S^1 as fixed. The greatest fixed point of mapping Φ^2 defined in (4.7) is

$$u^2_{\text{opt}} = \begin{bmatrix} e\delta^{8} \oplus 1\delta^{13} \oplus 2\delta^{18} \oplus 3\delta^{27} \oplus 4\delta^{30} \oplus 5\delta^{+\infty} \\ e\delta^{13} \oplus 1\delta^{18} \oplus 2\delta^{23} \oplus 3\delta^{32} \oplus 4\delta^{35} \oplus 5\delta^{+\infty} \end{bmatrix},$$

and the corresponding resource-allocation and release schedules are

$$\begin{aligned} x^2_{A_{\text{opt}}} &= x^2_{1_{\text{opt}}} &= e\delta^{13} \oplus 1\delta^{18} \oplus 2\delta^{23} \oplus 3\delta^{32} \oplus 4\delta^{35} \oplus 5\delta^{+\infty}, \\ x^2_{R_{\text{opt}}} &= x^2_{2_{\text{opt}}} &= e\delta^{17} \oplus 1\delta^{22} \oplus 2\delta^{27} \oplus 3\delta^{36} \oplus 4\delta^{39} \oplus 5\delta^{+\infty}. \end{aligned}$$

Finally, we calculate the optimal input for S^3, taking the optimal schedules of both S^1 and S^2 as hard restrictions. The solution is the greatest fixed point of mapping Φ^3 (as in (4.11), with $k = 3$), resulting in

$$u^3_{\text{opt}} = e\delta^{1} \oplus 1\delta^{3} \oplus 2\delta^{5} \oplus 3\delta^{9} \oplus 4\delta^{28} \oplus 5\delta^{+\infty}$$

and

$$\begin{aligned} x^3_{A_{\text{opt}}} &= x^3_{1_{\text{opt}}} &= e\delta^{1} \oplus 1\delta^{3} \oplus 2\delta^{5} \oplus 3\delta^{9} \oplus 4\delta^{28} \oplus 5\delta^{+\infty}, \\ x^3_{R_{\text{opt}}} &= x^3_{2_{\text{opt}}} &= e\delta^{4} \oplus 1\delta^{6} \oplus 2\delta^{8} \oplus 3\delta^{12} \oplus 4\delta^{31} \oplus 5\delta^{+\infty}. \end{aligned}$$

The obtained optimal outputs are

$$\begin{aligned} y^1_{\text{opt}} &= \mathcal{G}^1 u^1_{\text{opt}} &= e\delta^{22} \oplus 1\delta^{32} \oplus 2\delta^{42} \oplus 3\delta^{52} \oplus 4\delta^{+\infty}, \\ y^2_{\text{opt}} &= \mathcal{G}^2 u^2_{\text{opt}} &= e\delta^{17} \oplus 1\delta^{22} \oplus 2\delta^{27} \oplus 3\delta^{36} \oplus 4\delta^{39} \oplus 5\delta^{+\infty}, \\ y^3_{\text{opt}} &= \mathcal{G}^3 u^3_{\text{opt}} &= e\delta^{4} \oplus 1\delta^{6} \oplus 2\delta^{8} \oplus 3\delta^{12} \oplus 4\delta^{31} \oplus 5\delta^{+\infty}. \end{aligned}$$

The tracking of the respective output-references is shown in Fig. 4.3, and the schedule for usage of the resource by the three subsystems can be visualized in Fig. 4.4. ◇

Remark 4.1. The method presented in this section guarantees that the permission for a subsystem S^k to allocate the resource — represented by the firing of u^{k1} — is always granted exactly at the scheduled allocation times, and the allocation is never delayed by u^{k0}. More formally, the obtained just-in-time inputs u^k_{opt} are such that $x^k_{A_{\text{opt}}} = \mathcal{P}^k u^k_{\text{opt}} = u^{k1}_{\text{opt}}$, i. e., $u^{k1}_{\text{opt}} \succeq P^{k1} u^{k0}_{\text{opt}}$.

To show this by contradiction, assume $u^{k1}_{\text{opt}} \not\succeq P^{k1} u^{k0}_{\text{opt}}$ and consider

$$\tilde{u}^k = \begin{bmatrix} u^{k0}_{\text{opt}} \\ u^{k1}_{\text{opt}} \oplus P^{k1} u^{k0}_{\text{opt}} \end{bmatrix}.$$

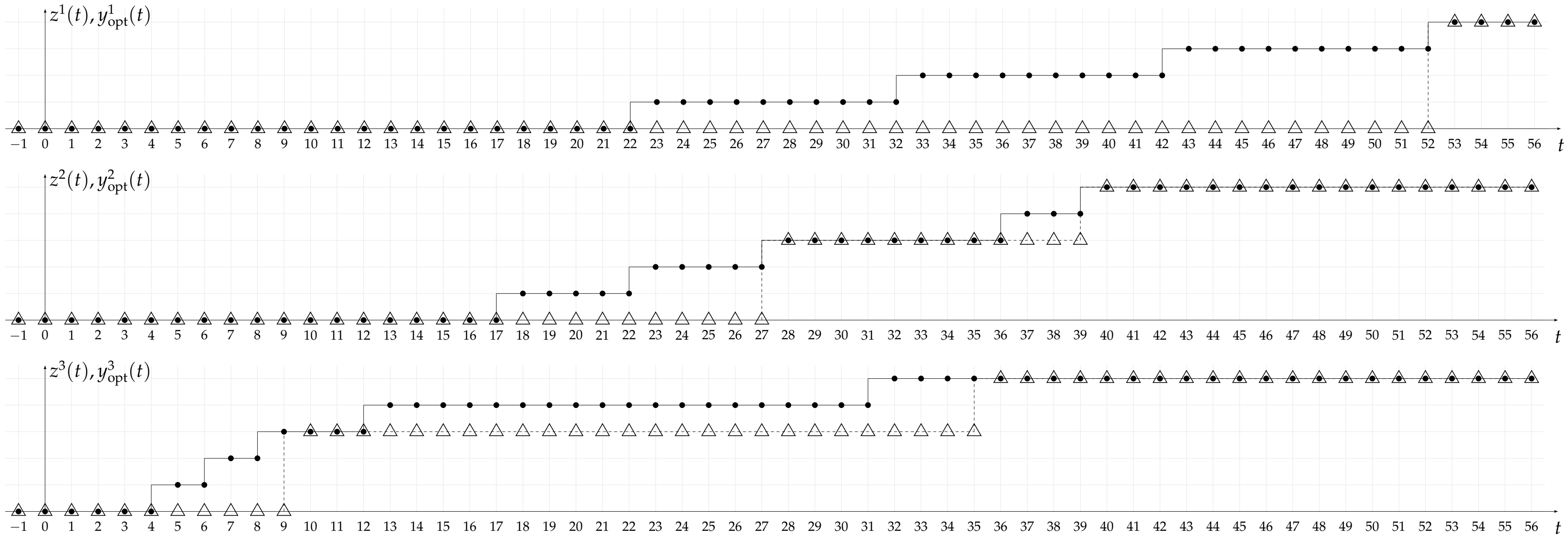

Figure 4.3: Tracking of the references z^k (denoted by $\triangle$) by the outputs y^k_{opt} (denoted by $\bullet$), $k \in \{1, 2, 3\}$, from Example 4.1.

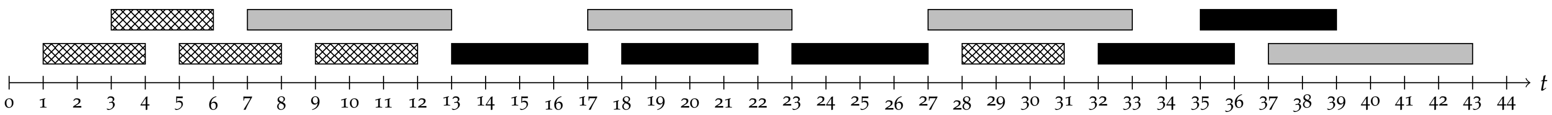

Figure 4.4: Schedule for the use of the shared resource, obtained in Example 4.1. The gray, black, and crosshatched bars represent the time windows during which an instance of the resource is held by S^1, S^2, and S^3, respectively.

Recalling that $\mathcal{P}^k = [P^{k1} \;\; s_e]$, we have

$$\mathcal{P}^k \widetilde{u}^k \;=\; P^{k1} u^{k0}_{\text{opt}} \oplus u^{k1}_{\text{opt}} \oplus P^{k1} u^{k0}_{\text{opt}} \;=\; P^{k1} u^{k0}_{\text{opt}} \oplus u^{k1}_{\text{opt}} \;=\; \mathcal{P}^k u^k_{\text{opt}} \,,$$

which implies $\widetilde{u}^k$ satisfies (4.9) and thus

$$\widetilde{u}^k \;\preceq\; H^k \mathcal{P}^k \diamond\!\!\!\!\lozenge \Big[\Big(\beta \diamond\!\!\!\!\lozenge \Big(\bigodot_{i=1}^{k-1} x^i_{A_{\text{opt}}} \odot \mathcal{P}^k \widetilde{u}^k \Big) \Big) \odot^\sharp \bigodot_{i=1}^{k-1} x^i_{R_{\text{opt}}} \Big] \,.$$

Furthermore, from Fig. 4.1 it is clear that $\mathcal{G}^{k0} = \mathcal{G}^{k1} P^{k1}$; therefore,

$$\begin{aligned} \mathcal{G}^k \widetilde{u}^k \;&=\; \mathcal{G}^{k0} u^{k0}_{\text{opt}} \oplus \mathcal{G}^{k1} (u^{k1}_{\text{opt}} \oplus P^{k1} u^{k0}_{\text{opt}}) \\ &=\; \mathcal{G}^{k0} u^{k0}_{\text{opt}} \oplus \mathcal{G}^{k1} u^{k1}_{\text{opt}} \oplus \mathcal{G}^{k1} P^{k1} u^{k0}_{\text{opt}} \\ &=\; \mathcal{G}^{k0} u^{k0}_{\text{opt}} \oplus \mathcal{G}^{k1} u^{k1}_{\text{opt}} \\ &=\; \mathcal{G}^k u^k_{\text{opt}} \,. \end{aligned}$$

Since u^k_{opt} is computed such that $\mathcal{G}^k u^k_{\text{opt}} \preceq z^k$, this implies $\mathcal{G}^k \widetilde{u}^k \preceq z^k$ or, equivalently, $\widetilde{u}^k \preceq \mathcal{G}^k \diamond\!\!\!\!\lozenge z^k$. We then conclude that $\widetilde{u}^k$ is a fixed point of Φ^k.

But note that $u^{k1}_{\text{opt}} \oplus P^{k1} u^{k0}_{\text{opt}} \succeq u^{k1}_{\text{opt}}$ and, due to our assumption that $u^{k1}_{\text{opt}} \not\succeq P^{k1} u^{k0}_{\text{opt}}$, also $u^{k1}_{\text{opt}} \oplus P^{k1} u^{k0}_{\text{opt}} \neq u^{k1}_{\text{opt}}$, implying $\widetilde{u}^k \succeq u^k_{\text{opt}}$ and $\widetilde{u}^k \neq u^k_{\text{opt}}$; this contradicts the fact that u^k_{opt} is the greatest fixed point of Φ^k. ◇

Remark 4.2. It is clear that the presence of resource sharing imposes additional restrictions for the firing of allocation transitions, besides the standard ones from the dynamics of the individual subsystems. Consequently, in general it might be the case that a subsystem S^k would not behave purely according to (3.2) for an arbitrary input u^k, and hence $y^k \neq \mathcal{G}^k u^k$. Nonetheless, in the presented method all just-in-time input firing schedules u^k_{opt} are computed so that the corresponding allocation schedules $x^k_{A_{\text{opt}}} = \mathcal{P}^k u^k_{\text{opt}}$ respect resource constraints (4.1). This means an allocation transition x^k_A is only going to be enabled when it is indeed the turn of S^k to allocate the resource; that is to say, all conflicts are resolved offline in the computation phase, and effectively there will be no nondeterministic dispute for the resource during the operation of the system. Thus, the obtained optimal inputs guarantee that the evolution of the subsystems will, in fact, follow (3.2), as if unaffected by the resource constraints, i. e., we have $y^k_{\text{opt}} = \mathcal{G}^k u^k_{\text{opt}}$ for every k. In conclusion, even though the overall resource-sharing system is not a TEG, we can still harness one of the main benefits of using TEG models — namely a linear algebraic representation which allows to extract a transfer relation for each subsystem — when computing and analyzing the behavior of the subsystems based on the optimal inputs yielded by the presented method.

Naturally, the same reasoning carries over to the case of multiple shared resources, to be discussed in Section 4.3. ◇

4.3 MODELING AND OPTIMAL CONTROL OF TEGS WITH MULTIPLE SHARED RESOURCES

Consider again a system comprising K TEGs $S^1, \dots, S^K$, but now suppose they share L resources, as shown in Fig. 4.5. Each subsystem S^k has $L+1$ input transitions, $u^{k0}, \dots, u^{kL}$, which are seen as control inputs, similarly to Section 4.1. The firing of u^{k0} can be thought of as the provisioning of raw material to be processed by S^k, whereas each $u^{k\ell}$ with $\ell \in \{1, \dots, L\}$ represents the permission for S^k to allocate resource ℓ. We assume that every $u^{k\ell}$ is connected to resource-allocation transition $x_A^{k\ell}$ via a place with no holding time and no initial tokens, meaning it is possible to decide in real time whether or not to grant a subsystem S^k access to each resource ℓ. We denote by $x_R^{k\ell}$ the transition — and associated counter — representing the release of resource ℓ by subsystem S^k, and by y^k the output transition of S^k.

$H^{k\ell}$ denotes the internal dynamics of S^k between allocation ($x_A^{k\ell}$) and release ($x_R^{k\ell}$) of resource ℓ. The dynamics between input transition u^{k0} and the resource-allocation transition for the first resource (x_A^{k1}) is denoted P^{k1}, whereas that between the resource-release transition for the last resource (x_R^{kL}) and output transition y^k is called $P^{k(L+1)}$. Finally, for each $\ell \in \{2, \dots, L\}$, the dynamics between the release of resource $\ell - 1$ and the allocation of resource ℓ by S^k (i. e., between $x_R^{k(\ell-1)}$ and $x_A^{k\ell}$) is denoted $P^{k\ell}$. As in Section 4.1, we assume there is no work in progress in any part of the system before the first firing of some input u^{k0}, in particular implying that the first firing of $x_R^{k\ell}$ cannot occur before that of $x_A^{k\ell}$. More formally, for each $\ell \in \{1, \dots, L\}$ there exists $\tau_\ell \geq 0$ such that $H^{k\ell}(t) = e$ for all $t \leq \tau_\ell$, so $H^{k\ell} \succeq s_e$ and hence $x_R^{k\ell} = H^{k\ell} x_A^{k\ell} \succeq x_A^{k\ell}$. Analogously for P^{k1} with respect to u^{k0} and x_A^{k1}, for $P^{k(L+1)}$ with respect to x_R^{kL} and y^k, as well as for $P^{k\ell}$ ($\ell \in \{2, \dots, L\}$) with respect to $x_R^{k(\ell-1)}$ and $x_A^{k\ell}$.

Each β^ℓ, $\ell \in \{1, \dots, L\}$, is modeled by a TEG (or possibly just a place) describing the capacity as well as the minimum delay between release and allocation of resource ℓ. Transitions $\alpha^{\ell 1}$ and $\alpha^{\ell 2}$ are again auxiliary, as explained in Section 4.1. We assume every resource has non-null capacity and imposes a non-null delay between release and allocation events. The assumptions made above on all $H^{k\ell}$ imply that the first resource is fully available before the first firing of u^{k0} and u^{k1} of the same subsystem S^k for some $k \in \{1, \dots, K\}$ and, for all $\ell \in \{2, \dots, L\}$, resource ℓ is fully available before the first firing of $x_R^{k(\ell-1)}$ and $u^{k\ell}$ of the same subsystem S^k for some $k \in \{1, \dots, K\}$.

Through the same reasoning as applied in Section 4.1, it is straightforward to conclude that, for any $k \in \{1, \dots, K\}$ and for each $\ell \in \{1, \dots, L\}$, the relationship among counters $x_A^{k\ell}$ and $x_R^{k\ell}$ must be such that

$$\beta^\ell \otimes \Big(\bigodot_{k=1}^{K} x_R^{k\ell} \Big) \preceq \bigodot_{k=1}^{K} x_A^{k\ell} \,. \tag{4.12}$$

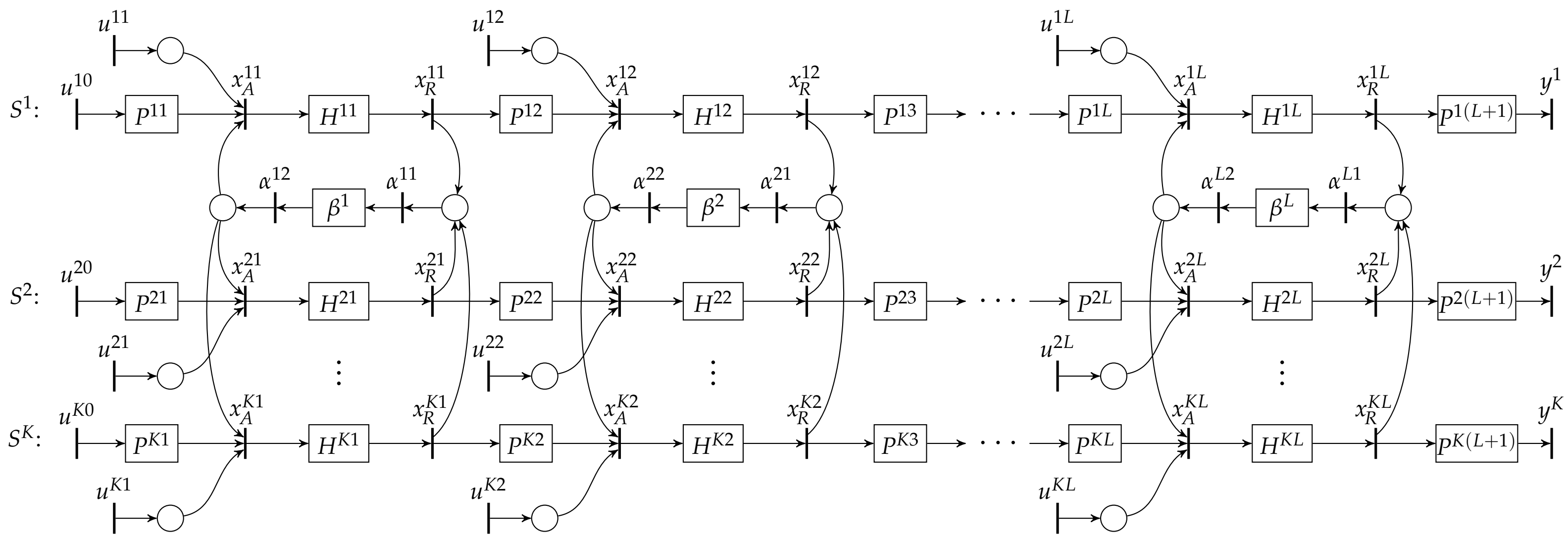

Figure 4.5: TEGs $S^1, \ldots, S^K$ with shared resources $\beta^1, \ldots, \beta^L$.

The optimal (just-in-time) schedule for the usage of the resources is sought under the same priority policy as in Section 4.2. Let the input-output behavior of each S^k, considering all resources and ignoring other subsystems, be described as usual by $y^k = \mathcal{G}^k u^k$, where

$$u^k = \begin{bmatrix} u^{k0} \\ u^{k1} \\ \vdots \\ u^{kL} \end{bmatrix} \in \Sigma^{(L+1)\times 1} \quad \text{and} \quad \mathcal{G}^k = \begin{bmatrix} \mathcal{G}^{k0} & \mathcal{G}^{k1} & \dots & \mathcal{G}^{kL} \end{bmatrix} \in \Sigma^{1\times(L+1)},$$

and let us again assume respective references z^k to be given.

For S^1, we can simply compute the optimal input by $u^1_{\text{opt}} = \mathcal{G}^1 \backslash\!\!\circ z^1$. Based on u^1_{opt}, the optimal firing schedules for the remaining transitions of S^1 can be obtained. For instance, we have $x^{11}_{A_{\text{opt}}} = P^{11} u^{10}_{\text{opt}} \oplus u^{11}_{\text{opt}}$ and $x^{11}_{R_{\text{opt}}} = H^{11} x^{11}_{A_{\text{opt}}}$. In general, for each $\ell \in \{2, \dots, L\}$ we can then successively compute $x^{1\ell}_{A_{\text{opt}}} = P^{1\ell} x^{1(\ell-1)}_{R_{\text{opt}}}$ and $x^{1\ell}_{R_{\text{opt}}} = H^{1\ell} x^{1\ell}_{A_{\text{opt}}}$.

In order to determine the optimal input u^2_{opt} for S^2 — i.e., the greatest u^2 such that $\mathcal{G}^2 u^2 \preceq z^2$ — while guaranteeing no interference with the optimal behavior of S^1, based on (4.12) we must have, for all $\ell \in \{1, \dots, L\}$,

$$\beta^\ell \otimes (x^{1\ell}_{R_{\text{opt}}} \odot x^{2\ell}_R) \preceq x^{1\ell}_{A_{\text{opt}}} \odot x^{2\ell}_A . \tag{4.13}$$

Notice that, for a just-in-time input u^2 computed so that (4.13) holds for $\ell = 1$, it follows that $x^{21}_A = P^{21} u^{20} \oplus u^{21}$, and hence $x^{21}_R = H^{21} x^{21}_A = H^{21} P^{21} u^{20} \oplus H^{21} u^{21}$. In fact, the optimal input we seek is such that (4.13) holds for every ℓ and, furthermore, such that a just-in-time behavior is enforced throughout the system, implying $x^{2\ell}_A = P^{2\ell} x^{2(\ell-1)}_R \oplus u^{2\ell}$ and $x^{2\ell}_R = H^{2\ell} x^{2\ell}_A$ for all $\ell \in \{2, \dots, L\}$. This means we can express any $x^{2\ell}_A$ and $x^{2\ell}_R$ in terms of u^2, as follows. Let us denote by $s_e^{\text{-}\ell\text{-}} \in \Sigma^{1\times(L+1)}$ the row vector such that

$$[s_e^{\text{-}\ell\text{-}}]_i = \begin{cases} s_e & \text{for } i = \ell + 1 , \\ s_\varepsilon & \text{for } i \neq \ell + 1 . \end{cases} \tag{4.14}$$

Now, define the terms $\mathcal{P}^{2\ell} \in \Sigma^{1\times(L+1)}$,

$$\mathcal{P}^{2\ell} = \begin{cases} [P^{21} \;\; s_e \;\; s_\varepsilon \;\; \dots \;\; s_\varepsilon] , & \text{if } \ell = 1 ; \\ P^{2\ell} H^{2(\ell-1)} \mathcal{P}^{2(\ell-1)} \oplus s_e^{\text{-}\ell\text{-}} , & \text{if } 2 \le \ell \le L . \end{cases} \tag{4.15}$$

One can see that, for a just-in-time input u_2, we have

$$\mathcal{P}^{21} u^2 = P^{21} u^{20} \oplus u^{21} = x^{21}_A ,$$

i.e., $\mathcal{P}^{21}$ represents the relation between the allocation schedule (by S^2) of resource 1 and the input transitions upstream from x^{21}_A, namely u^{20} and u^{21}. We also have

$$\mathcal{P}^{22} = P^{22} H^{21} \mathcal{P}^{21} \oplus s_e^{\text{-}2\text{-}} = [P^{22} H^{21} P^{21} \;\; P^{22} H^{21} \;\; s_e \;\; s_\varepsilon \;\; \dots \;\; s_\varepsilon] ,$$

so

$$\mathcal{P}^{22}u^2 = P^{22}H^{21}P^{21}u^{20} \oplus P^{22}H^{21}u^{21} \oplus u^{22} = x_A^{22};$$

again, $\mathcal{P}^{22}$ represents the relation between the allocation schedule (by S^2) of resource 2 and the input transitions upstream from x_A^{22}, , namely u^{20}, u^{21}, and u^{22}. In general, for any $\ell \in \{1, \dots, L\}$, $\mathcal{P}^{2\ell}$ represents the relation between the allocation schedule (by S^2) of resource ℓ and all input transitions upstream from $x_A^{2\ell}$, i.e., we have $x_A^{2\ell} = \mathcal{P}^{2\ell}u^2$. This also implies $x_R^{2\ell} = H^{2\ell}\mathcal{P}^{2\ell}u^2$, and hence we can rewrite (4.13) with u_2 as the only unknown:

$$\beta^\ell \otimes (x_{R_{\text{opt}}}^{1\ell} \odot H^{2\ell}\mathcal{P}^{2\ell}u^2) \preceq x_{A_{\text{opt}}}^{1\ell} \odot \mathcal{P}^{2\ell}u^2 . \tag{4.16}$$

Then, we have

$$\begin{aligned}
(4.16) \;\Leftrightarrow\;& x_{R_{\text{opt}}}^{1\ell} \odot H^{2\ell}\mathcal{P}^{2\ell}u^2 \preceq \beta^\ell \lozenge\!\!\backslash (x_{A_{\text{opt}}}^{1\ell} \odot \mathcal{P}^{2\ell}u^2) \\
\Leftrightarrow\;& H^{2\ell}\mathcal{P}^{2\ell}u^2 \preceq \big(\beta^\ell \lozenge\!\!\backslash (x_{A_{\text{opt}}}^{1\ell} \odot \mathcal{P}^{2\ell}u^2)\big) \odot^\sharp x_{R_{\text{opt}}}^{1\ell} \\
\Leftrightarrow\;& u^2 \preceq H^{2\ell}\mathcal{P}^{2\ell} \lozenge\!\!\backslash \big[\big(\beta^\ell \lozenge\!\!\backslash (x_{A_{\text{opt}}}^{1\ell} \odot \mathcal{P}^{2\ell}u^2)\big) \odot^\sharp x_{R_{\text{opt}}}^{1\ell}\big].
\end{aligned}$$

Define, for each $\ell \in \{1, \dots, L\}$, the mapping $\Phi^{2\ell} : \Sigma^{(L+1)\times 1} \to \Sigma^{(L+1)\times 1}$,

$$\Phi^{2\ell}(u^2) = H^{2\ell}\mathcal{P}^{2\ell} \lozenge\!\!\backslash \big[\big(\beta^\ell \lozenge\!\!\backslash (x_{A_{\text{opt}}}^{1\ell} \odot \mathcal{P}^{2\ell}u^2)\big) \odot^\sharp x_{R_{\text{opt}}}^{1\ell}\big].$$

We seek the greatest u^2 such that $u^2 \preceq \mathcal{G}^2 \lozenge\!\!\backslash z^2$ and $(\forall \ell \in \{1, \dots, L\})\; u^2 \preceq \Phi^{2\ell}(u^2)$. This amounts to looking for the greatest fixed point of the (isotone) mapping $\overline{\Phi}^2 : \Sigma^{(L+1)\times 1} \to \Sigma^{(L+1)\times 1}$,

$$\overline{\Phi}^2(u^2) = u^2 \wedge \mathcal{G}^2 \lozenge\!\!\backslash z^2 \wedge \bigwedge_{\ell=1}^{L} \Phi^{2\ell}(u^2).$$

The same arguments presented above can be applied to determine u_{opt}^k for an arbitrary k. Generalizing (4.15), define the terms

$$\mathcal{P}^{k\ell} = \begin{cases} [P^{k1} \;\; s_e \;\; s_\varepsilon \;\; \dots \;\; s_\varepsilon], & \text{if } \ell = 1, \\ P^{k\ell}H^{k(\ell-1)}\mathcal{P}^{k(\ell-1)} \oplus s_e^{\text{-}\ell\text{-}}, & \text{if } 2 \le \ell \le L, \end{cases} \tag{4.17}$$

with $s_e^{\text{-}\ell\text{-}}$ defined as in (4.14). Using (4.17) to express each $x_A^{k\ell}$ and $x_R^{k\ell}$ in terms of u^k, from (4.12) we obtain, for all $\ell \in \{1, \dots, L\}$,

$$\beta^\ell \otimes \Big(\bigodot_{i=1}^{k-1} x_{R_{\text{opt}}}^{i\ell} \odot H^{k\ell}\mathcal{P}^{k\ell}u^k\Big) \preceq \bigodot_{i=1}^{k-1} x_{A_{\text{opt}}}^{i\ell} \odot \mathcal{P}^{k\ell}u^k . \tag{4.18}$$

Then, proceeding as before and defining, for each $\ell \in \{1, \dots, L\}$, the mapping $\Phi^{k\ell} : \Sigma^{(L+1)\times 1} \to \Sigma^{(L+1)\times 1}$,

$$\Phi^{k\ell}(u^k) = H^{k\ell}\mathcal{P}^{k\ell} \lozenge\!\!\backslash \Big[\Big(\beta^\ell \lozenge\!\!\backslash \big(\bigodot_{i=1}^{k-1} x_{A_{\text{opt}}}^{i\ell} \odot \mathcal{P}^{k\ell}u^k\big)\Big) \odot^\sharp \bigodot_{i=1}^{k-1} x_{R_{\text{opt}}}^{i\ell}\Big],$$

the greatest u^k such that $u^k \preceq \mathcal{G}^k \diamond z^k$ and $u^k \preceq \Phi^{k\ell}(u^k)$ for all ℓ is given by the greatest fixed point of $\overline{\Phi}^k : \Sigma^{(L+1)\times 1} \to \Sigma^{(L+1)\times 1}$,

$$\overline{\Phi}^k(u^k) = u^k \wedge \mathcal{G}^k \diamond z^k \wedge \bigwedge_{\ell=1}^{L} \Phi^{k\ell}(u^k).$$

Remark 4.3. For the just-in-time inputs u^k_{opt}, it holds that $\mathcal{P}^{k\ell} u^k_{\text{opt}} = u^{k\ell}_{\text{opt}}$ for every $\ell \in \{1, \dots, L\}$. Intuitively, recalling that $x^{k\ell}_{A_{\text{opt}}} = \mathcal{P}^{k\ell} u^k_{\text{opt}}$ for every ℓ, this means the permission to allocate each resource is always granted exactly at the scheduled allocation times, and the allocation is never delayed by the influence of u^{k0} or of any other transition preceding $x^{k\ell}_A$.

This can be shown by induction on ℓ. Let us start by proving the base case $\ell = 1$. We want to show that $\mathcal{P}^{k1} u^k_{\text{opt}} = u^{k1}_{\text{opt}}$; since, recalling the definition of $\mathcal{P}^{k1}$ from (4.17), $\mathcal{P}^{k1} u^k_{\text{opt}} = P^{k1} u^{k0}_{\text{opt}} \oplus u^{k1}_{\text{opt}}$, this is equivalent to showing that $u^{k1}_{\text{opt}} \succeq P^{k1} u^{k0}_{\text{opt}}$. We shall do so by contradiction, by close analogy with Remark 4.1. Assume $u^{k1}_{\text{opt}} \not\succeq P^{k1} u^{k0}_{\text{opt}}$, and consider an input $\tilde{u}^k \in \Sigma^{(L+1)\times 1}$ with

$$[\tilde{u}^k]_i = \begin{cases} u^{k1}_{\text{opt}} \oplus P^{k1} u^{k0}_{\text{opt}}, & \text{for } i = 2, \\ u^{k(i-1)}_{\text{opt}}, & \text{for } i \neq 2. \end{cases}$$

We have

$$\mathcal{P}^{k1} \tilde{u}^k = P^{k1} u^{k0}_{\text{opt}} \oplus \left(u^{k1}_{\text{opt}} \oplus P^{k1} u^{k0}_{\text{opt}}\right) = P^{k1} u^{k0}_{\text{opt}} \oplus u^{k1}_{\text{opt}} = \mathcal{P}^{k1} u^k_{\text{opt}},$$

which implies $\tilde{u}^k$ satisfies (4.18) for $\ell = 1$ and thus $\tilde{u}^k \preceq \Phi^{k1}(\tilde{u}^k)$. Moreover,

$$\mathcal{P}^{k2} \tilde{u}^k = P^{k2} H^{k1} \mathcal{P}^{k1} \tilde{u}^k \oplus u^{k2}_{\text{opt}} = P^{k2} H^{k1} \mathcal{P}^{k1} u^k_{\text{opt}} \oplus u^{k2}_{\text{opt}} = \mathcal{P}^{k2} u^k_{\text{opt}};$$

progressing successively for $\ell = 3, \dots, L$, one can see that $\mathcal{P}^{k\ell} \tilde{u}^k = \mathcal{P}^{k\ell} u^k_{\text{opt}}$ — and hence $\tilde{u}^k \preceq \Phi^{k\ell}(\tilde{u}^k)$ — for every ℓ.

Furthermore, from Fig. 4.1 it is clear that $\mathcal{G}^{k0} = \mathcal{G}^{k1} P^{k1}$; therefore,

$$\mathcal{G}^{k1}\left(u^{k1}_{\text{opt}} \oplus P^{k1} u^{k0}_{\text{opt}}\right) = \mathcal{G}^{k1} u^{k1}_{\text{opt}} \oplus \mathcal{G}^{k1} P^{k1} u^{k0}_{\text{opt}} = \mathcal{G}^{k0} u^{k0}_{\text{opt}} \oplus \mathcal{G}^{k1} u^{k1}_{\text{opt}},$$

implying

$$\begin{aligned} \mathcal{G}^k \tilde{u}^k &= \mathcal{G}^{k0} u^{k0}_{\text{opt}} \oplus \mathcal{G}^{k1}\left(u^{k1}_{\text{opt}} \oplus P^{k1} u^{k0}_{\text{opt}}\right) \oplus \bigoplus_{\lambda=2}^{L} \mathcal{G}^{k\lambda} u^{k\lambda}_{\text{opt}} \\ &= \mathcal{G}^{k0} u^{k0}_{\text{opt}} \oplus \left(\mathcal{G}^{k0} u^{k0}_{\text{opt}} \oplus \mathcal{G}^{k1} u^{k1}_{\text{opt}}\right) \oplus \bigoplus_{\lambda=2}^{L} \mathcal{G}^{k\lambda} u^{k\lambda}_{\text{opt}} \\ &= \bigoplus_{\lambda=0}^{L} \mathcal{G}^{k\lambda} u^{k\lambda}_{\text{opt}} \\ &= \mathcal{G}^k u^k_{\text{opt}} \end{aligned}$$

and hence $\widetilde{u}^k \preceq \mathcal{G}^k \diamond z^k$. This shows that $\widetilde{u}^k$ is a fixed point of $\overline{\Phi}^k$, which, as $\widetilde{u}^k \succeq u^k_{\text{opt}}$ and $\widetilde{u}^k \neq u^k_{\text{opt}}$, contradicts the fact that u^k_{opt} is the greatest fixed point of $\overline{\Phi}^k$.

Now, as our inductive hypothesis, assume $\mathcal{P}^{k\ell} u^k_{\text{opt}} = u^{k\ell}_{\text{opt}}$ for an arbitrary $\ell \in \{2, \ldots, L-1\}$. We proceed to show that $\mathcal{P}^{k(\ell+1)} u^k_{\text{opt}} = u^{k(\ell+1)}_{\text{opt}}$. As

$$\mathcal{P}^{k(\ell+1)} u^k_{\text{opt}} = P^{k(\ell+1)} H^{k\ell} \mathcal{P}^{k\ell} u^k_{\text{opt}} \oplus u^{k(\ell+1)}_{\text{opt}} = P^{k(\ell+1)} H^{k\ell} u^{k\ell}_{\text{opt}} \oplus u^{k(\ell+1)}_{\text{opt}},$$

where the first equality follows directly from the definition of $\mathcal{P}^{k(\ell+1)}$ — see (4.17) — and the second follows from the inductive hypothesis, it suffices to show that $u^{k(\ell+1)}_{\text{opt}} \succeq P^{k(\ell+1)} H^{k\ell} u^{k\ell}_{\text{opt}}$. Arguing by contradiction, assume $u^{k(\ell+1)}_{\text{opt}} \not\succeq P^{k(\ell+1)} H^{k\ell} u^{k\ell}_{\text{opt}}$, and consider the input $\check{u}^k$ with

$$[\check{u}^k]_i = \begin{cases} u^{k(\ell+1)}_{\text{opt}} \oplus P^{k(\ell+1)} H^{k\ell} u^{k\ell}_{\text{opt}}, & \text{for } i = \ell+2, \\ u^{k(i-1)}_{\text{opt}}, & \text{for } i \neq \ell+2. \end{cases}$$

Note that this implies, in particular, that $\mathcal{P}^{k\lambda} \check{u}^k = \mathcal{P}^{k\lambda} u^k_{\text{opt}}$ for any $\lambda \in \{1, \ldots, \ell\}$. We then have

$$\begin{aligned} \mathcal{P}^{k(\ell+1)} \check{u}^k &= P^{k(\ell+1)} H^{k\ell} \mathcal{P}^{k\ell} \check{u}^k \oplus \left(u^{k(\ell+1)}_{\text{opt}} \oplus P^{k(\ell+1)} H^{k\ell} u^{k\ell}_{\text{opt}}\right) \\ &= P^{k(\ell+1)} H^{k\ell} \mathcal{P}^{k\ell} u^k_{\text{opt}} \oplus \left(u^{k(\ell+1)}_{\text{opt}} \oplus P^{k(\ell+1)} H^{k\ell} \mathcal{P}^{k\ell} u^k_{\text{opt}}\right) \\ &= P^{k(\ell+1)} H^{k\ell} \mathcal{P}^{k\ell} u^k_{\text{opt}} \oplus u^{k(\ell+1)}_{\text{opt}} \\ &= \mathcal{P}^{k(\ell+1)} u^k_{\text{opt}}, \end{aligned}$$

which implies $\check{u}^k$ satisfies (4.18) for $\ell+1$ and thus $\check{u}^k \preceq \Phi^{k(\ell+1)}(\check{u}^k)$. In fact, one can then easily check that $\mathcal{P}^{k\lambda} \check{u}^k = \mathcal{P}^{k\lambda} u^k_{\text{opt}}$ — and hence $\check{u}^k \preceq \Phi^{k\lambda}(\check{u}^k)$ — for all $\lambda \in \{1, \ldots, L\}$.

Furthermore, from Fig. 4.5 it is clear that $\mathcal{G}^{k\ell} = \mathcal{G}^{k(\ell+1)} P^{k(\ell+1)} H^{k\ell}$; therefore,

$$\begin{aligned} \mathcal{G}^{k(\ell+1)} \left(u^{k(\ell+1)}_{\text{opt}} \oplus P^{k(\ell+1)} H^{k\ell} u^{k\ell}_{\text{opt}}\right) &= \mathcal{G}^{k(\ell+1)} u^{k(\ell+1)}_{\text{opt}} \oplus \mathcal{G}^{k(\ell+1)} P^{k(\ell+1)} H^{k\ell} u^{k\ell}_{\text{opt}} \\ &= \mathcal{G}^{k\ell} u^{k\ell}_{\text{opt}} \oplus \mathcal{G}^{k(\ell+1)} u^{k(\ell+1)}_{\text{opt}}, \end{aligned}$$

implying

$$\begin{aligned} \mathcal{G}^k \check{u}^k &= \mathcal{G}^{k(\ell+1)} \left(u^{k(\ell+1)}_{\text{opt}} \oplus P^{k(\ell+1)} H^{k\ell} u^{k\ell}_{\text{opt}}\right) \oplus \bigoplus_{\substack{\lambda=0 \\ \lambda \neq \ell+1}}^{L} \mathcal{G}^{k\lambda} u^{k\lambda}_{\text{opt}} \\ &= \mathcal{G}^{k\ell} u^{k\ell}_{\text{opt}} \oplus \mathcal{G}^{k(\ell+1)} u^{k(\ell+1)}_{\text{opt}} \oplus \bigoplus_{\substack{\lambda=0 \\ \lambda \neq \ell+1}}^{L} \mathcal{G}^{k\lambda} u^{k\lambda}_{\text{opt}} \\ &= \bigoplus_{\lambda=0}^{L} \mathcal{G}^{k\lambda} u^{k\lambda}_{\text{opt}} \\ &= \mathcal{G}^k u^k_{\text{opt}} \end{aligned}$$

and hence $\check{u}^k \preceq \mathcal{G}^k \diamond z^k$. This shows that $\check{u}^k$ is a fixed point of $\overline{\Phi}^k$, which, as $\check{u}^k \succeq u^k_{\text{opt}}$ and $\check{u}^k \neq u^k_{\text{opt}}$, contradicts the fact that u^k_{opt} is the greatest fixed point of $\overline{\Phi}^k$. ◇

4.4 OPTIMAL CONTROL OF TEGS WITH SHARED RESOURCES AND WITH MULTIPLE INPUT TRANSITIONS

With the objective of making the method presented so far more broadly applicable, we now generalize the foregoing discussion to the case in which each subsystem may have an arbitrary number of input transitions. In the scope of this thesis, this will be particularly relevant in Chapter 10.

THE CASE OF A SINGLE SHARED RESOURCE

Starting with the case of a single shared resource (Sections 4.1 and 4.2), we consider a system with the same structure as the one from Fig. 4.1, in particular including input transitions u^{k0} and u^{k1}, but with the crucial difference that we hereafter assume there may be additional input transitions inside blocks P^{k1}, H^k, and P^{k2}. The firing of these additional input transitions can be interpreted, for instance, as provisioning of raw material or tools needed in intermediate steps of the production process, or as a direct permission for certain internal transitions to fire. Let us denote by m_k the total number of input transitions in subsystem S^k, i. e., $u^k \in \Sigma^{m_k \times 1}$ with $m_k \geq 2$ — the interesting case being studied here is, of course, that in which $m_k > 2$, since with $m_k = 2$ we have only the inputs u^{k0} and u^{k1} and are back to the case of Section 4.1.

We again make the assumption of there being initially "no work in progress" in the system, meaning, in particular, that the initial marking of block H^k in Fig. 4.1 is such that the first firing of x^k_R cannot occur before the first firing of x^k_A (i. e., $x^k_R \succeq x^k_A$). Analogously for block P^{k1} (resp. P^{k2}) with respect to the firings of u^{k0} and x^k_A (resp. x^k_R and y^k). We assume there are no input transitions inside the resource block β. Naturally, the same condition on the resource-allocation and release transitions established in Section 4.1 — inequality (4.1) — applies to the present case.

As in Section 4.2, suppose a reference z^k is given for every respective subsystem S^k. Our control objective is to obtain just-in-time firing schedules for input $u^k \in \Sigma^{m_k \times 1}$ with respect to z^k for each $k \in \{1, \dots, K\}$, while making sure that the capacity and dynamics of the resource are observed. In other words, we seek, for all $k \in \{1, \dots, K\}$, the greatest u^k leading to resource-allocation and release schedules satisfying inequality (4.1) and also such that $y^k = \mathcal{G}^k u^k \preceq z^k$ (where, as usual, $\mathcal{G}^k \in \Sigma^{1 \times m_k}$ is the transfer matrix of S^k, including the resource and ignoring all other subsystems).

The dispute for the resource is settled by adopting the same priority policy as in Section 4.2. Accordingly, we start by computing the optimal input for S^1, which can be done by neglecting all other subsystems (as they all have priority lower than that of S^1) and hence applying the method for a single TEG introduced in Section 3.2, i. e., $u^1_{\text{opt}} = \mathcal{G}^1 \diamond\!\!\!\!\!\! \setminus z^1$. In order to determine the corresponding resource-allocation and release schedules, suppose x^1_A and x^1_R occupy the i^{th} and j^{th} entries in vector x^1, respectively (i. e., $x^1_i = x^1_A$ and $x^1_j = x^1_R$), and let us denote, for convenience,

$$\mathcal{F}^1_A = \mathcal{F}^1_{[i\cdot]} \qquad \text{and} \qquad \mathcal{F}^1_R = \mathcal{F}^1_{[j\cdot]}\,, \tag{4.19}$$

recalling that $\mathcal{F}^1_{[\mu\cdot]}$ is the μ^{th} row of matrix $\mathcal{F}^1$ (cf. (3.4)). Then, we have

$$x^1_{A_{\text{opt}}} = \mathcal{F}^1_A u^1_{\text{opt}} \qquad \text{and} \qquad x^1_{R_{\text{opt}}} = \mathcal{F}^1_R u^1_{\text{opt}}\,.$$

For an arbitrary S^k with $k \in \{2, \dots, K\}$, the optimal input u^k_{opt} can be obtained by direct analogy with the method presented in Section 4.2, only replacing the terms $\mathcal{P}^k$ and $H^k\mathcal{P}^k$ respectively by $\mathcal{F}^k_A$ and $\mathcal{F}^k_R$ — the latter terms being defined by an obvious generalization of (4.19) to an arbitrary $\mathcal{F}^k$. This means we can write $x^k_A = \mathcal{F}^k_A u^k$ and $x^k_R = \mathcal{F}^k_R u^k$, and the sought just-in-time input u^k_{opt} is the greatest fixed point of mapping $\Phi^k_{\text{mi}} : \Sigma^{m_k \times 1} \to \Sigma^{m_k \times 1}$,

$$\Phi^k_{\text{mi}}(u^k) \;=\; \mathcal{F}^k_R \diamond\!\!\!\!\!\! \setminus \left[\left(\beta \diamond\!\!\!\!\!\! \setminus \left(\bigodot_{i=1}^{k-1} x^i_{A_{\text{opt}}} \odot \mathcal{F}^k_A u^k \right) \right) \odot^\sharp \bigodot_{i=1}^{k-1} x^i_{R_{\text{opt}}} \right] \wedge \mathcal{G}^k \diamond\!\!\!\!\!\! \setminus z^k \wedge u^k$$

(where "mi" stands for "multiple inputs").

THE CASE OF MULTIPLE SHARED RESOURCES

The above discussion carries over to the case of multiple shared resources (Section 4.3) in a straightforward manner. Consider a system with the same structure as the one from Fig. 4.5, in particular including input transitions $u^{k\ell}$ for all $\ell \in \{0, \dots, L\}$, but now suppose there may be additional input transitions inside any (possibly all) of the blocks $P^{k\ell}$ for $\ell \in \{1, \dots, L+1\}$ and $H^{k\ell}$ for $\ell \in \{1, \dots, L\}$. The interpretation of these additional input transitions is the same as in the previous case. Let us again denote by m_k the total number of input transitions in subsystem S^k, i. e., $u^k \in \Sigma^{m_k \times 1}$ with $m_k \geq L+1$ — the interesting case being studied here is, of course, that in which $m_k > L+1$, since with $m_k = L+1$ we have only the inputs $u^{k0}, \dots, u^{k\ell}$ and are back to the case of Section 4.3.

Assume, for every $\ell \in \{1, \dots, L\}$, that the initial marking of blocks $H^{k\ell}$ in Fig. 4.1 is such that the first firing of $x^{k\ell}_R$ cannot occur before the first firing of $x^{k\ell}_A$ (i. e., $x^{k\ell}_R \succeq x^{k\ell}_A$). Analogously for P^{k1} with respect to the firings of u^{k0} and x^{k1}_A, for $P^{k(L+1)}$ with respect to x^{kL}_R and y^k, as well as for $P^{k\ell}$ ($\ell \in \{2,\ldots,L\}$) with respect to $x_R^{k(\ell-1)}$ and $x_A^{k\ell}$. We assume there are no input transitions inside any of the resource blocks β^ℓ. It should be clear that the same conditions established in Section 4.3 on the allocation and release schedules of all resources — inequality (4.12) — apply to the present case.

Let a reference z^k be given for every subsystem S^k. Our goal is to obtain just-in-time firing schedules for input $u^k \in \Sigma^{m_k \times 1}$ with respect to z^k for each $k \in \{1,\ldots,K\}$, while making sure that the capacity and dynamics of all the resources are observed. More precisely, we seek, for all $k \in \{1,\ldots,K\}$, the greatest u^k leading to resource-allocation and release schedules satisfying (4.12) for all $\ell \in \{1,\ldots,L\}$ and also such that $y^k = \mathcal{G}^k u^k \preceq z^k$ (where $\mathcal{G}^k \in \Sigma^{1\times m_k}$ is the transfer matrix of S^k, including all resources but ignoring all other subsystems).

Adopting the usual priority policy, we start by computing the optimal input for S^1 while neglecting all lower-priority subsystems, which amounts to applying the method from Section 3.2, i. e., $u^1_{\text{opt}} = \mathcal{G}^1 \mathbin{\diamond\!\!\backslash} z^1$. In order to determine the corresponding resource-allocation and release schedules for all resources, suppose $x_A^{k\ell}$ and $x_R^{k\ell}$ occupy respectively the ${i_\ell}^{\text{th}}$ and ${j_\ell}^{\text{th}}$ entries in vector x^k, for all $\ell \in \{1,\ldots,L\}$ (i. e., $x^k_{i_\ell} = x_A^{k\ell}$ and $x^k_{j_\ell} = x_R^{k\ell}$), and let us denote

$$\mathcal{F}_A^{1\ell} = \mathcal{F}^1_{[i_\ell\,\cdot]} \qquad \text{and} \qquad \mathcal{F}_R^{1\ell} = \mathcal{F}^1_{[j_\ell\,\cdot]}\,. \tag{4.20}$$

Then, we have

$$x^{1\ell}_{A_{\text{opt}}} = \mathcal{F}_A^{1\ell} u^1_{\text{opt}} \qquad \text{and} \qquad x^{1\ell}_{R_{\text{opt}}} = \mathcal{F}_R^{1\ell} u^1_{\text{opt}}\,.$$

In order to obtain the optimal input u^k_{opt} for an arbitrary S^k with $k \in \{2,\ldots,K\}$, one can proceed by direct analogy with the method presented in Section 4.3. Define the terms $\mathcal{F}_A^{k\ell}$ and $\mathcal{F}_R^{k\ell}$ by an obvious generalization of (4.20) to an arbitrary $\mathcal{F}^k$, and replace $\mathcal{P}^{k\ell}$ and $H^{k\ell}\mathcal{P}^{k\ell}$ in the method from Section 4.3 by $\mathcal{F}_A^{k\ell}$ and $\mathcal{F}_R^{k\ell}$, respectively. Then, defining, for each $\ell \in \{1,\ldots,L\}$, the mapping $\Phi^{k\ell}_{\text{mi}} : \Sigma^{m_k\times 1} \to \Sigma^{m_k \times 1}$,

$$\Phi^{k\ell}_{\text{mi}}(u^k) \;=\; \mathcal{F}_R^{k\ell} \mathbin{\diamond\!\!\backslash} \Big[\Big(\beta^\ell \mathbin{\diamond\!\!\backslash} \big(\bigodot_{i=1}^{k-1} x^{i\ell}_{A_{\text{opt}}} \odot \mathcal{F}_A^{k\ell} u^k \big) \Big) \odot^\sharp \bigodot_{i=1}^{k-1} x^{i\ell}_{R_{\text{opt}}} \Big], \tag{4.21}$$

the just-in-time input u^k_{opt} is given by the greatest fixed point of $\overline{\Phi}^k_{\text{mi}} : \Sigma^{m_k\times 1} \to \Sigma^{m_k\times 1}$,

$$\overline{\Phi}^k_{\text{mi}}(u^k) \;=\; u^k \wedge \mathcal{G}^k \mathbin{\diamond\!\!\backslash} z^k \wedge \bigwedge_{\ell=1}^{L} \Phi^{k\ell}_{\text{mi}}(u^k)\,.$$

5

CONTROL OF TEGS WITH SHARED RESOURCES AND OUTPUT-REFERENCE UPDATE

In this chapter, we show how to determine the optimal (just-in-time) control inputs in face of changes in the output-references for TEGs that share resources under a given priority policy. Thus, we incorporate the ideas discussed in Section 3.3 to the class of systems studied in Chapter 4.

The structure is similar to Chapter 4, starting with the simple case of a single shared resource (Sections 5.1 and 5.2) and then generalizing to the case of multiple resources (Section 5.3). In order to avoid breaking the flow and improve readability, some proofs are postponed to Appendix A.

A preliminary version of part of the following material has appeared in [41, 44], which reflect original work from — and have as the main author and contributor — the author of this thesis.

5.1 PROBLEM FORMULATION — THE CASE OF A SINGLE SHARED RESOURCE

Consider the system from Fig. 4.1 and assume every subsystem S^k is operating optimally with respect to its own output-reference z^k, according to the priority-based strategy introduced in Sections 4.1 and 4.2. Now, suppose that at a certain time T each S^k has its reference z^k updated to $z^{k\prime}$ (with the possibility that $z^{k\prime} = z^k$ for some of them). Our goal is to determine, for each k, the input $u^{k\prime}_{\text{opt}}$ which leads the corresponding output to optimally track $z^{k\prime}$ while preserving the input u^k_{opt} up to time T. A crucial point is that the priority scheme for the use of the shared resource must continue to be observed. However, we assume that subsystems (even lower-priority ones) cannot be forced to interrupt their operation and release the resource before the scheduled time. Therefore, even though when updating the inputs we still aim at minimizing the influence of lower-priority subsystems on the performance of higher-priority ones, past resource allocations (i. e., those that have occurred before time T) by subsystems with lower priority must be taken into account, because the respective resource releases may take place after time T, thus irrevocably influencing the availability of the resource in the meantime.

According to the priority hierarchy, we must compute $u^{k\prime}_{\text{opt}}$ in decreasing order of priority, i. e., start from $k = 1$ and proceed up to $k = K$. Now, for the purpose of the discussion to follow, let us fix an arbitrary $k \in \{1, \ldots, K\}$. If $k > 1$, when updating the input of S^k we must consider the already-updated allocation and release schedules of higher-priority subsystems as hard restrictions, i. e., fix $x_A^i = x_{A_{\text{opt}}}^{i\prime}$ and $x_R^i = x_{R_{\text{opt}}}^{i\prime}$ for every $i \in \{1, \ldots, k-1\}$. It will be convenient to define the terms

$$\mathcal{H}_A^k = \bigodot_{i=1}^{k-1} x_{A_{\text{opt}}}^{i\prime}, \qquad \mathcal{H}_R^k = \bigodot_{i=1}^{k-1} x_{R_{\text{opt}}}^{i\prime},$$

where $\mathcal{H}$ stands for *higher* priority (with respect to k). $\mathcal{H}_A^k$ combines the counters $x_{A_{\text{opt}}}^{i\prime}$ of all subsystems S^i with priority higher than that of S^k, referring to the updated optimal schedules of resource-allocation transitions x_A^i with respect to the corresponding new references $z^{i\prime}$; similarly, $\mathcal{H}_R^k$ combines the counters $x_{R_{\text{opt}}}^{i\prime} = H^i x_{A_{\text{opt}}}^{i\prime}$ representing the respective resource-release events.

If $k < K$, when updating the input of S^k we require minimal interference from lower-priority subsystems (i. e., all S^j with $j \in \{k+1, \ldots, K\}$). This means that, although we have to respect past resource allocations in these subsystems, we may ignore future ones. To make the reasoning more precise, recall that $x_{A_{\text{opt}}}^j$ is the accumulated number of firings originally scheduled for x_A^j up to time t. Respecting past allocations means that the firings of x_A^j which have already occurred by time T (when the new references are received) cannot be revoked. On the other hand, the prospective firings that have not taken place by time T can still be postponed and hence, from the point of view of S^k, ignored. In other words, for the sake of determining $u_{\text{opt}}^{k\prime}$ while minimizing interference from S^j, we require the terms $x_A^j(t)$ to be preserved as $x_{A_{\text{opt}}}^j$ for $t \leq T$ and neglect all new firings by making $x_A^j(t) = x_{A_{\text{opt}}}^j(T)$ for $t > T$. Recalling Remark 3.1, this is precisely captured by the counter $r_T^\sharp(x_{A_{\text{opt}}}^j)$. Let us then define the additional terms

$$\mathcal{L}_A^k = \bigodot_{j=k+1}^{K} r_T^\sharp(x_{A_{\text{opt}}}^j), \qquad \mathcal{L}_R^k = \bigodot_{j=k+1}^{K} H^j r_T^\sharp(x_{A_{\text{opt}}}^j),$$

where $\mathcal{L}$ stands for *lower* priority (with respect to k). $\mathcal{L}_A^k$ combines the counters $r_T^\sharp(x_{A_{\text{opt}}}^j)$ of all subsystems S^j with priority lower than that of S^k, representing the past firings (up to and including time T) of resource-allocation transitions x_A^j and neglecting their firings after T. In turn, $\mathcal{L}_R^k$ gathers the respective resource-release events by combining the counters $H^j r_T^\sharp(x_{A_{\text{opt}}}^j)$. It should be emphasized once more that, even though we only consider the resource allocations by S^j up to time T, the respective resource-release events may take place after T; this explains why, in $\mathcal{L}_R^k$, we use the terms $H^j r_T^\sharp(x_{A_{\text{opt}}}^j)$ rather than $r_T^\sharp(x_{R_{\text{opt}}}^j)$.

Thus, based on (4.1) and on the foregoing discussion, in order to update u^k without compromising the performance of higher-priority subsystems and, at the same time, ensuring minimal interference of lower-priority subsystems while taking into account their past resource allocations, we must respect

$$\beta \otimes \left(\mathcal{H}_R^k \odot x_R^k \odot \mathcal{L}_R^k\right) \preceq \mathcal{H}_A^k \odot x_A^k \odot \mathcal{L}_A^k\,, \tag{5.1}$$

where it is understood that for $k = 1$ (resp. $k = K$), the degenerate terms $\mathcal{H}_A^1$ and $\mathcal{H}_R^1$ (resp. $\mathcal{L}_A^K$ and $\mathcal{L}_R^K$) are to be neglected. Arguing similarly to Section 4.2, for any just-in-time input $u^k = \begin{bmatrix} u^{k0} \\ u^{k1} \end{bmatrix}$ leading to schedules of x_A^k and x_R^k that satisfy (5.1), it holds that $x_A^k = P^{k1} u^{k0} \oplus u^{k1}$ and $x_R^k = H^k x_A^k$. Recalling that we denote $\mathcal{P}^k = [P^{k1} \;\; s_e]$, we have $x_A^k = \mathcal{P}^k u^k$, and (5.1) can be written in terms of u^k as

$$\beta \otimes \left(\mathcal{H}_R^k \odot H^k \mathcal{P}^k u^k \odot \mathcal{L}_R^k\right) \preceq \mathcal{H}_A^k \odot \mathcal{P}^k u^k \odot \mathcal{L}_A^k\,. \tag{$\star$}$$

The problem of determining the new optimal input $u_{\text{opt}}^{k\prime}$ with respect to a reference $z^{k\prime}$ given at time T can be formulated as follows: find the greatest element of the set

$$\mathcal{N}^k = \left\{u^k \in \Sigma^{2\times 1} \mid \mathcal{G}^k u^k \preceq z^{k\prime} \text{ and } (\star) \text{ and } r_T(u^k) = r_T(u_{\text{opt}}^k)\right\}. \tag{5.2}$$

5.2 OPTIMAL UPDATE OF THE INPUTS — THE CASE OF A SINGLE SHARED RESOURCE

We set out to look for the greatest element of set $\mathcal{N}^k$ (defined as in (5.2)) by proposing an adaptation of Theorem 3.1.

Proposition 5.1. *Let $\mathcal{D}$ be a complete idempotent semiring, $f : \mathcal{D} \to \mathcal{D}$ a residuated mapping, $\psi : \mathcal{D} \to \mathcal{D}$, and $c \in \mathcal{D}$. Consider the set*

$$\mathcal{S}_\psi = \{x \in \mathcal{D} \,|\, x \preceq \psi(x) \text{ and } f(x) = c\}$$

and the isotone mapping $\Omega : \mathcal{D} \to \mathcal{D}$,

$$\Omega(x) = x \wedge \psi(x) \wedge f^\sharp(c)\,.$$

If $\mathcal{S}_\psi \neq \emptyset$, we have $\bigoplus_{x \in \mathcal{S}_\psi} x = \bigoplus\{x \in \mathcal{D} \,|\, \Omega(x) = x\}$.

Now, let us once more fix an arbitrary $k \in \{1, \ldots, K\}$, and assume $u_{\text{opt}}^{i\prime}$— and hence also $x_{A_{\text{opt}}}^{i\prime}$ and $x_{R_{\text{opt}}}^{i\prime}$ — have been determined for each (if any) $i \in \{1, \ldots, k-1\}$. From Def. 2.3 and applying Remark 2.11 and Prop. 2.2, we have

$$\begin{aligned}
(\star) \;\Leftrightarrow\;& \mathcal{H}_R^k \odot H^k \mathcal{P}^k u^k \odot \mathcal{L}_R^k \preceq \beta \diamond\!\!\!\backslash (\mathcal{H}_A^k \odot \mathcal{P}^k u^k \odot \mathcal{L}_A^k) \\
\Leftrightarrow\;& (\mathcal{H}_R^k \odot \mathcal{L}_R^k) \odot H^k \mathcal{P}^k u^k \preceq \beta \diamond\!\!\!\backslash (\mathcal{H}_A^k \odot \mathcal{P}^k u^k \odot \mathcal{L}_A^k) \\
\Leftrightarrow\;& H^k \mathcal{P}^k u^k \preceq \left(\beta \diamond\!\!\!\backslash (\mathcal{H}_A^k \odot \mathcal{P}^k u^k \odot \mathcal{L}_A^k)\right) \odot^\sharp (\mathcal{H}_R^k \odot \mathcal{L}_R^k) \\
\Leftrightarrow\;& u^k \preceq H^k \mathcal{P}^k \diamond\!\!\!\backslash \left[\left(\beta \diamond\!\!\!\backslash (\mathcal{H}_A^k \odot \mathcal{P}^k u^k \odot \mathcal{L}_A^k)\right) \odot^\sharp (\mathcal{H}_R^k \odot \mathcal{L}_R^k)\right].
\end{aligned}$$

Then, defining the mapping $\Psi^k : \Sigma^{2\times 1} \to \Sigma^{2\times 1}$,

$$\Psi^k(u^k) \;=\; \mathcal{G}^k \diamond\!\!\!\!\backslash\, z^{k\prime} \;\wedge\; H^k\mathcal{P}^k \diamond\!\!\!\!\backslash \big[\big(\beta \diamond\!\!\!\!\backslash (\mathcal{H}_A^k \odot \mathcal{P}^k u^k \odot \mathcal{L}_A^k)\big) \odot^\sharp (\mathcal{H}_R^k \odot \mathcal{L}_R^k)\big], \tag{5.3}$$

set $\mathcal{N}^k$ can be equivalently defined as

$$\mathcal{N}^k = \big\{u^k \in \Sigma^{2\times 1} \mid u^k \preceq \Psi^k(u^k) \text{ and } r_T(u^k) = r_T(u^k_{\text{opt}})\big\}.$$

This reveals a correspondence between set $\mathcal{N}^k$ and set $\mathcal{S}_\psi$ from Prop. 5.1: take $\mathcal{D}$ as $\Sigma^{2\times 1}$, ψ as Ψ^k, f as r_T, and c as $r_T(u^k_{\text{opt}})$. So, as long as $\mathcal{N}^k \neq \emptyset$, the conditions from the proposition are met and, recalling that $r_T^\sharp \circ r_T = r_T^\sharp$, the optimal update of u^k, $u^{k\prime}_{\text{opt}}$, is the greatest fixed point of the (isotone) mapping $\Gamma^k : \Sigma^{2\times 1} \to \Sigma^{2\times 1}$,

$$\Gamma^k(u^k) \;=\; u^k \;\wedge\; \Psi^k(u^k) \;\wedge\; r_T^\sharp(u^k_{\text{opt}})\,. \tag{5.4}$$

Next, we must investigate under what conditions $\mathcal{N}^k$ is nonempty. Consider the set

$$\widetilde{\mathcal{N}}^k = \big\{u^k \in \Sigma^{2\times 1} \,|\, (\star) \text{ and } r_T(u^k) = r_T(u^k_{\text{opt}})\big\} \supseteq \mathcal{N}^k\,.$$

Our approach is to look for an element $\underline{u}^k$ of $\widetilde{\mathcal{N}}^k$ that leads to the fastest possible behavior of S^k, i.e., to the least possible output y^k. Clearly, if such an input does not lead to meeting the new reference $z^{k\prime}$, then no input respecting $(\star)$ and $r_T(u^k) = r_T(u^k_{\text{opt}})$ will; more precisely, as we shall conclude formally in the sequel (see Corollary 5.4), $\mathcal{N}^k \neq \emptyset \Leftrightarrow \mathcal{G}^k\underline{u}^k \preceq z^{k\prime}$.

A natural choice would be to set $\underline{u}^k$ as the least element of $\widetilde{\mathcal{N}}^k$. Unfortunately, in general $\widetilde{\mathcal{N}}^k$ may not possess a least element. Nevertheless, any input in $\widetilde{\mathcal{N}}^k$ — albeit not necessarily least or unique — leading to the least *allocation* schedule compatible with the resource constraints will result in the least possible y^k. Thus, we shall guide our quest for such an input by identifying all relevant constraints on the firing schedule of x_A^k and then checking whether there exists a least such schedule which satisfies these constraints and which can be attained by choosing an input in $\widetilde{\mathcal{N}}^k$.

First, we observe that a bound for the allocation schedule x_A^k can be obtained from (5.1), as

$$\begin{aligned}(5.1) \;&\Leftrightarrow\; \beta \otimes \big(\mathcal{H}_R^k \odot x_R^k \odot \mathcal{L}_R^k\big) \;\preceq\; (\mathcal{H}_A^k \odot \mathcal{L}_A^k) \odot x_A^k \\ &\Leftrightarrow\; \big(\beta \otimes (\mathcal{H}_R^k \odot x_R^k \odot \mathcal{L}_R^k)\big) \odot^\flat (\mathcal{H}_A^k \odot \mathcal{L}_A^k) \;\preceq\; x_A^k\,.\text{[1]}\end{aligned}$$

The left-hand side of the last inequality provides a bound for how small (in the sense of the order of Σ) x_A^k can be. It represents the

1 As $\mathcal{H}_A^k$ encodes the combined accumulated number of resource allocations by each time instant t of all subsystems with priority higher than that of S^k, it is reasonable (and entails no loss of generality) to assume that $\mathcal{H}_A^k(t) \notin \{-\infty, +\infty\}$ for any finite time $t \in \mathbb{Z}$. A similar argument applies to $\mathcal{L}_A^k$ and hence carries over to $\mathcal{H}_A^k \odot \mathcal{L}_A^k$; so, according to Prop. 2.3, mapping $\Pi_{\mathcal{H}_A^k \odot \mathcal{L}_A^k} : \Sigma \to \Sigma$ is dually residuated.

maximal availability of the resource for subsystem S^k, given the fixed optimal schedules of higher-priority subsystems ($\mathcal{H}_A^k$ and $\mathcal{H}_R^k$) and the truncated schedules of lower priority subsystems ($\mathcal{L}_A^k$ and $\mathcal{L}_R^k$); this availability also implicitly depends, of course, on x_A^k itself, since $x_R^k = H^k x_A^k$.

Another clear bound for x_A^k is imposed by u_{opt}^k, as any feasible allocation schedule cannot fire more often than enabled by the input. Since input firings that have occurred before time T cannot be changed, the most often u^k can possibly fire is encoded by the counter $r_T(u_{\text{opt}}^k)$, which represents the preservation of the firings up to time T and an infinite number of firings at T. The counter $\mathcal{P}^k r_T(u_{\text{opt}}^k)$ then limits how often x_A^k can fire, i.e., one must have $x_A^k \succeq \mathcal{P}^k r_T(u_{\text{opt}}^k)$.

Thus, any allocation schedule x_A^k must obey

$$x_A^k \succeq \left[(\beta \otimes (\mathcal{H}_R^k \odot H^k x_A^k \odot \mathcal{L}_R^k)) \odot^\flat (\mathcal{H}_A^k \odot \mathcal{L}_A^k)\right] \oplus \mathcal{P}^k r_T(u_{\text{opt}}^k)$$

or, equivalently, must be a fixed point of the (isotone) mapping $\Lambda^k : \Sigma \to \Sigma$,

$$\Lambda^k(\chi) = \left[(\beta \otimes (\mathcal{H}_R^k \odot H^k \chi \odot \mathcal{L}_R^k)) \odot^\flat (\mathcal{H}_A^k \odot \mathcal{L}_A^k)\right] \oplus \mathcal{P}^k r_T(u_{\text{opt}}^k) \oplus \chi\,. \tag{5.5}$$

It is easy to see that any input $\widetilde{u}^k \in \widetilde{\mathcal{N}}^k$ leads to an allocation schedule $\mathcal{P}^k \widetilde{u}^k$ which is a fixed point of Λ^k, as

$$\mathcal{P}^k \widetilde{u}^k \succeq \mathcal{P}^k r_T(\widetilde{u}^k) = \mathcal{P}^k r_T(u_{\text{opt}}^k)$$

and also $\widetilde{u}^k$ satisfies ($\star$), which is equivalent to

$$(\beta \otimes (\mathcal{H}_R^k \odot H^k \mathcal{P}^k u^k \odot \mathcal{L}_R^k)) \odot^\flat (\mathcal{H}_A^k \odot \mathcal{L}_A^k) \preceq \mathcal{P}^k u^k\,. \tag{5.6}$$

The remaining question then is whether the *least* fixed point of mapping Λ^k — which we shall denote $\underline{x}_A^k$ — is indeed feasible, i.e., whether there exists an input $\underline{u}^k$ which is an element of $\widetilde{\mathcal{N}}^k$ and such that $\mathcal{P}^k \underline{u}^k = \underline{x}_A^k$. We proceed to present a constructive proof that the answer is positive.

Consider the input

$$\underline{u}^k = \begin{bmatrix} r_T(u_{\text{opt}}^{k0}) \\ \underline{x}_A^k \end{bmatrix}. \tag{5.7}$$

To see that $\mathcal{P}^k \underline{u}^k = \underline{x}_A^k$, recalling that we denote $\mathcal{P}^k = [P^{k1}\ \ s_e]$, as $\underline{x}_A^k$ is a fixed point of Λ^k we have

$$\underline{x}_A^k \succeq \mathcal{P}^k r_T(u_{\text{opt}}^k) = P^{k1} r_T(u_{\text{opt}}^{k0}) \oplus r_T(u_{\text{opt}}^{k1}) \succeq P^{k1} r_T(u_{\text{opt}}^{k0})$$

and hence

$$\mathcal{P}^k \underline{u}^k = P^{k1} r_T(u_{\text{opt}}^{k0}) \oplus \underline{x}_A^k = \underline{x}_A^k\,.$$

Now, to prove that $\underline{u}^k \in \widetilde{\mathcal{N}}^k$, we begin by noticing that, because $\underline{x}_A^k$ is a fixed point of Λ^k,

$$(\beta \otimes (\mathcal{H}_R^k \odot H^k \underline{x}_A^k \odot \mathcal{L}_R^k)) \odot^\flat (\mathcal{H}_A^k \odot \mathcal{L}_A^k) \preceq \underline{x}_A^k\,.$$

Combined with the fact that $\mathcal{P}^k \underline{u}^k = \underline{x}_A^k$ as shown above, this implies taking $u^k = \underline{u}^k$ satisfies (5.6), which is equivalent to ($\star$).

It remains to show that $r_T(\underline{u}^k) = r_T(u^k_{\text{opt}})$, i.e., that

$$\begin{bmatrix} r_T\big(r_T(u^{k0}_{\text{opt}})\big) \\ r_T(\underline{x}_A^k) \end{bmatrix} = \begin{bmatrix} r_T(u^{k0}_{\text{opt}}) \\ r_T(u^{k1}_{\text{opt}}) \end{bmatrix}.$$

Since $r_T\big(r_T(u^{k0}_{\text{opt}})\big) = r_T(u^{k0}_{\text{opt}})$, all we need to prove is that $r_T(\underline{x}_A^k) = r_T(u^{k1}_{\text{opt}})$.

The fact that $\underline{x}_A^k$ is a fixed point of Λ^k implies

$$\underline{x}_A^k \succeq \mathcal{P}^k r_T(u^k_{\text{opt}}) = P^{k1} r_T(u^{k0}_{\text{opt}}) \oplus r_T(u^{k1}_{\text{opt}}) \succeq r_T(u^{k1}_{\text{opt}}),$$

so

$$r_T(\underline{x}_A^k) \succeq r_T\big(r_T(u^{k1}_{\text{opt}})\big) = r_T(u^{k1}_{\text{opt}}). \tag{5.8}$$

In order to conclude the argument by showing that $r_T(\underline{x}_A^k) \preceq r_T(u^k_{\text{opt}})$, we need the following result.

Proposition 5.2. *$r_T^\sharp(x^k_{A_{\text{opt}}})$ is a fixed point of mapping Λ^k.*

A consequence of Prop. 5.2 is that $\underline{x}_A^k \preceq r_T^\sharp(x^k_{A_{\text{opt}}}) = r_T^\sharp(\mathcal{P}^k u^k_{\text{opt}})$. We also know from Remark 4.1 that $\mathcal{P}^k u^k_{\text{opt}} = u^{k1}_{\text{opt}}$. Thus, as r_T is order-preserving and $r_T \circ r_T^\sharp = r_T$, we have

$$r_T(\underline{x}_A^k) \preceq r_T\big(r_T^\sharp(\mathcal{P}^k u^k_{\text{opt}})\big) = r_T\big(r_T^\sharp(u^{k1}_{\text{opt}})\big) = r_T(u^{k1}_{\text{opt}}).$$

Together with (5.8), this leads to $r_T(\underline{x}_A^k) = r_T(u^{k1}_{\text{opt}})$ and hence $r_T(\underline{u}^k) = r_T(u^k_{\text{opt}})$, concluding the proof that $\underline{u}^k \in \widetilde{\mathcal{N}^k}$.

This does not guarantee, however, that $\mathcal{N}^k \neq \emptyset$, as it is possible that $\mathcal{G}^k \underline{u}^k \not\preceq z^{k\prime}$ and hence $\underline{u}^k \notin \mathcal{N}^k$. Emptiness of $\mathcal{N}^k$ means $z^{k\prime}$ encodes an unachievable reference; ($\star$) and $r_T(u^k) = r_T(u^k_{\text{opt}})$ being hard restrictions, the only possibility is then to relax $z^{k\prime}$, i.e., to look for a new reference $z^{k\prime\prime} \succeq z^{k\prime}$ for which a solution exists. At the same time, we want to remain as close to the original reference as possible, meaning we seek the least possible such $z^{k\prime\prime}$. A natural approach is then to take $z^{k\prime\prime} = z^{k\prime} \oplus \mathcal{G}^k \underline{u}^k$ — as $\oplus$ is performed coefficient-wise on counters, this amounts to preserving the terms of $z^{k\prime}$ that can be achieved by taking $\underline{u}^k$ as input, and relaxing those that cannot only as much as necessary to be matched by the resulting output $y^k = \mathcal{G}^k \underline{u}^k$. The following proposition establishes that this is indeed the optimal way of relaxing $z^{k\prime}$ and, as a corollary, it also provides a simple way to check whether $\mathcal{N}^k$ is nonempty.

Proposition 5.3. *Let $\mathcal{N}^{k\prime\prime}$ denote the set defined as $\mathcal{N}^k$ in (5.2), only replacing $z^{k\prime}$ with $z^{k\prime\prime}$, and let $\underline{u}^k$ be defined as in (5.7). The least $z^{k\prime\prime} \succeq z^{k\prime}$ such that $\mathcal{N}^{k\prime\prime} \neq \emptyset$ is $z^{k\prime\prime} = z^{k\prime} \oplus \mathcal{G}^k \underline{u}^k$.*

Corollary 5.4. *With set $\mathcal{N}^k$ defined by (5.2) and $\underline{u}^k$ as in (5.7), it follows that $\mathcal{N}^k \neq \emptyset \Leftrightarrow \mathcal{G}^k \underline{u}^k \preceq z^{k\prime}$.*

In the case $\mathcal{N}^k$ turns out to be empty, define the mapping $\Psi^{k\prime\prime} : \Sigma^{2\times 1} \to \Sigma^{2\times 1}$ as Ψ^k in (5.3), only replacing $z^{k\prime}$ with $z^{k\prime\prime} = z^{k\prime} \oplus \mathcal{G}^k \underline{u}^k$. Following the same procedure as before, we can apply Prop. 5.1 — only now taking ψ as $\Psi^{k\prime\prime}$ instead of Ψ^k — to conclude that $u^{k\prime}_{\text{opt}}$ is the greatest fixed point of mapping $\Gamma^{k\prime\prime} : \Sigma^{2\times 1} \to \Sigma^{2\times 1}$,

$$\Gamma^{k\prime\prime}(u^k) = u^k \wedge \Psi^{k\prime\prime}(u^k) \wedge r^{\sharp}_T(u^k_{\text{opt}}) \,. \tag{5.9}$$

Remark 5.1. Since, according to the adopted priority policy, when determining the optimal inputs through the method from Section 4.2 we do not allow lower-priority subsystems to compromise the performance of higher-priority ones, it should be clear that, if the reference of S^1 is not changed (i. e., if $z^{1\prime} = z^1$), then the method above will yield $u^{1\prime}_{\text{opt}} = u^1_{\text{opt}}$, regardless of the changes in the references of other subsystems. Note that this implies $x^1_{A_{\text{opt}}} = x^{1\prime}_{A_{\text{opt}}}$ and $x^1_{R_{\text{opt}}} = x^{1\prime}_{R_{\text{opt}}}$. Hence, if it is also the case that $z^{2\prime} = z^2$, then the operation of S^2 will likewise remain unchanged, i. e., we will have $u^{2\prime}_{\text{opt}} = u^2_{\text{opt}}$. In general, for any $k \in \{2, \ldots, K\}$, if $z^{i\prime} = z^i$ for all $i \in \{1, \ldots, k-1\}$, then $u^{i\prime}_{\text{opt}} = u^i_{\text{opt}}$ for all $i \in \{1, \ldots, k-1\}$. In that case, supposing κ is the least index in $\{1, \ldots, K\}$ such that $z^{\kappa\prime} \neq z^\kappa$, the method presented in this section can be applied starting from S^κ and taking $x^{i\prime}_{A_{\text{opt}}} = x^i_{A_{\text{opt}}}$ and $x^{i\prime}_{R_{\text{opt}}} = x^i_{R_{\text{opt}}}$ for all $i \in \{1, \ldots, \kappa - 1\}$. Nonetheless, the input of *all* subsystems S^j with $j \in \{\kappa + 1, \ldots, K\}$ must be updated, even if $z^{j\prime} = z^j$. That is because, if the new reference $z^{\kappa\prime}$ implies $u^{\kappa\prime}_{\text{opt}} \neq u^\kappa_{\text{opt}}$ and thus $x^{\kappa\prime}_{A_{\text{opt}}} \neq x^\kappa_{A_{\text{opt}}}$ and $x^{\kappa\prime}_{R_{\text{opt}}} \neq x^\kappa_{R_{\text{opt}}}$, the availability of the resource for a lower-priority subsystem S^j is changed and the allocation schedule $x^j_{A_{\text{opt}}}$ resulting from its original input u^j_{opt} may no longer be compatible, in which case the method will yield $u^{j\prime}_{\text{opt}} \neq u^j_{\text{opt}}$. ◇

SUMMARY OF THE METHOD

In the following, we provide a step-by-step overview of how to apply the method discussed in the present chapter so far. For a system consisting of subsystems S^k, $k \in \{1, \ldots, K\}$, sharing a resource as in Figure 4.1, we assume that TEGs modeling all subsystems S^k are given. Assume also, for each $k \in \{1, \ldots, K\}$, the transfer relation $\mathcal{G}^k$ (see (3.4)) to have been precomputed and an output-reference to be provided in the form of a counter z^k. We consider here the case of a single shared resource; nonetheless, the generalization of the steps for the case of multiple shared resources (Section 5.3) is immediate.

I. Obtain the optimal input for the highest-priority subsystem, S^1, by computing $u^1_{\text{opt}} = \mathcal{G}^1 \backslash\!\!\circ z^1$. Then, compute the corresponding resource-allocation and release schedules, $x^1_{A_{\text{opt}}}$ and $x^1_{R_{\text{opt}}}$.

II. Obtain the optimal inputs for all other subsystems in decreasing order of priority — i.e., starting from S^2 and proceeding until covering S^K — by computing the greatest fixed points of the respective mappings Φ^k defined in (4.11).

III. If, at a certain time T, the reference signals z^k of one or more of the subsystems are altered, let κ denote the least index in $\{1, \ldots, K\}$ for which $z^{\kappa\prime} \neq z^\kappa$, i.e., $z^{i\prime} = z^i$ for all (if any) $i \in \{1, \ldots, \kappa - 1\}$. Set $k = \kappa$.

IV. Compute the terms $\mathcal{H}^k_A$, $\mathcal{H}^k_R$, $\mathcal{L}^k_A$, and $\mathcal{L}^k_R$, defined as in Section 5.1, and define the set $\mathcal{N}^k$ as in (5.2).

V. In order to check whether the new reference $z^{k\prime}$ is feasible (i.e., whether $\mathcal{N}^k \neq \emptyset$) based on Corollary 5.4, obtain $\underline{u}^k$ as in (5.7); as a prerequisite, compute $\underline{x}^k_A$, the least fixed point of mapping Λ^k defined in (5.5). Then, compute $\mathcal{G}^k \underline{u}^k$. If $\mathcal{G}^k \underline{u}^k \preceq z^{k\prime}$, go to step VI; otherwise, go to step VII.

VI. Obtain the optimal updated input $u^{k\prime}_{\text{opt}}$ by computing the greatest fixed point of mapping Γ^k defined in (5.4).

VII. According to Prop. 5.3, obtain the least-relaxed feasible reference $z^{k\prime\prime} = z^{k\prime} \oplus \mathcal{G}^k \underline{u}^k$. Then, obtain the optimal updated input $u^{k\prime}_{\text{opt}}$ by computing the greatest fixed point of mapping $\Gamma^{k\prime\prime}$ defined in (5.9).

VIII. Repeat steps IV–VII for every $k \in \{\kappa + 1, \ldots, K\}$, in this order.

Example 5.1. We now apply the method summarized above to the system from Fig. 4.5. Steps I and II have already been taken in Example 4.1, so let us assume the system is operating according to the obtained optimal schedules. Now, suppose the output-references are changed at time $T = 10$ as follows: one of the previously required output firings from S^1 is cancelled, so that now only 3 firings of y^1 are demanded by time 52; an additional firing of y^2 is required, so 3 firings are needed at time 39; the reference for S^3 remains unaltered. The counters encoding these updated references are

$$\begin{aligned} z^{1\prime} &= e\delta^{52} \oplus 3\delta^{+\infty}, \\ z^{2\prime} &= e\delta^{27} \oplus 3\delta^{39} \oplus 6\delta^{+\infty}, \\ z^{3\prime} &= e\delta^{9} \oplus 3\delta^{35} \oplus 5\delta^{+\infty}. \end{aligned}$$

Following step III, since in this case $z^{1\prime} \neq z^1$ we must start from $k = 1$. Through steps IV and V, we find out that $z^{1\prime}$ is feasible, as

$\mathcal{G}^1 \underline{u}^1 \preceq z^{1\prime}$ — which in this case is intuitively to be expected, as the new reference is less demanding than the original one. Hence, we proceed to step VI, and the greatest fixed point of mapping Γ^1 yields

$$u^{1\prime}_{\text{opt}} = \begin{bmatrix} e\delta^5 \oplus 1\delta^{25} \oplus 2\delta^{35} \oplus 3\delta^{+\infty} \\ e\delta^7 \oplus 1\delta^{27} \oplus 2\delta^{37} \oplus 3\delta^{+\infty} \end{bmatrix} .$$

The updated resource-allocation and release schedules are

$$\begin{aligned} x^{1\prime}_{A_{\text{opt}}} &= e\delta^7 \oplus 1\delta^{27} \oplus 2\delta^{37} \oplus 3\delta^{+\infty} , \\ x^{1\prime}_{R_{\text{opt}}} &= e\delta^{13} \oplus 1\delta^{33} \oplus 2\delta^{43} \oplus 3\delta^{+\infty} . \end{aligned}$$

Next, we update the input of S^2, i. e., we go back to step IV with $k = 2$. The new reference $z^{2\prime}$ is also achievable, so from step V we again go to step VI. The greatest fixed point of mapping Γ^2 is

$$u^{2\prime}_{\text{opt}} = \begin{bmatrix} e\delta^{12} \oplus 1\delta^{17} \oplus 3\delta^{22} \oplus 4\delta^{27} \oplus 5\delta^{30} \oplus 6\delta^{+\infty} \\ e\delta^{17} \oplus 1\delta^{22} \oplus 3\delta^{27} \oplus 4\delta^{32} \oplus 5\delta^{35} \oplus 6\delta^{+\infty} \end{bmatrix} ,$$

resulting in the new allocation and release schedules

$$\begin{aligned} x^{2\prime}_{A_{\text{opt}}} &= e\delta^{17} \oplus 1\delta^{22} \oplus 3\delta^{27} \oplus 4\delta^{32} \oplus 5\delta^{35} \oplus 6\delta^{+\infty} , \\ x^{2\prime}_{R_{\text{opt}}} &= e\delta^{21} \oplus 1\delta^{26} \oplus 3\delta^{31} \oplus 4\delta^{36} \oplus 5\delta^{39} \oplus 6\delta^{+\infty} . \end{aligned}$$

Finally, the input of S^3 is updated by continuing from step IV with $k = 3$. Even though reference z^3 has not changed, we must check whether it is still feasible given the new schedules of S^1 and S^2. In this case, the answer is affirmative, so once more we proceed to step VI and compute the greatest fixed point of mapping Γ^3, obtaining

$$u^{3\prime}_{\text{opt}} = e\delta^1 \oplus 1\delta^3 \oplus 2\delta^5 \oplus 3\delta^9 \oplus 4\delta^{18} \oplus 5\delta^{+\infty}$$

and

$$\begin{aligned} x^{3\prime}_{A_{\text{opt}}} &= e\delta^1 \oplus 1\delta^3 \oplus 2\delta^5 \oplus 3\delta^9 \oplus 4\delta^{18} \oplus 5\delta^{+\infty} , \\ x^{3\prime}_{R_{\text{opt}}} &= e\delta^4 \oplus 1\delta^6 \oplus 2\delta^8 \oplus 3\delta^{12} \oplus 4\delta^{21} \oplus 5\delta^{+\infty} . \end{aligned}$$

The resulting updated outputs are

$$\begin{aligned} y^{1\prime}_{\text{opt}} &= \mathcal{G}^1 u^{1\prime}_{\text{opt}} &&= e\delta^{22} \oplus 1\delta^{42} \oplus 2\delta^{52} \oplus 3\delta^{+\infty} , \\ y^{2\prime}_{\text{opt}} &= \mathcal{G}^2 u^{2\prime}_{\text{opt}} &&= e\delta^{21} \oplus 1\delta^{26} \oplus 3\delta^{31} \oplus 4\delta^{36} \oplus 5\delta^{39} \oplus 6\delta^{+\infty} , \\ y^{3\prime}_{\text{opt}} &= \mathcal{G}^3 u^{3\prime}_{\text{opt}} &&= e\delta^4 \oplus 1\delta^6 \oplus 2\delta^8 \oplus 3\delta^{12} \oplus 4\delta^{21} \oplus 5\delta^{+\infty} . \end{aligned}$$

Similarly to Example 4.1, the tracking of the new references can be visualized in Figure 5.1, and the updated resource-occupation schedule is shown in Figure 5.2. ◇

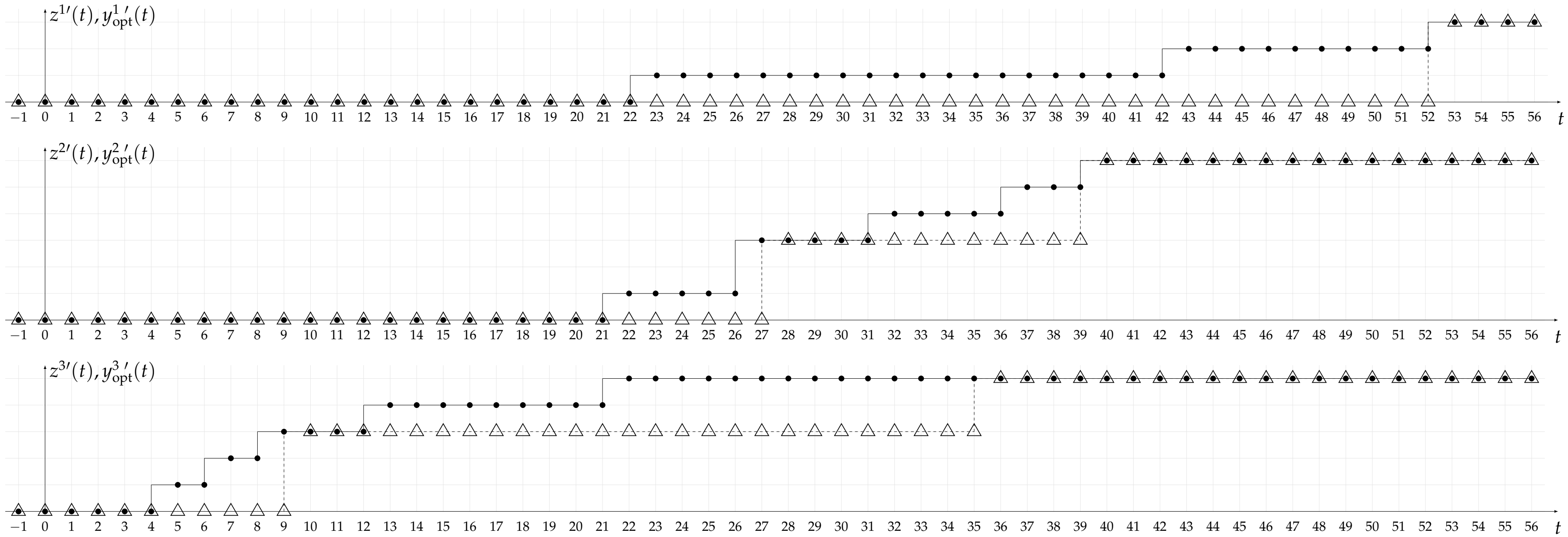

Figure 5.1: Tracking of the new references $z^{k\prime}$ ($\triangle$) by the updated outputs $y^{k\prime}_{\mathrm{opt}}$ ($\bullet$), $k \in \{1, 2, 3\}$, from Example 5.1.

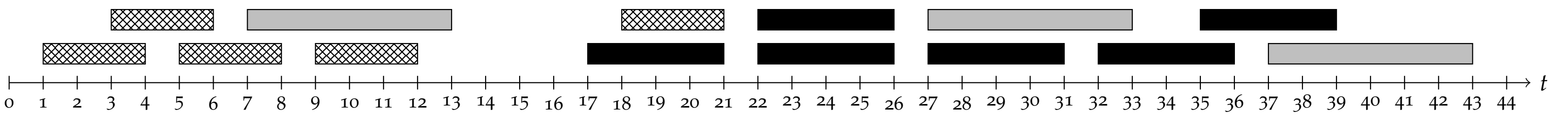

Figure 5.2: Updated schedule for the use of the shared resource, obtained in Example 5.1. The gray, black, and crosshatched bars represent the time windows during which an instance of the resource is held by S^1, S^2, and S^3, respectively.

Example 5.2. Consider once more the system from Fig. 4.5, only now operating according to the updated schedules obtained in Example 5.1. Since here we take these previously-updated schedules as the starting point, the extra notation is dropped, so e. g. we refer to the references received in Example 5.1 and to the obtained optimal updated inputs respectively as z^k and u^k_{opt}, $k \in \{1,2,3\}$.

Suppose that, at time $T = 15$, yet another demand for S^2 arrives: an additional firing of y^2 is required at time 39, so that the counter encoding the new reference for S^2 becomes

$$z^{2\prime} = e\delta^{27} \oplus 3\delta^{39} \oplus 7\delta^{+\infty},$$

whereas the references for S^1 and S^3 remain unchanged. According to Remark 5.1 and as instructed in step III of the procedure summarized above, we can then start updating the inputs by subsystem S^2. The updated reference $z^{2\prime}$ is still achievable, so from step V we proceed to step VI and compute the greatest fixed point of Γ^2, obtaining

$$u^{2\prime}_{\text{opt}} = \begin{bmatrix} e\delta^{12} \oplus 2\delta^{17} \oplus 4\delta^{22} \oplus 5\delta^{27} \oplus 6\delta^{30} \oplus 7\delta^{+\infty} \\ e\delta^{17} \oplus 2\delta^{22} \oplus 4\delta^{27} \oplus 5\delta^{32} \oplus 6\delta^{35} \oplus 7\delta^{+\infty} \end{bmatrix}.$$

The allocation and release schedules are updated to

$$\begin{aligned} x^{2\prime}_{A_{\text{opt}}} &= e\delta^{17} \oplus 2\delta^{22} \oplus 4\delta^{27} \oplus 5\delta^{32} \oplus 6\delta^{35} \oplus 7\delta^{+\infty}, \\ x^{2\prime}_{R_{\text{opt}}} &= e\delta^{21} \oplus 2\delta^{26} \oplus 4\delta^{31} \oplus 5\delta^{36} \oplus 6\delta^{39} \oplus 7\delta^{+\infty}. \end{aligned}$$

We proceed to update the input of S^3. However, in this case the new, tighter schedule of S^2 renders reference $z^{3\prime} = z^3$ infeasible. Hence, from step V we go to step VII and apply Prop. 5.3 to obtain its least-relaxed feasible version:

$$z^{3\prime\prime} = z^{3\prime} \oplus \mathcal{G}^3 \underline{u}^3 = e\delta^9 \oplus 3\delta^{35} \oplus 4\delta^{43} \oplus 5\delta^{+\infty}.$$

The greatest fixed point of mapping $\Gamma^{3\prime\prime}$ then provides the optimal updated input

$$u^{3\prime}_{\text{opt}} = e\delta^1 \oplus 1\delta^3 \oplus 2\delta^5 \oplus 3\delta^9 \oplus 4\delta^{40} \oplus 5\delta^{+\infty}$$

which results in the following resource-allocation and release schedules:

$$\begin{aligned} x^{3\prime}_{A_{\text{opt}}} &= e\delta^1 \oplus 1\delta^3 \oplus 2\delta^5 \oplus 3\delta^9 \oplus 4\delta^{40} \oplus 5\delta^{+\infty}, \\ x^{3\prime}_{R_{\text{opt}}} &= e\delta^4 \oplus 1\delta^6 \oplus 2\delta^8 \oplus 3\delta^{12} \oplus 4\delta^{43} \oplus 5\delta^{+\infty}. \end{aligned}$$

The resulting updated outputs (besides $y^{1\prime}_{\text{opt}} = y^1_{\text{opt}}$) are

$$\begin{aligned} y^{2\prime}_{\text{opt}} &= \mathcal{G}^2 u^{2\prime}_{\text{opt}} = e\delta^{21} \oplus 2\delta^{26} \oplus 4\delta^{31} \oplus 5\delta^{36} \oplus 6\delta^{39} \oplus 7\delta^{+\infty}, \\ y^{3\prime}_{\text{opt}} &= \mathcal{G}^3 u^{3\prime}_{\text{opt}} = e\delta^4 \oplus 1\delta^6 \oplus 2\delta^8 \oplus 3\delta^{12} \oplus 4\delta^{43} \oplus 5\delta^{+\infty}. \end{aligned}$$

Figures 5.3 and 5.4 show respectively the tracking of references $z^{2\prime}$ and $z^{3\prime\prime}$ and the newly updated schedule for resource usage. ◇

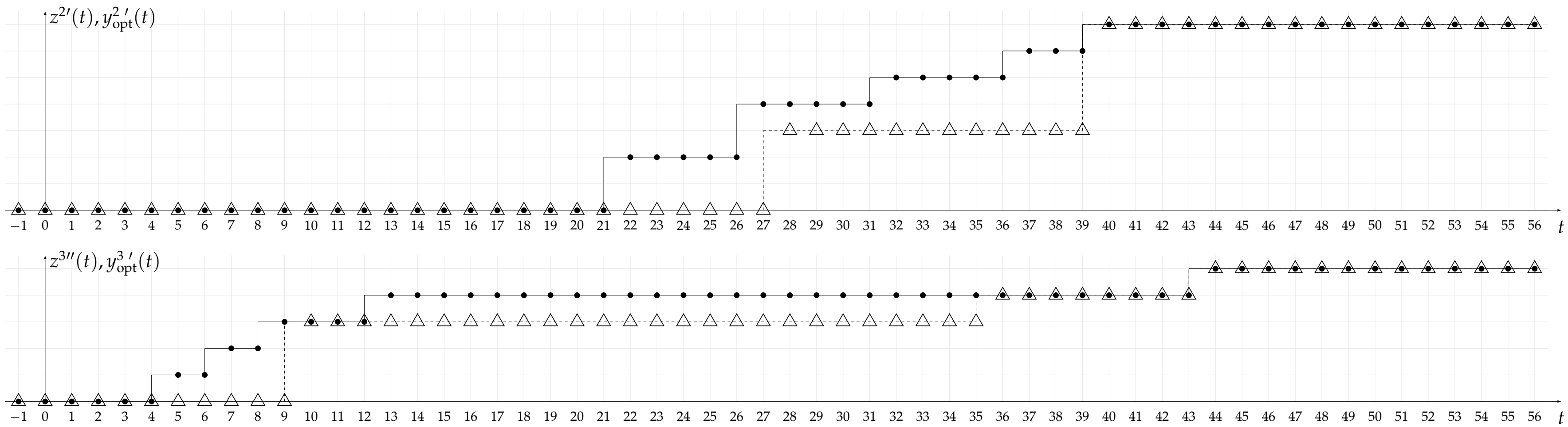

Figure 5.3: Tracking of the new references $z^{2\prime}$ and $z^{3\prime\prime}$ ($\triangle$) by the updated outputs $y^{2\prime}_{\text{opt}}$ and $y^{3\prime}_{\text{opt}}$ ($\bullet$) obtained in Example 5.2.

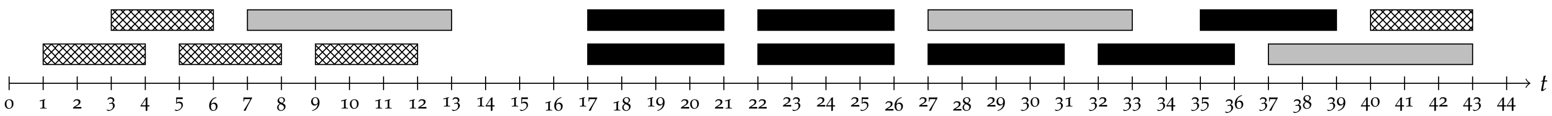

Figure 5.4: Updated schedule for the use of the shared resource, obtained in Example 5.2. The gray, black, and crosshatched bars represent the time windows during which an instance of the resource is held by S^1, S^2, and S^3, respectively.

5.3 PROBLEM FORMULATION AND OPTIMAL UPDATE OF THE INPUTS — THE CASE OF MULTIPLE SHARED RESOURCES

Consider the system from Fig. 4.5, with every subsystem S^k following the optimal schedule with respect to output-reference z^k, obtained according to Section 4.3. Suppose that each reference z^k is updated to $z^{k\prime}$ at time T (with perhaps $z^{k\prime} = z^k$ for some of them). In this section, we seek, for each k, the optimal input $u^{k\prime}_{\text{opt}}$ which preserves u^k_{opt} up to time T and results in the output $y^{k\prime}_{\text{opt}}$ that tracks $z^{k\prime}$ as closely as possible, without interfering with the operation of higher-priority subsystems and while respecting the past resource allocations of every resource by lower-priority subsystems.

As usual, we base the following discussion on a fixed but arbitrary $k \in \{1, \ldots, K\}$. Let us denote the counter representing the updated optimal firing schedule for a resource-allocation transition $x^{i\ell}_A$ by $x^{i\ell\,\prime}_{A_{\text{opt}}}$. Arguing as in Section 5.1, the task at hand can be summarized as follows: (i) we must compute $u^{k\prime}_{\text{opt}}$ in decreasing order of priority; (ii) when calculating $u^{k\prime}_{\text{opt}}$ for $k > 1$, we must consider $x^{i\ell}_A = x^{i\ell\,\prime}_{A_{\text{opt}}}$ for every $i \in \{1, \ldots, k-1\}$ and for all $\ell \in \{1, \ldots, L\}$; (iii) when calculating $u^{k\prime}_{\text{opt}}$ for $k < K$, we must take $x^{j\ell}_A = r^\sharp_T(x^{j\ell}_{A_{\text{opt}}}$ for every $j \in \{k+1, \ldots, K\}$ and for all $\ell \in \{1, \ldots, L\}$.

Still along the lines of Section 5.1, define the terms

$$\mathcal{H}^{k\ell}_A = \bigodot_{i=1}^{k-1} x^{i\ell\,\prime}_{A_{\text{opt}}}\,, \qquad \mathcal{H}^{k\ell}_R = \bigodot_{i=1}^{k-1} x^{i\ell\,\prime}_{R_{\text{opt}}}\,,$$

$$\mathcal{L}^{k\ell}_A = \bigodot_{j=k+1}^{K} r^\sharp_T(x^{j\ell}_{A_{\text{opt}}})\,, \qquad \mathcal{L}^{k\ell}_R = \bigodot_{j=k+1}^{K} H^{j\ell} r^\sharp_T(x^{j\ell}_{A_{\text{opt}}})\,,$$

which can be explained as in the referred section, only now for each resource ℓ. In order to achieve the goals stated above, based on (4.12) we must respect, for every $\ell \in \{1, \ldots, L\}$,

$$\beta^\ell \otimes \left(\mathcal{H}^{k\ell}_R \odot x^{k\ell}_R \odot \mathcal{L}^{k\ell}_R\right) \preceq \mathcal{H}^{k\ell}_A \odot x^{k\ell}_A \odot \mathcal{L}^{k\ell}_A\,, \tag{5.10}$$

where it is understood that for $k = 1$ (resp. $k = K$), the degenerate terms $\mathcal{H}^{1\ell}_A$ and $\mathcal{H}^{1\ell}_R$ (resp. $\mathcal{L}^{K\ell}_A$ and $\mathcal{L}^{K\ell}_R$) are to be neglected. Recall that, for a just-in-time input u^k leading to schedules of $x^{k\ell}_A$ and $x^{k\ell}_R$ that satisfy (5.10), using (4.17) we can write $x^{k\ell}_A = \mathcal{P}^{k\ell} u^k$ and $x^{k\ell}_R = H^{k\ell}\mathcal{P}^{k\ell} u^k$, so (5.10) becomes

$$\beta^\ell \otimes \left(\mathcal{H}^{k\ell}_R \odot H^{k\ell}\mathcal{P}^{k\ell} u^k \odot \mathcal{L}^{k\ell}_R\right) \preceq \mathcal{H}^{k\ell}_A \odot \mathcal{P}^{k\ell} u^k \odot \mathcal{L}^{k\ell}_A\,. \tag{$\star\star$}$$

We can then formulate the problem of optimally updating the input u^k with respect to a reference $z^{k\prime}$ given at time T as follows: find the greatest element of the set

$$\mathcal{M}^k = \big\{u^k \in \Sigma^{(L+1)\times 1} \,|\, \mathcal{G}^k u^k \preceq z^{k\prime} \text{ and } r_T(u^k) = r_T(u^k_{\text{opt}}) \\ \text{and } (\star\star) \text{ holds for all } \ell \in \{1, \ldots, L\}\big\}. \tag{5.11}$$

Define the mappings $\Psi^{k\ell} : \Sigma^{(L+1)\times 1} \to \Sigma^{(L+1)\times 1}$,

$$\Psi^{k\ell}(u^k) \;=\; (H^{k\ell}\mathcal{P}^{k\ell}) \backslash\!\!\circ \big[\big(\beta^{\ell} \backslash\!\!\circ (\mathcal{H}_A^{k\ell} \odot \mathcal{P}^{k\ell} u^k \odot \mathcal{L}_A^{k\ell})\big) \odot^{\sharp} (\mathcal{H}_R^{k\ell} \odot \mathcal{L}_R^{k\ell})\big]\,,$$

$\ell \in \{1, \dots, L\}$, and $\overline{\Psi}^k : \Sigma^{(L+1)\times 1} \to \Sigma^{(L+1)\times 1}$,

$$\overline{\Psi}^k(u^k) \;=\; \mathcal{G}^k \backslash\!\!\circ z^{k\prime} \,\wedge\, \bigwedge_{\ell=1}^{L} \Psi^{k\ell}(u^k)\,. \tag{5.12}$$

As u^k satisfying ($\star\star$) is equivalent to $u^k \preceq \Psi^{k\ell}(u^k)$, we can rewrite $\mathcal{M}^k$ as

$$\mathcal{M}^k = \{u^k \in \Sigma^{(L+1)\times 1} \,|\, u^k \preceq \overline{\Psi}^k(u^k) \text{ and } r_T(u^k) = r_T(u^k_{\text{opt}})\}$$

and then apply Prop. 5.1, taking $\mathcal{D}$ as $\Sigma^{(L+1)\times 1}$, ψ as $\overline{\Psi}^k$, f as r_T, and c as $r_T(u^k_{\text{opt}})$. Provided $\mathcal{M}^k \neq \emptyset$, the proposition entails that $u^{k\prime}_{\text{opt}}$ can be determined by computing the greatest fixed point of the (isotone) mapping $\overline{\Gamma}^k : \Sigma^{(L+1)\times 1} \to \Sigma^{(L+1)\times 1}$,

$$\overline{\Gamma}^k(u^k) \;=\; u^k \,\wedge\, \overline{\Psi}^k(u^k) \,\wedge\, r_T^{\sharp}(u^k_{\text{opt}})\,.$$

In order to check whether $\mathcal{M}^k$ is nonempty, consider the set

$$\widetilde{\mathcal{M}}^k = \{u^k \in \Sigma^{(L+1)\times 1} \,|\, (\star\star) \text{ holds for all } \ell \in \{1, \dots, L\} \text{ and } r_T(u^k) = r_T(u^k_{\text{opt}})\}. \tag{5.13}$$

We once more adopt the approach of looking for an element $\underline{u}^k$ of $\widetilde{\mathcal{M}}^k$ that leads to the fastest possible behavior of S^k, i. e., to the least possible output y^k. It shall then be formally concluded below (see Corollary 5.9) that $\mathcal{M}^k \neq \emptyset \Leftrightarrow \mathcal{G}^k \underline{u}^k \preceq z^{k\prime}$. Since $\widetilde{\mathcal{M}}^k$ does not necessarily possess a least element, we focus on finding an input in $\widetilde{\mathcal{M}}^k$ that leads to the least allocation schedules of all resources compatible with the respective constraints, which will result in the least possible y^k.

Let us begin the search for such an input by observing that, for the first resource, condition (5.10) provides a bound for how small (in the sense of the order of Σ) the allocation schedule x_A^{k1} can be, as

$$\begin{aligned} \beta^1 \otimes (\mathcal{H}_R^{k1} \odot x_R^{k1} \odot \mathcal{L}_R^{k1}) &\preceq \mathcal{H}_A^{k1} \odot x_A^{k1} \odot \mathcal{L}_A^{k1} \\ &\Leftrightarrow\; \beta^1 \otimes (\mathcal{H}_R^{k1} \odot x_R^{k1} \odot \mathcal{L}_R^{k1}) \preceq (\mathcal{H}_A^{k1} \odot \mathcal{L}_A^{k1}) \odot x_A^{k1} \\ &\Leftrightarrow\; (\beta^1 \otimes (\mathcal{H}_R^{k1} \odot x_R^{k1} \odot \mathcal{L}_R^{k1})) \odot^{\flat} (\mathcal{H}_A^{k1} \odot \mathcal{L}_A^{k1}) \;\preceq\; x_A^{k1}\,. \end{aligned}$$

The left-hand side of the last inequality represents the maximal availability of resource 1 for subsystem S^k, given the fixed optimal allocation and release schedules of higher-priority subsystems ($\mathcal{H}_A^{k1}$ and $\mathcal{H}_R^{k1}$) and of lower priority subsystems, truncated at time T ($\mathcal{L}_A^{k1}$ and $\mathcal{L}_R^{k1}$); naturally, this availability also implicitly depends on x_A^{k1} itself, since $x_R^{k1} = H^{k1} x_A^{k1}$.

Another bound for x_A^{k1} is imposed by u^k_{opt}. More precisely, since the most often u^k can possibly fire is encoded by the counter $r_T(u^k_{\text{opt}})$

and since any feasible allocation schedule cannot fire more often than enabled by the input, $\mathcal{P}^{k1} r_T(u^k_{\text{opt}})$ limits how often x^{k1}_A can fire, i. e., it must hold that $x^{k1}_A \succeq \mathcal{P}^{k1} r_T(u^k_{\text{opt}})$.

Thus, the allocation schedule x^{k1}_A must obey

$$x^{k1}_A \succeq \left[(\beta^1 \otimes (\mathcal{H}^{k1}_R \odot H^{k1} x^{k1}_A \odot \mathcal{L}^{k1}_R)) \odot^\flat (\mathcal{H}^{k1}_A \odot \mathcal{L}^{k1}_A)\right] \oplus \mathcal{P}^{k1} r_T(u^k_{\text{opt}})\,;$$

equivalently, it must be a fixed point of the mapping $\Lambda^{k1} : \Sigma \to \Sigma$,

$$\begin{aligned}\Lambda^{k1}(\chi) = {} & \left[(\beta^1 \otimes (\mathcal{H}^{k1}_R \odot H^{k1} \chi \odot \mathcal{L}^{k1}_R)) \odot^\flat (\mathcal{H}^{k1}_A \odot \mathcal{L}^{k1}_A)\right] \\ & \oplus \mathcal{P}^{k1} r_T(u^k_{\text{opt}}) \oplus \chi\,.\end{aligned} \tag{5.14}$$

Following a similar reasoning, for any $\ell \in \{2, \dots, L\}$, condition (5.10) implies $x^{k\ell}_A$ is subject to

$$x^{k\ell}_A \succeq (\beta^\ell \otimes (\mathcal{H}^{k\ell}_R \odot H^{k\ell} x^{k\ell}_A \odot \mathcal{L}^{k\ell}_R)) \odot^\flat (\mathcal{H}^{k\ell}_A \odot \mathcal{L}^{k\ell}_A)\,.$$

Evidently, it is also not possible for $x^{k\ell}_A$ to fire more often than $u^{k\ell}_{\text{opt}}$, which, taking into account that the past firings of $u^{k\ell}_{\text{opt}}$ must be preserved, translates to $x^{k\ell}_A \succeq r_T(u^{k\ell}_{\text{opt}})$.

Moreover, allocations of resource ℓ by S^k are also (indirectly) limited by the preceding inputs in the subsystem, i. e., by $u^{k\lambda}$ with $\lambda \in \{0, \dots, \ell - 1\}$. The effect of these inputs arrives at $x^{k\ell}_A$ through $x^{k(\ell-1)}_A$. For instance, since, as argued above, x^{k1}_A cannot fire more often than encoded by $\underline{x}^{k1}_A$, it must hold that $x^{k2}_A \succeq P^{k2} H^{k1} \underline{x}^{k1}_A$. We can then conclude that x^{k2}_A must obey

$$\begin{aligned}x^{k2}_A \succeq {} & \left[(\beta^2 \otimes (\mathcal{H}^{k2}_R \odot H^{k2} x^{k2}_A \odot \mathcal{L}^{k2}_R)) \odot^\flat (\mathcal{H}^{k2}_A \odot \mathcal{L}^{k2}_A)\right] \\ & \oplus r_T(u^{k2}_{\text{opt}}) \oplus P^{k2} H^{k1} \underline{x}^{k1}_A\,,\end{aligned}$$

which is equivalent to being a fixed point of the mapping $\Lambda^{k2} : \Sigma \to \Sigma$,

$$\begin{aligned}\Lambda^{k2}(\chi) = {} & \left[(\beta^2 \otimes (\mathcal{H}^{k2}_R \odot H^{k2} \chi \odot \mathcal{L}^{k2}_R)) \odot^\flat (\mathcal{H}^{k2}_A \odot \mathcal{L}^{k2}_A)\right] \\ & \oplus r_T(u^{k2}_{\text{opt}}) \oplus P^{k2} H^{k1} \underline{x}^{k1}_A \oplus \chi\,.\end{aligned}$$

Generalizing the argument, we conclude that, for every $\ell \in \{2, \dots, L\}$, $x^{k\ell}_A$ must be a fixed point of mapping $\Lambda^{k\ell} : \Sigma \to \Sigma$,

$$\begin{aligned}\Lambda^{k\ell}(\chi) = {} & \left[(\beta^\ell \otimes (\mathcal{H}^{k\ell}_R \odot H^{k\ell} \chi \odot \mathcal{L}^{k\ell}_R)) \odot^\flat (\mathcal{H}^{k\ell}_A \odot \mathcal{L}^{k\ell}_A)\right] \\ & \oplus r_T(u^{k\ell}_{\text{opt}}) \oplus P^{k\ell} H^{k(\ell-1)} \underline{x}^{k(\ell-1)}_A \oplus \chi\,,\end{aligned} \tag{5.15}$$

with $\underline{x}^{k(\ell-1)}_A$ denoting the least fixed point of $\Lambda^{k(\ell-1)}$.

We now state the following result.

Proposition 5.5. *Consider the set $\widetilde{\mathcal{M}}^k$ defined as in (5.13), the terms $\mathcal{P}^{k\ell}$ as in (4.17), and the mappings $\Lambda^{k\ell}$ as in (5.14)/(5.15). For any $\tilde{u}^k \in \widetilde{\mathcal{M}}^k$, $\mathcal{P}^{k\ell} \tilde{u}^k$ is a fixed point of $\Lambda^{k\ell}$ for all $\ell \in \{1, \dots, L\}$.*

This means that, for every $\ell \in \{1, \ldots, L\}$, any schedule for the allocation of resource ℓ by subsystem S^k which is reachable from the inputs and which is also compatible with the resource constraints and with past input firings is a fixed point of $\Lambda^{k\ell}$. Then, what remains to be investigated is whether the least fixed points of mappings $\Lambda^{k\ell}$ are all simultaneously feasible, i. e., whether there exists an input $\underline{u}^k$ which is an element of $\widetilde{\mathcal{M}}^k$ and such that $\mathcal{P}^{k\ell}\underline{u}^k = \underline{x}_A^{k\ell}$ for all ℓ. Similarly to Section 5.2, we prove constructively that the answer is positive. As the proof is analogous to the corresponding discussion in Section 5.2, only more intricate and with a heavier notation, we state the two key facts as propositions and omit their proofs from the present discussion. The interested reader can find the proofs in Appendix A.

Consider the input

$$\underline{u}^k = \begin{bmatrix} r_T(u_{\text{opt}}^{k0}) \\ \underline{x}_A^{k1} \\ \vdots \\ \underline{x}_A^{kL} \end{bmatrix} \in \Sigma^{(L+1)\times 1}. \tag{5.16}$$

Proposition 5.6. *For every $\ell \in \{1, \ldots, L\}$, it holds that $\mathcal{P}^{k\ell}\underline{u}^k = \underline{x}_A^{k\ell}$, where $\underline{x}_A^{k\ell}$ denotes the least fixed point of mapping $\Lambda^{k\ell}$ defined as in (5.14)/(5.15), and $\mathcal{P}^{k\ell}$ is defined as in (4.17).*

Proposition 5.7. *For $\underline{u}^k$ defined as in (5.16) and $\widetilde{\mathcal{M}}^k$ as in (5.13), it follows that $\underline{u}^k \in \widetilde{\mathcal{M}}^k$.*

This does not guarantee, however, that $\mathcal{M}^k \neq \emptyset$, as one might still have $\mathcal{G}^k\underline{u}^k \npreceq z^{k\prime}$ and hence $\underline{u}^k \notin \mathcal{M}^k$. In the case $\mathcal{M}^k = \emptyset$, reference $z^{k\prime}$ is not achievable; since conditions $r_T(u^k) = r_T(u_{\text{opt}}^k)$ and ($\star\star$) for all ℓ are irrevocable, in order to find a solution we must then relax $z^{k\prime}$ into a new reference $z^{k\prime\prime} \succeq z^{k\prime}$. We want, nonetheless, to remain as close to the original reference as possible, meaning we seek the least possible such $z^{k\prime\prime}$. As argued in Section 5.2, it is then natural to take $z^{k\prime\prime} = z^{k\prime} \oplus \mathcal{G}^k\underline{u}^k$. The following proposition shows that this is indeed the optimal way of relaxing $z^{k\prime}$, and its corollary provides a way to check whether $\mathcal{M}^k$ is nonempty.

Proposition 5.8. *Let $\mathcal{M}^{k\prime\prime}$ denote the set defined as $\mathcal{M}^k$ in (5.11), only replacing $z^{k\prime}$ with $z^{k\prime\prime}$, and let $\underline{u}^k$ be defined as in (5.16). The least $z^{k\prime\prime} \succeq z^{k\prime}$ such that $\mathcal{M}^{k\prime\prime} \neq \emptyset$ is $z^{k\prime\prime} = z^{k\prime} \oplus \mathcal{G}^k\underline{u}^k$.*

Corollary 5.9. *With set $\mathcal{M}^k$ defined by (5.11) and $\underline{u}^k$ as in (5.16), it follows that $\mathcal{M}^k \neq \emptyset \Leftrightarrow \mathcal{G}^k\underline{u}^k \preceq z^{k\prime}$.*

If $\mathcal{M}^k$ is empty, define the mapping $\overline{\Psi}^{k\prime\prime} : \Sigma^{(L+1)\times 1} \to \Sigma^{(L+1)\times 1}$ as $\overline{\Psi}^k$ in (5.12), only replacing $z^{k\prime}$ with $z^{k\prime\prime}$. Following the same procedure as before, we can apply Prop. 5.1 — only now taking ψ as $\overline{\Psi}^{k\prime\prime}$ instead of $\overline{\Psi}^k$ — to conclude that $u^{k\prime}_{\text{opt}}$ is the greatest fixed point of mapping $\overline{\Gamma}^{k\prime\prime} : \Sigma^{(L+1)\times 1} \to \Sigma^{(L+1)\times 1}$,

$$\overline{\Gamma}^{k\prime\prime}(u^k) \;=\; u^k \,\wedge\, \overline{\Psi}^{k\prime\prime}(u^k) \,\wedge\, r^{\sharp}_{T}(u^k_{\text{opt}})\,.$$

Remark 5.2. Similarly to Remark 5.1, supposing the output reference z^k of each subsystem S^k is updated to $z^{k\prime}$ at time T, let κ be the least index in $\{1,\ldots,K\}$ such that $z^{\kappa\prime} \neq z^{\kappa}$, i.e., we have $z^{i\prime} = z^i$ for all $i \in \{1,\ldots,\kappa-1\}$. The method presented in this section will then result in $u^{i\prime}_{\text{opt}} = u^{i}_{\text{opt}}$ for all $i \in \{1,\ldots,\kappa-1\}$. We can, therefore, apply the method starting from S^κ and taking $x^{i\ell\,\prime}_{A_{\text{opt}}} = x^{i\ell}_{A_{\text{opt}}}$ and $x^{i\ell\,\prime}_{R_{\text{opt}}} = x^{i\ell}_{R_{\text{opt}}}$ for all $i \in \{1,\ldots,\kappa-1\}$ and for all $\ell \in \{1,\ldots,L\}$. However, the inputs of all subsystems S^j with $j \in \{\kappa+1,\ldots,K\}$ must be updated, even if $z^{j\prime} = z^j$ for some of them.

5.4 ON THE FLEXIBILITY OF THE METHOD REGARDING PRIORITY POLICY AND SYSTEM STRUCTURE

In Chapter 4 as well as in the present chapter, a fixed priority hierarchy for access to the resource has been assumed among the subsystems. Notwithstanding, it should be pointed out that the method discussed in this chapter for the case of varying output-reference signals can also be applied to a situation in which the priority order of the subsystems is rearranged during the operation. To make the idea more palpable, consider for instance the system from Figure 4.2 and assume it is operating according to the optimal schedules obtained in Example 4.1. Now, suppose that, at a certain time T, subsystems S^1 and S^2 swap priority levels, i.e., S^2 assumes the highest priority and S^1 takes the second-highest. Naturally, this makes it necessary to update the input schedules, while (as in the case of updates in the output-reference) maintaining any transition firings that occurred before time T. Then, the method from Sections 5.1 and 5.2 can be directly applied, by simply taking references $z^{k\prime}$ as the original references z^k and updating the inputs in decreasing order of priority with respect to the new priority hierarchy. Concretely, in the above example one would start by updating the input of S^2, taking into account the current occupancy of the resource by S^1 and S^3 due to past firings, but neglecting their prospective resource-allocations that have not taken place by time T. The resulting optimal updated schedule of S^2 is then considered as fixed, and the input of S^1 is updated in the standard way for a subsystem with the second-highest priority level. Finally, the inputs of S^3 must also be updated, as the availability of the resource may be affected by the readjustments in the operations of S^1 and S^2.

Another aspect that, at first glance, may seem rigid in the presented method is the structure of the system, particularly in the case of multiple shared resources (Figure 4.5). In order to simplify the presentation, we have considered so far that all subsystems require access to all resources, which may, of course, not necessarily be the case in practical scenarios. However, the case in which each subsystem only requires a subset of the total number of resources can be handled in a straightforward manner. For instance, based on Figure 4.5, suppose an arbitrary subsystem S^κ does not require access to a certain resource β^λ. One can then simply remove the arcs connecting transitions $x_A^{\kappa\lambda}$ and $x_R^{\kappa\lambda}$ to that resource, and consider blocks $P^{\kappa\lambda}$ and $H^{\kappa\lambda}$ as single places with zero holding time and no initial tokens, so that the corresponding counters become $P^{\kappa\lambda} = H^{\kappa\lambda} = s_e$. By omitting the counters $x_A^{\kappa\lambda}$ and $x_R^{\kappa\lambda}$ from inequality (4.12) for $\ell = \lambda$, the method from Section 4.3 can be directly applied. Similarly, in Section 5.3 we omit the counters $x_{A_{\text{opt}}}^{\kappa\lambda\prime}$, $x_{R_{\text{opt}}}^{\kappa\lambda\prime}$, $r_T^\sharp(x_{A_{\text{opt}}}^{\kappa\lambda})$, and $H^{\kappa\lambda} r_T^\sharp(x_{R_{\text{opt}}}^{\kappa\lambda})$ respectively from the terms $\mathcal{H}_A^{k\lambda}$, $\mathcal{H}_R^{k\lambda}$, $\mathcal{L}_A^{k\lambda}$, and $\mathcal{L}_R^{k\lambda}$ for all $k \in \{1, \ldots, K\} \backslash \{\kappa\}$. The method can then be applied normally to obtain the optimal updated inputs, only ignoring inequality ($\star\star$) for $\ell = \lambda$ when updating the input of S^κ.

6 RELATED WORK ON SYSTEMS WITH SHARED RESOURCES

The method presented in Chapter 4 is largely based on the one proposed in [36]. More specifically, the strategy of using the Hadamard product to express the global relationship among resource-allocation and release schedules as an inequality in the semiring of counters, explained in Section 4.1, was introduced in [36], inspired, in turn, by the ideas from [21]. The control approach discussed in Section 4.2 also preserves many of the characteristics from the one of [36], with the difference that in [36] the authors consider only the case in which each subsystem is a single-input-single-output TEG, whereas we start by treating the case of TEGs with two inputs (Section 4.2) and then generalize to the case of TEGs with an arbitrary number of inputs (Section 4.4). Furthermore, in Section 4.3 we extend the approach to the case of an arbitrary number of resources shared by an arbitrary number of subsystems; in [36], only the case of two resources shared by two subsystems is explicitly treated. The most significant novelty in this thesis with respect to [36], however, is the method proposed in Chapter 5, where the whole control approach for systems with resource-sharing is generalized to the case in which the output-references of the subsystems may unexpectedly change while the system is running.

Different groups of authors have invested their effort in several attempts to handle resource-sharing phenomena within a tropical-algebraic framework. In [12], a modeling approach for TEGs with shared resources is proposed where the constraints due to resource-sharing are expressed as inequalities over the min-plus algebra. There are similarities with the way we express these constraints in inequality (4.1), although the use of the Hadamard product arguably makes our expression simpler and easier to interpret and manipulate. Moreover, in [12] no systematic control method is proposed for systems with resource-sharing. Taking a different approach, the authors of [53] use so-called switching max-plus-linear systems — i. e., discrete-event systems that can switch between different operation modes — to model systems with concurrency. They show that the model predictive control problem for this class of systems can be reduced to a mixed integer optimization problem. One limitation of the method is that it does not scale well for systems with multiple shared resources, as each possible distribution of the resources among the users leads to a new mode of operation, which may cause a dramatic (possibly exponential) increase in the number of modes. In [1], conflicting TEGs are modeled by time-varying equations in the max-plus algebra; the technique is restricted to safe conflict places, i. e., it can only handle shared resources with single capacity. This restriction is then relaxed in [8], where the authors study cycle time evaluation of TEGs with shared resources modeled by conflict places; the obtained models are not employed for control purposes.

Finally, we should point out that the optimal control method presented in Chapter 4 has the advantage of being extensible to systems exhibiting both resource-sharing and partial-synchronization phenomena by being naturally combined with the method from Chapter 7, as shown in Chapter 10. This, to the best of the author's knowledge, is not the case of any of the previously proposed control approaches for systems with shared resources.

Part III

SYSTEMS WITH PARTIAL SYNCHRONIZATION

7

MODELING AND CONTROL OF TEGS UNDER PARTIAL SYNCHRONIZATION

The behavior of TEGs with partial synchronization cannot be modeled solely by equations like (3.2). In this chapter, we propose a strategy to incorporate the PS phenomenon into the model of the system, which naturally leads to a way of expressing PS restrictions in the context of counters (Section 7.2). This paves the way for a method to obtain optimal (just-in-time) inputs for TEGs with PS, presented first for the simpler case of a single partially-synchronized transition (Section 7.3) and then generalized to the case of multiple partially-synchronized transitions (Section 7.4).

A preliminary version of part of the following material has appeared in [42, 43], which reflect original work from — and have as the main author and contributor — the author of this thesis.

7.1 THE CONCEPT OF PARTIAL SYNCHRONIZATION

A general way of characterizing the partial synchronization phenomenon is the following: the firings of a TEG's partially-synchronized (internal) transition x_ι are subject to a predefined synchronizing signal $\mathcal{S} : \overline{\mathbb{Z}} \to \mathbb{Z}^+_{\min}$, where

$$\mathbb{Z}^+_{\min} = \{a \in \overline{\mathbb{Z}}_{\min} \mid \varepsilon \prec a \preceq e\} \subset \overline{\mathbb{Z}}_{\min}$$

is the set of finite nonnegative (in the standard sense) elements of $\overline{\mathbb{Z}}_{\min}$. More precisely, an additional condition for the firing of x_ι — besides the ones from standard synchronization as expressed in (3.2) — is imposed; namely, at any time $t \in \overline{\mathbb{Z}}$, x_ι can only fire if $\mathcal{S}(t) \neq e$, in which case it can fire at most $\mathcal{S}(t)$ times. If $\mathcal{S}(t) = e$, x_ι is not allowed to fire at time t. Note that limiting $\mathcal{S}$ to only assume finite values is not restrictive, as they can be arbitrarily large. In $\overline{\mathbb{Z}}_{\min}$, this condition on x_ι reads as

$$(\forall t \in \overline{\mathbb{Z}}) \;\; x_\iota(t) \succeq \mathcal{S}(t) \otimes x_\iota(t-1)\,. \tag{7.1}$$

Signal $\mathcal{S}$ as above assumes values $\mathcal{S}(t)$ which are not necessarily nonincreasing (in the order of $\overline{\mathbb{Z}}_{\min}$) over time, and thus it cannot, in general, be encoded as a counter. In the sequel, we present a way to capture the effects of PS within the domain of Σ.

7.2 MODELING OF TEGS UNDER PARTIAL SYNCHRONIZATION

We now propose an alternative perspective to model PS in TEGs. The method consists in appending to any partially-synchronized transition

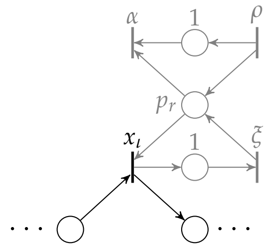

Figure 7.1: Appended structure (in gray) to represent PS of internal transition x_ι in a TEG.

x_ι the structure shown in Fig. 7.1. At any given time t, the number of tokens in place p_r corresponds to how many firings PS allows for x_ι at t. For this to correctly represent the restrictions on x_ι due to PS, the number of tokens in p_r needs to be managed accordingly, which is made possible by assigning appropriate firing schedules to transitions ρ and α. Suppose x_ι is to be conceded k firings at time t. Then, ρ will fire k times at t, inserting k tokens in p_r. These will remain available for only one time unit, during which they enable up to k firings of x_ι. Note that the number of tokens inserted in p_r provides only an upper bound to the number of times x_ι can fire at time t, but it is not known a priori how many firings (if any) x_ι will actually perform. The role of transition ξ is to make the mechanism independent of how often x_ι fires by returning to p_r at time $t+1$ all the tokens consumed by x_ι at t. In fact, as the earliest firing rule is assumed, based on Fig. 7.1 we have $\xi(t) = x_\iota(t-1)$ for all $t \in \overline{\mathbb{Z}}$ (or simply $\xi = e\delta^1 x_\iota$, cf. Remark 2.7). Then, at time $t+1$, x_ι's "right to fire" is revoked, which is carried out by scheduling k firings for α so that p_r becomes empty. Formally, $\alpha = e\delta^1\rho$. In order to avoid any (nondeterministic) dispute between α and x_ι for the tokens residing in p_r at $t+1$, the final touch is to assume that α has higher priority than x_ι, meaning the firing schedule of x_ι must be determined under the hard restriction that it cannot interfere with that of α. The described mechanism is initialized as follows: if x_ι is first granted the right to fire at time τ, define $\rho(t) = e$ for all $t \leq \tau$.

Example 7.1. Consider the TEG from Fig. 3.1 and suppose transition x_2 is partially synchronized, with the following restrictions: it may only fire at times

$$t \in \mathcal{T} = \big\{[4,6] \cup [10,12] \cup [18,19] \cup [24,27] \cup [31,32]\big\} \subset \mathbb{Z}\,,$$

and at most once at each $t \in \mathcal{T}$. This PS is modeled through the structure described above, as shown in Fig. 7.2, with

$$\rho(t) = \begin{cases} e & \text{if } t \leq 4\,; \\ 1 \otimes \rho(t-1) & \text{if } t-1 \in \mathcal{T}\,; \\ \rho(t-1) & \text{if } t-1 \notin \mathcal{T} \text{ and } t > 4\,. \end{cases}$$

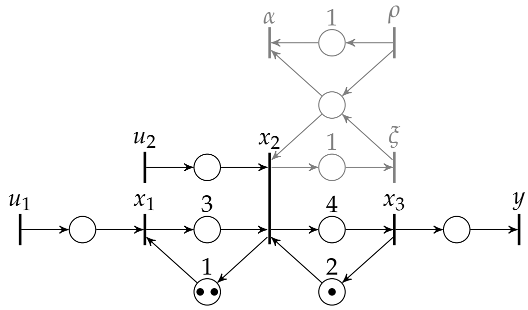

Figure 7.2: TEG from Fig. 3.1 with internal transition x_2 under PS.

Explicitly, we have

$$\rho = e\delta^4 \oplus 1\delta^5 \oplus 2\delta^6 \oplus 3\delta^{10} \oplus 4\delta^{11} \oplus 5\delta^{12} \oplus 6\delta^{18} \oplus 7\delta^{19} \oplus 8\delta^{24} \oplus 9\delta^{25} \oplus 10\delta^{26} \oplus 11\delta^{27} \oplus 12\delta^{31} \oplus 13\delta^{32} \oplus 14\delta^{+\infty}.$$

Recall that the schedule for α is then determined as $\alpha = e\delta^1\rho$, i.e., by shifting that of ρ backwards by one time unit. ◇

It should be clear that the overall system resulting from the method described above is no longer a TEG, as place p_r has two upstream and two downstream transitions. As a consequence, it cannot be modeled solely by linear equations such as (3.2). In order to capture the restrictions imposed by PS on a transition x_ι, we need to be able to express the relationship among transitions (and corresponding counters) ρ, α, x_ι, and ξ. For this, the Hadamard product of counters is used.

Recall from Def. 2.4 that the Hadamard product amounts to the coefficient-wise standard sum of counters. From the structure of Fig. 7.1 one can see that, at any time instant t, the combined accumulated number of firings of α and x_ι cannot exceed (in the conventional sense) that of ρ and ξ. The Hadamard product allows us to translate this into the following condition:

$$\rho \odot \xi \preceq \alpha \odot x_\iota\,. \tag{7.2}$$

With ρ, α, and ξ defined as described in this section, inequality (7.2) fully captures the restrictions imposed by PS on a transition x_ι.

Remark 7.1. The formulation presented in this section does not entail any loss of generality with respect to that of Section 7.1. If transition x_ι is partially synchronized based on a synchronizing signal $\mathcal{S}$, the structure of Fig. 7.1 can be adopted to implement the same PS for x_ι by defining, for all $t \in \overline{\mathbb{Z}}$, $\rho(t) = \bigotimes_{\tau \le t} \mathcal{S}(\tau)$. Hence, the accumulated number of firings of ρ by any time t is equal to the total number of firings of x_ι allowed by $\mathcal{S}$ up to t — naturally, not all such firings are necessarily performed by x_ι, i.e., in general we have $x_\iota \succeq \rho$. Recall that α is then automatically defined as $\alpha = e\delta^1\rho$.

Remark 7.2. We shall henceforth assume that the firings of a partially-synchronized transition x_ι can be allowed or prevented in real time, i.e., that there is a control input transition u_η with a single downstream place which is initially empty, has zero holding time, and is an upstream place of x_ι. This is illustrated in Fig. 7.3 for a general TEG, and it is the case, in particular, for the system from Example 7.1 (see input u_2 in Fig. 7.2). Note that this assumption is compatible with the real-world examples mentioned in the introduction; it is natural to assume that one is capable of deciding (through a direct control signal) whether or not a machine or piece of equipment should be turned on, the same being true about granting permission for a train/vehicle to enter a shared track segment.

We should emphasize that, even though from the point of view of the model structure both transitions ρ and u_η characterize "inputs" (in the sense that both have no upstream place), their roles are conceptually very different. Whereas u_η is indeed a control input whose firing schedule can be freely assigned, the firings of ρ are assumed to be predetermined based on external factors, thus enforcing the restrictions from PS, as explained above.

Remark 7.3. The modeling method presented in this section naturally applies to the case of TEGs with multiple transitions under PS. Suppose that, in a given TEG, out of the n internal transitions, I are partially synchronized, with $I \leq n$. PS is modeled by appending an independent structure like the one from Fig. 7.1 to each partially-synchronized transition x_ι, accordingly adding subscripts to transitions — and corresponding counters — ρ_ι, ξ_ι, and α_ι. It is then straightforward to generalize the previous discussion leading to condition (7.2), namely every x_ι must obey

$$\rho_\iota \odot \xi_\iota \preceq \alpha_\iota \odot x_\iota \, . \tag{7.3}$$

Based on Remark 7.2, we assume there is an input transition u_η connected to each partially-synchronized transition x_ι via a place with zero holding time and no initial tokens.

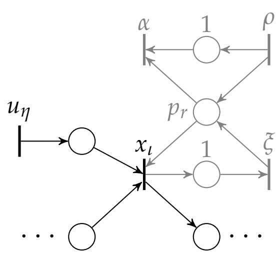

Figure 7.3: Illustration of the assumption that there is an input transition u_η directly connected to partially-synchronized transition x_ι (cf. Remark 7.2).

7.3 OPTIMAL CONTROL OF TEGS WITH A SINGLE PARTIALLY-SYNCHRONIZED TRANSITION

Consider a TEG modeled by linear equations (3.2), and suppose one of its internal transitions, x_ι, is partially synchronized. We represent the PS phenomenon through the structure shown in Fig. 7.3, as discussed in Section 7.2, including input transition u_η according to Remark 7.2. Recall that counters ρ and $\alpha = e\delta^1\rho$ are predetermined. Given an output reference z, our objective is to obtain the optimal input u_{opt} which leads to tracking the reference as closely as possible while respecting the partial synchronization of x_ι described by ρ, i. e., we seek the largest u such that $y = \mathcal{G}u \preceq z$ and such that (7.2) holds.

Let us start by noting that, as (3.3) describes the behavior of the TEG operating under the earliest firing rule, for an arbitrary input $u \in \Sigma^{m\times 1}$ leading to a schedule of x_ι that respects (7.2), the schedule of all internal transitions can be uniquely determined through matrix $\mathcal{F} = A^*B \in \Sigma^{n\times m}$, where n is the number of internal transitions and m the number of inputs in the TEG. Denoting the ι^{th} row of $\mathcal{F}$ by $\mathcal{F}_{[\iota\cdot]}$, we have $x_\iota = \mathcal{F}_{[\iota\cdot]}u$. Applying this to (7.2), together with the fact that $\xi = e\delta^1 x_\iota$ and $\alpha = e\delta^1\rho$ (cf. Section 7.2), we can write

$$\rho \odot e\delta^1\mathcal{F}_{[\iota\cdot]}u \preceq e\delta^1\rho \odot \mathcal{F}_{[\iota\cdot]}u\,. \tag{7.4}$$

Recalling Proposition 2.2, inequality (7.4) is equivalent to

$$e\delta^1\mathcal{F}_{[\iota\cdot]}u \preceq (e\delta^1\rho \odot \mathcal{F}_{[\iota\cdot]}u) \odot^\sharp \rho\,,$$

which, in turn, is equivalent to (cf. Remark 2.11)

$$u \preceq e\delta^1\mathcal{F}_{[\iota\cdot]}\diamond\!\!\!\!\backslash\left[(e\delta^1\rho \odot \mathcal{F}_{[\iota\cdot]}u) \odot^\sharp \rho\right]. \tag{7.5}$$

Finding an input which leads to tracking the reference while respecting (7.2) thus amounts to simultaneously solving $u \preceq \mathcal{G}\diamond\!\!\!\!\backslash z$ and (7.5), i. e., solving

$$u \preceq e\delta^1\mathcal{F}_{[\iota\cdot]}\diamond\!\!\!\!\backslash\left[(e\delta^1\rho \odot \mathcal{F}_{[\iota\cdot]}u) \odot^\sharp \rho\right] \wedge \mathcal{G}\diamond\!\!\!\!\backslash z\,,$$

which is equivalent to

$$u = e\delta^1\mathcal{F}_{[\iota\cdot]}\diamond\!\!\!\!\backslash\left[(e\delta^1\rho \odot \mathcal{F}_{[\iota\cdot]}u) \odot^\sharp \rho\right] \wedge \mathcal{G}\diamond\!\!\!\!\backslash z \wedge u\,.$$

The optimal input u_{opt} is, therefore, the greatest fixed point of the isotone mapping $\Phi : \Sigma^{m\times 1} \to \Sigma^{m\times 1}$,

$$\Phi(u) = e\delta^1\mathcal{F}_{[\iota\cdot]}\diamond\!\!\!\!\backslash\left[(e\delta^1\rho \odot \mathcal{F}_{[\iota\cdot]}u) \odot^\sharp \rho\right] \wedge \mathcal{G}\diamond\!\!\!\!\backslash z \wedge u\,. \tag{7.6}$$

Example 7.2. Let us revisit Example 3.1, only now with transition x_2 partially synchronized as in Example 7.1. For the TEG from Fig. 3.1, from (3.5) we have $\mathcal{F}_{[2\cdot]} = \begin{bmatrix} e\delta^3(1\delta^6)^* & (1\delta^6)^* \end{bmatrix}$. With ρ and α defined

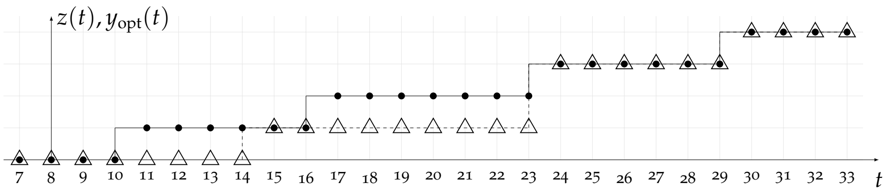

Figure 7.4: Tracking of the reference z ($\triangle$) by the optimal output y_{opt} ($\bullet$) obtained in Example 7.2.

as in Example 7.1, we compute the greatest fixed point of mapping Φ to get

$$u_{\text{opt}} = \begin{bmatrix} u_{1_{\text{opt}}} \\ u_{2_{\text{opt}}} \end{bmatrix} = \begin{bmatrix} e\delta^3 \oplus 1\delta^9 \oplus 2\delta^{16} \oplus 3\delta^{22} \oplus 4\delta^{+\infty} \\ e\delta^6 \oplus 1\delta^{12} \oplus 2\delta^{19} \oplus 3\delta^{25} \oplus 4\delta^{+\infty} \end{bmatrix}.$$

The corresponding optimal output (see Remark 7.4, below) is

$$y_{\text{opt}} = \mathcal{G}u_{\text{opt}} = e\delta^{10} \oplus 1\delta^{16} \oplus 2\delta^{22} \oplus 3\delta^{29} \oplus 4\delta^{+\infty}.$$

The resulting reference tracking is illustrated in Fig. 7.4; as expected, performance is clearly degraded due to the additional restrictions imposed by PS, meaning the reference cannot be tracked as closely as in the case without PS (compare with Fig. 3.2). ◇

Remark 7.4. Due to the additional restrictions for the firing of a partially-synchronized transition, in general it may be the case that a TEG under PS does not behave purely according to (3.2), and hence $y \neq \mathcal{G}u$. Nonetheless, since in the presented method the firing schedules of all input transitions are computed so as to respect condition (7.2) and to be just-in-time, a partially-synchronized transition x_ι is only going to be enabled when PS indeed allows it to fire. That is to say, the additional restrictions are dealt with offline in the computation phase, and the obtained optimal inputs guarantee that the evolution of the system will follow (3.2), as if unaffected by PS constraints. To put it in a formal way, as $x_{\iota_{\text{opt}}} = \mathcal{F}_{[\iota\cdot]}u_{\text{opt}}$ and as $x_{\iota_{\text{opt}}}$ satisfies (7.2), we have $x_{\text{opt}} = \mathcal{F}u_{\text{opt}}$ and hence $y_{\text{opt}} = \mathcal{G}u_{\text{opt}}$. Naturally, the same reasoning carries over to the case of multiple partially-synchronized transitions, to be discussed in Section 7.4. ◇

Remark 7.5. For the just-in-time input u_{opt} obtained through the method presented in this section, it holds that $\mathcal{F}_{[\iota\cdot]}u_{\text{opt}} = u_{\eta_{\text{opt}}}$. Intuitively, as u_{opt} is computed so that $x_{\iota_{\text{opt}}} = \mathcal{F}_{[\iota\cdot]}u_{\text{opt}}$ respects condition (7.2), this means the control input u_η enabling x_ι to fire is always provided exactly within the time windows in which PS allows x_ι to fire.

To show this, first note that, since $\mathcal{F}_{[\iota\cdot]}u_{\mathrm{opt}} = x_{\iota_{\mathrm{opt}}} \succeq u_{\eta_{\mathrm{opt}}}$, it suffices to prove that $\mathcal{F}_{[\iota\cdot]}u_{\mathrm{opt}} \preceq u_{\eta_{\mathrm{opt}}}$. The proof goes by contradiction. Assume $\mathcal{F}_{[\iota\cdot]}u_{\mathrm{opt}} \not\preceq u_{\eta_{\mathrm{opt}}}$, and consider the input $\tilde{u} \in \Sigma^{m\times 1}$ with

$$\tilde{u}_\kappa = \begin{cases} u_{\eta_{\mathrm{opt}}} \oplus \mathcal{F}_{[\iota\cdot]}u_{\mathrm{opt}}\,, & \text{for } \kappa = \eta\,, \\ u_{\kappa_{\mathrm{opt}}}\,, & \text{for } \kappa \neq \eta\,. \end{cases}$$

Because input transition u_η is connected to x_ι via a place with no initial tokens and null holding time (see Remark 7.2 and Fig. 7.3), for matrix $B \in \Sigma^{n\times m}$ as in (3.2) it follows that, for all $\mu \in \{1,\dots,n\}$,

$$B_{\mu\eta} = \begin{cases} s_e\,, & \text{for } \mu = \iota\,, \\ s_\varepsilon\,, & \text{for } \mu \neq \iota\,. \end{cases} \tag{7.7}$$

So, denoting the η^{th} column of B by $B_{[\cdot\eta]}$, for any $j \in \{1,\dots,n\}$ we have

$$\mathcal{F}_{j\eta} = [A^*B]_{j\eta} = [A^*]_{[j\cdot]}B_{[\cdot\eta]} = [A^*]_{j\iota}\,. \tag{7.8}$$

Moreover, as x_{opt} is a solution of (3.2) and, therefore, $x_{\mathrm{opt}} = A^*x_{\mathrm{opt}}$ (cf. Remark 2.4), we have

$$x_{j_{\mathrm{opt}}} = [A^*]_{[j\cdot]}x_{\mathrm{opt}} = \bigoplus_{\mu=1}^{n}[A^*]_{j\mu}x_{\mu_{\mathrm{opt}}} \succeq [A^*]_{j\iota}x_{\iota_{\mathrm{opt}}}\,.$$

Combined with (7.8), this means

$$x_{j_{\mathrm{opt}}} \succeq \mathcal{F}_{j\eta}x_{\iota_{\mathrm{opt}}} \tag{7.9}$$

for all $j \in \{1,\dots,n\}$. Then,

$$\begin{aligned} \mathcal{F}_{[j\cdot]}\tilde{u} &= \bigoplus_{\substack{\kappa=1\\ \kappa\neq\eta}}^{m}\mathcal{F}_{j\kappa}u_{\kappa_{\mathrm{opt}}} \oplus \mathcal{F}_{j\eta}\left(u_{\eta_{\mathrm{opt}}} \oplus \mathcal{F}_{[\iota\cdot]}u_{\mathrm{opt}}\right) \\ &= \bigoplus_{\substack{\kappa=1\\ \kappa\neq\eta}}^{m}\mathcal{F}_{j\kappa}u_{\kappa_{\mathrm{opt}}} \oplus \mathcal{F}_{j\eta}u_{\eta_{\mathrm{opt}}} \oplus \mathcal{F}_{j\eta}\mathcal{F}_{[\iota\cdot]}u_{\mathrm{opt}} \\ &= \bigoplus_{\kappa=1}^{m}\mathcal{F}_{j\kappa}u_{\kappa_{\mathrm{opt}}} \oplus \mathcal{F}_{j\eta}\mathcal{F}_{[\iota\cdot]}u_{\mathrm{opt}} \\ &= \mathcal{F}_{[j\cdot]}u_{\mathrm{opt}} \oplus \mathcal{F}_{j\eta}\mathcal{F}_{[\iota\cdot]}u_{\mathrm{opt}} \\ &= \mathcal{F}_{[j\cdot]}u_{\mathrm{opt}}\,, \end{aligned}$$

where the last equality follows from (7.9) and the fact that $\mathcal{F}_{[j\cdot]}u_{\mathrm{opt}} = x_{j_{\mathrm{opt}}}$ for all $j \in \{1,\dots,n\}$ (which includes, of course, the case $j = \iota$). This implies $\mathcal{F}\tilde{u} = \mathcal{F}u_{\mathrm{opt}}$ and thus, recalling from (3.4) that $\mathcal{G} = C\mathcal{F}$, also $\mathcal{G}\tilde{u} = \mathcal{G}u_{\mathrm{opt}} \preceq z$, so $\tilde{u} \preceq \mathcal{G} \backslash\!\!\!\circ\, z$.

Furthermore, the fact that $\mathcal{F}_{[\iota\cdot]}\tilde{u} = \mathcal{F}_{[\iota\cdot]}u_{\mathrm{opt}}$ as shown above implies $\tilde{u}$ satisfies (7.5), so we conclude that $\tilde{u}$ is a fixed point of mapping Φ. But $\tilde{u} \succeq u_{\mathrm{opt}}$ and $\tilde{u} \neq u_{\mathrm{opt}}$, contradicting the fact that u_{opt} is the greatest fixed point of Φ. ◇

7.4 OPTIMAL CONTROL OF TEGS WITH MULTIPLE PARTIALLY-SYNCHRONIZED TRANSITIONS

Consider a TEG modeled by linear equations (3.2), and suppose I out of its n internal transitions are partially synchronized. We assume, for ease of discussion and without loss of generality, that the corresponding counters x_ι are the first I entries of vector $x \in \Sigma^{n\times 1}$. The PS of each partially-synchronized transition x_ι, $\iota \in \{1,\dots,I\}$, is again represented by a structure like the one from Fig. 7.3, accordingly adding subscripts to transitions — and corresponding counters — ρ_ι, ξ_ι, and α_ι. The assumptions from Remark 7.3 concerning input transitions u_η connected to each x_ι are in place.

Besides tracking a given reference z as closely as possible, the optimal input must now be computed ensuring that (7.3) holds for every $\iota \in \{1,\dots,I\}$. Following the same arguments as in Section 7.3, one can see that inequality (7.3) is equivalent to

$$u \preceq e\delta^1 \mathcal{F}_{[\iota\cdot]} \Lsh \left[(e\delta^1 \rho_\iota \odot \mathcal{F}_{[\iota\cdot]} u) \odot^\sharp \rho_\iota \right] . \tag{7.10}$$

Recall that $\mathcal{F}_{[\iota\cdot]}$ is the ι^{th} row of $\mathcal{F} = A^*B$ as in (3.3), i. e., for an input u that leads to respecting (7.2) for every $\iota \in \{1,\dots,I\}$ we have $x_\iota = \mathcal{F}_{[\iota\cdot]} u$.

Defining the collection of mappings $\Phi_\iota : \Sigma^{m\times 1} \to \Sigma^{m\times 1}$,

$$\Phi_\iota(u) = e\delta^1 \mathcal{F}_{[\iota\cdot]} \Lsh \left[(e\delta^1 \rho_\iota \odot \mathcal{F}_{[\iota\cdot]} u) \odot^\sharp \rho_\iota \right] ,$$

where m is the number of input transitions in the system, an input $u \in \Sigma^{m\times 1}$ satisfying (7.10) simultaneously for all $\iota \in \{1,\dots,I\}$ while respecting reference z is such that

$$u \preceq \bigwedge_{\iota=1}^{I} \Phi_\iota(u) \qquad \text{and} \qquad u \preceq \mathcal{G} \Lsh z$$

or, again through a reasoning similar to the one put forth in Section 7.3,

$$u = \bigwedge_{\iota=1}^{I} \Phi_\iota(u) \wedge \mathcal{G} \Lsh z \wedge u .$$

Hence, the input u_{opt} which optimally tracks the reference while respecting (7.10) for all $\iota \in \{1,\dots,I\}$ is the greatest fixed point of the (isotone) mapping $\overline{\Phi} : \Sigma^{m\times 1} \to \Sigma^{m\times 1}$,

$$\overline{\Phi}(u) = \bigwedge_{\iota=1}^{I} \Phi_\iota(u) \wedge \mathcal{G} \Lsh z \wedge u .$$

Remark 7.6. Similarly to Remark 7.5, the method presented in this section yields a just-in-time input u_{opt} such that $\mathcal{F}_{[\iota\cdot]} u_{\text{opt}} = u_{\iota_{\text{opt}}}$ for every $\iota \in \{1,\dots,I\}$. Again the intuition behind this fact is that, as the method guarantees that $x_{\iota_{\text{opt}}} = \mathcal{F}_{[\iota\cdot]} u_{\text{opt}}$ obeys (7.3) for all $\iota \in \{1,\dots,I\}$, no partially-synchronized transition x_ι is ever enabled to fire by the corresponding control input u_ι unless it is also allowed to fire by the PS restrictions.

To show this, let us first recall from Remark 7.3 that we can assume, without loss of generality, that $\eta = \iota$ whenever u_η is connected to x_ι. As $\mathcal{F}_{[\iota \cdot]} u_{\text{opt}} = x_{\iota_{\text{opt}}} \succeq u_{\iota_{\text{opt}}}$ for every $\iota \in \{1, \ldots, I\}$, all that needs to be proved is that $\mathcal{F}_{[\iota \cdot]} u_{\text{opt}} \preceq u_{\iota_{\text{opt}}}$ for all ι. The proof is again done by contradiction. Note that negating the claim "$\mathcal{F}_{[\iota \cdot]} u_{\text{opt}} \preceq u_{\iota_{\text{opt}}}$ for all $\iota \in \{1, \ldots, I\}$" implies assuming there exists $\tilde{\iota} \in \{1, \ldots, I\}$ such that $\mathcal{F}_{[\tilde{\iota} \cdot]} u_{\text{opt}} \npreceq u_{\tilde{\iota}_{\text{opt}}}$. Now, seeing as the arguments from Remark 7.5 are valid for an arbitrary ι, the remainder of the proof proceeds identically to the referred remark, only replacing ι and η with $\tilde{\iota}$. ◇

ON THE SIMILARITY BETWEEN THE METHODS FOR RESOURCE-SHARING AND PARTIAL SYNCHRONIZATION

The attentive reader will have noticed a strong similarity between the method presented in this chapter and the one from Chapter 4; the same will be true between Chapters 8 and 5. This, of course, is no coincidence.

In fact, from one perspective, note that the structure from Fig. 7.1 (including transition x_ι) resembles that of a system comprising two subsystems sharing a single resource, represented by place p_r. The TEG in question plays the role of a lower-priority subsystem, allocating the (fictitious) resource via transition x_ι and releasing it through the added auxiliary transition ξ. In turn, transitions α and ρ can be seen respectively as allocation and release transitions of a "resource manager", a higher-priority user whose role is to moderate the availability of the resource — i. e., the number of tokens available in place p_r — according to predetermined allocation and release schedules that cannot be interfered with by the lower-priority user.

On the other hand, in the method from Chapter 4, once the optimal inputs for a higher-priority subsystem S^i have been obtained, they are taken as fixed. Hence, from the point of view of a lower-priority subsystem S^j, the resource-allocation and release schedules of S^i can be seen as hard restrictions that limit the time instants at which its allocation transition, x_A^j, is allowed to fire, much like PS restrictions.

Granted the consequent repetitive nature of the mathematical formulation may make the reading of the thesis somewhat tedious, this similarity can, nonetheless, be considered a side contribution, as it reveals a (not necessarily self-evident) correspondence between the two phenomena. It also makes the merging of the two methods (Chapter 10) rather natural and elegant.

8 CONTROL OF TEGS UNDER VARYING PARTIAL SYNCHRONIZATION

In this chapter, we extend the results presented in Chapter 7 to the case of varying PS, i.e., where the restrictions on partially-synchronized transitions may change during run-time. As before, we start with the simpler case of TEGs with a single partially-synchronized transition (Sections 8.1 and 8.2) and then proceed to generalize to the case of multiple partially-synchronized transitions (Section 8.3). In order to avoid breaking the flow and improve readability, some proofs along the chapter are postponed to Appendix B.

A preliminary version of part of the following material has appeared in [42, 43], which reflect original work from — and have as the main author and contributor — the author of this thesis.

8.1 PROBLEM FORMULATION — THE CASE OF A SINGLE PARTIALLY-SYNCHRONIZED TRANSITION

Consider a system modeled as a TEG with n internal transitions — one of which, x_ι, is partially synchronized — and m input transitions — one of which, u_η, is connected to x_ι via a place with no initial tokens and null holding time, according to Remark 7.2. Assume the system is operating optimally with respect to a given output-reference z, with optimal input u_{opt} obtained according to the method presented in Chapter 7.

Now, suppose that at a certain time T the restrictions due to PS are altered, which, in terms of the modeling technique introduced in Section 7.2, means the firing schedule of transition ρ is updated to a new one, ρ'. Naturally, as past firings cannot be altered, it must be the case that $\rho'(t) = \rho(t)$ for all $t \leq T$, i.e., recalling mapping r_T defined in (3.7) we have $r_T(\rho') = r_T(\rho)$ — and thus, as $\alpha = e\delta^1\rho$, the schedule of transition α is also updated to α' with $r_{(T+1)}(\alpha') = r_{(T+1)}(\alpha)$. Based on (7.2), the new restrictions imposed by PS on x_ι can be expressed by

$$\rho' \odot \xi \preceq \alpha' \odot x_\iota \,. \tag{8.1}$$

Our goal is to determine the input u'_{opt} which preserves u_{opt} up to time T and which results in an output that tracks reference z as closely as possible, while guaranteeing that the resulting firing schedule for x_ι, denoted $x'_{\iota_{\text{opt}}}$, observes the restrictions from PS expressed by (8.1). Recall, as argued in Section 7.3, that we can express the firing schedule of x_ι in terms of u as $x_\iota = \mathcal{F}_{[\iota\cdot]}u$, where $\mathcal{F}_{[\iota\cdot]}$ is the ι^{th} row of $\mathcal{F} = A^*B$

as in (3.3). Combined with the fact that $\xi = e\delta^1 x_\iota$ and $\alpha' = e\delta^1\rho'$ (cf. Section 7.2), this means we can write (8.1) as

$$\rho' \odot e\delta^1 \mathcal{F}_{[\iota\cdot]} u \preceq e\delta^1 \rho' \odot \mathcal{F}_{[\iota\cdot]} u \,. \tag{⌘}$$

The problem described above can then be stated as follows: find the greatest element of the set

$$\mathcal{Q} = \{u \in \Sigma^{m\times 1} \mid \mathcal{G}u \preceq z \text{ and } (⌘) \text{ and } r_T(u) = r_T(u_{\text{opt}})\}. \tag{8.2}$$

8.2 OPTIMAL UPDATE OF THE INPUTS — THE CASE OF A SINGLE PARTIALLY-SYNCHRONIZED TRANSITION

With the objective of determining the greatest element of set $\mathcal{Q}$ defined in (8.2), notice that

$$\begin{aligned}(⌘) \;\Leftrightarrow\;& e\delta^1 \mathcal{F}_{[\iota\cdot]} u \preceq (e\delta^1\rho' \odot \mathcal{F}_{[\iota\cdot]} u) \odot^\sharp \rho' \\ \Leftrightarrow\;& u \preceq e\delta^1 \mathcal{F}_{[\iota\cdot]} \backslash\!\!\circ \left[(e\delta^1\rho' \odot \mathcal{F}_{[\iota\cdot]} u) \odot^\sharp \rho'\right].\end{aligned}$$

So, defining the mapping $\Psi : \Sigma^{m\times 1} \to \Sigma^{m\times 1}$,

$$\Psi(u) = \mathcal{G} \backslash\!\!\circ z \wedge e\delta^1 \mathcal{F}_{[\iota\cdot]} \backslash\!\!\circ \left[(e\delta^1\rho' \odot \mathcal{F}_{[\iota\cdot]} u) \odot^\sharp \rho'\right], \tag{8.3}$$

set $\mathcal{Q}$ can be equivalently written as

$$\mathcal{Q} = \{u \in \Sigma^{m\times 1} \mid u \preceq \Psi(u) \text{ and } r_T(u) = r_T(u_{\text{opt}})\}.$$

We can then solve the problem stated in Section 8.1 by applying Prop. 5.1, taking $\mathcal{D}$ as $\Sigma^{m\times 1}$, ψ as Ψ, f as r_T, and c as $r_T(u_{\text{opt}})$. Therefore, as long as set $\mathcal{Q}$ is nonempty, recalling that mapping r_T is residuated (cf. Remark 3.1) and $r_T^\sharp \circ r_T = r_T^\sharp$, the sought optimal update of the input, u'_{opt}, is the greatest fixed point of mapping $\Gamma : \Sigma^{m\times 1} \to \Sigma^{m\times 1}$,

$$\Gamma(u) = u \wedge \Psi(u) \wedge r_T^\sharp(u_{\text{opt}})\,. \tag{8.4}$$

The next step is to investigate whether set $\mathcal{Q}$ is nonempty. With that in mind, let us define the set

$$\widetilde{\mathcal{Q}} = \{u \in \Sigma^{m\times 1} \mid (⌘) \text{ and } r_T(u) = r_T(u_{\text{opt}})\} \supseteq \mathcal{Q}\,. \tag{8.5}$$

We look for an element $\underline{u}$ of $\widetilde{\mathcal{Q}}$ that leads to the fastest possible behavior of the system, i. e., to the least possible firing schedule of the output. If such an input does not lead to respecting reference z, then clearly no input satisfying (⌘) and $r_T(u) = r_T(u_{\text{opt}})$ will. Formally, as shall be concluded in Corollary 8.3, $\mathcal{Q} \neq \emptyset \Leftrightarrow \mathcal{G}\underline{u} \preceq z$.

Even though $\widetilde{\mathcal{Q}}$ may not possess a least element, any input in $\widetilde{\mathcal{Q}}$ which leads to the fastest possible schedule of the internal transitions while guaranteeing that the restrictions due to PS are respected will result in the least possible schedule for the output y.

In the quest for such an input, we observe that a bound for the firing schedule of x_ι can be obtained from (8.1), as, recalling from Section 7.2 that $\alpha' = e\delta^1\rho'$ and $\xi = e\delta^1 x_\iota$,

$$(8.1) \;\Leftrightarrow\; (\rho' \odot e\delta^1 x_\iota) \odot^\flat e\delta^1\rho' \preceq x_\iota\,.$$ [1]

The left-hand side of the latter inequality provides a bound for how small (in the sense of the order of Σ) x_ι can be. It represents the maximal number of firings allowed for x_ι under the PS-restrictions.

Furthermore, naturally no internal transition can fire more often than enabled by the inputs. Considering that input firings that have occurred before time T cannot be changed, the most often each input u_κ can possibly fire from time T onward is encoded by the counter $r_T(u_{\kappa_{\text{opt}}})$, which represents the preservation of the firings up to T and an infinite number of firings at time $T+1$. Thus, $\mathcal{F} r_T(u_{\text{opt}})$ imposes a bound for x, limiting how often each internal transition can fire, i. e., we must have $x \succeq \mathcal{F} r_T(u_{\text{opt}})$; in particular, this implies $x_\iota \succeq \mathcal{F}_{[\iota\cdot]} r_T(u_{\text{opt}})$.

We also require x to be a solution of (3.2), which, according to Remark 2.4, implies $x = A^*x$. In particular, this means we must have $x_\iota = [A^*]_{[\iota\cdot]} x \succeq [A^*]_{\iota\iota} x_\iota$. But recall from (7.8) that $[A^*]_{\iota\iota} = \mathcal{F}_{\iota\eta}$, so the above condition can be written as $x_\iota \succeq \mathcal{F}_{\iota\eta} x_\iota$.

Based on the foregoing discussion, any schedule for x_ι must obey

$$x_\iota \succeq \left[(\rho' \odot e\delta^1 x_\iota) \odot^\flat e\delta^1\rho'\right] \oplus \mathcal{F}_{[\iota\cdot]} r_T(u_{\text{opt}}) \oplus \mathcal{F}_{\iota\eta} x_\iota\,,$$

which is equivalent to saying x_ι must be a fixed point of the (isotone) mapping $\Lambda : \Sigma \to \Sigma$,

$$\Lambda(\chi) = \left[(\rho' \odot e\delta^1 \chi) \odot^\flat e\delta^1\rho'\right] \oplus \mathcal{F}_{[\iota\cdot]} r_T(u_{\text{opt}}) \oplus \mathcal{F}_{\iota\eta}\chi \oplus \chi\,. \tag{8.6}$$

Remark 8.1. One can easily see that, for any $\widetilde{u} \in \widetilde{Q}$, $\mathcal{F}_{[\iota\cdot]}\widetilde{u}$ is a fixed point of Λ, because

– $\widetilde{u}$ satisfies (⌘), which is equivalent to

$$(\rho' \odot e\delta^1 \mathcal{F}_{[\iota\cdot]} u) \odot^\flat e\delta^1\rho' \preceq \mathcal{F}_{[\iota\cdot]} u\,; \tag{8.7}$$

– $\mathcal{F}_{[\iota\cdot]}\widetilde{u} \succeq \mathcal{F}_{[\iota\cdot]} r_T(\widetilde{u}) = \mathcal{F}_{[\iota\cdot]} r_T(u_{\text{opt}})$;

– $\widetilde{x} = \mathcal{F}\widetilde{u}$ is a solution of (3.2), so $\widetilde{x} = A^*\widetilde{x}$ (cf. Remark 2.4) and hence

$$\mathcal{F}_{[\iota\cdot]}\widetilde{u} = \widetilde{x}_\iota = [A^*]_{[\iota\cdot]}\widetilde{x} \succeq [A^*]_{\iota\iota}\widetilde{x}_\iota = \mathcal{F}_{\iota\eta}\widetilde{x}_\iota\,.$$

◇

1 As ρ' encodes the accumulated number of firings of transition ρ by each time instant t, which corresponds to the accumulated number of firings granted to x_ι up to t, it is reasonable (and entails no loss of generality) to assume that $\rho'(t) \notin \{-\infty, +\infty\}$ for any finite time $t \in \mathbb{Z}$. The same holds, of course, for $e\delta^1\rho'$, as $[e\delta^1\rho'](t) = \rho'(t-1)$ for all t. Hence, according to Prop. 2.3, mapping $\Pi_{e\delta^1\rho'} : \Sigma \to \Sigma$ is dually residuated.

Remark 8.1 implies that any firing schedule of x_ι which is reachable from the inputs and which is compatible with the restrictions due to PS and with past input firings is in fact a fixed point of Λ. What remains to be investigated then is whether the least fixed point of mapping Λ — which we shall denote $\underline{x}_\iota$ — is indeed feasible, i.e., whether there exists an input $\underline{u}$ which is an element of $\widetilde{\mathcal{Q}}$ and such that $\mathcal{F}_{[\iota\cdot]}\underline{u} = \underline{x}_\iota$. In the following, we present a constructive proof that the answer is positive.

Define the vector $\theta \in \Sigma^{m\times 1}$,

$$\theta_\mu = \begin{cases} s_\varepsilon\,, & \text{for } \mu \neq \eta\,, \\ \underline{x}_\iota\,, & \text{for } \mu = \eta\,, \end{cases}$$

and consider the input

$$\underline{u} \;=\; r_T(u_{\text{opt}}) \oplus \theta \;=\; \begin{bmatrix} r_T(u_{1_{\text{opt}}}) \\ \vdots \\ r_T(u_{\eta_{\text{opt}}}) \oplus \underline{x}_\iota \\ \vdots \\ r_T(u_{m_{\text{opt}}}) \end{bmatrix}. \tag{8.8}$$

In order to show that $\mathcal{F}_{[\iota\cdot]}\underline{u} = \underline{x}_\iota$, first note that, as

$$A^* \;=\; \bigoplus_{\kappa\geq 0} A^\kappa \;=\; \mathcal{I}^{n\times n} \oplus \bigoplus_{\kappa\geq 1} A^\kappa \;\succeq\; \mathcal{I}^{n\times n}\,, \tag{8.9}$$

where $A^0 = \mathcal{I}^{n\times n}$ is the identity matrix in $\Sigma^{n\times n}$ (see Remark 2.3), it follows that $\left[A^*\right]_{\iota\iota} \succeq \left[\mathcal{I}^{n\times n}\right]_{\iota\iota} = s_e$, so $\mathcal{F}_{\iota\eta}\underline{x}_\iota = \left[A^*\right]_{\iota\iota}\underline{x}_\iota \succeq \underline{x}_\iota$. On the other hand, the fact that $\underline{x}_\iota$ is a fixed point of Λ implies $\underline{x}_\iota \succeq \mathcal{F}_{\iota\eta}\underline{x}_\iota$, and hence

$$\mathcal{F}_{\iota\eta}\underline{x}_\iota \;=\; \underline{x}_\iota\,. \tag{8.10}$$

Then, we have

$$\begin{aligned}
\mathcal{F}_{[\iota\cdot]}\underline{u} \;&=\; \bigoplus_{\substack{\mu=1\\ \mu\neq\eta}}^{m} \mathcal{F}_{\iota\mu} r_T(u_{\mu_{\text{opt}}}) \;\oplus\; \mathcal{F}_{\iota\eta}\big(r_T(u_{\eta_{\text{opt}}}) \oplus \underline{x}_\iota\big) \\
&=\; \bigoplus_{\substack{\mu=1\\ \mu\neq\eta}}^{m} \mathcal{F}_{\iota\mu} r_T(u_{\mu_{\text{opt}}}) \;\oplus\; \mathcal{F}_{\iota\eta} r_T(u_{\eta_{\text{opt}}}) \;\oplus\; \mathcal{F}_{\iota\eta}\underline{x}_\iota \\
&=\; \bigoplus_{\mu=1}^{m} \mathcal{F}_{\iota\mu} r_T(u_{\mu_{\text{opt}}}) \;\oplus\; \mathcal{F}_{\iota\eta}\underline{x}_\iota \\
&=\; \mathcal{F}_{[\iota\cdot]} r_T(u_{\text{opt}}) \;\oplus\; \underline{x}_\iota \qquad\qquad \text{(because of (8.10))} \\
&=\; \underline{x}_\iota \qquad\qquad\qquad\qquad\qquad \text{(as } \underline{x}_\iota \text{ is a fixed point of } \Lambda\text{)}\,.
\end{aligned}$$

Now, to prove that $\underline{u} \in \widetilde{\mathcal{Q}}$, we begin by noticing that, because $\underline{x}_\iota$ is a fixed point of Λ,

$$(\rho' \odot e\delta^1 \underline{x}_\iota) \odot^\flat e\delta^1 \rho' \preceq \underline{x}_\iota \,.$$

Combined with the fact that $\mathcal{F}_{[\iota\cdot]}\underline{u} = \underline{x}_\iota$ as shown above, this implies taking $u = \underline{u}$ satisfies (8.7), which is equivalent to (⌘).

It remains to show that $r_T(\underline{u}) = r_T(u_{\text{opt}})$. Note that, as $r_T \circ r_T = r_T$, for $\mu \neq \eta$ it trivially holds that $r_T(\underline{u}_\mu) = r_T(u_{\mu_{\text{opt}}})$. The problem is then reduced to showing that $r_T(\underline{u}_\eta) = r_T\big(r_T(u_{\eta_{\text{opt}}}) \oplus \underline{x}_\iota\big) = r_T(u_{\eta_{\text{opt}}})$, which, in turn, as r_T distributes over $\oplus$, is equivalent to $r_T(u_{\eta_{\text{opt}}}) \oplus r_T(\underline{x}_\iota) = r_T(u_{\eta_{\text{opt}}})$, or $r_T(\underline{x}_\iota) \preceq r_T(u_{\eta_{\text{opt}}})$. Our argument will be based on the following result.

Proposition 8.1. *$r_T^\sharp(x_{\iota_{\text{opt}}})$ is a fixed point of mapping Λ.*

A consequence of Prop. 8.1 is that $\underline{x}_\iota \preceq r_T^\sharp(x_{\iota_{\text{opt}}}) = r_T^\sharp(\mathcal{F}_{[\iota\cdot]}u_{\text{opt}})$. We also know from Remark 7.5 that $\mathcal{F}_{[\iota\cdot]}u_{\text{opt}} = u_{\eta_{\text{opt}}}$. Thus, as r_T is isotone and recalling that $r_T \circ r_T^\sharp = r_T$,

$$r_T(\underline{x}_\iota) \preceq r_T\big(r_T^\sharp(u_{\eta_{\text{opt}}})\big) = r_T(u_{\eta_{\text{opt}}}) \,,$$

concluding the proof that $\underline{u} \in \widetilde{\mathcal{Q}}$.

This does not guarantee, however, that $\mathcal{Q} \neq \emptyset$, as it is possible that $\mathcal{G}\underline{u} \not\preceq z$ and hence $\underline{u} \notin \mathcal{Q}$. Intuitively, if the new restrictions from PS on x_ι are more stringent than the original ones, since up to time T we implemented just-in-time inputs based on the original restrictions, it may be impossible to respect both reference z and the new restrictions after T. As we assume PS-restrictions to be hard ones, this means we have no choice but to relax z, i. e., look for a new reference $z' \succeq z$ for which a solution exists. In fact, we seek the least possible such z', in order to remain as close as possible to the original reference. A natural choice is then to take $z' = z \oplus \mathcal{G}\underline{u}$; as $\oplus$ is performed coefficient-wise on counters, this amounts to preserving the terms of z that can still be achieved by taking $\underline{u}$ as input, and relaxing those that cannot only as much as necessary to be matched by the resulting output $y = \mathcal{G}\underline{u}$. The following proposition establishes that this is indeed the optimal way of relaxing z.

Proposition 8.2. *Let $\mathcal{Q}'$ denote the set defined as $\mathcal{Q}$ in (8.2), only replacing z with z', and let $\underline{u}$ be defined as in (8.8). The least $z' \succeq z$ such that $\mathcal{Q}' \neq \emptyset$ is $z' = z \oplus \mathcal{G}\underline{u}$.*

Prop. 8.2 also provides a simple way to check whether set $\mathcal{Q}$ is nonempty.

Corollary 8.3. *Let $\mathcal{Q}$ be defined as in (8.2) and $\underline{u}$ as in (8.8). Then, $\mathcal{Q} \neq \emptyset \Leftrightarrow \mathcal{G}\underline{u} \preceq z$.*

In the case $\mathcal{Q}$ turns out to be empty, define the mapping $\Psi' : \Sigma^{m\times 1} \to \Sigma^{m\times 1}$ as Ψ in (8.3), only replacing z with $z' = z \oplus \mathcal{G}\underline{u}$.

Following the same procedure as before, we can apply Prop. 5.1 — only now taking ψ as Ψ' instead of Ψ — to conclude that u'_{opt} is the greatest fixed point of mapping $\Gamma' : \Sigma^{m\times 1} \to \Sigma^{m\times 1}$,

$$\Gamma'(u) = u \wedge \Psi'(u) \wedge r_T^{\sharp}(u_{\text{opt}}) . \tag{8.11}$$

SUMMARY OF THE METHOD

Let us now provide a step-by-step overview of how to apply the method discussed in this chapter. We assume that a TEG modeling the system to be controlled is given, as are the external signals describing PS restrictions on some of its internal transitions. Assume also the transfer relations $\mathcal{F}$ and $\mathcal{G}$ (see (3.4)) to have been precomputed and an output-reference to be provided in the form of a counter z. Here, we consider the case of a single transition under PS, the generalization to the case of multiple partially-synchronized transitions (after the discussion in Section 8.3) being straightforward.

I. Model the PS restrictions by appending to the partially-synchronized transition x_ι a structure like the one shown in Fig. 7.1, and obtain the counter ρ according to the given external signals, as described in Section 7.2. Recall that this implicitly provides counters $\alpha = e\delta^1\rho$.

II. Obtain the optimal input u_{opt} by computing the greatest fixed point of mapping Φ defined as in (7.6), according to Section 7.3.

III. If, at a certain time T, the PS restrictions are altered, update the corresponding counters ρ and α to ρ' and α'.

IV. Obtain the input $\underline{u}$ defined as in (8.8). As a prerequisite, compute $\underline{x}$, the least fixed point of mapping Λ defined in (8.6).

V. Based on Corollary 8.3, check whether set $\mathcal{Q}$ — defined as in (8.2) — is nonempty by checking if the inequality $\mathcal{G}\underline{u} \preceq z$ holds.

VI. In the case $\mathcal{Q} \neq \emptyset$, obtain the optimal updated input u'_{opt} by computing the greatest fixed point of mapping Γ defined in (8.4).

VII. If $\mathcal{Q} = \emptyset$, obtain the least feasible reference z' according to Prop. 8.2 and then obtain the optimal updated input u'_{opt} by computing the greatest fixed point of mapping Γ' defined in (8.11).

Example 8.1. Consider, once more, the system from Example 3.1, with transition x_2 partially synchronized as in Example 7.1, and assume it is operating optimally according to the input obtained in Example 7.2. This means steps I and II have already been taken. Now, suppose that at time $T = 14$ the restrictions from PS are updated as follows:

transition x_2 is no longer allowed to fire at times 18 and 19. This means that now x_2 may only fire at times

$$t \in \mathcal{T}' = \{[4,6] \cup [10,12] \cup [24,27] \cup [31,32]\} \subset \mathbb{Z}.$$

Proceeding to step III, the new schedule ρ' for transition ρ is defined similarly as in Example 7.1:

$$\rho'(t) = \begin{cases} e & \text{if } t \leq 4; \\ 1 \otimes \rho'(t-1) & \text{if } t-1 \in \mathcal{T}'; \\ \rho'(t-1) & \text{if } t-1 \notin \mathcal{T}' \text{ and } t > 4. \end{cases}$$

The explicit counter thus obtained is

$$\begin{aligned} \rho' &= e\delta^4 \oplus 1\delta^5 \oplus 2\delta^6 \oplus 3\delta^{10} \oplus 4\delta^{11} \oplus 5\delta^{12} \oplus 6\delta^{24} \oplus 7\delta^{25} \oplus 8\delta^{26} \\ &\oplus 9\delta^{27} \oplus 10\delta^{31} \oplus 11\delta^{32} \oplus 12\delta^{+\infty}. \end{aligned}$$

In order to apply Corollary 8.3 to check whether reference z is still achievable — i.e., whether $\mathcal{Q} \neq \emptyset$ — we compute input $\underline{u}$, as instructed in step IV; for that, we first need to compute $\underline{x}_2$, which is the least fixed point of mapping Λ defined in (8.6). Note that, as the total number of output firings required by reference z is 4, we know the computed just-in-time inputs will not fire more than 4 times in total, and consequently the same is true for transition x_2. Thus, in order to simplify computations, the initial counter χ for computing the least fixed point of Λ may be chosen such that $\chi(t) \succeq 4$ for all t. As we also know that the obtained least fixed point $\underline{x}_2$ will be such that $\underline{x}_2 \succeq \mathcal{F}_{[2\cdot]} r_T(u_{\text{opt}})$, a natural choice for the starting point of the fixed point algorithm is $\chi = \mathcal{F}_{[2\cdot]} r_T(u_{\text{opt}}) \oplus 4\delta^{+\infty}$; the first term in the sum represents the maximal (in the standard sense) possible number of firings of x_2, and the second truncates counter χ so that the total number of firings does not exceed 4. We obtain

$$\underline{x}_2 = e\delta^6 \oplus 1\delta^{12} \oplus 2\delta^{24} \oplus 3\delta^{31} \oplus 4\delta^{+\infty}$$

and then

$$\underline{u} = \begin{bmatrix} e\delta^3 \oplus 1\delta^9 \oplus 2\delta^{14} \oplus \varepsilon\delta^{+\infty} \\ e\delta^6 \oplus 1\delta^{12} \oplus 2\delta^{24} \oplus 3\delta^{31} \oplus 4\delta^{+\infty} \end{bmatrix}.$$

This yields (step V)

$$\mathcal{G}\underline{u} = e\delta^{10} \oplus 1\delta^{16} \oplus 2\delta^{28} \oplus 3\delta^{35} \oplus 4\delta^{+\infty} \not\preceq z,$$

implying $\mathcal{Q} = \emptyset$. Thus, going to step VII, we need to relax reference z according to Prop. 8.2, which gives

$$z' = e\delta^{14} \oplus 1\delta^{23} \oplus 2\delta^{28} \oplus 3\delta^{35} \oplus 4\delta^{+\infty}.$$

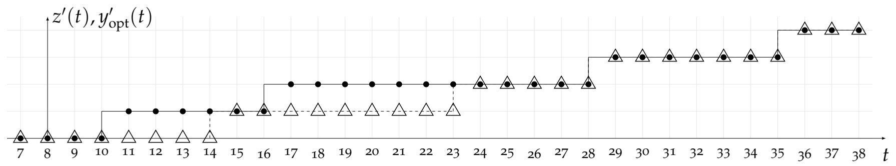

Figure 8.1: Tracking of the new reference z' (△) by the updated optimal output y'_{opt} (•) obtained in Example 8.1.

Finally, the updated optimal input u'_{opt} is obtained by computing the greatest fixed point of mapping Γ', resulting in

$$u'_{\mathrm{opt}} = \begin{bmatrix} e\delta^3 \oplus 1\delta^9 \oplus 2\delta^{21} \oplus 3\delta^{28} \oplus 4\delta^{+\infty} \\ e\delta^6 \oplus 1\delta^{12} \oplus 2\delta^{24} \oplus 3\delta^{31} \oplus 4\delta^{+\infty} \end{bmatrix}$$

and

$$y'_{\mathrm{opt}} = \mathcal{G}u'_{\mathrm{opt}} = e\delta^{10} \oplus 1\delta^{16} \oplus 2\delta^{28} \oplus 3\delta^{35} \oplus 4\delta^{+\infty} .$$

The tracking of the new reference z' by the updated output y'_{opt} is shown in Fig. 8.1. ◇

8.3 PROBLEM FORMULATION AND OPTIMAL UPDATE OF THE INPUTS — THE CASE OF MULTIPLE PARTIALLY-SYNCHRONIZED TRANSITIONS

Consider a system modeled as a TEG with n internal transitions — I of which are partially synchronized — and m input transitions. As in Section 7.4, for ease of discussion and without loss of generality let us assume that the corresponding counters x_ι are the first I entries of vector $x \in \Sigma^{n\times 1}$. Based on Remark 7.3, we also assume there is an input transition u_η connected to each partially-synchronized transition x_ι via a place with zero holding time and no initial tokens. Moreover, again to facilitate the discussion and without loss of generality, let these inputs be the first I entries of the input vector $u \in \Sigma^{m\times 1}$, and let $\eta = \iota$ whenever u_η is connected to x_ι. Suppose the system is operating optimally with respect to a given output-reference z, with optimal input u_{opt} obtained according to the method presented in Chapter 7.

Now, suppose that at a certain time T the restrictions due to PS are altered for some (possibly all) x_ι, $\iota \in \{1, \ldots, I\}$. In terms of the modeling technique introduced in Section 7.2, this means that, for each $\iota \in \{1, \ldots, I\}$, the firing schedule of transition ρ_ι is updated to a new one, ρ'_ι, with $r_T(\rho'_\iota) = r_T(\rho_\iota)$ (and with the possibility that $\rho'_\iota = \rho_\iota$). Recalling that we have $\alpha_\iota = e\delta^1\rho_\iota$, the schedule of transition α_ι is thus also updated to α'_ι with $r_{(T+1)}(\alpha'_\iota) = r_{(T+1)}(\alpha_\iota)$. Based on (7.3), the new restrictions imposed by PS on each partially-synchronized transition x_ι can be expressed by

$$\rho'_\iota \odot \xi_\iota \preceq \alpha'_\iota \odot x_\iota . \tag{8.12}$$

Our goal is to determine the input u'_{opt} which preserves u_{opt} up to time T and which results in an output that tracks reference z as closely as possible, while guaranteeing, for every $\iota \in \{1, \dots, I\}$, that the resulting firing schedule for x_ι, denoted $x'_{\iota_{\text{opt}}}$, observes the restrictions from PS expressed by (8.12).

Recall that we can express the firing schedule of each x_ι in terms of u as $x_\iota = \mathcal{F}_{[\iota\cdot]}u$, where $\mathcal{F}_{[\iota\cdot]}$ is the ι^{th} row of $\mathcal{F} = A^*B$ as in (3.3). Combined with the fact that $\xi_\iota = e\delta^1 x_\iota$ and $\alpha'_\iota = e\delta^1 \rho'_\iota$ (cf. Section 7.2), this means we can write (8.12) as

$$\rho'_\iota \odot e\delta^1 \mathcal{F}_{[\iota\cdot]}u \;\preceq\; e\delta^1 \rho'_\iota \odot \mathcal{F}_{[\iota\cdot]}u \,. \qquad (⌘⌘)$$

The problem described above can then be stated as follows: find the greatest element of the set

$$\mathcal{V} \;=\; \big\{u \in \Sigma^{m\times 1} \mid \mathcal{G}u \preceq z \text{ and } r_T(u) = r_T(u_{\text{opt}}) \text{ and } (⌘⌘) \text{ holds for all } \iota \in \{1,\dots,I\}\big\}. \qquad (8.13)$$

Along the lines of Section 8.2, we set out to look for the greatest element of set $\mathcal{V}$ defined in (8.13) by noticing that

$$\begin{aligned} (⌘⌘) \;&\Leftrightarrow\; e\delta^1 \mathcal{F}_{[\iota\cdot]}u \;\preceq\; (e\delta^1 \rho'_\iota \odot \mathcal{F}_{[\iota\cdot]}u) \odot^\sharp \rho'_\iota \\ &\Leftrightarrow\; u \;\preceq\; e\delta^1 \mathcal{F}_{[\iota\cdot]} \lozenge \big[(e\delta^1 \rho'_\iota \odot \mathcal{F}_{[\iota\cdot]}u) \odot^\sharp \rho'_\iota\big] \,. \end{aligned}$$

Let us define, for each $\iota \in \{1, \dots, I\}$, the mapping $\Psi_\iota : \Sigma^{m\times 1} \to \Sigma^{m\times 1}$,

$$\Psi_\iota(u) \;=\; e\delta^1 \mathcal{F}_{[\iota\cdot]} \lozenge \big[(e\delta^1 \rho'_\iota \odot \mathcal{F}_{[\iota\cdot]}u) \odot^\sharp \rho'_\iota\big] \,,$$

and also the mapping $\overline{\Psi} : \Sigma^{m\times 1} \to \Sigma^{m\times 1}$,

$$\overline{\Psi}(u) \;=\; \mathcal{G} \lozenge z \wedge \bigwedge_{\iota=1}^{I} \Psi_\iota(u) \,. \qquad (8.14)$$

Note that u satisfying (⌘⌘) is equivalent to $u \preceq \Psi_\iota(u)$, so we can write set $\mathcal{V}$ equivalently as

$$\mathcal{V} = \big\{u \in \Sigma^{m\times 1} \mid u \preceq \overline{\Psi}(u) \text{ and } r_T(u) = r_T(u_{\text{opt}})\big\}.$$

The problem stated above can then be solved by applying Prop. 5.1, taking $\mathcal{D}$ as $\Sigma^{m\times 1}$, ψ as $\overline{\Psi}$, f as r_T, and c as $r_T(u_{\text{opt}})$. Thus, as long as set $\mathcal{V}$ is nonempty, recalling that mapping r_T is residuated (cf. Remark 3.1) and $r_T^\sharp \circ r_T = r_T^\sharp$, the sought optimal update of the input, u'_{opt}, is the greatest fixed point of mapping $\overline{\Gamma} : \Sigma^{m\times 1} \to \Sigma^{m\times 1}$,

$$\overline{\Gamma}(u) \;=\; u \wedge \overline{\Psi}(u) \wedge r_T^\sharp(u_{\text{opt}}) \,.$$

Next, we must investigate whether set $\mathcal{V}$ is nonempty. To that end, let us define the set

$$\widetilde{\mathcal{V}} = \big\{u \in \Sigma^{m\times 1} \mid (⌘⌘) \text{ holds for all } \iota \in \{1,\dots,I\} \text{ and } r_T(u) = r_T(u_{\text{opt}})\big\} \,. \qquad (8.15)$$

We look for an element $\underline{u}$ of $\widetilde{\mathcal{V}}$ that leads to the fastest possible behavior of the system, i. e., to the least possible firing schedule of the output. If such an input does not ensure that reference z is respected, then clearly there does not exist any input that does so while satisfying (⌘⌘) for all $\iota \in \{1, \ldots, I\}$ and $r_T(u) = r_T(u_{\text{opt}})$. This means, as shall be concluded formally in Corollary 8.7, $\mathcal{V} \neq \emptyset \Leftrightarrow \mathcal{G}\underline{u} \preceq z$.

In general, set $\widetilde{\mathcal{V}}$ may not possess a least element. Nevertheless, our goal is to find an input in $\widetilde{\mathcal{V}}$, not necessarily least of unique, which leads to the fastest possible schedule of the internal transitions while guaranteeing that the restrictions on all partially-synchronized transitions are respected, as this will result in the least possible schedule for the output y.

Note that, for any $\iota \in \{1, \ldots, I\}$, a bound for the firing schedule of x_ι can be obtained from (8.12), as, recalling from Section 7.2 that $\alpha'_\iota = e\delta^1\rho'_\iota$ and $\xi_\iota = e\delta^1 x_\iota$,

$$(8.12) \;\Leftrightarrow\; (\rho'_\iota \odot e\delta^1 x_\iota) \odot^\flat e\delta^1\rho'_\iota \preceq x_\iota \,.$$

In the latter inequality, the left-hand side establishes a bound for how small (in the sense of the order of Σ) x_ι can be, representing the maximal number of firings allowed for x_ι under the PS-restrictions.

Additionally, as no internal transition can fire more often than enabled by the inputs and, since past firings must be preserved, the most often each input u_κ can possibly fire from time T onward is encoded by the counter $r_T(u_{\kappa_{\text{opt}}})$, one can see that $\mathcal{F}r_T(u_{\text{opt}})$ imposes a bound for x, i. e., it must hold that $x \succeq \mathcal{F}r_T(u_{\text{opt}})$. In particular, for each $\iota \in \{1, \ldots, I\}$, this implies $x_\iota \succeq \mathcal{F}_{[\iota\cdot]} r_T(u_{\text{opt}})$.

It is also natural to require that x be a solution of (3.2), which, according to Remark 2.4, implies $x = A^* x$. In particular, for each $\iota \in \{1, \ldots, I\}$, this means we must have $x_\iota = [A^*]_{[\iota\cdot]} x \succeq [A^*]_{\iota j} x_j$ for all $j \in \{1, \ldots, I\}$. But note that (7.8) implies $[A^*]_{\iota j} = \mathcal{F}_{\iota j}$ for any $\iota, j \in \{1, \ldots, I\}$; hence, we can rewrite the above condition as $x_\iota \succeq \mathcal{F}_{\iota j} x_j$.

In conclusion, for every $\iota \in \{1, \ldots, I\}$, any schedule for x_ι must obey

$$x_\iota \succeq \left[(\rho'_\iota \odot e\delta^1 x_\iota) \odot^\flat e\delta^1\rho'_\iota\right] \oplus \mathcal{F}_{[\iota\cdot]} r_T(u_{\text{opt}}) \oplus \bigoplus_{j=1}^{I} \mathcal{F}_{\iota j} x_j \,. \tag{8.16}$$

Note that the inequality above — in particular, its last term — implies the schedules of all partially-synchronized transitions are interdependent. Therefore, we must look for the fastest feasible schedule of all such transitions simultaneously. With that in mind, define, for each $\iota \in \{1, \ldots, I\}$, the mapping $\Lambda_\iota : \Sigma^{n\times 1} \to \Sigma$,

$$\Lambda_\iota(x) = \left[(\rho'_\iota \odot e\delta^1 x_\iota) \odot^\flat e\delta^1\rho'_\iota\right] \oplus \mathcal{F}_{[\iota\cdot]} r_T(u_{\text{opt}}) \oplus \bigoplus_{j=1}^{I} \mathcal{F}_{\iota j} x_j \;\oplus x_\iota \,,$$

and then define the mapping $\overline{\Lambda} : \Sigma^{n\times 1} \to \Sigma^{n\times 1}$,

$$\left[\overline{\Lambda}(x)\right]_\kappa = \begin{cases} \Lambda_\kappa(x)\,, & \text{if } 1 \leq \kappa \leq I\,, \\ x_\kappa \oplus \mathcal{F}_{[\kappa\cdot]} r_T(u_{\text{opt}})\,, & \text{if } I+1 \leq \kappa \leq n\,. \end{cases} \tag{8.17}$$

Based on the foregoing discussion, it is clear that any vector $x \in \Sigma^{n\times 1}$ whose entries are feasible schedules for the internal transitions x_κ, $\kappa \in \{1, \ldots, n\}$, must be a fixed point of mapping $\overline{\Lambda}$.

In fact, it is straightforward to see that, for any $\widetilde{u} \in \widetilde{\mathcal{V}}$, $\mathcal{F}\widetilde{u}$ is a fixed point of $\overline{\Lambda}$. First, for any $\iota \in \{1, \ldots, I\}$, $\widetilde{u}$ satisfies (⌘⌘), which is equivalent to

$$(\rho'_\iota \odot e\delta^1 \mathcal{F}_{[\iota\cdot]} u) \odot^\flat e\delta^1 \rho'_\iota \preceq \mathcal{F}_{[\iota\cdot]} u\,. \tag{8.18}$$

Moreover, $\widetilde{x} = \mathcal{F}\widetilde{u}$ is a solution of (3.2), so

$$\widetilde{x}_\iota = [A^*]_{[\iota\cdot]}\widetilde{x} \succeq [A^*]_{\iota j}\widetilde{x}_j = \mathcal{F}_{\iota j}\widetilde{x}_j$$

for all $j \in \{1, \ldots, I\}$. Finally, for all $\kappa \in \{1, \ldots, n\}$, we have

$$\mathcal{F}_{[\kappa\cdot]}\widetilde{u} \succeq \mathcal{F}_{[\kappa\cdot]} r_T(\widetilde{u}) = \mathcal{F}_{[\kappa\cdot]} r_T(u_{\text{opt}})\,.$$

This implies that any $x \in \Sigma^{n\times 1}$ comprising firing schedules of internal transitions which are compatible with past input firings and such that the schedules x_ι of partially-synchronized transitions are reachable from the inputs and are compatible with the restrictions due to PS is in fact a fixed point of $\overline{\Lambda}$. Thus, what remains to be checked is whether the least fixed point of mapping $\overline{\Lambda}$ — which we shall denote $\underline{x}$ — is indeed feasible, i. e., whether there exists an input $\underline{u}$ which is an element of $\widetilde{\mathcal{V}}$ and such that $\mathcal{F}\underline{u} = \underline{x}$. Similarly to Section 8.2, we prove constructively that the answer is affirmative. As the proof is analogous to the corresponding discussion in Section 8.2, we state the two key facts as propositions and omit their proofs from the present discussion. The interested reader can find the proofs in Appendix B.

Let us denote the μ^{th} entry of $\underline{x}$ by $\underline{x}_\mu$, and define the vector $\theta \in \Sigma^{m\times 1}$,

$$\theta_\mu = \begin{cases} \underline{x}_\mu\,, & \text{if } 1 \leq \mu \leq I\,, \\ s_\varepsilon\,, & \text{if } I+1 \leq \mu \leq m\,. \end{cases}$$

Now, consider the input

$$\underline{u} = r_T(u_{\text{opt}}) \oplus \theta = \begin{bmatrix} r_T(u_{1_{\text{opt}}}) \oplus \underline{x}_1 \\ \vdots \\ r_T(u_{I_{\text{opt}}}) \oplus \underline{x}_I \\ r_T(u_{(I+1)_{\text{opt}}}) \\ \vdots \\ r_T(u_{m_{\text{opt}}}) \end{bmatrix}. \tag{8.19}$$

Proposition 8.4. *Let $\underline{u}$ be defined as in (8.19), $\underline{x}$ the least fixed point of mapping $\overline{\Lambda}$ defined in (8.17), and $\mathcal{F} = A^*B$ as in (3.3). Then, it holds that $\mathcal{F}\underline{u} = \underline{x}$.*

Proposition 8.5. *Vector $\underline{u}$ defined as in (8.19) is an element of set $\widetilde{\mathcal{V}}$ defined in (8.15).*

This does not guarantee, however, that $\mathcal{V} \neq \emptyset$, as it is possible that $\mathcal{G}\underline{u} \npreceq z$ and hence $\underline{u} \notin \mathcal{V}$. Intuitively, if the updated restrictions from PS on some partially-synchronized transitions are more stringent than the original ones, since up to time T we implemented just-in-time inputs based on the original restrictions, it may be impossible to respect both reference z and the new restrictions after T. As we assume PS-restrictions to be hard ones, this means we have no choice but to relax z, i. e., look for a new reference $z' \succeq z$ for which a solution exists. In fact, we seek the least possible such z', in order to remain as close as possible to the original reference. A natural choice is then to take $z' = z \oplus \mathcal{G}\underline{u}$; as $\oplus$ is performed coefficient-wise on counters, this amounts to preserving the terms of z that can still be achieved by taking $\underline{u}$ as input, and relaxing those that cannot only as much as necessary to be matched by the resulting output $y = \mathcal{G}\underline{u}$. The following proposition establishes that this is indeed the optimal way of relaxing z.

Proposition 8.6. *Let $\mathcal{V}'$ denote the set defined as $\mathcal{V}$ in (8.13), only replacing z with z', and let $\underline{u}$ be defined as in (8.19). The least $z' \succeq z$ such that $\mathcal{V}' \neq \emptyset$ is $z' = z \oplus \mathcal{G}\underline{u}$.*

Prop. 8.6 also provides a simple way to check whether set $\mathcal{V}$ is nonempty.

Corollary 8.7. *Let $\mathcal{V}$ be defined as in (8.13) and $\underline{u}$ as in (8.19). Then, $\mathcal{V} \neq \emptyset \Leftrightarrow \mathcal{G}\underline{u} \preceq z$.*

If $\mathcal{V}$ turns out to be empty, define the mapping $\overline{\Psi}' : \Sigma^{m\times 1} \to \Sigma^{m\times 1}$ as $\overline{\Psi}$ in (8.14), only replacing z with $z' = z \oplus \mathcal{G}\underline{u}$. Following the same procedure as before, we can apply Prop. 5.1 — only now taking ψ as $\overline{\Psi}'$ instead of $\overline{\Psi}$ — to conclude that u'_{opt} is the greatest fixed point of mapping $\overline{\Gamma}' : \Sigma^{m\times 1} \to \Sigma^{m\times 1}$,

$$\overline{\Gamma}'(u) = u \wedge \overline{\Psi}'(u) \wedge r_T^{\sharp}(u_{\text{opt}}) \,.$$

9 RELATED WORK ON SYSTEMS WITH PARTIAL SYNCHRONIZATION

TEGs with PS were originally studied in [15–17], where they are modeled by recursive equations with additional constraints over the max-plus and the min-plus algebra; the authors develop a method for optimal feedforward control and MPC for this class of systems. In [47], a specific semiring of operators is introduced to model the subclass of TEGs under periodic PS, where PS restrictions are determined by periodic signals. An advantage of the operatorial representation is the possibility to obtain a direct input-output relation (i.e., a transfer function or transfer matrix) for the system, which allows to efficiently compute the response to periodic inputs over an infinite horizon and solve output-reference and model-reference control problems. In the method presented in Chapters 7 and 8, we make no periodicity assumption on the PS signals and propose a method entirely based on the well-established semiring of counters. We believe this makes our model more intuitive and easier to interpret than that in [47] and, most importantly, it allows us to harness the benefits of having a transfer relation for the system while encompassing the general class of TEGs under (not necessarily periodic) PS restrictions treated in [15–17].

Other classes of systems somewhat related to TEGs with PS have been investigated in the past decades. Katz [25] and Maia et al. [30, 32] consider (A, B)-invariant and semimodule subspaces in order to compute a control enforcing certain restrictions on the state of the system. This can be applied, for instance, to ensure that the sojourn time of tokens through the system belongs to a given interval. Note that this models a different phenomenon from that of TEGs with PS, where the permission to fire certain transitions is successively granted and revoked according to external signals but no upper bound for their firing times is directly imposed. In [18], the authors study TEGs with soft synchronization, where the synchronization between certain transitions in the system can be broken at a cost. For example, an operation may be allowed to start without waiting for the conclusion of delayed predecessor operations, hence preventing the propagation of delays but incurring penalty costs. Dually to PS, where external signals impose additional restrictions, in this case external decisions can overrule standard synchronization constraints based on a trade-off between performance criteria and penalty costs. Finally, it is worth mentioning that a phenomenon analogous to PS has been studied by the scheduling community, where the external restrictions for the occurrence of certain events are often referred to as *availability* *constraints* (see, e.g., [39] and references therein). A closer comparison of the results presented here with such scheduling methods is beyond the scope of this thesis and remains as an interesting subject for future work.

Part IV

SYSTEMS WITH SHARED RESOURCES AND PARTIAL SYNCHRONIZATION

10 CONTROL OF TEGS WITH SHARED RESOURCES AND PARTIAL SYNCHRONIZATION

In this chapter, we merge the methods from Chapters 4 and 7 into a unified framework for the optimal control of systems exhibiting both treated phenomena, namely resource sharing and partial synchronization.

A preliminary version of part of the material from Section 10.1 has appeared in [42], which reflects original work from — and has as the main author and contributor — the author of this thesis.

10.1 OPTIMAL CONTROL OF TEGS WITH A SINGLE SHARED RESOURCE AND WITH PARTIAL SYNCHRONIZATION

Consider a system consisting of K subsystems — modeled as TEGs $S^1, \ldots, S^K$ — that share a resource with finite but arbitrary capacity. We shall here adopt the same notation as in Section 4.1, to which the reader is referred for a detailed description. The structure of the system is illustrated in Fig. 4.1, but we assume the setting of Section 4.4, i. e., each subsystem S^k may have an arbitrary number m_k of input transitions, including u^{k0} and u^{k1} as in Fig. 4.1 (hence $m_k \geq 2$). The additional input transitions (besides u^{k0} and u^{k1}) do not appear in Fig. 4.1 and are here, as in Section 4.4, assumed to be inside blocks P^{k1}, H^k, and/or P^{k2}.

Recall that Section 4.1 culminates in the following inequality (copied below from (4.1) for convenience) capturing the restrictions imposed by the dynamics and the finite capacity of the resource on the combined allocation (x_A^k) and release (x_R^k) schedules of all subsystems:

$$\beta \otimes \Big(\bigodot_{k=1}^{K} x_R^k \Big) \preceq \bigodot_{k=1}^{K} x_A^k \, . \tag{10.1}$$

Condition (10.1) also applies in the context of Section 4.4 and, therefore, in that of the present discussion.

Now, suppose an internal transition $x_{\iota_k}^k$ — which might, in particular, be x_A^k or x_R^k — of each subsystem S^k is partially synchronized (for ease of discussion, in this section we treat the simpler case of a single partially-synchronized transition per TEG; the more general case of multiple such transitions per TEG is considered in Section 10.2). For each such transition, PS is modeled through an independent structure like the one from Fig. 7.1, as described in Section 7.2, with the appropriate indexing of transitions (and related counters) ρ^k, α^k, and ξ^k. The assumptions from Remark 7.2 are also in place, i. e., there is an input $u^k_{\eta_k}$ connected to $x^k_{\iota_k}$ via a place with zero holding time and no initial tokens. Based on (7.2), each partially-synchronized transition $x^k_{\iota_k}$ is subject to

$$\rho^k \odot \xi^k \preceq \alpha^k \odot x^k_{\iota_k} . \qquad (10.2)$$

As in Chapter 4, let us assume each subsystem S^k is assigned a respective output-reference z^k. According to the priority policy introduced in the referred chapter, when computing the optimal inputs we must start from the subsystem with highest priority, S^1, and proceed successively through the lower-priority ones until covering S^K. Subsystem S^1 is free to use the resource at will, and we can compute its optimal input neglecting any dispute with other subsystems. We must, nonetheless, take the restrictions due to PS into account. Computing the optimal input for S^1 thus amounts to the case discussed in Section 7.3. Recalling that we can write $x^1_{\iota_1} = \mathcal{F}^1_{[\iota_1\cdot]}u^1$, where $\mathcal{F}^1_{[\iota_1\cdot]}$ is the $\iota_1{}^{\text{th}}$ row of matrix $\mathcal{F}^1 = A^{1*}B^1$ (cf. (3.4)), and that we have $\xi^1 = e\delta^1 x^1_{\iota_1}$ and $\alpha^1 = e\delta^1\rho^1$ (cf. Section 7.2), from (10.2) we obtain[1]

$$\rho^1 \odot e\delta^1 \mathcal{F}^1_{[\iota_1\cdot]}u^1 \preceq e\delta^1\rho^1 \odot \mathcal{F}^1_{[\iota_1\cdot]}u^1 . \qquad (10.3)$$

Following similar steps as in Section 7.3, we can then conclude that the optimal input for S^1 respecting (10.3) and $\mathcal{G}^1 u^1 \preceq z^1$ is given by the greatest fixed point of mapping $\widehat{\Phi}^1 : \Sigma^{m_1\times 1} \to \Sigma^{m_1\times 1}$,

$$\widehat{\Phi}^1(u^1) = e\delta^1\mathcal{F}^1_{[\iota_1\cdot]} \backslash\!\!\circ \left[(e\delta^1\rho^1 \odot \mathcal{F}^1_{[\iota_1\cdot]}u^1) \odot^\sharp \rho^1 \right] \wedge \mathcal{G}^1 \backslash\!\!\circ z^1 \wedge u^1 , \qquad (10.4)$$

where m_1 is the number of input transitions in subsystem S^1. Recall from Section 4.4 that we denote

$$\mathcal{F}^1_A = \mathcal{F}^1_{[i\cdot]} \qquad \text{and} \qquad \mathcal{F}^1_R = \mathcal{F}^1_{[j\cdot]} ,$$

where i and j indicate the entries in vector x^1 occupied by x^1_A and x^1_R, respectively (i.e., $x^1_i = x^1_A$ and $x^1_j = x^1_R$). Then, the resulting resource-allocation and release schedules for S^1 are

$$x^1_{A_{\text{opt}}} = \mathcal{F}^1_A u^1_{\text{opt}} \qquad \text{and} \qquad x^1_{R_{\text{opt}}} = \mathcal{F}^1_R u^1_{\text{opt}} .$$

For S^2, we must compute the optimal input under the restriction that the optimal behavior of S^1 is unaffected; based on (10.1) and neglecting all lower-priority subsystems (i.e., all S^j with $2 < j \le K$), this means we must respect

$$\beta \otimes (x^1_{R_{\text{opt}}} \odot x^2_R) \preceq x^1_{A_{\text{opt}}} \odot x^2_A . \qquad (10.5)$$

1 Notation at this point becomes a bit unfortunate. Please note that the superscript on δ (as in $e\delta^1$) means the exponent of δ in the compact representation of that counter, whereas the superscripts on any other symbols (as in u^1 or $\mathcal{F}^1_{[\iota_1\cdot]}$) denote the index of the corresponding subsystem, in this case S^1.

Following similar steps as in Sections 4.2 and 4.4, we can write $x_A^2 = \mathcal{F}_A^2 u^2$ and $x_R^2 = \mathcal{F}_R^2 u^2$, and (10.5) can then be rewritten as

$$\beta \otimes (x_{R_{\mathrm{opt}}}^1 \odot \mathcal{F}_R^2 u^2) \preceq x_{A_{\mathrm{opt}}}^1 \odot \mathcal{F}_A^2 u^2 ,$$

which, in turn, is equivalent to

$$u^2 \preceq \mathcal{F}_R^2 \diamond \left[\left(\beta \diamond (x_{A_{\mathrm{opt}}}^1 \odot \mathcal{F}_A^2 u^2) \right) \odot^\sharp x_{R_{\mathrm{opt}}}^1 \right] . \tag{10.6}$$

Additionally, the restrictions on the partially-synchronized transition $x_{\iota_2}^2$ must be observed. From condition (10.2) and applying arguments similar to the ones for S^1 above, these restrictions can be expressed as

$$\rho^2 \odot e\delta^1 \mathcal{F}_{[\iota_2 \cdot]}^2 u^2 \preceq e\delta^1 \rho^2 \odot \mathcal{F}_{[\iota_2 \cdot]}^2 u^2$$

or, equivalently,

$$u^2 \preceq e\delta^1 \mathcal{F}_{[\iota_2 \cdot]}^2 \diamond \left[(e\delta^1 \rho^2 \odot \mathcal{F}_{[\iota_2 \cdot]}^2 u^2) \odot^\sharp \rho^2 \right] . \tag{10.7}$$

We look for the greatest input satisfying inequalities (10.6) and (10.7), as well as $\mathcal{G}^2 u^2 \preceq z^2$, i. e., $u^2 \preceq \mathcal{G}^2 \diamond z^2$. The fact that all three conditions are expressed as upper bounds on u^2 makes it straightforward to combine them into a single expression using operator $\wedge$. The sought optimal solution u_{opt}^2 is the greatest fixed point of mapping $\widehat{\Phi}^2 : \Sigma^{m_2 \times 1} \to \Sigma^{m_2 \times 1}$,

$$\begin{aligned} \widehat{\Phi}^2(u^2) = \; & \mathcal{F}_R^2 \diamond \left[\left(\beta \diamond (x_{A_{\mathrm{opt}}}^1 \odot \mathcal{F}_A^2 u^2) \right) \odot^\sharp x_{R_{\mathrm{opt}}}^1 \right] \\ & \wedge \; e\delta^1 \mathcal{F}_{[\iota_2 \cdot]}^2 \diamond \left[(e\delta^1 \rho^2 \odot \mathcal{F}_{[\iota_2 \cdot]}^2 u^2) \odot^\sharp \rho^2 \right] \\ & \wedge \; \mathcal{G}^2 \diamond z^2 \; \wedge \; u^2 . \end{aligned} \tag{10.8}$$

An entirely analogous reasoning can be applied to determine the optimal input for an arbitrary subsystem S^k with $k \in \{2, \ldots, K\}$. Writing $x_A^k = \mathcal{F}_A^k u^k$ and $x_R^k = \mathcal{F}_R^k u^k$, from (10.1) we obtain

$$u^k \preceq \mathcal{F}_R^k \diamond \left[\left(\beta \diamond \left(\bigodot_{i=1}^{k-1} x_{A_{\mathrm{opt}}}^i \odot \mathcal{F}_A^k u^k \right) \right) \odot^\sharp \bigodot_{i=1}^{k-1} x_{R_{\mathrm{opt}}}^i \right] . \tag{10.9}$$

Moreover, writing $x_{\iota_k}^k = \mathcal{F}_{[\iota_k \cdot]}^k u^k$, from (10.2) we obtain

$$u^k \preceq e\delta^1 \mathcal{F}_{[\iota_k \cdot]}^k \diamond \left[(e\delta^1 \rho^k \odot \mathcal{F}_{[\iota_k \cdot]}^k u^k) \odot^\sharp \rho^k \right] . \tag{10.10}$$

Combining inequalities (10.9) and (10.10) with the condition that reference z^k must be respected, i. e., $u^k \preceq \mathcal{G}^k \diamond z^k$, we conclude that the optimal solution u_{opt}^k is the greatest fixed point of $\widehat{\Phi}^k : \Sigma^{m_k \times 1} \to \Sigma^{m_k \times 1}$,

$$\begin{aligned} \widehat{\Phi}^k(u^k) = \; & \mathcal{F}_R^k \diamond \left[\left(\beta \diamond \left(\bigodot_{i=1}^{k-1} x_{A_{\mathrm{opt}}}^i \odot \mathcal{F}_A^k u^k \right) \right) \odot^\sharp \bigodot_{i=1}^{k-1} x_{R_{\mathrm{opt}}}^i \right] \\ & \wedge \; e\delta^1 \mathcal{F}_{[\iota_k \cdot]}^k \diamond \left[(e\delta^1 \rho^k \odot \mathcal{F}_{[\iota_k \cdot]}^k u^k) \odot^\sharp \rho^k \right] \\ & \wedge \; \mathcal{G}^k \diamond z^k \; \wedge \; u^k . \end{aligned} \tag{10.11}$$

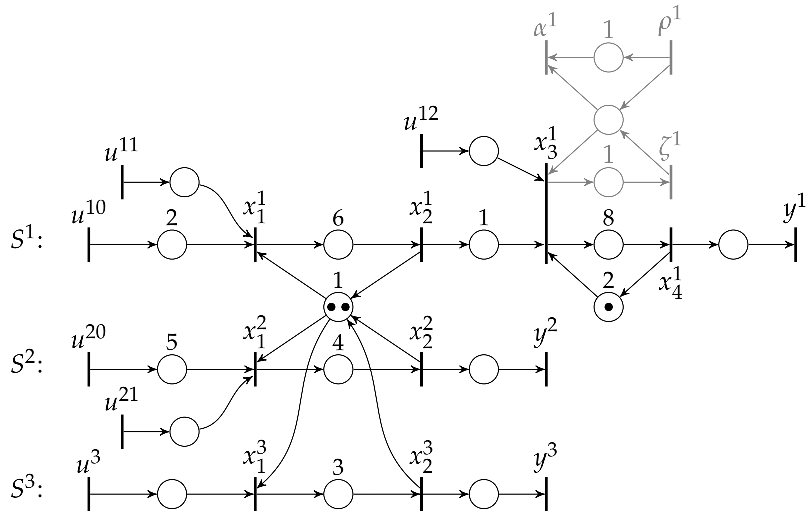

Figure 10.1: Three TEGs S^1, S^2, and S^3 with a shared resource, and with transition x_3^1 partially synchronized.

Example 10.1. Consider the system from Figure 4.2, described as in Example 4.1. Now, suppose transition x_3^1 is partially synchronized; following Section 7.2, we append to it a structure like the one from Figure 7.3, including an extra input u^{12} connected to x_3^1 via a place with zero initial tokens and no holding time, according to Remark 7.2. The resulting model is shown in Figure 10.1.

The transfer matrix for subsystem S^1 is

$$\mathcal{G}^1 = \begin{bmatrix} e\delta^{17}(1\delta^{10})^* & e\delta^{15}(1\delta^{10})^* & e\delta^{8}(1\delta^{10})^* \end{bmatrix},$$

whereas those of S^2 and S^3 are the same as in Example 4.1.

The PS-restrictions dictate that x_3^1 can only fire at times

$$t \in \mathcal{T} = \{[11,15] \cup [21,23] \cup [32,36] \cup [44,47]\} \subset \mathbb{Z},$$

and at most once at each such instant. This is encoded in counter ρ^1 as follows:

$$\rho^1(t) = \begin{cases} e & \text{if } t \leq 11; \\ 1 \otimes \rho^1(t-1) & \text{if } t-1 \in \mathcal{T}; \\ \rho^1(t-1) & \text{if } t-1 \notin \mathcal{T} \text{ and } t > 11. \end{cases}$$

Let the same references as in Example 4.1 be given, namely

$$\begin{aligned} z^1 &= e\delta^{52} \oplus 4\delta^{+\infty}, \\ z^2 &= e\delta^{27} \oplus 3\delta^{39} \oplus 5\delta^{+\infty}, \\ z^3 &= e\delta^{9} \oplus 3\delta^{35} \oplus 5\delta^{+\infty}. \end{aligned}$$

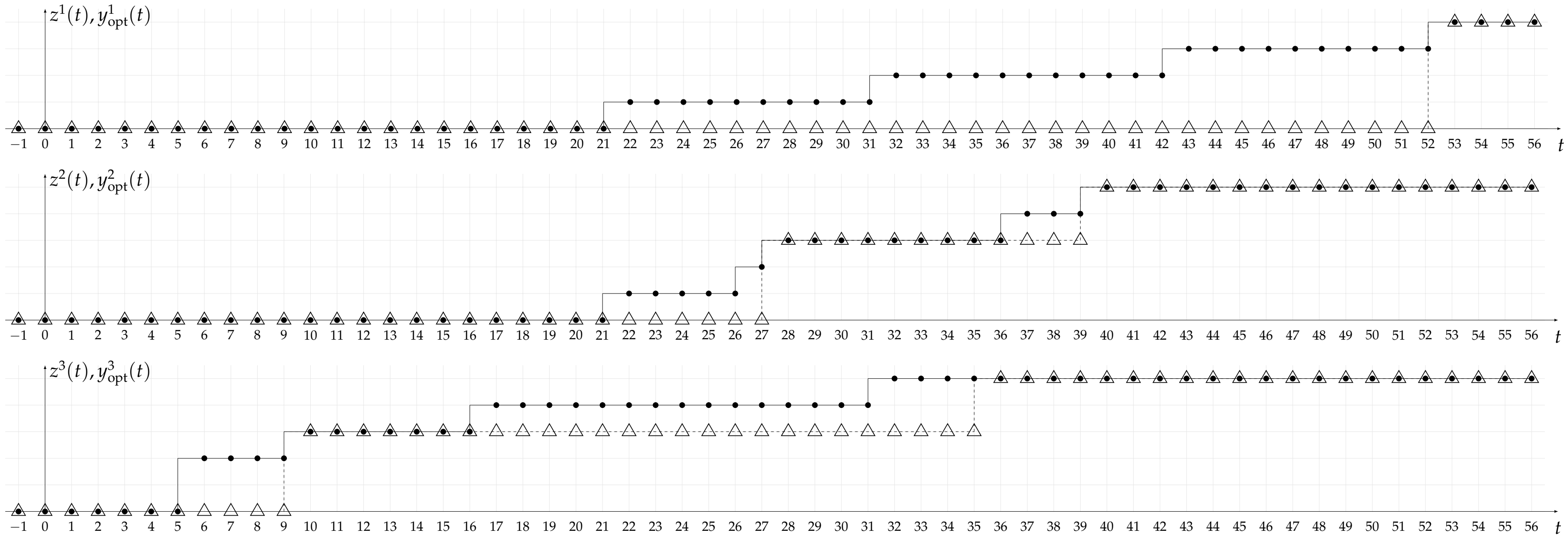

Figure 10.2: Tracking of the references z^k (denoted by $\triangle$) by the outputs y^k_{opt} (denoted by $\bullet$), $k \in \{1, 2, 3\}$, from Example 10.1.

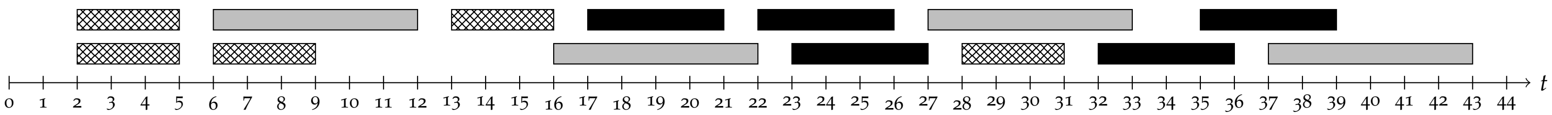

Figure 10.3: Schedule for the use of the shared resource, obtained in Example 10.1. The gray, black, and crosshatched bars represent the time windows during which an instance of the resource is held by S^1, S^2, and S^3, respectively.

Following the procedure laid down in the present section, the greatest fixed points of mappings $\widehat{\Phi}^k$, $k \in \{1, 2, 3\}$, yield

$$u^1_{\mathrm{opt}} = \begin{bmatrix} e\delta^4 \oplus 1\delta^{14} \oplus 2\delta^{25} \oplus 3\delta^{35} \oplus 4\delta^{+\infty} \\ e\delta^6 \oplus 1\delta^{16} \oplus 2\delta^{27} \oplus 3\delta^{37} \oplus 4\delta^{+\infty} \\ e\delta^{13} \oplus 1\delta^{23} \oplus 2\delta^{34} \oplus 3\delta^{44} \oplus 4\delta^{+\infty} \end{bmatrix},$$

$$u^2_{\mathrm{opt}} = \begin{bmatrix} e\delta^8 \oplus 1\delta^{13} \oplus 2\delta^{18} \oplus 3\delta^{27} \oplus 4\delta^{30} \oplus 5\delta^{+\infty} \\ e\delta^{17} \oplus 1\delta^{22} \oplus 2\delta^{23} \oplus 3\delta^{32} \oplus 4\delta^{35} \oplus 5\delta^{+\infty} \end{bmatrix},$$

$$u^3_{\mathrm{opt}} = e\delta^2 \oplus 2\delta^6 \oplus 3\delta^{13} \oplus 4\delta^{28} \oplus 5\delta^{+\infty}.$$

The resulting optimal outputs are

$$y^1_{\mathrm{opt}} = e\delta^{21} \oplus 1\delta^{31} \oplus 2\delta^{42} \oplus 3\delta^{52} \oplus 4\delta^{+\infty},$$
$$y^2_{\mathrm{opt}} = e\delta^{21} \oplus 1\delta^{26} \oplus 2\delta^{27} \oplus 3\delta^{36} \oplus 4\delta^{39} \oplus 5\delta^{+\infty},$$
$$y^3_{\mathrm{opt}} = e\delta^{5} \oplus 2\delta^{9} \oplus 3\delta^{16} \oplus 4\delta^{31} \oplus 5\delta^{+\infty}.$$

Figure 10.2 shows the tracking of the corresponding references, and in Figure 10.3 the distribution of the resource among the three subsystems over time is illustrated. One can see that, while (as expected) the effect of PS slightly degrades de performance of subsystem S^1, in the sense that it cannot track its output-reference as closely as in the case without PS, subsystems S^2 and S^3 benefit from the additional resource availability and can track their references more closely than before (compare Figures 10.2 and 4.3).

10.2 OPTIMAL CONTROL OF TEGS WITH MULTIPLE SHARED RESOURCES AND WITH PARTIAL SYNCHRONIZATION

In this section, we extend the ideas discussed in Section 10.1 to the case of TEGs sharing multiple resources and with multiple partially-synchronized transitions. We start from the setting and the notation from Section 4.3, i. e., a system consisting of K subsystems — modeled as TEGs $S^1, \dots, S^K$ — that share L resources, each with finite but arbitrary capacity. The structure of the system is illustrated in Fig. 4.5, but let us now assume the case investigated in Section 4.4, i. e., each subsystem S^k may have an arbitrary number m_k of input transitions, including $u^{k\ell}$ for all $\ell \in \{0, \dots, L\}$ as in Fig. 4.5 (hence $m_k \geq L + 1$).

In Section 4.3, we established condition (4.12) — which also applies to the case of Section 4.4 and hence to that of the present section — capturing the restrictions imposed by the dynamics and the finite capacity of each resource ℓ on the combined allocation ($x^{k\ell}_A$) and release ($x^{k\ell}_R$) schedules of all subsystems; we repeat inequality (4.12) here for ease of reference:

$$\beta^\ell \otimes \Big(\bigodot_{k=1}^{K} x^{k\ell}_R \Big) \preceq \bigodot_{k=1}^{K} x^{k\ell}_A. \tag{10.12}$$

Now, for each subsystem S^k, suppose I_k out of its n_k internal transitions are partially synchronized. As in Section 7.4, we assume, for notational convenience and without loss of generality, that the corresponding counters x^k_ι are the first I_k entries of vector $x^k \in \Sigma^{n_k \times 1}$. The PS of each partially-synchronized transition x^k_ι, $\iota \in \{1, \ldots, I_k\}$, is represented by a structure like the one from Fig. 7.3, with the appropriate indexing of transitions (and related counters) ρ^k_ι, ξ^k_ι, and α^k_ι. The assumptions from Remark 7.3 concerning input transitions u^k_η connected to each x^k_ι are in place. Based on (7.3), each partially-synchronized transition x^k_ι, for every $\iota \in \{1, \ldots, I_k\}$, is subject to

$$\rho^k_\iota \odot \xi^k_\iota \preceq \alpha^k_\iota \odot x^k_\iota \,. \tag{10.13}$$

As usual, assume that each subsystem S^k is assigned a respective output-reference z^k and that the priority policy introduced in Chapter 4 is to be observed when computing the optimal inputs. The optimal input for S^1 can be computed ignoring all other subsystems, thus amounting to the case from Section 7.4. Writing, for each $\iota \in \{1, \ldots, I_1\}$, $x^1_\iota = \mathcal{F}^1_{[\iota\cdot]} u^1$, $\xi^1_\iota = e\delta^1 x^1_\iota$, and $\alpha^1_\iota = e\delta^1 \rho^1_\iota$, inequality (10.13) assumes the form

$$\rho^1_\iota \odot e\delta^1 \mathcal{F}^1_{[\iota\cdot]} u^1 \preceq e\delta^1 \rho^1_\iota \odot \mathcal{F}^1_{[\iota\cdot]} u^1 \,. \tag{10.14}$$

Now, define, for $\iota \in \{1, \ldots, I_1\}$, the collection of mappings $\widehat{\Phi}^1_\iota : \Sigma^{m_1 \times 1} \to \Sigma^{m_1 \times 1}$,

$$\widehat{\Phi}^1_\iota(u^1) = e\delta^1 \mathcal{F}^1_{[\iota\cdot]} \backslash\!\!\circ \left[(e\delta^1 \rho^1_\iota \odot \mathcal{F}^1_{[\iota\cdot]} u^1) \odot^\sharp \rho^1_\iota \right] .$$

Trough similar steps as in Section 7.4, the optimal input u^1_{opt} respecting (10.14) for all $\iota \in \{1, \ldots, I_1\}$ and also $\mathcal{G}^1 u^1 \preceq z^1$ is then given by the greatest fixed point of mapping $\overline{\widehat{\Phi}}{}^1 : \Sigma^{m_1 \times 1} \to \Sigma^{m_1 \times 1}$,

$$\overline{\widehat{\Phi}}{}^1(u^1) = \bigwedge_{\iota=1}^{I_1} \widehat{\Phi}^1_\iota(u^1) \wedge \mathcal{G}^1 \backslash\!\!\circ z^1 \wedge u^1 \,.$$

Using the notation from Section 4.4 (see (4.20)), we obtain the resulting schedules for allocation and release of each resource ℓ by S^1 as

$$x^{1\ell}_{A_{\mathrm{opt}}} = \mathcal{F}^{1\ell}_A u^1_{\mathrm{opt}} \qquad \text{and} \qquad x^{1\ell}_{R_{\mathrm{opt}}} = \mathcal{F}^{1\ell}_R u^1_{\mathrm{opt}} \,.$$

The method then proceeds by computing the optimal inputs for $S^2, \ldots, S^K$ in decreasing order of priority. For an arbitrary $k \in \{2, \ldots, K\}$, we must compute the optimal input of S^k under the restriction that the optimal behavior of higher-priority subsystems (i. e., all S^i with $i < k$) is unaffected. Based on (10.12) and neglecting all lower-priority subsystems (i. e., all S^j with $k < j \leq K$), this means we must respect

$$u^k \preceq \mathcal{F}^{k\ell}_R \backslash\!\!\circ \left[\left(\beta^\ell \backslash\!\!\circ \left(\bigodot_{i=1}^{k-1} x^{i\ell}_{A_{\mathrm{opt}}} \odot \mathcal{F}^{k\ell}_A u^k \right) \right) \odot^\sharp \bigodot_{i=1}^{k-1} x^{i\ell}_{R_{\mathrm{opt}}} \right] . \tag{10.15}$$

We must also observe the restrictions on all partially-synchronized transitions x_ι^k, $\iota \in \{1, \dots, I_k\}$. From condition (10.13) and applying arguments similar to the ones for S^1 above, these restrictions can be expressed, for each $\iota \in \{1, \dots, I_k\}$, as

$$u^k \preceq e\delta^1 \mathcal{F}^k_{[\iota\cdot]} \,\phi\, \left[(e\delta^1 \rho_\iota^k \odot \mathcal{F}^k_{[\iota\cdot]} u^k) \odot^\sharp \rho_\iota^k \right] . \tag{10.16}$$

We look for the greatest input satisfying inequalities (10.15) for all $\ell \in \{1, \dots, L\}$ and (10.16) for all $\iota \in \{1, \dots, I_k\}$, as well as $\mathcal{G}^k u^k \preceq z^k$, i. e., $u^k \preceq \mathcal{G}^k \phi z^k$. Consider, for each $\ell \in \{1, \dots, L\}$, mapping $\Phi^{k\ell}_{\text{mi}}$ defined in (4.21), and also, for each $\iota \in \{1, \dots, I_k\}$, define the mapping $\widehat{\Phi}^k_\iota : \Sigma^{m_k \times 1} \to \Sigma^{m_k \times 1}$,

$$\widehat{\Phi}^k_\iota(u^k) = e\delta^1 \mathcal{F}^k_{[\iota\cdot]} \,\phi\, \left[(e\delta^1 \rho_\iota^k \odot \mathcal{F}^k_{[\iota\cdot]} u^k) \odot^\sharp \rho_\iota^k \right] ,$$

The sought optimal solution, u^k_{opt}, is the greatest fixed point of mapping $\overline{\widehat{\Phi}}^k : \Sigma^{m_k \times 1} \to \Sigma^{m_k \times 1}$,

$$\overline{\widehat{\Phi}}^k(u^k) = \bigwedge_{\ell=1}^{L} \Phi^{k\ell}_{\text{mi}}(u^k) \wedge \bigwedge_{\iota=1}^{I_k} \widehat{\Phi}^k_\iota(u^k) \wedge \mathcal{G}^k \phi z^k \wedge u^k .$$

11 CONCLUSION

The results presented in this thesis contribute to a well-established framework for the control of discrete-event systems based on tropical algebra. The main contributions are two-fold, as the proposed method enhances the existing framework by encompassing two different phenomena of practical relevance, namely resource-sharing and partial synchronization.

In the resource-sharing front, the considered scenario is that of a number of subsystems, each modeled as a TEG, competing for access to one or more shared resources. Note that, in this scenario, the overall system cannot be modeled as a single TEG. The proposed method first shows that it is possible to express the additional constraints on certain transitions in the system due to the limited capacity of the shared resources as inequalities in the semiring of counters, with the help of an operation called Hadamard product. In order to settle the dispute for the resources, a priority hierarchy is enforced among the users (subsystems), so that each subsystem strives to track its own output-reference while being prohibited from interfering with the operation of any of the higher-priority subsystems. The optimal control input for each of the subsystems is then sought under the *just-in-time* paradigm, meaning the input-events must occur as late as possible while guaranteeing that the demand for output-events, encoded by a reference signal, is met at all times. We formulate the problem such that these optimal inputs can be obtained by computing greatest fixed points of appropriate (isotone) mappings, defined so as to capture all relevant constraints — namely, global constraints coming from the shared resources as well as local ones from the respective output-reference signals.

The method for systems with shared resources is also extended to the case in which the output-references of the subsystems are subject to unforeseen changes during the operation of the system. This makes it necessary to perform on-line updates in (possibly all) the control inputs, which must be done while still observing the adopted priority policy. Nevertheless, as past event occurrences (obviously) cannot be revoked, the behavior of a given subsystem may not only be affected by that of higher-priority subsystems, but also by lower-priority ones if they happen to be in possession of the resources at the time the reference signals are updated. In order to check whether the new references are feasible, we show how to determine the fastest possible behavior (i. e., the one providing the earliest possible outputs) for each subsystem from the time the new references are received, by computing least fixed points of certain mappings. If this fastest behavior cannot meet the new reference for a given subsystem, we provide a way to relax the reference as little as possible so as to make it attainable. Then, as before the just-in-time inputs are computed as greatest fixed points of appropriately-defined mappings.

The partial synchronization (PS) phenomenon consists in the existence of external signals restricting the occurrence of certain events in the system, and it cannot be modeled by a TEG alone. In this thesis, an approach is proposed to model this phenomenon entirely within the context of the semiring of counters. Then, similarly to the case of resource-sharing, the external constraints can be expressed in the form of inequalities. Still seeking optimality in a just-in-time sense, the problem of obtaining the optimal input for the system can again be formulated as a fixed-point problem. We further consider the case in which the external signals encoding PS-restrictions may change while the system is running; similarly to the case of varying output-references, this requires an on-line update in the input schedules, and the newly imposed restrictions may render the output-reference unachievable. The proposed method detects whether that is the case by calculating the fastest behavior of the system after receiving the updated PS signal, provides the least-relaxed feasible reference (if necessary), and optimally updates the inputs by computing greatest fixed points of mappings tailored for that purpose.

Every part of the method mentioned above is developed in a formal and systematic way. In particular, this means that, as long as the pertinent TEG models, reference signals, and (in the case of PS) external restrictions are provided, the optimal inputs can be automatically computed. Similarly, if the reference or PS signals are changed during the operation, it is possible to obtain the least-relaxed feasible versions of the references (whenever necessary) and then the optimally-updated inputs in an algorithmic fashion.

The similarities in the mathematical formulation of the methods for dealing with resource-sharing and with PS make it natural to merge them into a unified method, capable of providing optimal inputs (based on output-references) for systems exhibiting both phenomena. The extension of this combined method to the cases of varying output-references and varying PS signals requires further investigation and remains as a promising topic for future work. Furthermore, as a side contribution of the method proposed in this thesis, the fact that these two phenomena can be studied through such similar mathematical lenses reveals a correspondence between them, which — at lease to this author — was previously not self-evident. It is then natural to wonder whether a method of similar flavor can be applied to deal with yet other interesting phenomena; this also remains as an open question for future work.

Part V

APPENDIX

A

PROOFS FROM CHAPTER 5

Proof of Proposition 5.1. Define the set

$$\widetilde{\mathcal{S}}_\psi = \{x \in \mathcal{D} \,|\, x \preceq \psi(x) \text{ and } f(x) \preceq c\}$$

and denote $\chi = \bigoplus_{x \in \mathcal{S}_\psi} x$ and $\widetilde{\chi} = \bigoplus_{x \in \widetilde{\mathcal{S}}_\psi} x$. Note that

$$\begin{aligned} x \preceq \psi(x) \text{ and } f(x) \preceq c \;&\Leftrightarrow\; x \preceq \psi(x) \text{ and } x \preceq f^\sharp(c) \qquad \text{(see Def. 2.3)} \\ &\Leftrightarrow\; x \preceq \psi(x) \wedge f^\sharp(c) \\ &\Leftrightarrow\; x = x \wedge \psi(x) \wedge f^\sharp(c) = \Omega(x)\,. \end{aligned}$$

So, set $\widetilde{\mathcal{S}}_\psi$ can be equivalently defined as $\widetilde{\mathcal{S}}_\psi = \{x \in \mathcal{D} \,|\, x = \Omega(x)\}$, clearly implying $\widetilde{\chi} = \bigoplus\{x \in \mathcal{D} \,|\, \Omega(x) = x\}$. Then, it also follows from Remark 2.9 that $\widetilde{\chi} \in \widetilde{\mathcal{S}}_\psi$.

Now, assume $\mathcal{S}_\psi \neq \emptyset$. As $\mathcal{S}_\psi \subseteq \widetilde{\mathcal{S}}_\psi$, this implies $(\exists \widetilde{x} \in \widetilde{\mathcal{S}}_\psi)\, f(\widetilde{x}) = c$. Taking such an $\widetilde{x}$, we have $\widetilde{x} \preceq \widetilde{\chi}$ and so $c = f(\widetilde{x}) \preceq f(\widetilde{\chi})$ (as f is isotone). But we saw above that $\widetilde{\chi} \in \widetilde{\mathcal{S}}_\psi$, meaning $f(\widetilde{\chi}) \preceq c$, so $f(\widetilde{\chi}) = c$. Therefore, $\widetilde{\chi} \in \mathcal{S}_\psi$ and hence $\widetilde{\chi} \preceq \chi$. On the other hand, $\mathcal{S}_\psi \subseteq \widetilde{\mathcal{S}}_\psi$ implies $\chi \preceq \widetilde{\chi}$, showing that $\chi = \widetilde{\chi}$. □

Proof of Proposition 5.2. First, note that the assumptions made about β in Section 4.1 imply that there exist $t_\beta > 0$ and $b \prec e$ (which, recall, in the standard sense means $b > 0$) such that $\beta(t) = b$ for all $t \leq t_\beta$. Therefore, for any $t \leq T$ we have

$$\begin{aligned} \left[\beta \otimes \left(H^1 r_T^\sharp(x^1_{A_\text{opt}}) \odot \mathcal{L}_R^1\right)\right](t) &= \left[\beta \otimes \bigodot_{k=1}^{K} H^k r_T^\sharp(x^k_{A_\text{opt}})\right](t) \\ &= \bigoplus_{\tau \in \overline{\mathbb{Z}}} \beta(\tau) \otimes \left[\bigodot_{k=1}^{K} H^k r_T^\sharp(x^k_{A_\text{opt}})\right](t-\tau) \\ &= \bigoplus_{\tau \geq t_\beta} \beta(\tau) \otimes \left[\bigodot_{k=1}^{K} H^k r_T^\sharp(x^k_{A_\text{opt}})\right](t-\tau) \\ \text{(as } t-\tau < T\text{)} \qquad &= \bigoplus_{\tau \geq t_\beta} \beta(\tau) \otimes \left[\bigodot_{k=1}^{K} H^k x^k_{A_\text{opt}}\right](t-\tau) \\ &= \left[\beta \otimes \bigodot_{k=1}^{K} H^k x^k_{A_\text{opt}}\right](t) \\ &\preceq \left[\bigodot_{k=1}^{K} x^k_{A_\text{opt}}\right](t) \\ \text{(because } t \leq T\text{)} \qquad &= \left[\bigodot_{k=1}^{K} r_T^\sharp(x^k_{A_\text{opt}})\right](t) \end{aligned}$$

$$= \left[r_T^\sharp(x^1_{A_{\text{opt}}}) \odot \mathcal{L}^1_A \right](t)\,.$$

Moreover, for $t > T$ we have

$$\begin{aligned}\left[\beta \otimes \left(H^1 r_T^\sharp(x^1_{A_{\text{opt}}}) \odot \mathcal{L}^1_R\right)\right](t) &\preceq \left[\beta \otimes \left(H^1 r_T^\sharp(x^1_{A_{\text{opt}}}) \odot \mathcal{L}^1_R\right)\right](T) \\ &\preceq \left[r_T^\sharp(x^1_{A_{\text{opt}}}) \odot \mathcal{L}^1_A \right](T) \\ &= \left[r_T^\sharp(x^1_{A_{\text{opt}}}) \odot \mathcal{L}^1_A \right](t)\,.\end{aligned}$$

This shows that

$$\beta \otimes \left(H^1 r_T^\sharp(x^1_{A_{\text{opt}}}) \odot \mathcal{L}^1_R\right) \preceq r_T^\sharp(x^1_{A_{\text{opt}}}) \odot \mathcal{L}^1_A$$

or, equivalently,

$$\left(\beta \otimes \left(H^1 r_T^\sharp(x^1_{A_{\text{opt}}}) \odot \mathcal{L}^1_R\right)\right) \odot^\flat \mathcal{L}^1_A \preceq r_T^\sharp(x^1_{A_{\text{opt}}})\,. \tag{A.1}$$

For any $k \in \{2, \ldots, K\}$, assuming $x^{i\prime}_{A_{\text{opt}}}$ to be given for each $i \in \{1, \ldots, k-1\}$, from ($\star$) and since $\mathcal{P}^k u^{(k-1)\prime}_{\text{opt}} = x^{(k-1)\prime}_{A_{\text{opt}}}$ we know

$$\beta \otimes \left(\mathcal{H}^{(k-1)}_R \odot H^{(k-1)} x^{(k-1)\prime}_{A_{\text{opt}}} \odot \mathcal{L}^{(k-1)}_R\right) \preceq \mathcal{H}^{(k-1)}_A \odot x^{(k-1)\prime}_{A_{\text{opt}}} \odot \mathcal{L}^{(k-1)}_A\,. \tag{A.2}$$

But note that

$$\begin{aligned}&\mathcal{H}^{(k-1)}_R \odot H^{(k-1)} x^{(k-1)\prime}_{A_{\text{opt}}} = \mathcal{H}^k_R\,,\\ &\mathcal{L}^{(k-1)}_R = H^k r_T^\sharp(x^k_{A_{\text{opt}}}) \odot \mathcal{L}^k_R\,,\\ &\mathcal{H}^{(k-1)}_A \odot x^{(k-1)\prime}_{A_{\text{opt}}} = \mathcal{H}^k_A\,, \quad \text{and}\\ &\mathcal{L}^{(k-1)}_A = r_T^\sharp(x^k_{A_{\text{opt}}}) \odot \mathcal{L}^k_A\,,\end{aligned}$$

so (A.2) is equivalent to

$$\beta \otimes \left(\mathcal{H}^k_R \odot H^k r_T^\sharp(x^k_{A_{\text{opt}}}) \odot \mathcal{L}^k_R\right) \preceq \mathcal{H}^k_A \odot r_T^\sharp(x^k_{A_{\text{opt}}}) \odot \mathcal{L}^k_A$$

which, in turn, implies

$$\left(\beta \otimes (\mathcal{H}^k_R \odot H^k r_T^\sharp(x^k_{A_{\text{opt}}}) \odot \mathcal{L}^k_R)\right) \odot^\flat (\mathcal{H}^k_A \odot \mathcal{L}^k_A) \preceq r_T^\sharp(x^k_{A_{\text{opt}}})\,. \tag{A.3}$$

Finally, for any $k \in \{1, \ldots, K\}$, we have

$$r_T^\sharp(x^k_{A_{\text{opt}}}) = r_T^\sharp(\mathcal{P}^k u^k_{\text{opt}}) \succeq \mathcal{P}^k u^k_{\text{opt}} \succeq \mathcal{P}^k r_T(u^k_{\text{opt}})\,.$$

This, together with (A.1) and (A.3), concludes the proof. □

Proof of Proposition 5.3. Taking $z^{k\prime\prime} = z^{k\prime} \oplus \mathcal{G}^k \underline{u}^k$ implies $\mathcal{G}^k \underline{u}^k \preceq z^{k\prime\prime}$ and, as $\underline{u}^k \in \widetilde{\mathcal{N}}^k$, it follows that $\underline{u}^k \in \mathcal{N}^{k\prime\prime}$ and hence $\mathcal{N}^{k\prime\prime} \neq \emptyset$. Now, take $\zeta \succeq z^{k\prime}$ such that $\mathcal{N}^k_\zeta \neq \emptyset$ (where $\mathcal{N}^k_\zeta$ is defined like $\mathcal{N}^k$, only replacing $z^{k\prime}$ with ζ), and take $v \in \mathcal{N}^k_\zeta$. As $v \in \widetilde{\mathcal{N}}^k$, it satisfies ($\star$) and thus also

$$\left(\beta \otimes (\mathcal{H}^k_R \odot H^k \mathcal{P}^k v \odot \mathcal{L}^k_R)\right) \odot^\flat (\mathcal{H}^k_A \odot \mathcal{L}^k_A) \preceq \mathcal{P}^k v\,;$$

besides, as $v \succeq r_T(v)$, we have $\mathcal{P}^k v \succeq \mathcal{P}^k r_T(v) = \mathcal{P}^k r_T(u^k_{\text{opt}})$, so clearly $\mathcal{P}^k v$ is a fixed point of Λ^k, which implies $\mathcal{P}^k v \succeq \underline{x}^k_A = \mathcal{P}^k \underline{u}^k$. Recalling, as argued in Remark 4.1, that $G^{k0} = G^{k1} P^{k1}$ and so $\mathcal{G}^k = G^{k1} \mathcal{P}^k$, we have $\mathcal{G}^k \underline{u}^k = G^{k1} \mathcal{P}^k \underline{u}^k \preceq G^{k1} \mathcal{P}^k v = \mathcal{G}^k v \preceq \zeta$, so $z^{k\prime\prime} = z^{k\prime} \oplus \mathcal{G}^k \underline{u}^k \preceq z^{k\prime} \oplus \zeta = \zeta$. □

Proof of Proposition 5.5. First, note that, as $\widetilde{u}^k$ satisfies ($\star\star$), we have

$$\left(\beta^\ell \otimes (\mathcal{H}^{k\ell}_R \odot H^{k\ell} \mathcal{P}^{k\ell} \widetilde{u}^k \odot \mathcal{L}^{k\ell}_R)\right) \odot^\flat (\mathcal{H}^{k\ell}_A \odot \mathcal{L}^{k\ell}_A) \preceq \mathcal{P}^{k\ell} \widetilde{u}^k$$

for every ℓ. We then proceed by induction on ℓ. For any $\widetilde{u}^k \in \widetilde{\mathcal{M}}^k$, $\mathcal{P}^{k1} \widetilde{u}^k \succeq \mathcal{P}^{k1} r_T(\widetilde{u}^k) = \mathcal{P}^{k1} r_T(u^k_{\text{opt}})$, proving the base case $\ell = 1$. Assuming $\mathcal{P}^{k\ell} \widetilde{u}^k$ is a fixed point of $\Lambda^{k\ell}$ for a fixed but arbitrary $\ell \in \{1, \dots, L-1\}$, we have

$$\mathcal{P}^{k(\ell+1)} \widetilde{u}^k \succeq \mathcal{P}^{k(\ell+1)} r_T(\widetilde{u}^k) = \mathcal{P}^{k(\ell+1)} r_T(u^k_{\text{opt}}) = r_T(u^{k(\ell+1)}_{\text{opt}})$$

(where the last equality follows from Remark 4.3) and

$$\begin{aligned} \mathcal{P}^{k(\ell+1)} \widetilde{u}^k &= P^{k(\ell+1)} H^{k\ell} \mathcal{P}^{k\ell} \widetilde{u}^k \oplus \widetilde{u}^{k(\ell+1)} && \text{(check (4.17))} \\ &\succeq P^{k(\ell+1)} H^{k\ell} \mathcal{P}^{k\ell} \widetilde{u}^k \\ &\succeq P^{k(\ell+1)} H^{k\ell} \underline{x}^{k\ell}_A\,, && \text{(from the induction hypothesis)} \end{aligned}$$

which proves that $\mathcal{P}^{k(\ell+1)} \widetilde{u}^k$ is a fixed point of $\Lambda^{k(\ell+1)}$. □

Proof of Proposition 5.6. Denote by $\mathcal{P}^{k\ell}_i$ the i^{th} entry of $\mathcal{P}^{k\ell}$ (defined as in (4.17)). As $\underline{x}^{k1}_A$ is a fixed point of Λ^{k1},

$$\underline{x}^{k1}_A \succeq \mathcal{P}^{k1} r_T(u^k_{\text{opt}}) \succeq \mathcal{P}^{k1}_1 r_T(u^{k0}_{\text{opt}})\,,$$

so, as $\mathcal{P}^{k1}_2 = s_e$ and $\mathcal{P}^{k1}_i = s_\varepsilon$ for $i > 2$ (cf. (4.17)),

$$\mathcal{P}^{k1} \underline{u}^k = \mathcal{P}^{k1}_1 r_T(u^{k0}_{\text{opt}}) \oplus \mathcal{P}^{k1}_2 \underline{x}^{k1}_A = \mathcal{P}^{k1}_1 r_T(u^{k0}_{\text{opt}}) \oplus \underline{x}^{k1}_A = \underline{x}^{k1}_A\,.$$

This proves the case $\ell = 1$.

Now, for any $\ell \in \{2, \dots, L\}$, because $\underline{x}^{k\ell}_A$ is a fixed point of $\Lambda^{k\ell}$ we have

$$\underline{x}^{k\ell}_A \succeq r_T(u^{k\ell}_{\text{opt}}) = \mathcal{P}^{k\ell} r_T(u^k_{\text{opt}}) \succeq \mathcal{P}^{k\ell}_1 r_T(u^{k0}_{\text{opt}})\,, \tag{A.4}$$

where the equality follows from Remark 4.3.

Furthermore, since $\underline{x}^{k2}_A$ is a fixed point of Λ^{k2},

$$\underline{x}^{k2}_A \succeq P^{k2} H^{k1} \underline{x}^{k1}_A = \mathcal{P}^{k2}_2 \underline{x}^{k1}_A$$

and, similarly, as $\underline{x}^{k3}_A$ is a fixed point of Λ^{k3}

$$\underline{x}^{k3}_A \succeq P^{k3} H^{k2} \underline{x}^{k2}_A = \mathcal{P}^{k3}_3 \underline{x}^{k2}_A\,,$$

so also

$$\underline{x}_A^{k3} \succeq P^{k3} H^{k2} P^{k2} H^{k1} \underline{x}_A^{k1} = \mathcal{P}_2^{k3} \underline{x}_A^{k1};$$

more generally, for any $\ell \in \{2, \dots, L\}$,

$$\underline{x}_A^{k\ell} \succeq \mathcal{P}_i^{k\ell} \underline{x}_A^{k(i-1)} \text{ for all } i \in \{2 \dots, \ell\}. \tag{A.5}$$

Therefore, recalling that the $(\ell+1)^{\text{st}}$ entry of $\mathcal{P}^{k\ell}$ is s_e and the i^{th} entry is s_ε for $i > \ell+1$ (check (4.17)), for all $\ell \in \{2, \dots, L\}$ we have

$$\begin{aligned}
\mathcal{P}^{k\ell} \underline{u}^k &= \mathcal{P}_1^{k\ell} r_T(u_{\text{opt}}^{k0}) \oplus \bigoplus_{i=2}^{\ell+1} \mathcal{P}_i^{k\ell} \underline{x}_A^{k(i-1)} \\
&= \mathcal{P}_1^{k\ell} r_T(u_{\text{opt}}^{k0}) \oplus \bigoplus_{i=2}^{\ell} \mathcal{P}_i^{k\ell} \underline{x}_A^{k(i-1)} \oplus \underline{x}_A^{k\ell} \\
&= \underline{x}_A^{k\ell},
\end{aligned}$$

where the last equality follows from (A.4) and (A.5). □

Lemma A.1 (of Proposition 5.7)**.** *$r_T^\sharp(x_{A_{\text{opt}}}^{k\ell})$ is a fixed point of mapping $\Lambda^{k\ell}$, for all $\ell \in \{1, \dots, L\}$.*

Proof. It follows by direct analogy with the proof of Prop. 5.2. □

Proof of Proposition 5.7. For any $\ell \in \{1, \dots, L\}$, the fact $\underline{x}_A^{k\ell}$ is a fixed point of $\Lambda^{k\ell}$ implies

$$(\beta^\ell \otimes (\mathcal{H}_R^{k\ell} \odot H^{k\ell} \underline{x}_A^{k\ell} \odot \mathcal{L}_R^{k\ell})) \odot^\flat (\mathcal{H}_A^{k\ell} \odot \mathcal{L}_A^{k\ell}) \preceq \underline{x}_A^{k\ell}.$$

This, combined with Prop. 5.6, implies that taking $u^k = \underline{u}^k$ satisfies

$$(\beta^\ell \otimes (\mathcal{H}_R^{k\ell} \odot H^{k\ell} \mathcal{P}^{k\ell} u^k \odot \mathcal{L}_R^{k\ell})) \odot^\flat (\mathcal{H}_A^{k\ell} \odot \mathcal{L}_A^{k\ell}) \preceq \mathcal{P}^{k\ell} u^k,$$

which is equivalent to ($\star\star$).

It remains to show that $r_T(\underline{u}^k) = r_T(u_{\text{opt}}^k)$. Since $r_T\big(r_T(u_{\text{opt}}^{k0})\big) = r_T(u_{\text{opt}}^{k0})$, all we need to prove is that $r_T(\underline{x}_A^{k\ell}) = r_T(u_{\text{opt}}^{k\ell})$ for all $\ell \in \{1, \dots, L\}$.

The fact that $\underline{x}_A^{k1}$ is a fixed point of Λ^{k1} implies

$$\underline{x}_A^{k1} \succeq \mathcal{P}^{k1} r_T(u_{\text{opt}}^k) = r_T(u_{\text{opt}}^{k1}),$$

where the equality follows from Remark 4.3. Moreover, $\underline{x}_A^{k\ell} \succeq r_T(u_{\text{opt}}^{k\ell})$ for all $\ell \in \{2, \dots, L\}$ (because $\underline{x}_A^{k\ell}$ is a fixed point of $\Lambda^{k\ell}$). Thus, as r_T is order-preserving and $r_T \circ r_T = r_T$, we can conclude that

$$r_T(\underline{x}_A^{k\ell}) \succeq r_T\big(r_T(u_{\text{opt}}^{k\ell})\big) = r_T(u_{\text{opt}}^{k\ell})$$

for all $\ell \in \{1, \dots, L\}$.

In order to show that the converse inequality holds, note that, as a consequence of Lemma A.1,

$$\underline{x}_A^{k\ell} \preceq r_T^\sharp(x_{A_{\text{opt}}}^{k\ell}) = r_T^\sharp(\mathcal{P}^{k\ell} u_{\text{opt}}^k)$$

for all $\ell \in \{1, \ldots, L\}$. We also know from Remark 4.3 that $\mathcal{P}^{k\ell} u_{\text{opt}}^{k} = u_{\text{opt}}^{k\ell}$. Thus, as r_T is order-preserving and $r_T \circ r_T^{\sharp} = r_T$, we have

$$r_T(\underline{x}_A^{k\ell}) \preceq r_T\big(r_T^{\sharp}(\mathcal{P}^{k\ell} u_{\text{opt}}^{k})\big) = r_T\big(r_T^{\sharp}(u_{\text{opt}}^{k\ell})\big) = r_T(u_{\text{opt}}^{k\ell})$$

for all $\ell \in \{1, \ldots, L\}$, concluding the proof. □

Proof of Proposition 5.8. It follows by direct analogy with the proof of Prop. 5.3. □

B

PROOFS FROM CHAPTER 8

Proof of Proposition 8.1. For any $t \leq T$, we have

$$\begin{aligned}
\left[\rho' \odot e\delta^1 r_T^\sharp(x_{\iota_\text{opt}})\right](t) &= \rho'(t) \otimes \left[e\delta^1 r_T^\sharp(x_{\iota_\text{opt}})\right](t) \\
&= \rho(t) \otimes \left[r_T^\sharp(x_{\iota_\text{opt}})\right](t-1) \\
&= \rho(t) \otimes x_{\iota_\text{opt}}(t-1) && \text{(because } t-1 < T\text{)} \\
&= \rho(t) \otimes \left[e\delta^1 x_{\iota_\text{opt}}\right](t) \\
&= \left[\rho(t) \odot e\delta^1 x_{\iota_\text{opt}}\right](t) \\
&\preceq \left[e\delta^1 \rho \odot x_{\iota_\text{opt}}\right](t) && \text{(as } x_{\iota_\text{opt}} \text{ satisfies (7.2))} \\
&= \left[e\delta^1 \rho\right](t) \otimes x_{\iota_\text{opt}}(t) \\
&= \left[e\delta^1 \rho'\right](t) \otimes \left[r_T^\sharp(x_{\iota_\text{opt}})\right](t) && \text{(again as } t-1 < T\text{)} \\
&= \left[e\delta^1 \rho' \odot r_T^\sharp(x_{\iota_\text{opt}})\right](t)\,.
\end{aligned}$$

Moreover, for $t > T$,

$$\begin{aligned}
\left[\rho' \odot e\delta^1 r_T^\sharp(x_{\iota_\text{opt}})\right](t) &= \rho'(t) \otimes \left[e\delta^1 r_T^\sharp(x_{\iota_\text{opt}})\right](t) \\
&= \rho'(t) \otimes \left[r_T^\sharp(x_{\iota_\text{opt}})\right](t-1) \\
&= \rho'(t) \otimes \left[r_T^\sharp(x_{\iota_\text{opt}})\right](T) && \text{(because } t-1 \geq T\text{)} \\
&= \rho'(t) \otimes \left[r_T^\sharp(x_{\iota_\text{opt}})\right](t) && \text{(because } t > T\text{)} \\
&\preceq \left[e\delta^1 \rho'\right](t) \otimes \left[r_T^\sharp(x_{\iota_\text{opt}})\right](t) \\
&= \left[e\delta^1 \rho' \odot r_T^\sharp(x_{\iota_\text{opt}})\right](t)\,.
\end{aligned}$$

This shows that $\rho' \odot e\delta^1 r_T^\sharp(x_{\iota_\text{opt}}) \preceq e\delta^1 \rho' \odot r_T^\sharp(x_{\iota_\text{opt}})$ or, equivalently,

$$\left(\rho' \odot e\delta^1 r_T^\sharp(x_{\iota_\text{opt}})\right) \odot^\flat e\delta^1 \rho' \preceq r_T^\sharp(x_{\iota_\text{opt}})\,.$$

We also have

$$r_T^\sharp(x_{\iota_\text{opt}}) = r_T^\sharp(\mathcal{F}_{[\iota\cdot]} u_\text{opt}) \succeq \mathcal{F}_{[\iota\cdot]} u_\text{opt} \succeq \mathcal{F}_{[\iota\cdot]} r_T(u_\text{opt})\,.$$

Finally, as x_{ι_opt} is a solution of (3.2), we have

$$x_{\iota_\text{opt}} = [A^*]_{[\iota\cdot]} x_\text{opt} \succeq [A^*]_{\iota\iota} x_{\iota_\text{opt}} = \mathcal{F}_{\iota\eta} x_{\iota_\text{opt}}\,,$$

which implies $r_T^\sharp(x_{\iota_\text{opt}}) \succeq r_T^\sharp(\mathcal{F}_{\iota\eta} x_{\iota_\text{opt}}) \succeq \mathcal{F}_{\iota\eta} r_T^\sharp(x_{\iota_\text{opt}})$. □

Proof of Proposition 8.2. Taking $z' = z \oplus \mathcal{G}\underline{u}$ implies $\mathcal{G}\underline{u} \preceq z'$ and, as $\underline{u} \in \widetilde{\mathcal{Q}}$, it follows that $\underline{u} \in \mathcal{Q}'$ and hence $\mathcal{Q}' \neq \emptyset$. Now, take $\zeta \succeq z$ such that $\mathcal{Q}_\zeta \neq \emptyset$ (where $\mathcal{Q}_\zeta$ is defined like $\mathcal{Q}$, only replacing z with

ζ), and take $v \in \mathcal{Q}_\zeta$. As $\mathcal{F}v$ is a solution of (3.2), from Remark 2.4 is follows that $\mathcal{F}v = A^*\mathcal{F}v$, which implies

$$\mathcal{F}_{[j\cdot]}v \;=\; [A^*]_{[j\cdot]}\mathcal{F}v \;=\; \bigoplus_{\kappa=1}^{n}[A^*]_{j\kappa}[\mathcal{F}v]_\kappa \;\succeq\; [A^*]_{j\iota}[\mathcal{F}v]_\iota \;=\; [A^*]_{j\iota}\mathcal{F}_{[\iota\cdot]}v$$

for all $j \in \{1,\dots,n\}$. Moreover, as $v \in \widetilde{\mathcal{Q}}$, we know from Remark 8.1 that $\mathcal{F}_{[\iota\cdot]}v$ is a fixed point of mapping Λ, which implies $\mathcal{F}_{[\iota\cdot]}v \succeq \underline{x}_\iota$. Hence, recalling from (7.8) that $\mathcal{F}_{j\eta} = [A^*]_{j\iota}$ for any $j \in \{1,\dots,n\}$, we have

$$\mathcal{F}_{[j\cdot]}v \;\succeq\; [A^*]_{j\iota}\mathcal{F}_{[\iota\cdot]}v \;=\; \mathcal{F}_{j\eta}\mathcal{F}_{[\iota\cdot]}v \;\succeq\; \mathcal{F}_{j\eta}\underline{x}_\iota\,. \tag{B.1}$$

The fact that $v \in \widetilde{\mathcal{Q}}$ also implies $r_T(v) = r_T(u_{\text{opt}})$, so $v \succeq r_T(u_{\text{opt}})$ and hence

$$(\forall j \in \{1,\dots,n\})\;\; \mathcal{F}_{[j\cdot]}v \;\succeq\; \mathcal{F}_{[j\cdot]}r_T(u_{\text{opt}})\,. \tag{B.2}$$

Thus, for every $j \in \{1,\dots,n\}$ we have

$$\begin{aligned}
\mathcal{F}_{[j\cdot]}\underline{u} \;&=\; \bigoplus_{\substack{\mu=1\\ \mu\neq\eta}}^{m}\mathcal{F}_{j\mu}r_T(u_{\mu_{\text{opt}}}) \;\oplus\; \mathcal{F}_{j\eta}\big(r_T(u_{\mu_{\text{opt}}})\oplus\underline{x}_\iota\big)\\
&=\; \bigoplus_{\substack{\mu=1\\ \mu\neq\eta}}^{m}\mathcal{F}_{j\mu}r_T(u_{\mu_{\text{opt}}}) \;\oplus\; \mathcal{F}_{j\eta}r_T(u_{\mu_{\text{opt}}}) \;\oplus\; \mathcal{F}_{j\eta}\underline{x}_\iota\\
&=\; \bigoplus_{\mu=1}^{m}\mathcal{F}_{j\mu}r_T(u_{\mu_{\text{opt}}}) \;\oplus\; \mathcal{F}_{j\eta}\underline{x}_\iota\\
&=\; \mathcal{F}_{[j\cdot]}r_T(u_{\text{opt}}) \oplus \mathcal{F}_{j\eta}\underline{x}_\iota\\
&\preceq\; \mathcal{F}_{[j\cdot]}v\,,
\end{aligned}$$

where the last inequality is a consequence of (B.1) and (B.2). This means $\mathcal{F}\underline{u} \preceq \mathcal{F}v$. But, recalling from (3.4) that $\mathcal{G} = C\mathcal{F}$, we then have $\mathcal{G}\underline{u} = C\mathcal{F}\underline{u} \preceq C\mathcal{F}v = \mathcal{G}v \preceq \zeta$, so $z' = z \oplus \mathcal{G}\underline{u} \preceq z \oplus \zeta = \zeta$. □

Proof of Corollary 8.3. First note that, if $\mathcal{G}\underline{u} \preceq z$, then $\underline{u} \in \mathcal{Q}$ and hence $\mathcal{Q} \neq \emptyset$. Conversely, if $\mathcal{Q} \neq \emptyset$, then obviously the least $z' \succeq z$ such that $\mathcal{Q}' \neq \emptyset$ is z itself; Prop. 8.2 then implies $z = z \oplus \mathcal{G}\underline{u}$ or, equivalently, $z \succeq \mathcal{G}\underline{u}$. □

Proof of Proposition 8.4. We want to show that $\mathcal{F}_{[\iota\cdot]}\underline{u} = \underline{x}_\iota$ for all $\iota \in \{1,\dots,I\}$. First, from (8.9) it follows, for all $\iota \in \{1,\dots,I\}$, that $[A^*]_{\iota\iota} \succeq [\mathcal{I}^{n\times n}]_{\iota\iota} = s_e$, so $\mathcal{F}_{\iota\iota}\underline{x}_\iota = [A^*]_{\iota\iota}\underline{x}_\iota \succeq \underline{x}_\iota$. On the other hand, the fact that $\underline{x}$ is a fixed point of $\overline{\Lambda}$ implies $\underline{x}_\iota \succeq \mathcal{F}_{\iota\iota}\underline{x}_\iota$, and hence

$$\mathcal{F}_{\iota\iota}\underline{x}_\iota \;=\; \underline{x}_\iota\,; \tag{B.3}$$

it further implies that

$$\underline{x}_\iota \;\succeq\; \bigoplus_{\substack{j=1\\ j\neq\iota}}^{I}\mathcal{F}_{\iota j}\underline{x}_j \qquad\text{and}\qquad \underline{x}_\iota \;\succeq\; \mathcal{F}_{[\iota\cdot]}r_T(u_{\text{opt}})\,. \tag{B.4}$$

Then, for any $\iota \in \{1, \ldots, I\}$, we have

$$\begin{aligned}
\mathcal{F}_{[\iota\cdot]}\underline{u} &= \bigoplus_{\mu=I+1}^{m} \mathcal{F}_{\iota\mu} r_T(u_{\mu_{\text{opt}}}) \oplus \bigoplus_{j=1}^{I} \mathcal{F}_{\iota j}\big(r_T(u_{j_{\text{opt}}}) \oplus \underline{x}_j\big) \\
&= \bigoplus_{\mu=I+1}^{m} \mathcal{F}_{\iota\mu} r_T(u_{\mu_{\text{opt}}}) \oplus \bigoplus_{j=1}^{I} \mathcal{F}_{\iota j} r_T(u_{j_{\text{opt}}}) \oplus \bigoplus_{j=1}^{I} \mathcal{F}_{\iota j}\underline{x}_j \\
&= \bigoplus_{\mu=1}^{m} \mathcal{F}_{\iota\mu} r_T(u_{\mu_{\text{opt}}}) \oplus \bigoplus_{j=1}^{I} \mathcal{F}_{\iota j}\underline{x}_j \\
&= \mathcal{F}_{[\iota\cdot]} r_T(u_{\text{opt}}) \oplus \bigoplus_{\substack{j=1\\ j\neq\iota}}^{I} \mathcal{F}_{\iota j}\underline{x}_j \oplus \mathcal{F}_{\iota\iota}\underline{x}_\iota \\
&= \mathcal{F}_{[\iota\cdot]} r_T(u_{\text{opt}}) \oplus \bigoplus_{\substack{j=1\\ j\neq\iota}}^{I} \mathcal{F}_{\iota j}\underline{x}_j \oplus \underline{x}_\iota \qquad \text{(because of (B.3))} \\
&= \underline{x}_\iota \qquad \text{(due to (B.4))}\,.
\end{aligned}$$

□

Lemma B.1 (of Proposition 8.5). *$r_T^\sharp(x_{\text{opt}})$ is a fixed point of mapping $\overline{\Lambda}$.*

Proof. It follows as a straightforward generalization of the proof of Prop. 8.1. □

Proof of Proposition 8.5. Because $\underline{x}$ is a fixed point of $\overline{\Lambda}$, for all $\iota \in \{1, \ldots, I\}$ it follows that

$$(\rho' \odot e\delta^1 \underline{x}_\iota) \odot^\flat e\delta^1 \rho' \preceq \underline{x}_\iota\,.$$

Combined with the fact that $\mathcal{F}_{[\iota\cdot]}\underline{u} = \underline{x}_\iota$ for all such ι, as shown in Prop. 8.4, this implies taking $u = \underline{u}$ satisfies (8.18), which is equivalent to (⌘⌘).

It remains to show that $r_T(\underline{u}) = r_T(u_{\text{opt}})$. Note that, as $r_T \circ r_T = r_T$, for $\mu \in \{I+1, \ldots, m\}$ it trivially holds that $r_T(\underline{u}_\mu) = r_T(u_{\mu_{\text{opt}}})$. The problem is then reduced to showing that, for all $\iota \in \{1, \ldots, I\}$, $r_T(\underline{u}_\iota) = r_T\big(r_T(u_{\iota_{\text{opt}}}) \oplus \underline{x}_\iota\big) = r_T(u_{\iota_{\text{opt}}})$, which, in turn, as r_T distributes over $\oplus$, is equivalent to $r_T(u_{\iota_{\text{opt}}}) \oplus r_T(\underline{x}_\iota) = r_T(u_{\iota_{\text{opt}}})$, or $r_T(\underline{x}_\iota) \preceq r_T(u_{\iota_{\text{opt}}})$. From Prop. B.1 we know that $\underline{x}_\iota \preceq r_T^\sharp(x_{\iota_{\text{opt}}}) = r_T^\sharp(\mathcal{F}_{[\iota\cdot]}u_{\text{opt}})$ for every ι. We also know from Remark 7.6 that $\mathcal{F}_{[\iota\cdot]}u_{\text{opt}} = u_{\iota_{\text{opt}}}$. Thus, as r_T is isotone and recalling that $r_T \circ r_T^\sharp = r_T$, for all $\iota \in \{1, \ldots, I\}$ we have

$$r_T(\underline{x}_\iota) \preceq r_T\big(r_T^\sharp(u_{\iota_{\text{opt}}})\big) = r_T(u_{\iota_{\text{opt}}})\,.$$

□

Proof of Proposition 8.6. It follows by direct analogy with the proof of Prop. 8.2. □

Proof of Corollary 8.7. It follows by direct analogy with the proof of Corollary 8.3. □

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DECLARATION

I hereby declare that this thesis has been composed by myself, that the work contained herein is my own except where explicitly stated otherwise, and that this work has not been submitted for any other degree or professional qualification.

Berlin, July 2024

Germano Schafaschek