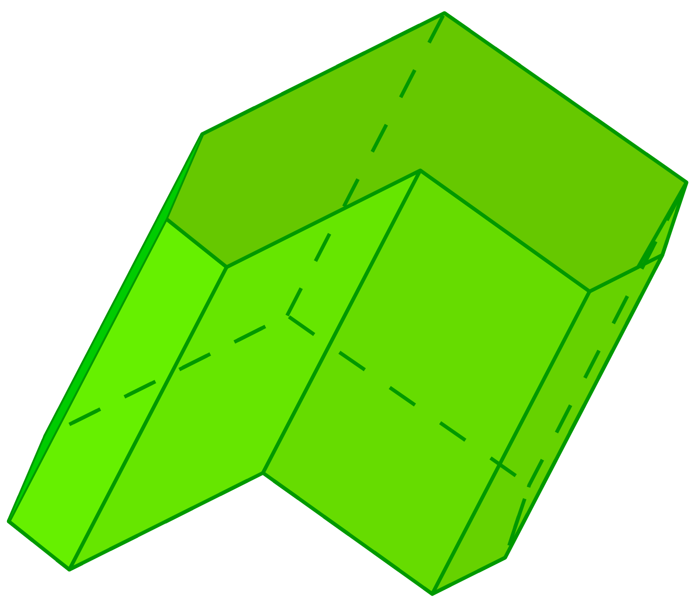
TROPICAL ALGORITHMS FOR LINEAR ALGEBRA AND LINEAR EVENT-INVARIANT DYNAMICAL SYSTEMS

ALGORITMOS TROPICAIS PARA ÁLGEBRA LINEAR E SISTEMAS DINÂMICOS LINEARES INVARIANTES A EVENTOS



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"Tropical Algorithms for Linear Algebra and Linear Event-invariant Dynamical Systems"

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Abstract

Tropical Algebra is a relatively-recent algebraic structure, named in honor of a hungarian-born brazilian mathematician, that has raised interest in many different areas including linear algebra, geometry, economics, optimization, biology and, in special, discrete event systems. In this context, the contributions of this thesis are twofold and independent: in a linear algebra/optimization perspective, this thesis proposes algorithms for solving tropical linear equations (a classical problem in linear tropical algebra) and tropical analogues of linear programs (a problem which received attention from the scientific community only very recently). In the perspective of discrete event systems, closer to engineering, this thesis presents a mathematical analysis and proposes algorithms for solving problems which are tropical analogues of those found in the linear dynamical system theory, in particular, the *regulation* and *observation* problems (also studied by the scientific community only very recently, especially the latter). These problems frequently appear in the control of some systems which have characteristics of synchronization. In order to show the applicability of the approach, the algorithms were implemented in a real plant, an implementation that, to the author's knowledge, the present work is pioneer.

Keywords: Tropical Algebra, Max-Plus Algebra, Control, Observer, Petri Nets, Linear Programming

Resumo

Álgebra tropical é uma estrutura algébrica relativamente recente, nomeada em honra a um matemático húngaro-brasileiro, que tem atraído interesse em diversas áreas incluindo álgebra linear, geometria, economia, otimização, biologia e, em especial, sistemas a eventos discretos. Nesse contexto, as contribuições desta tese são duas e independentes: na perspectiva da álgebra linear/otimização, esta tese propõe algoritmos para a resolução de equações tropicais lineares (um problema clássico na álgebra linear tropical) e análogos tropicais de programas lineares (um problema que tem recebido atenção da comunidade científica apenas bem recentemente). Na perspectiva de sistemas a eventos discretos, mais próxima da engenharia, esta tese apresenta uma análise matemática e propõe algoritmos para a resolução de problemas que são análogos tropicais daqueles encontrados na teoria de sistemas dinâmicos lineares tradicionais, em particular, os problemas de *regulação* e *observação* (também estudados pela comunidade científica apenas bem recentemente, especialmente esse último). Esses problemas frequentemente aparecem no controle de alguns sistemas que apresentam características de sincronização. Para mostrar a aplicabilidade do método, esses algoritmos foram implementados em um sistema real, uma implementação que, pelo conhecimento dos autores, o presente trabalho é pioneiro.

Keywords: Álgebra Tropical, Álgebra Max-Plus, Controle, Observador, Redes de Petri, Programação Linear

Résumé

Algèbre Tropicale est une nom générique donné à des structures algébriques relativement récentes, baptisées en l'honneur d'un mathématicien brésilien d'origine hongroise, qui a suscité l'intérêt dans de nombreux domaines, parmi lesquels on peut citer l'algèbre linéaire, la géométrie, l'économie, l'optimisation, la biologie et l'étude des systèmes à événements discrets.

Dans ce contexte, les contributions de cette thèse sont de deux ordres et peuvent être considérées de manière indépendante: Un premier axe est une contribution à l'algèbre et à l'optimisation linéaire. Il propose des algorithmes pour résoudre des équations linéaires tropicales (un problème classique en algèbre linéaire tropicale) et pour résoudre l'analogue tropical de programmes linéaires (ce point a reçu l'attention de la communauté scientifique concernée que très récemment). Un second axe s'intéresse à l'étude des systèmes à événements discrets, plus proche de l'ingénierie. Cette thèse présente une analyse mathématique et propose des algorithmes pour résoudre les problèmes qui sont des analogues tropicaux à ceux rencontrés dans le cadre de la théorie des systèmes dynamiques linéaires, en particulier, les problèmes de *régulation* et *d'observation* (également étudiés par le communauté scientifique que très récemment). Ces problèmes apparaissent fréquemment dans le contrôle de certains systèmes qui sont caractérises par des phénomènes de synchronisation. Afin de montrer l'applicabilité de l'approche, les algorithmes ont été mis en oeuvre sur une véritable plateforme automatisée au sein de l'université d'Angers, une telle mise oeuvre, à la connaissance de l'auteur, est une première.

Mots-clés: Algèbre Tropical , Algèbre Max-Plus, Contrôle, Observateur, Réseaux de Petri, Programmation Linéaire

List of Symbols

Symbol	Description	Example
$\mathbb{T}_{\mathtt{max}}$	Tropical dioid	$\mathbb{T}_{\mathtt{max}} \equiv \{\mathbb{Z} \cup \{-\infty\}, \oplus, \otimes\}$
$\overline{\mathbb{T}}_{\mathtt{max}}$	Complete tropical dioid	$\overline{\mathbb{T}}_{\mathtt{max}} \equiv \{\mathbb{Z} \cup \{-\infty, \infty\}, \oplus, \otimes\}$
		Convention: $-\infty \otimes \infty = \infty \otimes -\infty = -\infty$
$\preceq$	Less or equal than (natural order)	$(1\ 2) \preceq (3\ 4)$
$\succeq$	Greater or equal than (natural order)	$(3\ 4) \succeq (1\ 2)$
$\oplus$	Matrix lowest upper bound	$(1\ 4) \oplus (2\ 3) = (2\ 4)$
$\wedge$	Matrix greatest lower bound	$(1\ 4) \wedge (2\ 3) = (1\ 3)$
$\otimes$	Tropical matrix product (usually omitted)	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$
$+$	Traditional matrix sum	$(1\ 4) + (2\ 3) = (3\ 7)$
$\cdot$	Traditional matrix product	$2 \cdot 7 = 14$
$/$	Traditional division	$14/2 = 7$
$\frac{x}{y}$	Traditional division	$\frac{14}{2} = 7$
$\perp$	Null element of tropical sum	$\perp = -\infty$
A^T	Transpose of A	$\begin{pmatrix} 3 \\ 4 \end{pmatrix}^T = (3\ 4)$
$\top$	Absorbing element of tropical sum	$\top = \infty$
$-$	Matrix opposite	$-(1\ 2) = (-1\ \ -2)$
$-$	Matrix subtraction	$(1\ 2) - (3\ 4) = (-2\ \ -2)$
I	Tropical identity matrix (appropriate order)	$I = \begin{pmatrix} 0 & \perp \\ \perp & 0 \end{pmatrix}$
$\perp$	Tropical null matrix (appropriate order)	$\perp = \begin{pmatrix} \perp & \perp \\ \perp & \perp \end{pmatrix}$

Symbol	**Description**	**Example**
$\top$	Tropical absorbing matrix (appropriate order)	$\top = \begin{pmatrix} \top & \top \\ \top & \top \end{pmatrix}$
A^n	n^{th} tropical power of A	$A^n \equiv AA^{n-1}, A^0 \equiv I$
α^{-1}	Tropical inverse for scalars	$\alpha^{-1} \equiv -\alpha$
A^*	Kleene Closure of A	$A^* \equiv \bigoplus_{i=0}^{\infty} A^i$
$\{A\}_{ij}$ or A_{ij}	Indexing of A	$\{I\}_{12} = I_{12} = \bot$
$\{A\}_{i\bullet}$ or $A_{i\bullet}$	i^{th} row of A	$\{I\}_{1\bullet} = (0 \;\; \bot)$
$\{A\}_{\bullet j}$ or $A_{\bullet j}$	j^{th} column of A	$\{I\}_{\bullet 1} = \begin{pmatrix} 0 \\ \bot \end{pmatrix}$
$\rho(A)$	Spectral radius of A	$\rho(I) = 0$
$\backslash\!\!\circ$	Left residuation of the product	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \backslash\!\!\circ \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
$/\!\!\circ$	Right residuation of the product	$(5 \; 6) /\!\!\circ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = (3 \; 2)$
$/\!\!\circ$	Pointwise subtraction	$(1 \; 2) /\!\!\circ 8 = (-7 \; -6)$
$-\!\!\!\ominus$	Dual residuation of the sum	$(1 \; 2) -\!\!\!\ominus (0 \; 3) = (1 \;\; \bot)$
$Im\{A\}$	Image of a matrix	$\{x \mid \exists \; y \; , \; x = Ay\}$
$Ker\{A\}$	Kernel of a matrix	$\{\{x, \overline{x}\} \mid Ax = A\overline{x}\}$
$A[n:m]$	Row concatenation of matrices	$A[n:m] = (A[n]^T \; A[n+1]^T \; ... \; A[m]^T)^T$
$A^T[n:m]$	Column concatenation of matrices	$A^T[n:m] = (A[n] \; A[n+1] \; ... \; A[m])$

The picture in the front page of this thesis, a non-convex (in the traditional sense) polyhedron, is a slice (cut at bottom and at the top) of a specific three-dimensional semimodule, that is, the set of all points such that a tropical linear equation holds. It is convex in the tropical sense.

Chapter 1

Introduction

1.1 Two branches of results

This thesis presents results in two different branches of linear algebra in the so-called **Tropical Algebra** (or **Max-Plus Algebra**) [1]. The first branch concerns the solution of tropical linear equations and also the tropical analogue of fractional linear programs. The second branch, closer to engineering, concerns the dynamical system theory in tropical linear algebra.

The two contributions can be considered as independent. This branching, and later independence, was not planned *a priori* by the author, but is a natural result of the changes that happened during the research. In the beginning, there was a close relationship between the two.

The author's advisor made his PHD research in, besides other related things, the topic of control of tropical linear event invariant dynamical systems (see Maia et al. (2003)). Back to 2010, he came into contact with the work of Katz (2007), which deals with a specific kind of control problems for this class of systems. The example problem presented in the paper was, apparently, very simple and he was puzzled with the fact that the proposed methodology, that had a very large computational complexity (possibly doubly exponential), was thus far the only way available to solve it. He then adapted and tried to apply the methodology that he developed in his PHD research to solve the example problem and was even more puzzled with the fact that his solution was also very complicated (it generated a very complex control topology for the system) and took more time to compute than he deemed it was necessary. His intuition was that a simpler methodology for the general problem

[1] The "tropical" adjective is given in honor of the hungarian born brazilian Mathematician Imre Simon (Speyer and Sturmfels (2004)), although the name is usually given to the isomorphic dioid Min-Plus. The use of "Max-Plus Algebra" seems to be more common in the system control community. For particular reasons of the author, the moniker "Tropical" was adopted.

could be developed, and by consequence this example in particular could be solved in a much simpler way. This pursuit generated some work Maia et al. (2011a,b), and he was eventually able to solve that particular example in a simpler way using sufficient conditions he and his collaborators derived. However, he still deemed that there were a lot of unanswered questions of both theoretical and practical nature, specially because the published work in that subject was (and still is) few and far between. This is when he posed to the author of this thesis the challenge of studying this problem (the definition will be posed formally in Chapter 4), of understanding its essence and of developing a practical and efficient method for solving it that was applicable to as many situations as possible.

The earliest result of the author concerning control of tropical linear event invariant dynamical systems was the design of a feedback matrix F. By the author's result, this feedback can be obtained by solving a specific tropical affine equation. The personal choice of the author to solve this equation, the so-called Dual Method (see Chapter 2), led naturally the matrix F to have large entries. This frequently slows down the closed loop, which can be undesirable in practical situations. Although the author expected this, since the Dual Method has the characteristic of finding the greatest solution to tropical affine equations, he wondered if it was possible to find an algorithm that worked in a *dual* manner to the Dual Method: one that finds the *smallest*, not the greatest solution. He quickly found that such pursuit was deemed to failure, since the smallest solution of a tropical affine equation does not exist, in general. Decided to continue despite this setback, the author found that, although it was impossible to find (in general) the smallest solution in all the space of solutions, he was able to find the smallest solution in some specific sets. This is the genesis of the *Primal Method* described in Chapter 2 (this is why the author gave the two names: "dual" and "primal" methods). He was successful in applying this method to "improve" feedbacks F into others that generated a faster closed-loop. This discussion can be seen in the published paper Gonçalves et al. (2012).

While he was continuing working in the control problem, he came into contact with the papers Butkovic and Aminu (2008); Gaubert et al. (2012). While reading, he was fascinated with the concept of tropical analogue of linear (fractional) programs. Linear (fractional) programs is a personal favourite of the author in his work as a system engineer in industry. He then realized that the concepts he learned/developed while working with the Primal Method could be put in use for solving tropical linear (fractional) programs. This is the genesis of the results in Chapter 3.

The author later discovered that the control problem he was working with could be solved by an open-loop approach by solving a specific equation. Interestingly, this equation - the *control characteristic equation*- has an unknown λ which controls the rate of the system under the action of the (open-loop) controller. The author then discovered that this open-loop rule can be implemented in a closed-loop way with a feedback matrix F, bringing a lot of advantages. In this case, the parameter λ controls the rate of the closed-loop system. This was the moment when the two branches of research effectively became independent: there was longer the need of a method for improving the closed-loop rate induced by F, because this rate could be controlled directly by using this parameter λ (the characteristic equation by itself tells the designer which rates he/she can choose). All of these results can be seen in Chapter 4.

At last, when trying to implement the results presented in Chapter 4 in an automatic plant in his internship at the *Laboratoire Angevin de Recherche en Ingénierie des Systèmes (LARIS)* of the *Université d'Angers*, which was supervised by his co-advisor (which, in turn, was the co-advisor of his advisor in his internship at the same university), he noticed that the assumption that all the states are measured made in Chapter 4 was unapplicable. This encouraged the author to research the observer problem. In this way, he would be able to implement the state feedback controller using only the measured outputs. He then realized that some concepts from the control problem he was dealing with could be translated to the observer problem. This is the genesis of Chapter 5. This implementation can be seen in Chapter 6.

At the beginning of each chapter, an introductory text will explain the *state of the art* of the research in the respective chapter, as well as the importance of the subject. One should keep in mind, however, that the subjects of Chapters 3, 4 and 5, in special, are highly incipient. Indeed, to the author's knowledge, the oldest paper that deals with one of these subjects (which would be the regulator problem) was published 10 years ago. So, while there are already some (engineering) applications, they are in fact scarce. The author hopes that the results published in this thesis, specially regarding the second branch (system theory), allow a tangible increase in the applicability of the subject as well as its dissemination among engineers/researchers.

1.2 Publications

The publications associated with this thesis are:

- Gonçalves et al. (2012): V. M. Gonçalves , C. A. Maia, L. Hardouin, *On the solution of Max-plus linear equations with application on the control of Timed Event Graphs*, WODES 2012, Guadalajara, Mexico .

- Gonçalves et al. (2013c): V. M. Gonçalves , C. A. Maia, L. Hardouin, *Solving tropical linear* *equations with weak dual residuations*, Linear Algebra and Applications.

• Gonçalves et al. (2013b): V. M. Gonçalves , C. A. Maia, L. Hardouin, *On Tropical Fractional Linear Programming*, Linear Algebra and Applications.

• Gonçalves et al. (2014a): V. M. Gonçalves , C. A. Maia, L. Hardouin, *Avanços na teoria de controle para sistemas lineares na álgebra tropical*, Congresso Brasileiro de Automática 2014.

• Gonçalves et al. (2014b): V. M. Gonçalves , C. A. Maia, L. Hardouin and Ying Shang, *An Observer for Tropical Linear Event-Invariant Dynamical Systems* , Conference on Decision and Control 2014.

1.3 Organization

This thesis is organized as follows:

• Chapter 1 is this introduction;

• Chapter 2 presents the first part of the contributions in the first branch. It establishes an algorithm for solving tropical linear equations. This algorithm, which is a sort of "dual" of another algorithm described in literature, focus on the order concepts of tropical algebra to solve, partially, the question of finding the "smallest" solution of a tropical affine equation;

• Chapter 3 presents the second part of the contributions in the first branch. Using the results in Chapter 2, this chapter discusses tropical analogue of fractional linear programs and presents algorithms for solving them;

• Chapter 4 presents the first part of the contributions in the second branch. It discusses a class of control problems (regulation problems) for tropical linear event-invariant systems, which describes the dynamics of timed event graphs with fixed timings. It establishes the important concept of (controllable) non-critical problems, and a necessary and sufficient condition for solving them is presented;

• Chapter 5 presents the second part of the contributions in the second branch. This chapter, which is in some sense "dual" to Chapter 4, discusses the observation problem for tropical linear event-invariant systems. It establishes the important concept of (observable) non-critical problems, and a necessary and sufficient condition for solving them is presented;

• Chapter 6 presents the practical implementation of the algorithms discussed in the second branch of results;

• Chapter 7 presents directions for future research.

All the propositions/lemmas in this thesis are results obtained during the author's PHD research.

Minor results or results that can be already found in other sources are given in footnote (with the appropriate references, when it is the case). All the definitions that are presented in other works not by the author are also referenced properly in the statement. Otherwise, they are new definitions (at least as far as the author's knowledge goes) proposed in this research.

In order to follow this thesis, it is also necessary to have basic concepts in Tropical Algebra (residuation, Kleene closures, semimodules) and also in the modelling of Timed Event Graphs using this algebra. Introduction to (the basics) of these concepts can be seen in the Appendix A. The author also recommends the references Baccelli et al. (1992); Heidergott et al. (2006); Gaubert and M.Plus (1997). Along the text, other specific references will also be mentioned.

Chapter 2

On Tropical Linear Equations and Weak Dual Residuations

A method for solving tropical linear equations, named Primal Method, is presented. This method generates a non-decreasing sequence which converges to the smallest solution inside a special semimodule. It is shown that the proposed method is related to a specialization of the Alternating Method of R.A. Cuninghame-Green and P. Butkovic (Cuninghame-Green and Butkovic (2003)). In addition, it is also shown that both methods come from an extension of the general algorithm for solving equations with residuated functions presented by R.A. Cuninghame-Green and K. Zimmermann. This extension relies on the concept of weak residuation and in the so-called "strong property".

Tropical affine equations (which are kindred to tropical linear equations, as it will be shown in this chapter) are ubiquitous in many problems of Tropical Algebra. Specifically, the method developed in this chapter (the Primal Method) is essential for one of the algorithms for solving Tropical Fractional Linear Programs (Chapter 3).

Before all the formalism, it is maybe beneficial to understand the idea which underlies the Primal Method. Allegorically, suppose all the solutions (vectors x) of a tropical linear equation $Ex = Dx$ are split in "families" (formally, a family is a *semimodule*). All members of a given family will have in common the same "DNA" (formally, the same *dominances*, as it will be explained formally later in this chapter). Members of a family can merge together to generate a "descendant", which is also inside the same family (formally, tropical linear combinations of members of a semimodule generate another member of the same semimodule). It is also possible that some members for different families merge together to generate a "son" with a different genetic code, and thus in another family (potentially, tropical linear combination of solutions with different dominances can generate solutions with a mixed dominance, and thus with a different dominance than all of their antecessors). In this family, there are more primitive - *primal* - members which are in some ways ancestors of them all. They are characterized by their simplicity (formally, they have more $\perp$ entries) and the fact they can alone generate, by merging, the entire family again (they form a *basis* of this semimodule). The key point of the Primal Method is that each member of a given family has in themselves the DNA code necessary to recreate all their primal ancestors. Finishing the allegory, the Primal Method works by extracting this DNA (dominance) and then using a cloning technology (some matricial products/sums and a Kleene Closure) to recreate all the ancestors of that solution (and thus of its family).

This chapter is a slightly modified version of the paper published in *Linear Algebra and its Applications*: Gonçalves et al. (2013c). The conference paper Gonçalves et al. (2012) also has some initial results that foreshadow the results of this chapter.

2.1 Introduction

An important problem in the tropical algebra concerns the solution of two-sided linear equations

$$Ex = Dx. \tag{2.1.1}$$

Cuninghame-Green and Zimmermann (2001) introduced a general iterative algorithm for solving equations of the form

$$f(x) = g(y) \tag{2.1.2}$$

when f and g are residuated functions. A specialization of this algorithm to the linear tropical equation $Ax = By$ can be further adapted to the (equivalent) equation $Ex = Dx$. Then, it has the important property of generating a non-increasing sequence which converges to the greatest solution x smaller than or equal to the initial condition x_0.

Algorithms for solving tropical linear equations can also solve their affine counterparts $Rp \oplus r = Sp \oplus s$, by introducing an auxiliary scalar variable y (see Cuninghame-Green and Butkovic (2003))

$$Rp \oplus ry = Sp \oplus sy. \tag{2.1.3}$$

Equation (2.1.3) is linear in the extended vector $x = (p^T \ \ y)^T$ if one sets $E = (R \ \ r)$ and $D = (S \ \ s)$. If one employs the method of Cuninghame-Green and Zimmerman with the initial condition $p[0] = p_0$ and $y[0] = 0$, and a lower bounded solution to the original affine equation $Rp \oplus r = Sp \oplus s$ such that $p \preceq p_0$ exists, the vector p will converge to a solution and y will remain equal to 0 [1]. Due to the algorithm properties, the resulting solution will be the greatest one of the original affine equation which is smaller than or equal to p_0. Thus - provided that the solution set is not empty - the greatest solution of an affine tropical equation exists and can be found by using the greatest possible initial condition $p[0] = \top$.

However, in general, the smallest solution does not exist. This is a consequence to the fact that the product in $\overline{\mathbb{T}}_{\mathtt{max}}^{n \times m}$ is not dually residuated (in general). Since seeking for the smallest solution is futile in general, one can weaken the problem asking for a solution in a special set. As an example, a special *semimodule* $\mathscr{S}$ can be considered. Then, according to this constraint, the proposed problem may have the smallest solution.

To this end, the concept of *weak residuation* and *strong residuation for an element* will be introduced. Then, one can weaken the requirement of residuated functions f and g in Cuninghame-Green and Zimmermann (2001), and instead require that f and g have a weak residuation which has the strong property for a previously found solution. Thus, one can use this general algorithm in the dual dioid $\mathbb{T}_{min}$ (so the minimum becomes the maximum and $\preceq$ becomes $\succeq$) and obtain a method for generating other solutions, which are "small", to the tropical affine equation. In fact, the method can find the smallest solution in a particular semimodule $\mathscr{S}$ using a special initial condition. So, the proposed method uses an already known solution for finding other solutions with a special property.

The aforementioned method, which will be called *Primal Method* hereafter, is closely related to the specialization of the method of Cuninghame-Green and Zimmerman for equations of the form $Ex = Dx$, which will be called in this thesis *Dual Method*. It is also closely related - and this will be explicitly addressed later in Subsection 2.3.5 - to the cellular decomposition of Develin and Sturmfels (2004), the mean payoff games and the algorithms presented in Truffet (2010), Lorenzo and de la Puente (2011) and Gaubert et al. (2012) .

[1] Suppose this is not true: a solution p_{sol} which is smaller than or equal to p_0 to the affine equation $Rp \oplus r = Sp \oplus s$ exists, and the algorithm in the homogenized equation $Rp \oplus ry = Sp \oplus sy$ beginning with $p[0] = p_0$ and $y[0] = 0$ converges to $y = y_{neg} < 0$ (y must be negative since the sequence is non-increasing and $y \neq 0$, by hypothesis) and $p = p_{nsol}$. This is a contradiction to the fact that the method of Cuninghame-Green and Zimmerman converges to the greatest solution to the homogenized equation which is smaller than or equal to the initial condition $(p_0^T \ \ 0)^T$, since $(p_{sol}^T \ \ 0)^T \preceq (p_0^T \ \ 0)^T$ is a solution but $(p_{nsol}^T \ \ y_{neg})^T \not\succeq (p_{sol}^T \ \ 0)^T$ (since $y_{neg} \not\succeq 0$).

Equation (2.1.1) has also been studied in several other works other than the previously mentioned ones. Baccelli et al. (1992) provides a method for finding solutions using the symmetrized tropical algebra, which introduces a weak form of subtraction (in a weak inequality, the *balance*) and therefore allows analogous algorithms from traditional algebra to be adapted to the problem. Following this idea, many algorithms were also discussed in Gaubert (1992). Butkovic and Hegedus (1984) provided a method, called the *Elimination Method*, which can generate the entire set of solutions by solving the system of equations row-by-row. As a consequence of this method, it was proved that this set has a finite (albeit possibly very large) representation. Using concepts of residuation, Cuninghame-Green and Butkovic (2003) proposed the *Alternating Method*, which generates a non-increasing sequence (after the first step) converging to a solution. Butkovic and Zimmermann (2006) provided an algorithm for finding a single solution, the *Stepping Stone Method*. It works by checking at each step which equalities hold and the ones which do not. Then, it decreases the values of the current vector in a way that the non-achieved equalities began to hold while keeping the ones which were already satisfied. Akian et al. (2010) shows that the existence of a non-trivial solution is related to the problem of solving mean payoff games. The *Tropical Double-Description Method* in Allamigeon et al. (2010) is conceptually similar to the one proposed in Butkovic and Hegedus (1984), being capable of generating the entire set of solutions by solving the system row-by-row. It uses, however, a more elaborated approach for solving each equation, using the concept of *extreme rays*. This leads to a more compact representation of the intermediate solutions set and thus the method has a substantially better average complexity than the Elimination Method. Finally, the analogue of Equation (2.1.1) to the interval of dioids was established in Hardouin et al. (2009) and the related problem of solving inequations of the form $Ax \preceq x \preceq B \diamond\!\!\!\!\!\!\setminus x$ in Brunsch et al. (2012).

In summary, the contributions of this chapter are:

(i) An extension of the algorithm presented in Cuninghame-Green and Zimmermann (2001), considering the concepts of weak residuation and the strong property for it, which is presented in Section 2.2.

(ii) A method for generating solutions to tropical linear equations, the Primal Method, which is presented in Section 2.3. This method is a contextualization of the the aforementioned extension to tropical linear equations. The similarities of this method with previously published works are also discussed.

2.2 Solving equations with weak residuated functions

Cuninghame-Green and Zimmermann (2001) proposed an algorithm for solving Equation (2.1.2), when f and g are residuated. It can be stated as iterating the sequences

$$\begin{aligned} x[k+1] &= f^{\sharp}(g(y[k])) \wedge x[k]; \\ y[k+1] &= g^{\sharp}(f(x[k])) \wedge y[k] \end{aligned} \tag{2.2.1}$$

for an initial pair $x[0]$, $y[0]$. Here $f^{\sharp}$ and $g^{\sharp}$ are the residuation for f and for g, respectively.

Using a similar reasoning, the following sequence can be derived, which converges to a solution of $f(x) = g(x)$:

$$x[k+1] = f^{\sharp}(g(x[k])) \wedge g^{\sharp}(f(x[k])) \wedge x[k]. \tag{2.2.2}$$

This algorithm can be extended if the residuation is relaxed to a weaker form.

Definition 2.2.1. *(Weak residuation)* A non-decreasing function f is said to have a weak residuation if there is a non-decreasing function $f^{\natural}$ such that

$$f(f^{\natural}(x)) \preceq x \quad \forall x. \tag{2.2.3}$$

□

For a given function, many, or none at all, weak residuations may exist [2]. Further, as the name suggests, the requirement in Equation (2.2.3) by itself is not very useful. So, it is important to introduce another definition.

Definition 2.2.2. *(Weak residuation with strong property)* For an element z, a weak residuation $f_z^{\natural}$ with the additional property

$$f_z^{\natural}(f(z)) \succeq z \tag{2.2.4}$$

is said to have the strong property for z.

□

[2] For instance, for $f(x) = Ax$, $f^{\natural}(x) = A \backslash x$ is a weak dual residuation. But it is not the only one: for any $M \succeq A$, $f^{\natural}(x) = M \backslash x$ also is, since $A(M \backslash x) \preceq A(A \backslash x) \preceq x$. Conversely, the non-decreasing function $f(x) = x \oplus a$, $a \neq \perp$, has none (since for $x \not\succeq a$, the statement in Equation (2.2.3) cannot hold).

Remember that the usual residuation is such that $f(f^\sharp(x)) \preceq x$ and $f^\sharp(f(x)) \succeq x$, *both* holding for all x. Then, clearly, the usual residuation is a weak residuation which is strong for each element z. It is also important to remark that one can, in analogy, define weak *dual* residuations with a strong property with an element. This kind of residuations will be used later on this chapter.

Now, the *residuated* requirement in the algorithm presented in Cuninghame-Green and Zimmermann (2001) can be replaced by *weak residuated with a residuation which is strong for a solution z of the equation*. If z is lower bounded, the fact that z is a solution guarantees the convergence to a lower bounded solution.

Proposition 2.2.1. *(Convergence with weak residuation with a strong property for a solution)*: The sequence generated by Equation (2.2.2) with initial condition $x[0] \succeq z$ converges for a lower bounded solution of Equation (2.1.2) if weak residuation functions with the strong property to a lower bounded solution z are used (that is, switching $f^\sharp$, $g^\sharp$ to $f_z^\natural$, $g_z^\natural$, respectively, in Equation (2.2.2)).

Proof. It is straightforward to see that the sequence generated by Equation (2.2.2) is non-increasing (due to the minimum with $x[k]$). Therefore, it either degenerates to the trivial solution $\perp$ or stabilizes.

Suppose it stabilizes, thus

$$\begin{aligned} x &\preceq f_z^\natural(g(x)); \\ x &\preceq g_z^\natural(f(x)). \end{aligned} \tag{2.2.5}$$

Then, using Equation (2.2.3) (that is, after applying f and g in both sides of the top and bottom inequalities of Equation (2.2.5), respectively)

$$\begin{aligned} f(x) &\preceq g(x); \\ g(x) &\preceq f(x). \end{aligned} \tag{2.2.6}$$

and thus it stabilizes to a solution.

The concern is that this solution can be the trivial one, $\perp$. This is addressed by the fact that $f_z^\natural$ has the strong residuation property for a lower bounded solution z. Thus, by Equation (2.2.4)

$$f_z^\natural(f(z)) \succeq z \;\Rightarrow\; f_z^\natural(g(z)) \succeq z \tag{2.2.7}$$

where the fact that $f(z) = g(z)$ was used. Then, also

$$g_z^\natural(f(z)) \succeq z. \tag{2.2.8}$$

And thus, combining Equations (2.2.7) and (2.2.8)

$$z = f_z^\natural(g(z)) \wedge g_z^\natural(f(z)) \wedge z. \tag{2.2.9}$$

Then, z is a fixed point for the iteration map in Equation (2.2.2). Hence, the function

$$h(x) = f_z^\natural(g(x)) \wedge g_z^\natural(f(x)) \wedge x \tag{2.2.10}$$

is monotonic. As $x[0] \succeq z$, by induction and using Equation (2.2.9), it can be shown that $x[k] \succeq z$. Thus, as z is lower bounded, the sequence will converge to a lower bounded solution.

□

Remark 2.2.1. When the residuated functions $f(x) = Ex$, and $g(x) = Dx$ (that is, one is dealing with Equation (2.1.1)) are used in the original algorithm of Cuninghame-Green and K. Zimmermann, the resulting algorithm iterates the function

$$h(x) = E \backslash\!\!\!\circ (Dx) \wedge D \backslash\!\!\!\circ (Ex) \wedge x \tag{2.2.11}$$

starting from an initial $x[0] = x_0$.

This algorithm is well known and has been exploited in literature. For example, Equation (2.2.11) appears in Dhingra and Gaubert (2006) and then in Gaubert and Sergeev (2013), in connection with mean payoff games (that will be discussed in Subsection 2.3.5). It is also related to the Alternating Method of Cuninghame-Green and Butkovic (2003).

This method enjoys many important properties, such as for example generating a non-increasing sequence $x[k]$ which converges to the greatest solution of Equation (2.1.1) smaller than or equal to x_0 (see Cuninghame-Green and Zimmermann (2001)). In the present thesis, this method will be called **Dual Method**, in contrast with one that will be presented further that shares many (almost) dualized properties with them (and also the same origin, as a particular case of the proposed extended algorithm in Proposition 2.2.1), the **Primal Method**.

□

2.3 The Primal Method

As a particular case of the proposed extended algorithm, the *Primal Method* will be presented. It concerns tropical linear equations like Equation (2.1.1). First, the method will be established in an independent way. Then, the connection with the extended algorithm will be given.

It will be assumed from now on that the matrices E and D have their entries only in $\mathbb{T}_{\mathtt{max}}$, so no $\top$ entries are allowed. This is a weak assumption that permits to avoid some technicalities concerning the expression $\top \otimes \perp$ in the proposed results.

2.3.1 Introduction

The Primal Method will now be presented by the following sequence of definitions and propositions.

Definition 2.3.1. *(Dominance)* A *dominance* is a mapping $\Upsilon : \{1,2,...,n\} \mapsto \{1,2,..,m\}$. □

The reason behind this name will be clear later.

Definition 2.3.2. *(Matrix generated by the dominance)* Let $E \in \mathbb{T}_{\mathtt{max}}^{n\times m}$ and Υ be a dominance. The matrix $\mathsf{W}(\Upsilon, E) \in \mathbb{T}_{\mathtt{max}}^{n\times m}$ is defined as the matrix constructed in the following way:

$$\begin{aligned} \{\mathsf{W}(\Upsilon,E)\}_{ij} &\equiv E_{ij} \text{ if } \Upsilon(i) = j; \\ \{\mathsf{W}(\Upsilon,E)\}_{ij} &\equiv \perp \text{ otherwise.} \end{aligned} \tag{2.3.1}$$

□

The matrix $\mathsf{W}(\Upsilon, E)$ generated by a dominance is simply a matrix constructed from E, such that all rows have at most one non-$\perp$ entry, and the only (possible) non-$\perp$ entry on row i is exactly $j = \Upsilon(i)$.

Property 2.3.1. *(Dual residuation)* The map $x \mapsto \mathsf{W}(\Upsilon, E)x$ is *dually residuated* if all rows of the matrix have exactly one non-$\perp$ entry (by Definition 2.3.2, it can have one or zero non-$\perp$ entries). This means that there exists a matrix, that will be denoted by $\mathsf{W}^\flat(\Upsilon, E) \in \mathbb{T}_{\mathtt{max}}^{m\times n}$, such that for any $x \in \mathbb{T}_{\mathtt{max}}^{m}$, $y \in \mathbb{T}_{\mathtt{max}}^{n}$

$$\mathsf{W}(\Upsilon,E)x \succeq y \Longleftrightarrow x \succeq \mathsf{W}^\flat(\Upsilon,E)y. \tag{2.3.2}$$

□

Remark 2.3.1. Hereafter, it will be assumed, without loss of generality, that all rows of the matrix $\mathsf{W}(\Upsilon, E)$ have at least one non-$\perp$ entry (it is *row G-astic*, see Butkovic and Hegedus (1984)). The same must hold for $\mathsf{W}(\Upsilon, D)$. It will be discussed later why this assumption is not restrictive. □

It can be seen, by inspection, that this matrix $\mathsf{W}^\flat(\Upsilon, E)$ is obtained by switching the sign of all non-$\perp$ entries of $\mathsf{W}(\Upsilon, E)$ and transposing the resulting matrix. Thus, one introduces the following definitions.

Definition 2.3.3. *(Dual residuation matrix)* If $\mathsf{W}(\Upsilon, E) \in \mathbb{T}_{\mathtt{max}}^{n\times m}$ has one non-$\perp$ entry per row, then $\mathsf{W}^\flat(\Upsilon, E)$ denotes the matrix obtained from $\mathsf{W}(\Upsilon, E)$ by switching the sign of all non-$\perp$ entries and transposing the result. □

Definition 2.3.4. *(Induced dominance)* A dominance Υ_E^z can be induced by a vector z in a matrix E as follows

$$\Upsilon_E^z(i) \equiv \arg\max_j \{\bigoplus_{j=1}^m E_{ij} z_j\}. \tag{2.3.3}$$

with the additional constraint that, for all i, if $j = \Upsilon_E^z(i)$ then $E_{ij} \neq \perp$.

□

Remark 2.3.2. Note that an induced dominance exists on a given matrix E if and only if it is row G-astic. Otherwise, the additional constraint that $E_{ij} \neq \perp$ cannot hold. This constraint guarantees that $\mathsf{W}(\Upsilon_E^z, E)$ is row G-astic (as assumed without loss of generality in Remark 2.3.1). Then, if one considers *only* induced dominances, the assumption that without loss of generality one can assume $\mathsf{W}(\Upsilon, E)$ to be row G-astic can be transferred to the assumption that without loss of generality E is row G-astic. The same must hold for D. This new assumption, that is, E is row G-astic, will be discussed further. □

A given vector z can induce multiple dominances, since two or more indexes can lead to the maximum values, as in the sum $2 \oplus 1 \oplus 2$ in which the first and third entry achieve the greatest value. Thus, in this case, Υ^z is multiple-defined and can be any of those dominances.

Property 2.3.2. Now the meaning of the label "dominance" can be made clear. If Υ_E^z is an induced dominance from a given z, then it can be shown by inspection that

$$Ez = \mathsf{W}(\Upsilon_E^z, E) z. \tag{2.3.4}$$

Thus, the mapping Υ_E^z maps to each row the dominating index in the product Ez in this row (see Equation (2.3.3)).

□

In addition:

Property 2.3.3. It is straightforward by the structure of the matrix $\mathsf{W}(\Upsilon, E)$ that, for any Υ

$$E \succeq \mathsf{W}(\Upsilon, E). \tag{2.3.5}$$

□

Definition 2.3.5. *(H matrix)* Let $E, D \in \mathbb{T}_{\max}^{n\times m}, z \in \mathbb{T}_{\max}^{m}$ and $\Upsilon_E^z, \Upsilon_D^z$ be induced dominances. Then, the H matrix is defined as

$$\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z) \equiv \mathsf{W}^\flat(\Upsilon_E^z, E)D \oplus \mathsf{W}^\flat(\Upsilon_D^z, D)E. \tag{2.3.6}$$

□

The H matrix has an important property.

Proposition 2.3.1. *(Obtaining solutions)* Any linear combination x of columns of $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^* \in \overline{\mathbb{T}}_{\max}^{m\times m}$ is a solution to the equation $Ex = Dx$.

Proof. Let $x = \mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^* y$. Due to the properties of Kleene Closure, this is equivalent to the following statement [3]

$$x \succeq \mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)x. \tag{2.3.7}$$

Which is equivalent to:

$$\begin{aligned} x &\succeq \mathsf{W}^\flat(\Upsilon_E^z, E)Dx; \\ x &\succeq \mathsf{W}^\flat(\Upsilon_D^z, D)Ex. \end{aligned} \tag{2.3.8}$$

Due to the fact that the maps are dually residuated (Property 2.3.1), this is equivalent to

[3] If $x = H^* y$, then $H^* x = H^* H^* y = H^* y$ since $H^* H^* = H^*$ holds for Kleene Closures. Thus $x = H^* x$, and it can be stated that $x \succeq H^* x \succeq Hx$.

$$\mathsf{W}(\Upsilon_E^z, E)x \succeq Dx;$$
$$\mathsf{W}(\Upsilon_D^z, D)x \succeq Ex. \tag{2.3.9}$$

By using Property 2.3.3, it can be deduced that $Ex \succeq Dx$ and $Dx \succeq Ex$. Then, the statement is proved.

□

Proposition 2.3.1, however, does not guarantee that the Kleene Closure of $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)$ will be upper bounded (meaning that the Kleene Closure is, generally, in the complete dioid $\in \overline{\mathbb{T}}_{\mathtt{max}}^{m\times m}$). Indeed, dominances $\Upsilon_E^z, \Upsilon_D^z$ in which the closure is a matrix full of $\top$'s can be chosen. One must note that these are degenerate solutions, but solutions nonetheless.

The next Proposition ensures how dominances that guarantee at least partial upper boundedness of the Kleene Closure can be chosen, thus guaranteeing that non-trivial solutions are found.

Proposition 2.3.2. *(Upper bounded Kleene Closure)* Let z be an upper bounded solution of $Ex = Dx$. Then, for all the entries j in which z is non-$\perp$, the j^{th} column of the matrix $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^*$ is upper bounded.

Proof. By hypothesis

$$Ez = Dz. \tag{2.3.10}$$

Using Property 2.3.2, on the left side

$$\mathsf{W}(\Upsilon_E^z, E)z = Dz. \tag{2.3.11}$$

Then

$$\mathsf{W}(\Upsilon_E^z, E)z \succeq Dz. \tag{2.3.12}$$

Using Property 2.3.1

$$z \succeq \mathsf{W}^\flat(\Upsilon_E^z, E)Dz. \tag{2.3.13}$$

Similarly

$$z \succeq \mathsf{W}^{\flat}(\Upsilon_D^z, D)Ez. \tag{2.3.14}$$

And then, by summing the statements in Equations (2.3.13) and (2.3.14)

$$z \succeq \mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)z. \tag{2.3.15}$$

Using the property of Kleene Closures [4], Equation (2.3.15) is equivalent to

$$z = \mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^* z. \tag{2.3.16}$$

Then the conclusion of Proposition is clear: if the j^{th} entry of z is non-$\bot$ and z is upper bounded, then the j^{th} column of $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^*$ is also upper bounded. □

Propositions 2.3.1 and 2.3.2 together constitute a method for computing more solutions from the equation $Ex = Dx$ from a given known one z. First, find z (using any method). Then, find the dominance induced by z and then compute $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^*$. This procedure is what is called ***Primal Method***.

Algorithm 2.3.1. *Primal Method for tropical linear equations*

1. Solve Equation (2.1.1), using any method, obtaining a solution z;
2. Use this solution to induce dominances, $\Upsilon_E^z, \Upsilon_D^z$ (see Definition 2.3.4);
3. Construct the matrix $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)$, as in Definition 2.3.5;
4. Any linear combination of columns of $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^*$ is a solution.

Remark 2.3.3. It was mentioned in Remark 2.3.2 that, without loss of generality, one can assume both E and D row G-astic . No generality is lost because, if this is not the case, it is possible to rewrite the equation removing these rows and appropriate columns/corresponding entries in E, D and x (these will be fixed to $\bot$) such that the new system has this property. If both rows of E and D

[4] If $x \succeq Hx$, by pre-multiplying by H one concludes that $Hx \succeq H^2x$ and thus $x \succeq H^2x$. By induction, $x \succeq H^k x$ for any natural k. By adding all these inequalities for all k, $x \succeq H^*x$. Since $H^*x \succeq x$, one can finally conclude that $x = H^*x$. Further, by Footnote 3, $x = H^*y$ for some y.

are $\perp$, they can be removed without any problem. If it is only in E or D, say the i^{th} of D, there is a situation

$$\bigoplus_{j=1}^{m} E_{ij} x_j = \perp \tag{2.3.17}$$

and then, for all j such that $E_{ij} \neq \perp$ necessarily $x_j = \perp$. These variables can be set to $\perp$, then they can be removed from the vector x along with the i^{th} row and j^{th} column of both E and D. One can proceed in that way till there is nothing to remove and no row in E or D is $\perp$.

□

2.3.2 Connection with the extended Cuninghame-Green and Zimmerman algorithm

The Dual Method (a specialization of the Cuninghame-Green and Zimmerman algorithm to linear equations, see Remark 2.2.1) is an iterative algorithm. Regardless of the method used to compute the Kleene Closure, the Primal Method *as presented* is not iterative. However, one can note that $z = \mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^* x_0$ can be implemented by the sequence

$$x[k+1] = (\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z) \oplus I) x[k], \tag{2.3.18}$$

for the initial $x[0] = x_0$. Then, the Primal Method can also be seen (implemented) as an iterative method. This is not the most computationally efficient way to implement it, since computing powers of $(A \oplus I)$ is not the best algorithm for computing A^*.

If the dioid is swapped from $\mathbb{T}_{\mathtt{max}}$ to $\mathbb{T}_{min}$, the usual residuation in the dual dioid is just the dual residuation in the original dioid (to avoid confusion, everything will be nominated by the reference of $\mathbb{T}_{\mathtt{max}}$). Then, it can be seen that the map $f(x) = Ex$ has a weak dual residuation with a strong property for an element z: the map $f_z^\dagger(x) = \mathsf{W}^\flat(\Upsilon^z, E)x$ as defined in Definition 2.3.2 using an induced dominance.

Proposition 2.3.3. *(Dominances induce a weak dual residuation with a strong property)* $f(x) = Ex$ has as weak dual residuation the map $f_z^\dagger(x) = \mathsf{W}^\flat(\Upsilon^z, E)x$ with strong property for z.

Proof. It can be noted that

$$Ey \succeq \mathsf{W}(\Upsilon^z, E)y \ \forall y \tag{2.3.19}$$

(see Property 2.3.3). Thus, taking $y = \mathsf{W}^\flat(\Upsilon^z, E)x$ and the fact that

$$\mathsf{W}(\Upsilon^z, E)\mathsf{W}^\flat(\Upsilon^z, E) \succeq I \tag{2.3.20}$$

(since $x \mapsto \mathsf{W}^\flat(\Upsilon^z, E)x$ is a dual residuation for $x \mapsto \mathsf{W}(\Upsilon^z, E)x$), it can be concluded that

$$E\mathsf{W}^\flat(\Upsilon^z, E)x \succeq x \ \forall x \tag{2.3.21}$$

which is exactly the requirement for a weak *dual* residuation, that is, Equation (2.2.3) with $f(x) = Ex$, $f^\natural(x) = \mathsf{W}^\flat(\Upsilon^z, E)x$ and the inequality swapped from $\preceq$ to $\succeq$ (since the definition concerns weak residuations, and the current proposition concerns weak *dual* residuations). The strong property comes from Property 2.3.2:

$$\mathsf{W}(\Upsilon^z, E)z = Ez. \tag{2.3.22}$$

Thus

$$\mathsf{W}(\Upsilon^z, E)z \succeq Ez \iff z \succeq \mathsf{W}^\flat(\Upsilon^z, E)Ez \tag{2.3.23}$$

using the dual residuation of the map $x \mapsto \mathsf{W}(\Upsilon^z, E)x$. Hence, again using $f(x) = Ex$, $f_z^\natural(x) = \mathsf{W}^\flat(\Upsilon^z, E)x$ and exchanging $\succeq$ for $\preceq$ (since the current proposition concerns weak *dual* residuations), one concludes by the virtue of (the modified) Equation (2.2.4) that this weak dual residuation has the strong property for z. □

Then, if Equation (2.2.2) is contextualized to $\mathbb{T}_{min}$ and also weak residuations who have a strong property for a solution are used, it can be concluded that this equation reduces to

$$x[k+1] = \mathsf{W}^\flat(\Upsilon^z, E)Dx[k] \oplus \mathsf{W}^\flat(\Upsilon^z, D)Ex[k] \oplus x[k]. \tag{2.3.24}$$

It is straightforward to see that the resulting Equation (2.3.24) is the iterative form of the Primal Method, as in Equation (2.3.18). Thus, as claimed, the Dual Method and Primal Method share the same origin.

It will now be proved that they also share (almost) dualized properties.

2.3.3 Properties of the method

Some results concerning the properties of the solutions found by the Primal Method are presented now. For this, it is useful to consider the iterative form of the method, Equation (2.3.18). Thus, this form will be the one considered in this subsection.

First, a *dominance space* is defined.

Definition 2.3.6. *(Dominance space)* Given a dominance Υ and a matrix E, it is called $\mathscr{D}(\Upsilon, E)$, the dominance space of Υ under E, the sets of all x such that

$$Ex = \mathsf{W}(\Upsilon, E)x. \tag{2.3.25}$$

□

It is important to remark that $\mathscr{D}(\Upsilon, E)$ is a semimodule, since it is the solution set of a linear tropical equation which can be given as the image of a finite matrix (see Butkovic and Hegedus (1984)). In fact, by Property 2.3.3, Equation (2.3.25) is equivalent to $\mathsf{W}(\Upsilon, E)x \succeq Ex$. Then, by using Property 2.3.1, $x \succeq \mathsf{W}^\flat(\Upsilon, E)Ex$. This fact has two interesting implications. The first one is that this implies that the semimodule is convex in the traditional sense. The second one is that this semimodule is generated by the matrix $(\mathsf{W}^\flat(\Upsilon, E)E)^*$ (see Footnote 4).

Proposition 2.3.4. *(Exhaustion of dominance space)* Any solution x generated by the Primal Method using a matrix $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^*$ is such that $x \in \mathscr{D}(\Upsilon_E^z, E) \cap \mathscr{D}(\Upsilon_D^z, D)$. Furthermore, $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^*$ is a generator matrix for all such x's.

Proof. Consider x a solution. For the first part, using Property 2.3.3 and post multiplying by x

$$Ex \succeq \mathsf{W}(\Upsilon_E^z, E)x. \tag{2.3.26}$$

Now, from the first Equation in (2.3.9), and using the fact that $Ex = Dx$

$$\mathsf{W}(\Upsilon_E^z, E)x \succeq Ex. \tag{2.3.27}$$

Thus $Ex = \mathsf{W}(\Upsilon_E^z, E)x$, and then $x \in \mathscr{D}(\Upsilon_E^z, E)$. A similar result holds for D and the first part is proved.

For the second part, suppose that $x \in \mathscr{D}(\Upsilon_E^z, E) \cap \mathscr{D}(\Upsilon_D^z, D)$. Then

$$\mathsf{W}(\Upsilon_E^z, E)x = Ex;$$
$$\mathsf{W}(\Upsilon_D^z, D)x = Dx. \tag{2.3.28}$$

See Property 2.3.2. Using the fact that $Ex = Dx$, and also using the fact that the equality implies, in particular, the inequality.

$$\mathsf{W}(\Upsilon_E^z, E)x \succeq Dx;$$
$$\mathsf{W}(\Upsilon_D^z, D)x \succeq Ex. \tag{2.3.29}$$

Then, using the same manipulations as in Proposition 2.3.1, it can be concluded that $x = \mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^* x$. Then, clearly x is a linear combination of columns of $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^*$ and the proof is completed.

□

Using the iterative form of the Primal Method, one may show the following result concerning the evolution of the sequence and its final value.

Proposition 2.3.5. (*Sequence characterization*) The sequence $x[k]$ is non-decreasing, and converges to the smallest solution x such that $x \in \mathscr{D}(\Upsilon_E^z, E) \cap \mathscr{D}(\Upsilon_D^z, D)$ and $x \succeq x_0$.

Proof. The fact that it is a non-decreasing sequence is straightforward by the sum of $x[k]$ in Equation (2.3.18).

For the second part, let $\mathscr{X}$ be the set of all solutions x ($Ex = Dx$) in $\mathscr{D}(\Upsilon_E^z, E) \cap \mathscr{D}(\Upsilon_D^z, D)$ and $\mathscr{X}^{\succeq}(x_0)$ all $x \in \mathscr{X}$ such that $x \succeq x_0$. Then, by the conclusions of Proposition 2.3.4, if $x \in \mathscr{X}$

$$x = \mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^* x \;\Rightarrow\; x \succeq \mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^* x. \tag{2.3.30}$$

With this, if also $x \succeq x_0$ then one has $x \succeq \mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^* x_0$. Thus, any member of $\mathscr{X}^{\succeq}(x_0)$ is lower bounded by $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^* x_0$, which is the value in which the sequence $x[k]$ converges and also a member of $\mathscr{X}^{\succeq}(x_0)$. Then, the proof is completed. □

$\mathscr{S} = \mathscr{D}(\Upsilon_E^z, E) \cap \mathscr{D}(\Upsilon_D^z, D)$ is a semimodule, being an intersection of two semimodules. This semimodule has the special property that the smallest solution of $Ex = Dx$ which is greater than or equal to a given x_0 in it exists, as one of the conclusions of Proposition 2.3.5 states.

Property 2.3.4. It can be seen that the non-iterating Primal Method (computing $\mathsf{H}(E,D,\Upsilon_E^z,\Upsilon_D^z)^*$) and iterating Primal Method (iterating the map in Equation (2.3.18)) are related by using the initial conditions formed by the columns of the identity matrix in the iterating Primal Method. □

Note that the Dual Method generates a non-increasing sequence which converges to the greatest solution smaller than or equal to the initial x_0. The Primal Method has an "almost" dual property in that regard: it generates a non-decreasing sequence which converges to the smallest solution greater than or equal to the initial x_0 which is in the dominance spaces. The "...which is in the dominance space" part, not present in the Dual, is what justifies the "almost".

At last, an important result concerning the affine equation $Rp \oplus r = Sp \oplus s$ will be proposed. For this, it is necessary the following definition.

Definition 2.3.7. *(Affine projection)* Given a set $\mathscr{S} \subseteq \mathbb{T}_{\mathtt{max}}^{m+1}$, the set $\mathscr{A}(\mathscr{S}) \subseteq \mathbb{T}_{\mathtt{max}}^{m}$ is defined as

$$\mathscr{A}(\mathscr{S}) \equiv \{(x_1 \ x_2 \ \dots \ x_m)^T \mid x \in \mathscr{S} \ , \ x_{m+1} = 0\}. \tag{2.3.31}$$

□

Then

Proposition 2.3.6. (*Smallest solution to affine equations*) Consider the affine equation $Rp \oplus r = Sp \oplus s$, $R, S \in \mathbb{T}_{\mathtt{max}}^{n\times m}$, $r, s \in \mathbb{T}_{\mathtt{max}}^{n}$ and its respective homogenized equation $Rp \oplus ry = Sp \oplus sy$, which will be written as $Ex = Dx$, $x \in \mathbb{T}_{\mathtt{max}}^{m+1}$. Consider a vector z satisfying $Ex = Dx$ such that $z_{m+1} \neq \perp$. Then, the sequence $x[k]$ generated by the iterative Primal Method (Equation (2.3.18)) with the initial condition $x[0] = (\perp \quad 0)^T$ converges to $(p_{sol}^T \quad 0)^T$ (the last entry remains in 0), in which p_{sol} is the smallest solution of the affine equation in $\mathscr{A}(\mathscr{D}(\Upsilon_E^z, E) \cap \mathscr{D}(\Upsilon_D^z, D))$.

Proof. Any solution of the affine equation in $\mathscr{A}(\mathscr{D}(\Upsilon_E^z, E) \cap \mathscr{D}(\Upsilon_D^z, D))$ can be obtained from a solution of the homogenized equation in the set $\mathscr{X}$ composed of all vectors in $\mathscr{D}(\Upsilon_E^z, E) \cap \mathscr{D}(\Upsilon_D^z, D)$ with the last component $x_{m+1} = 0$. Then, it remains to prove that the smallest member of $\mathscr{X}$ can be obtained by the iterative Primal Method with the initial condition $x_0 = (\perp \quad 0)^T$.

The set $\mathscr{X}$ is a subset of the set $\mathscr{X}^{\succeq}(x_0)$ of all the solutions of the homogenized equation in $\mathscr{D}(\Upsilon_E^z, E) \cap \mathscr{D}(\Upsilon_D^z, D)$ such that $x \succeq x_0$. Proposition 2.3.5 states that the smallest member of $\mathscr{X}^{\succeq}(x_0)$ can be obtained by using the iterative Primal Method with the initial condition x_0, that is, $x_{sm} = \mathsf{H}(E,D,\Upsilon_E^z,\Upsilon_D^z)^* x_0$.

As $x_0 = (\perp \quad 0)^T$, x_{sm} is the last column ($(m+1)^{th}$ column) of the matrix $\mathsf{H}(E,D,\Upsilon_E^z,\Upsilon_D^z)^*$. Since by hypothesis the last entry of the vector z which induced the dominances is non-$\perp$, Proposition

2.3.2 states that this last column is upper bounded. Since $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^*$ is a Kleene Closure, this implies that the respective member of the diagonal, in this case $\{\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^*\}_{m+1,m+1}$, is 0. This readily implies that x_{sm} can be written as $x_{sm} = (p_{sol}^T \;\; 0)^T$. Then x_{sm} is a member of $\mathscr{X}$, and since it is the smallest member of a superset of this set ($\mathscr{X}^{\succeq}(x_0)$), it must be the smallest member of $\mathscr{X}$ itself. This concludes the proof.

□

2.3.4 Numerical example

Let

$$Rp \oplus r = Sp \oplus s; \tag{2.3.32}$$

be an equation in which R, r, S and s are given by

$$\begin{pmatrix} 2 & -3 & -2 \\ -4 & -3 & 4 \\ 0 & 3 & -3 \\ -3 & 1 & -2 \end{pmatrix} p \oplus \begin{pmatrix} -2 \\ -3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & -1 & -3 \\ -3 & 2 & -3 \end{pmatrix} p \oplus \begin{pmatrix} 1 \\ -3 \\ 2 \\ -5 \end{pmatrix}. \tag{2.3.33}$$

Using the Dual Method, the augmented solution $z = (3 \;\; -2 \;\; 0 \;\; 0)^T$ can be found, thus $p_{solD} = (3 \;\; -2 \;\; 0)^T$ is a solution.

By inspection of the products Ez and Dz, it is possible to conclude that

$$\begin{aligned} \Upsilon_E^z(1) = 1; \;\; \Upsilon_E^z(2) = 3; \\ \Upsilon_E^z(3) = 1; \;\; \Upsilon_E^z(4) = 1. \end{aligned} \tag{2.3.34}$$

$$\begin{aligned} \Upsilon_D^z(1) = 1; \;\; \Upsilon_D^z(2) = 1; \\ \Upsilon_D^z(3) = 1; \;\; \Upsilon_D^z(4) = 1, \end{aligned} \tag{2.3.35}$$

and thus

$$\mathsf{W}(\Upsilon_E^z, E) = \begin{pmatrix} 2 & \perp & \perp & \perp \\ \perp & \perp & 4 & \perp \\ 0 & \perp & \perp & \perp \\ -3 & \perp & \perp & \perp \end{pmatrix}; \; \mathsf{W}(\Upsilon_D^z, D) = \begin{pmatrix} 2 & \perp & \perp & \perp \\ 1 & \perp & \perp & \perp \\ 0 & \perp & \perp & \perp \\ -3 & \perp & \perp & \perp \end{pmatrix}; \tag{2.3.36}$$

$$\mathsf{W}^\flat(\Upsilon_E^z, E) = \begin{pmatrix} -2 & \perp & 0 & 3 \\ \perp & \perp & \perp & \perp \\ \perp & -4 & \perp & \perp \\ \perp & \perp & \perp & \perp \end{pmatrix}; \; \mathsf{W}^\flat(\Upsilon_D^z, D) = \begin{pmatrix} -2 & -1 & 0 & 3 \\ \perp & \perp & \perp & \perp \\ \perp & \perp & \perp & \perp \\ \perp & \perp & \perp & \perp \end{pmatrix}. \tag{2.3.37}$$

And also

$$\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^* = \begin{pmatrix} 0 & 5 & 3 & 2 \\ \perp & 0 & \perp & \perp \\ -3 & 2 & 0 & -1 \\ \perp & \perp & \perp & 0 \end{pmatrix}. \tag{2.3.38}$$

The first, second and fourth columns of $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)^*$ are linearly independent. The last column generates the solution $p_{solP} = (2 \;\perp\; -1)^T$, which is smaller than or equal to p_{solD}.

It is also remarkable that this solution p_{solP} is the smallest solution in $\mathscr{A}(\mathscr{D}(\Upsilon_E^z, E) \cap \mathscr{D}(\Upsilon_D^z, D))$. See Proposition 2.3.6.

As an illustration of the practical behavior of the method, an experiment (using the computer package ScicosLab 4.4.1) in which the Primal Method was applied to systems of the form of Equation (2.3.32) was done. The matrices $E, D \in \mathbb{T}_{\mathtt{max}}^{n\times n}$ were square, with random entries between -10 and 10 or $\perp$, with 20% of entries equal to $\perp$. The experiment was done with values of n being 50, 100, 200 and 300, in an Intel Core I5 with 2.50 GHz and 4GB of RAM. For each n the experiment was repeated 50 times. The results are shown in Table 2.1. From this table, it is possible to infer that the Primal Method seems to yield a high number of linearly independent solutions and also with a high sparsity. Further, that it has a considerable speed even with a relatively large dimension as $n = 300$.

2.3.5 Connection with other works

The proposed method has similarities with some other previously published results.

Table 2.1: Shown in the table: the mean of number of linearly independent solutions found in the Primal Method, the standard deviation of LI_p, the mean of time taken to end the Primal Method (seconds, and including the time for solving the equation with the Dual Method), the standard deviation of t_p, the mean sparsity of matrix $\mathsf{H}(E, D, \Upsilon_E^z, \Upsilon_D^z)$ (proportion of $\perp$ entries), the standard deviation of $n_\perp$.

n	$\overline{LI_p}$	$\sigma(LI_p)$	$\bar{t}_p$	$\sigma(t_p)$	$\bar{n}_\perp$	$\sigma(n_\perp)$
50	35.45	2.03	0.04	0.005	71%	3%
100	77.65	2.49	0.15	0.005	76%	3%
200	166.25	2.75	0.74	0.082	82%	2%
300	254.15	3.16	1.88	0.060	85%	1%

Connection to the Cellular Decomposition of Develin and Sturmfels (2004): The Develin-Sturmfels cellular decomposition decomposes a tropically convex polytope $tconv(V)$, considered as the image of a finite matrix $V \in \mathbb{T}_{\mathtt{max}}^{n \times m}$, in a finite number of convex (in the traditional sense) polytopes. It does so using the concept of *type*: a type of a vector x relative to V, $type_V^x$, is a set of n subsets of $\{1, 2, ..., m\}$ defined such that [5]:

$$type_V^x(i) = \{j \in \{1, 2, ..., m\} \text{ such that } \bigoplus_{k=1}^{n} (-x_k) V_{kj} = (-x_i) V_{ij}\}. \tag{2.3.39}$$

A type and an induced dominance are closely related. In fact, if $\Upsilon_M^x(i) = j$ then $i \in type_{M^T}^{-x}(j)$ (note that the type must act on $-x$ and the matrix M must be transposed).

If one defines for a type S the set $\mathscr{X}_S$ of all points $x \in \mathbb{T}_{\mathtt{max}}^n$ such that their type $type_V^x$ contains S, then (i) $\mathscr{X}_S$ is convex in the traditional sense (ii) $\mathscr{X}_S$ is bounded if and only if $S(j) \neq \emptyset$ for all j and (iii) $tconv(V)$ is the union of all bounded $\mathscr{X}_S$. Further, $\mathscr{X}_S$ can be completely characterized as the image of a Kleene Closure matrix $C(S)^*$ (see Equation (6) in Lemma 10 of Develin and Sturmfels (2004)) [6].

The solution set of Equation (2.1.1) - an *implicit* characterization of the semimodule of solutions - is a tropically convex polytope, that is, there exists a finite matrix G (see Butkovic and Hegedus

[5]The original work of Develin and Sturmfels (2004) uses $\mathbb{T}_{min}$ instead of $\mathbb{T}_{\mathtt{max}}$. Further, it assumes that the linear span of the rows of V generates the tropically convex polytope. The definition was adapted to the settings of this thesis: $\mathbb{T}_{\mathtt{max}}$ and column linear span.

[6]It is necessary to swap the sign of the inequation, to comply with the Max-Plus tropical algebra context adopted in this thesis (as opposed to the context of Min-Plus tropical algebra adopted in Develin and Sturmfels (2004)).

(1984)) such that all solutions can be written as $x = Gy$ for a vector y - an *explicit* characterization of the semimodule of solutions. Thus, in possession of a solution z, one can compute its type $type^{z}_{G}$ and with it characterize a convex set of solutions as the image of the matrix $C(type^{z}_{G})^{*}$. The Primal Method works similarly, but uses an implicit characterization (instead of the explicit), that is Equation (2.1.1), to compute a convex set of solutions as the image of the matrix $\mathsf{H}(E, D, \Upsilon^{z}_{E}, \Upsilon^{z}_{D})^{*}$. Further, the Primal Method also induces a decomposition by convex cells of the semimodule of solutions if one enumerates all dominances Υ and considers all the sets generated by the image of the upper bounded matrices $\mathsf{H}(E, D, \Upsilon_E, \Upsilon_D)^{*}$. This approach was the one taken in the work described below.

Connections with the Algorithm of Truffet (2010): That work deals with equations of the form $Ax \preceq Bx$, $A, B \in \mathbb{T}^{n' \times m}_{\mathtt{max}}$. $\bot$-full rows considerations aside (the author considers explicitly the $\bot$-full rows of B when constructing the solution, as opposed to this chapter in which the $\bot$ rows are considered to be, without loss of generality, non existent by Remark 2.3.1), the author enumerates a set of n'-tuples with values ranging from 1 to m, that is, a set of functions $\underline{j} : \{1, 2, ..., n'\} \mapsto \{1, 2, ..., m\}$. Then, they are used to construct the entire semimodule of solutions to the tropical linear equation as an image of a matrix G by the augmentation of individual matrices

$$G^{\underline{j}} = \left(\bigoplus_{i=1}^{n} Q^{i\underline{j}(i)} \right)^{*} \tag{2.3.40}$$

for all n'-tuples $\underline{j}$ in the set.

These results can be interpreted by using the notations of this chapter. But since in this chapter equations of the form of Equation (2.1.1) are considered, it is necessary to use $A = (E^T D^T)^T$, $B = (D^T E^T)^T$ so $Ax \preceq Bx \Rightarrow Ex = Dx$ (so, $n' = 2n$ since $E, D \in \mathbb{T}^{n \times m}_{\mathtt{max}}$). Then, in the notation of this thesis, a mapping $\underline{j}$ is a dominance on the matrix B and thus a dominance on E (denoted by Υ_E) augmented with a dominance in D (denoted by Υ_D). Then, the matrix inside the Kleene Closure of Equation (2.3.40) can be written as $\mathsf{H}(E, D, \Upsilon_E, \Upsilon_D) \oplus I$ and, by consequence, $G^{\underline{j}} = \mathsf{H}(E, D, \Upsilon_E, \Upsilon_D)^{*}$.

The main difference of the approach of this chapter and that one is that the former presents a guidance for choosing dominances in which the "usefulness" is guaranteed (that is, those induced by solutions with appropriate properties). The latter work generates all solutions by enumerating a set of "promising " dominances and computing the $G^{\underline{j}}$ for all of them. These promising dominances are obtained by discarding the n'-tuples $\underline{j}$ that would surely generate an unbounded (and therefore useless) $G^{\underline{j}}$. However, (in general) it is possible that even the promising dominances generate unbounded solutions, and these must be removed from the matrix G later.

Connection with Mean Payoff Games: Consider a directed bipartite graph with two disjoint sets of nodes, say “CIRCLE” nodes (n nodes $i = 1, 2, .., n$) and “SQUARE” nodes (m nodes $j = 1, 2, .., m$). A game is played in which, initially, a pawn is in one SQUARE node j. A player, MIN, plays by moving the pawn to a CIRCLE node i and receives from the other player, MAX, an integer amount A_{ij}. Then, it is time for the player MAX to move the pawn to a SQUARE node j' and then receiving from MIN an integer amount $B_{ij'}$. Then, a *turn* ends and this zero-sum game proceeds again with a move from MIN player, and so on.

Given a number of turns k, one defines $v_j[k]$ as the *value* [7] of the finite horizon game for player MAX in which k turns are played and the starting SQUARE node is j. Then, it is of special interest the mean payoff version of this game (called *mean payoff game*), in which the payoff of an infinite trajectory ($k \to \infty$) is defined as the average payment (payments received minus payments made) per turn received by player MAX. In this case, the value of this game at the starting SQUARE node j, the scalar χ_j, is the limsup of the ratio (in the traditional algebra) $v_j[k]/k$ as k goes to infinity.

There is a close connection between tropical linear equations, written on the form $Ax \preceq Bx$ $A, B \in \mathbb{T}_{\max}^{n\times m}$ (which can be formulated as Equation (2.1.1) and *vice-versa*, and thus are equivalent in terms of what they can describe) and mean payoff games described above. In fact, the map $f(x) = A\natural(Bx)$ can be seen as a dynamic programming operator of the described game. Then, solving the tropical linear equation equation can be reduced to the problem of finding an *invariant half-line* to this map. These ideas have been developed in Dhingra and Gaubert (2006); Akian et al. (2010); Gaubert et al. (2012); Gaubert and Sergeev (2013) and in the references therein.

The Primal Method can be also interpreted by what is denoted in the literature as *one player mean payoff game* (Dhingra and Gaubert (2006)), which establishes mean payoff games as another common ground to both Primal and Dual, other than the Extended Cuninghame-Green and K. Zimmermann algorithm presented in Section 2.2. In an (MAX) one player mean payoff game, the MAX player uses a *positional strategy* $\sigma : \{1, 2, .., n\} \mapsto \{1, 2, ..., m\}$, that is, he chooses an *a priori* strategy that he will chose SQUARE node $j = \sigma(i)$ when it is at the CIRCLE node i. The player MIN then aims to minimize the rewards of player MAX based on a more general strategy. This positional strategy is simply what is called *dominance* is this chapter. With this strategy, the equation $Bx \succeq Ax$ is reduced to an Equation $B^\sigma x \succeq Ax$, in which B^σ is dually residuated, that is $B^\sigma x \succeq Ax \iff x \succeq (B^\sigma)^\flat Ax$. Then, a standard Kleene Closure can be used. This step is essentially an application of the Primal Method, but adapted to Equation $Ax \preceq Bx$ instead of Equation (2.3.32). However, not all strategies

[7] In the sense of the MINIMAX theorem, see Osborne and Rubinstein (1994).

σ generate "useful" dominances (upper bounded solutions): one needs that all $\chi_j \preceq 0$ (or equivalently, $\rho((B^\sigma)^\flat A) \preceq 0$) for this to happen. The fact that the Primal Method presented in this chapter uses another solution (with special characteristics) to generate dominances addresses this problem.

Connections with the Algorithm of Lorenzo and de la Puente (2011): The concept of dominance is closely related to the concept of *Winning sequences* in Lorenzo and de la Puente (2011). Following this work, given a system as Equation (2.1.1), with $E, D \in \mathbb{T}_{\max}^{n\times m}$, a *winning pair* is a pair $(i, j) \in \{1, 2, ..., m\} \times \{1, 2, ..., m\}$ of indexes. A winning sequence is a set of n winning pairs such that a compatibility requirement for the matrices E, D holds. This compatibility is a necessary condition for the proposed algorithm in Lorenzo and de la Puente (2011) to be successful in returning solutions when this winning sequence is used.

The winning sequences are replaced in this work by a pair of induced dominances, $\{\Upsilon_E^z, \Upsilon_D^z\}$. The compatibility requirement then comes naturally from the fact that z is a solution, by hypothesis. Using a winning sequence, the authors derive from Equation (2.1.1) a set of bivariate equalities and inequalities. Then, a specialized Gaussian elimination is used to generate the entire set of solutions that are induced by that winning sequence. The reasoning here is similar, but a previously found solution is used for finding adequate dominances and Kleene Closures are used instead of the Gaussian elimination. As it was proved in Proposition 2.3.4, it also generates the entire set of solutions in that particular dominance.

Connections with the Algorithm of Gaubert et al. (2012): In Gaubert et al. (2012), algorithms for solving tropical fractional linear programs (tropical analogues of fractional linear programs) are presented. One of the algorithms (Algorithm 2) concerns minimization problems, and an idea closely related to the Primal Method was used for solving them. At each step, the current suboptimal solution $x[k]$ is used to transform the constraint equation (which determines the feasible set of the optimization problem)

$$Rx \oplus r \succeq Sx \oplus s \tag{2.3.41}$$

(which is equivalent to equations of the form Equation (2.3.32) discussed in this chapter) in a "simplified form" $R^\sigma x \succeq Sx \oplus s$. The matrix R^σ is, in the notations of this chapter, the matrix generated by the dominance of $x[k]$ in R, that is, $R^\sigma = \mathsf{W}(\Upsilon_R^{x[k]}, R)$. Then, the equation is reduced to the form $x \succeq \mathsf{W}^\flat(\Upsilon_R^{x[k]}, R)(Sx \oplus s)$ in which, as discussed in this chapter, the smallest solution $x[k+1] \preceq x[k]$ exists and can be computed using Kleene Closures. This step is essentially an application of the Primal Method, but adapted to Equation (2.3.41) instead of Equation (2.3.32) (see the connection with mean payoff games above). Then, after this procedure one has either a smaller (therefore better) objective function - and thus the iteration must continue - or the algorithm converged to (one possible) optimal solution.

Among the works cited, this one is probably the closest to the proposed Primal Method. This is due to the fact that it contains implicitly an important feature of the algorithm: the idea that a solution is useful to guarantee the convergence (upper boundedness) of the Kleene Closure of a specially constructed matrix.

2.4 Conclusion

This chapter presented an algorithm for solving two sided equations, introducing the concept of *weak residuation which is stronger with respect to an element*. As a subproduct of this method, the *Primal Method* was derived. In the next chapter, it will be clear that this method can be useful to solve another problem that appeared recently in the Tropical Algebra literature.

Chapter 3

On Tropical Fractional Linear Programs

Recently, tropical counterparts of fractional linear programs have been studied. Some algorithms were proposed for solving them, with techniques ranging from bisection methods to homeomorphisms to formal power series. In this chapter, some algorithms are also proposed. They mainly rely in the ability of finding the greatest and smallest solutions of tropical equations, a subject that was discussed in the previous chapter.

The content of this chapter is the same of the one presented in Gonçalves et al. (2013b), with minor adaptations.

3.1 Introduction

Tropical Fractional Linear Programs (denoted as **TFLP** hereafter) are problems of the form (Gaubert et al. (2012))

$$\begin{aligned} \max / \min \ \ (w^T p \oplus \alpha) / (f^T p \oplus \beta) \ \ \text{such that} \\ Rp \oplus r = Sp \oplus s. \end{aligned} \tag{3.1.1}$$

This formulation can be used to compute the tightest inequality of the form $p_i - p_j \geq K$ if p is inside a tropical polyhedron, which finds applications in static analysis (see Gaubert et al. (2012)). It can also be used to check if a set of equalities $Rp \oplus r = Sp \oplus s$ implies another equality $w^T p \oplus \alpha = f^T p \oplus \beta$ without the burden of finding all solutions explicitly [1]. This is true if and only if both max and min versions of Problem 3.1.1 have optimal value 0. This is worthy of mentioning because, in contrast

[1] Finding all solutions of a tropical linear equation can be a very time consuming task, so, it is advantageous to avoid it whenever possible.

with the traditional algebra, in the tropical setting there are equalities that can be logically deduced from a set of other equalities, but *cannot* be obtained by taking tropical linear combinations of these (see Gaubert and Katz (2009)). Hence, it is not always possible to expect to claim that $Rp \oplus r = Sp \oplus s$ does not imply $w^T p \oplus \alpha = f^T p \oplus \beta$ by verifying the solvability of the equations $z^T R = w$, $z^T r = \alpha$, $z^T S = f^T$ and $z^T s = \beta$ for z [2]. With an efficient algorithm for solving TFLPs, one can check the validity of this proposition in an easy manner.

These kind of optimization problems have begun receiving attention recently from scientific community. By the authors' knowledge, the first published work that has solved a particular case of Problem 3.1.1 (save the very particular cases in which it can be solved by a direct application of residuation theory, such as $Ax \preceq b$) of Problem 3.1.1 was (Butkovic and Aminu (2008)). This special case is when $f = \perp$ and $\beta = 0$ (Tropical Linear Programs, denoted as **TLP** hereafter)

$$\max / \min \;\; w^T p \oplus \alpha \;\; \text{such that}$$
$$Rp \oplus r = Sp \oplus s \tag{3.1.2}$$

and an algorithm was presented to solve them. This formulation can be used to solve optimization problems for multiprocessor systems (see Butkovic and Aminu (2008)). The idea is that it is possible to check whether a value of objective function in Problem 3.1.1 is achievable by solving a tropical affine equation. Thus, if a lower and an upper bounds for the objective function are derived, it is possible to use a bisection method to search for the optimal value. The recent paper (Butkovic and MacCaig (2013)) pursues an integer solution to the problem when the entries are real numbers, also using a similar bisection approach.

(Gaubert et al. (2012)) studied the complete problem, and derived a Newton-like algorithm which works by solving a sequence of mean-payoff games. More recently, (Allamigeon et al. (2013)) used the field of *generalized Puiseux series* over $\mathbb{R}$, $\mathbb{K}$ (formal power series in one variable in which the exponents can be any real number) to develop an alternative approach to the problem. It explores the idea of *valuation*, a function which maps each Puiseux series to the opposite of its smallest exponent with a non-zero coefficient. In a special subset of $\mathbb{K}$, $\mathbb{K}_+$ (the set of "non-negative" Puiseux series), this valuation function is a homeomorphism between $\mathbb{K}_+$ and the tropical semiring $\mathbb{T}_{\mathtt{max}}$. Since many of the results used in conventional linear programming rely only in axioms of ordered fields, such as $\mathbb{K}$, the classical Simplex algorithm can be adapted to solve linear programs over $\mathbb{K}$ (instead of the conventional $\mathbb{R}$) and hence, thanks to the valuation homeomorphism, tropical linear

[2] The analogue of this affirmation for traditional algebra, i.e. $Ax = b$ implies $c^T x = d$ only if (the "if" part is trivial) there exists y such that $y^T A = c^T$ and $y^T b = d$, is a consequence of the Farkas' Lemma.

programs as well.

In this chapter, some algorithms will be proposed to solve the general Problem 3.1.1. The first algorithm, to solve max type TLPs, comes directly from a remarkable result about tropical affine equations: they do have a greatest solution. For min type TLPs, a more sophisticated approach using the recent developments in (Gonçalves et al. (2013c)) is presented. The connection between TLPs and TFLPs is made by using a tropical version of the classical Charnes-Cooper method (Charnes and Cooper (1962)) for converting (traditional) fractional linear programs to (traditional) linear programs. As a byproduct of the derivation of these methods, some interesting conclusions can be obtained. Mainly that, as far as the solution p is concerned, the objective function does not matter for max TLPs and hence that the numerator and denominator of the objective functions in max and min TFLPs, respectively, also does not.

3.2 Solving TLPs and TFLPs

3.2.1 Max type programs

Max type TLPs are problems of the form

$$\begin{aligned} \max \quad & w^T p \oplus \alpha \quad \text{such that} \\ & Rp \oplus r = Sp \oplus s. \end{aligned} \tag{3.2.1}$$

As discussed in Chapter 2, the greatest solution to Equation (2.3.32) always exists (see Cuninghame-Green and Zimmermann (2001)), that is, there exists a solution $p_{\mathtt{max}}$ such that for all other solutions p of this affine equation $p_{\mathtt{max}} \succeq p$.

Thus, since the objective function is non-decreasing with p, a Max-type TLP can be solved disregarding w and α, by finding this greatest solution of Equation (2.3.32). The Dual Method can be used for this purpose. Thus, the algorithm is very straightforward.

Algorithm 3.2.1. *Solving max type TLP*

1. Find the greatest solution of Equation (2.3.32) using the Dual Method.

3.2.2 Min type programs

Min type TLPs are problems of the form

$$\begin{aligned} \min \quad & f^T p \oplus \beta \quad \text{such that} \\ & Rp \oplus r = Sp \oplus s. \end{aligned} \tag{3.2.2}$$

The method for solving Min type TLPs is not as straightforward as solving Max type ones since, in general, the smallest solution of Equation (2.3.32) does not exist. The Primal Method addresses this problem partially by finding the smallest solution inside a special set, the *dominance space* (see Chapter 2, in special Proposition 2.3.6). For that, it is necessary to find a solution to the equation first .

Given a solution $p_{\texttt{sol}}$ to Equation (2.3.32), the optimality of it can be ensured if and only if there is no solution to the following equation

$$\begin{aligned} & f^T p \oplus \beta \preceq \gamma \\ & \text{with } \gamma = (-1)(f^T p_{\texttt{sol}} \oplus \beta) \end{aligned} \tag{3.2.3}$$

inside the solution set of Equation (2.3.32). This is due to the fact that all the numbers used in the TLP are integers and thus an integer solution p exists (see Corollary 3.1 in Butkovic and Aminu (2008)). Hence the value of the objective function is also an integer. Since Equation (3.2.3) can be written as an equality

$$f^T p \oplus (\beta \oplus \gamma) = \gamma \tag{3.2.4}$$

it is possible to check the optimality of $p_{\texttt{sol}}$ by checking if there is a solution to the augmented equation

$$\begin{aligned} Rp \oplus r &= Sp \oplus s \\ f^T p \oplus (\beta \oplus \gamma) &= \gamma. \end{aligned} \tag{3.2.5}$$

Further, if a solution is found, it is guaranteed that this solution improves (decreases) the objective function value by at least one unit. Thus, one can apply the following procedure.

Algorithm 3.2.2. *Solving min type TLP*

1. Set k=0;
2. Find a solution $p_{\mathtt{dl}}[0]$ to Equation (2.3.32), using as initial condition of the Dual Method $p[0] = \top$, $y = 0$;
3. Use the Primal Method to solve Equation (2.3.32), using the solution $p_{\mathtt{dl}}[k]$ to induce a dominance, to reduce this solution to a smallest solution in this dominance space, $p_{\mathtt{pr}}[k]$ (see Chapter 2, in special Proposition 2.3.6);
4. Check for optimality of $p_{\mathtt{pr}}[k]$ by solving Equation (3.2.5), using as initial condition of the Dual Method $p[0] = \top$, $y = 0$;
5. If optimality is found (there is no solution for Equation (3.2.5), see Chapter 2, in special Section 2.1), end the algorithm with the solution $p_{\mathtt{sol}} = p_{\mathtt{pr}}[k]$;
6. Else, obtain the solution $p_{\mathtt{dl}}[k+1]$ of Step 4, set k to $k+1$ and go to Step 3.

One can note that Step 3 can be avoided by replacing $p_{\mathtt{pr}}[k]$ with $p_{\mathtt{dl}}[k]$. However, the fact that the Primal Method finds a "small" solution (even if it is not the smallest) can substantially reduce the number of steps taken for convergence (the algorithm can naively just reduce the objective function one unit at each step, see Subsection 3.3.2).

It is important to remark that there is a deep connection between Algorithm 3.2.2 and Algorithm 2 presented in (Gaubert et al. (2012)). In this paper, the problem of solving Min type Problem 3.1.1 is shown to be equivalent to finding the smallest $\lambda \in \mathbb{R}$ such that $\phi(\lambda) \geq 0$ for a function $\phi : \mathbb{R} \mapsto \mathbb{R}$ constructed from all the parameters of the problem.

At each step, the authors find a so-called *left optimal strategy* σ, constructing a simplified function from this strategy, $\phi^\sigma(\lambda)$, and finding the smallest λ such that $\phi^\sigma(\lambda) \geq 0$. The latter problem can be solved by rather straightforward means using Kleene Closures. Due to the properties of left strategies, it is guaranteed that at each step $\lambda[k+1] \leq \lambda[k]$.

The concept of (max player) strategies is very close to the concept of dominances defined in Chapter 2. Then, the problem of finding the smallest λ such that $\phi^\sigma(\lambda) \geq 0$ can be interpreted as the problem of finding the smallest solution inside the dominance space, which is exactly Step 3 in Algorithm 3.2.2. Thus, the essential difference between the algorithms lies in the way that the monotonic convergence is guaranteed (left optimal strategies for Algorithm 2 of (Gaubert et al. (2012)) and extra constraint Equation (3.2.4) for Algorithm 3.2.2). Also, Algorithm 3.2.2 requires that the inputs are integers, while Algorithm 2 of (Gaubert et al. (2012)) does not (albeit one which requires this property is also presented).

Finally, Algorithm 3.2.2 solves Min-type TLPs, while Algorithm 2 of (Gaubert et al. (2012)) solves the more general Min type TFLPs. It will be shown in Subsection 3.2.3 that Algorithm 3.2.2 can also be used to solve this more general kind of problem.

It is also very important to remark that the method *may* take an infinite number of steps to converge if the optimal objective function value is $\bot$. If $\beta \neq \bot$, this bound is evident. If this does not hold, it is helpful to introduce such a bound as a constraint or adding it as a new β' to ensure that the number of steps will be finite. In practice such a bound can be inherent to the structure of the problem. Nevertheless, (Butkovic and Aminu (2008)) shows how to compute lower and upper bounds which are finite in some situations. Basically, assuming that $r \succeq s$ (this can be always assumed, since the equation can be reordered in a way that this holds true) and $\beta = \bot$ (otherwise, β itself is a bound) this bound can be written as

$$V_{\mathrm{LB}} = (f^T \mathbin{/\!\!\!\circ} S)(r \mathbin{-\!\!\!\!\ominus} s). \tag{3.2.6}$$

Indeed, with a very similar argument of those present in (Butkovic and Aminu (2008)), a symmetric version of Equation (3.2.6) (without the assumption $r \succeq s$) can be found

$$V_{\mathrm{LB}} = (f^T \mathbin{/\!\!\!\circ} S)(r \mathbin{-\!\!\!\!\ominus} s) \oplus (f^T \mathbin{/\!\!\!\circ} R)(s \mathbin{-\!\!\!\!\ominus} r). \tag{3.2.7}$$

Finally, one can note that (as opposed to Max type programs) the solution of Min type programs depends on f and β.

3.2.3 Solving TFLPs

The complete problem (TFLPs) is defined as

$$\begin{aligned} \max/\min \;\; (w^T p \oplus \alpha) \mathbin{/\!\!\!\circ} (f^T p \oplus \beta) \;\; \text{such that} \\ Rp \oplus r = Sp \oplus s. \end{aligned} \tag{3.2.8}$$

It is straightforward to adapt the Charnes-Cooper transformation (Charnes and Cooper (1962)) to the tropical setting. Set

$$\begin{aligned} q &= p \not{/} (f^T p \oplus \beta); \\ \hat{q} &= 0 \not{/} (f^T p \oplus \beta). \end{aligned} \tag{3.2.9}$$

Then, by dividing (in Tropical Algebra) both sides of the affine equation in Equation (3.2.8) by $f^T p \oplus \beta$, it can be rewritten as

$$\begin{aligned} \max / \min \ w^T q \oplus \alpha \hat{q} \text{ such that} \\ Rq \oplus r\hat{q} = Sq \oplus s\hat{q}; \\ f^T q \oplus \beta \hat{q} = 0 \end{aligned} \tag{3.2.10}$$

in which the equation $f^T q \oplus \beta \hat{q} = 0$ condenses Equation (3.2.9). Thus, this problem in the transformed variables is a Max / Min type TLP. Once q and $\hat{q}$ are obtained, in order to come back to the original variable one just needs to revert Equation (3.2.9)

$$p = (f^T p \oplus \beta) q = q \not{/} \hat{q}. \tag{3.2.11}$$

It is necessary, however, to consider that *not all* the feasible space of Problem 3.2.10 (transformed problem) can be mapped back to a member of the feasible space of Problem 3.2.8 (original problem). This is the case for some problems in which the transformed problem has $\hat{q} = \top$. For instance, suppose that r and s are finite (no $\bot$ entries), $r \neq s$ and $\beta = \bot$. In this case, $\hat{q} = \top$ and any vector q with no $\top$ entries such that $f^T q = 0$ are in the feasible space of the transformed problem (since $r\hat{q} = s\hat{q} = \top$ and hence the equation $Rq \oplus r\hat{q} = Sq \oplus s\hat{q}$ clearly holds, regardless of q). Transforming back to the original variables using Equation (3.2.11) yields to $p = \bot$ (since $\hat{q} = \bot$ and no entry of q is $\top$), which is not in the original feasible space since by hypothesis $r \neq s$.

For Max type TFLPs, if one uses the method presented in Subsection 3.2.1 this consideration is crucial. This is due to the fact that the Dual Method will initialize the vector $(q^T \ \hat{q})^T$ in $\top$, and hence $\hat{q}$ will begin - and may stay - in $\top$. Thus, it is *necessary* to give an upper bound $\hat{q}$ (by any finite amount). This can be done by lower bounding $f^T p \oplus \beta$ by a finite amount. As mentioned previously, (Butkovic and Aminu (2008)) shows how to compute lower and upper bounds which are finite in some situations (Equation (3.2.6)). Indeed, a cursory observation of Equation (3.2.6) is sufficient to conclude that as long as f has no $\bot$ entries and $r \neq s$ then this bound is finite.

Thus

Algorithm 3.2.3. *Solving Max type TFLPs using Max type TLPs*

1. Use the Charnes-Cooper transformation to transform a Max type TFLP to a Max type TLP;

2. Find a finite lower bound V_{LB} for $f^T p \oplus \beta$ in the feasible space of Equation (2.3.32). That is, find $V_{\mathrm{LB}} \succ \bot$ such that for all p such that $Rp \oplus r = Sp \oplus s$ one has $f^T p \oplus \beta \succeq V_{\mathrm{LB}}$. See (Butkovic and Aminu (2008)) for how to obtain this lower bound for certain problems;

3. Solve the Max type TLP using Algorithm 3.2.1 with the additional constraint $\hat{q} \preceq -V_{\mathrm{LB}}$. This can be done by either setting the constraint $\hat{q} \oplus (-V_{\mathrm{LB}}) = (-V_{\mathrm{LB}})$ explicitly for the transformed problem or initializing the Dual Method with $\hat{q}[0] = -V_{\mathrm{LB}}$ instead of $\top$;

4. Return to the original variables using Equation (3.2.11).

Also, one can solve Min type TFLPs using Algorithm 3.2.2. Hence

Algorithm 3.2.4. *Solving Min type TFLPs using Min type TLPs*

1. Use the Charnes-Cooper transformation to transform a Min type TFLP to a Min type TLP;

2. Solve the Min type TLP using Algorithm 3.2.2 ;

3. Return to the original variables.

Now, a Min type TFLP is simply a Max type TFLP with the inverse of the objective function

$$\begin{aligned} \min\ & (w^T p \oplus \alpha) \not{/} (f^T p \oplus \beta) = \\ -\max\ & (f^T p \oplus \beta) \not{/} (w^T p \oplus \alpha). \end{aligned} \tag{3.2.12}$$

Thus, Min type programs can be transformed in conventional Max type programs and *vice versa*. In summary, any of Algorithms 3.2.3 and 3.2.4 can be used to solve either Max or Min type TFLPs. See Figure 3.1.

It is important to note that, due to the discussion presented in this subsection, Max type TFLPs solutions (that is, the value of p) are independent of f and β. Dually, Min type programs are independent of w and α.

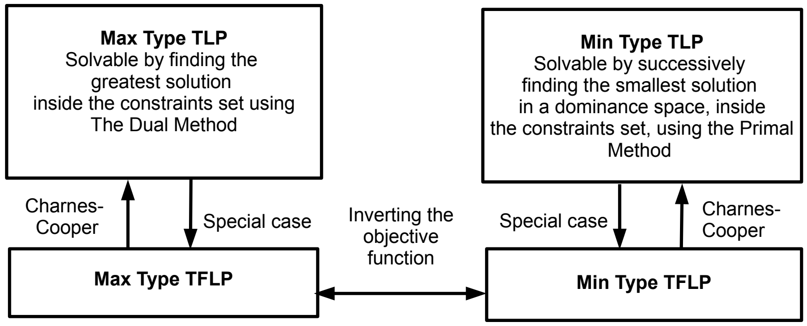

Figure 3.1: Connection between the problems.

3.3 Example

The following example was taken from (Butkovic and Aminu (2008)).

$$\min \quad 3p_1 \oplus 1p_2 \oplus 4p_3 \oplus (-2)p_4 \oplus p_5 \text{ such that}$$

$$\begin{pmatrix} 17 & 12 & 9 & 4 & 9 \\ 9 & 0 & 7 & 9 & 10 \\ 19 & 4 & 3 & 7 & 11 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} \oplus \begin{pmatrix} 12 \\ 15 \\ 13 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & 11 & 8 & 10 & 9 \\ 11 & 0 & 12 & 20 & 3 \\ 2 & 13 & 5 & 16 & 4 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} \oplus \begin{pmatrix} 12 \\ 12 \\ 3 \end{pmatrix}. \tag{3.3.1}$$

The solution using Algorithms 3.2.3 and 3.2.2 will be presented.

3.3.1 Solving using Algorithm 3.2.3

For Algorithm 3.2.3, one must transform the original Min type TLP in a Max type TFLP, that is

$$\min \quad 3p_1 \oplus 1p_2 \oplus 4p_3 \oplus (-2)p_4 \oplus p_5 =$$

$$\max 0 / (3p_1 \oplus 1p_2 \oplus 4p_3 \oplus (-2)p_4 \oplus p_5). \tag{3.3.2}$$

Then, it is necessary to compute a lower bound for $f^T p \oplus \beta$. Using Lemma 3.2 of (Butkovic and Aminu (2008)) (see Equation (3.2.6), note that $r \succeq s$), one can obtain $V_{\text{LB}} = -5$. Thus, $\hat{q} \preceq 5$. The program in the modified variables is

$$\max \hat{q} \text{ such that}$$

$$\begin{pmatrix} 17 & 12 & 9 & 4 & 9 & 12 \\ 9 & 0 & 7 & 9 & 10 & 15 \\ 19 & 4 & 3 & 7 & 11 & 13 \\ 3 & 1 & 4 & -2 & 0 & \bot \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ \hat{q} \end{pmatrix} \oplus \begin{pmatrix} \bot \\ \bot \\ \bot \\ \bot \end{pmatrix} =$$

$$\begin{pmatrix} 2 & 11 & 8 & 10 & 9 & 12 \\ 11 & 0 & 12 & 20 & 3 & 12 \\ 2 & 13 & 5 & 16 & 4 & 3 \\ \bot & \bot & \bot & \bot & \bot & \bot \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ \hat{q} \end{pmatrix} \oplus \begin{pmatrix} \bot \\ \bot \\ \bot \\ 0 \end{pmatrix}. \tag{3.3.3}$$

Now, in order to solve Problem 3.3.1, it is sufficient to find the greatest solution to the affine constraint equation. Using the Dual Method with $(q_1[0]\ q_2[0]\ q_3[0]\ q_4[0]\ q_5[0]\ \hat{q}[0])^T = (\top\ \top\ \top\ \top\ \top\ 5)^T$ (note that $\hat{q}$ was initialized at 5 instead of $\top$. Another possibility is initializing in 0 and adding the constraint $\hat{q} \oplus 5 = 5$), it is possible to find, after 8 iterations of the Dual Method (see Equation (2.2.11))

$$(q_1\ q_2\ q_3\ q_4\ q_5\ \hat{q}) = (-7\ -1\ -4\ -6\ 0\ -1). \tag{3.3.4}$$

Thus, the objective function value is $-\hat{q} = 1$, and $(p_1\ \ p_2\ \ p_3\ \ p_4\ \ p_5)^T = (-6\ 0\ -3\ 5\ 1)^T$ is a possible p. The value obtained for the objective function is, obviously, the same as the one obtained in (Butkovic and Aminu (2008)). Also, the resulting solution p is exactly the same.

3.3.2 Solving using Algorithm 3.2.2

Solving Equation (3.3) with the Dual Method and initial condition

$$(p_1[0]\ p_2[0]\ p_3[0]\ p_4[0]\ p_5[0])^T = (0\ 0\ 0\ 0\ 0)^T, \tag{3.3.5}$$

it is possible to find after 2 iterations

$$p_{\mathtt{d1}}[0] = (-6\ 0\ 0\ -5\ 0)^T \tag{3.3.6}$$

which has objective function value 4. For notational simplicity, let $U = (R\ r)$ and $V = (R\ s)$ and $z[k] = (p_{\mathtt{d1}}[k]^T\ \ 0)^T$ for any k.

Using the Primal Method with the dominances

$$\Upsilon_U^{z[0]}(i) = \begin{cases} 6, & \text{if } i = 1 \\ 4, & \text{if } i = 2 \\ 2, & \text{if } i = 3 \end{cases},\ \Upsilon_V^{z[0]}(i) = \begin{cases} 2, & \text{if } i = 1 \\ 6, & \text{if } i = 2 \\ 1, & \text{if } i = 3 \end{cases} \tag{3.3.7}$$

(this is not the unique pair of dominances possible to be induced by $p_{\mathtt{d1}}[0]$, and were chosen at random) and with these it is possible to obtain

$$p_{\mathtt{pr}}[0] = (-6\ 0\ \perp\ -5\ \perp)^T. \tag{3.3.8}$$

The new objective function value is 1. Equation (3.2.5) has no solution and the algorithm halts. It took a single iteration of Algorithm 3.2.2 to do so.

If Step 3 in Algorithm 3.2.2 is avoided, the algorithm takes 4 steps to converge, beginning from objective function value equal to 4 and decreasing one unit by iteration.

3.4 Conclusion

This chapter presented two different algorithms: one for solving Max and other Min type TLPs. It was shown that the classical Charnes-Cooper transformation can be adapted to transform TFLPs to TLPs. In this case, a connection between Max and Min type TLPs was created, and any algorithm that solves the latter can solve the former and *vice-versa*. As a subproduct of these observations, it was argued that for Max type TFLPs the numerator of the objective function does not matter at all as far as the solution is concerned, and dually the same holds for the denominator of Min type TFLPs.

A careful comparison between the algorithms proposed in this chapter and as well with the others published in literature (Gaubert et al. (2012); Butkovic and Aminu (2008); Allamigeon et al. (2013)) is planned for future works. The author conjectures, however, that Algorithm 3.2.3 (transform to Max TFLP, if necessary then apply Charnes-Cooper and then find the greatest solution using the Dual

Method), other than being conceptually simple and straightforward to implement, is very efficient for average cases.

Chapter 4

On the Regulator Problem for Tropical Linear Event-Invariant Dynamical Systems: Non-Critical Case

This chapter deals with a control problem that very recently (apparently not before than 2004, since Amari et al. (2004) was the first, by the author's knowledge, to propose a particular case of this problem) has drawn the attention of researchers, proposing a necessary and sufficient condition for solving it for a particular class of problems.

Some results concerning this problem were presented in Gonçalves et al. (2012). However, the results shown in this paper (a sufficient condition which is not robust) are supplanted by those shown in this chapter, and thus will not be presented. Nevertheless, the author believes that some concepts introduced in that paper, mainly the concept of system equivalence, could be useful for solving/understanding some control problems.

4.1 Introduction

4.1.1 Overview

Control theory for linear time invariant systems (time sampled, in this case)

$$x[k+1] = Ax[k] + Bu[k] \tag{4.1.1}$$

was largely studied. Its results, based in strong and elegant concepts of linear algebra, are ubiquitous in curricular grades of system engineers. Its importance is undoubtable, either being a direct application or as a basis for more general results (non-linear theory). Some discrete event systems, specifically Timed Event Graphs (a subclass of Petri Nets, see Baccelli et al. (1992)), **TEGs** henceforth, admit a representation in state space curiously similar to the one in Equation (4.1.1) when the timings are event-invariant

$$x[k+1] = Ax[k] \oplus Bu[k] \tag{4.1.2}$$

in which $x_i[k]$; $u_i[k]$ represent the time of the k^{th} firing of the i^{th} transition of state and controller, respectively (see Baccelli et al. (1992) for the modelling details). However, in the context of Equation (4.1.2), the matricial sums and products are performed in the Tropical Algebra. Recently, an effort was put in the pursuit of a solid theory for *tropical linear event-invariant dynamical systems* as in Equation (4.1.2). Due to the peculiar structure of the tropical algebra, few results of the well developed conventional linear dynamical system theory can be easily transposed for this new algebra.

This chapter deals with a problem in which the closest analogue in the traditional theory of linear dynamical systems is the problem of regulation. In this problem, in one of its formulations, one asks for a controller $u[k]$ which guarantees the convergence of the state in steady state to a vector space $\mathscr{V}$ and then its permanence there. In this way, the analogue problem in tropical algebra - subject of this chapter - searches for a controller $u[k]$ that guarantees that $x[k]$ converges in steady state to a (finitely generated) semimodule, which is the analogue of vector spaces in the context of semirings (the algebraic class of tropical algebra), and its permanence there. Such problem finds application in, for instance, train scheduling (Heidergott et al. (2006)), manufacturing systems (Atto et al. (2011)), semiconductors manufacturing processes (Attia et al. (2010)), protein synthesis (Brackley et al. (2011)) also some automatized processes for the discovery of new drugs (T. Brunsch and Hardouin (2012)).

This chapter proposes a necessary and sufficient condition, which induces a not computationally expensive (in average) procedure, for solving this problem for a specific class of problems: the so-called *(coupled) controllable non-critical problems* and also the *(coupled) controllable faux-critical problems* (these definitions are new and will be presented further in this text).

4.1.2 State of the art

One can pose the problem, informally, as:

"Given a system as in Equation (4.1.2) *and a (finitely generated) semimodule* $\mathscr{X}_{\texttt{cons}}$*, find a controller* $u[k]$ *that guarantees, for any initial condition* $x[0]$*, that* $x[k]$ *eventually converges to* $\mathscr{X}_{\texttt{cons}}$ *in a finite number of steps and then stays there indefinitely."*

one has that, by the author's knowledge, this problem was never stated and treated (except in a previous work of the authors which is currently under review, Gonçalves et al. (2013a)) in the literature exactly in this formulation in what concerns two specific points: the broadness of the initial condition and the broadness of the constraint.

Chronologically, by the author's knowledge, the first published paper that treats a problem similar to this one was Amari et al. (2004) in 2004, which suggests that the problem is very recent (this in some form explains the low number of papers published in the subject). The authors work in the counter domain (a dual representation to the one adopted in this chapter, event domain). The aforementioned paper, however, solves a very specific problem (there is no genericness in the considered problem).

Going further, the next paper seems to be Amari et al. (2005), also dealing in the counter domain. The authors deal with a specific class of semimodules $\mathscr{X}_{\texttt{cons}}$, but consider an arbitrary initial condition. They provide a sufficient condition.

For now on all the works deal with the dater domain. Katz (2007) presented a method to solve the control problem for an arbitrary finitely generated semimodule of constraints, but the initial conditions are required to be in a specific semimodule which is also computed in the methodology. This work is based in the concepts of geometric control. However, while powerful, the approach suffers from the facts that it requires an intensive computational effort and also that the termination of the algorithm is not guaranteed for all classes of constraints (one of the main contributions of the paper is a guarantee of finite-time convergence for a very reasonable kind of constraints). This work and the presented one share many ideological similarities (and, indeed, it was very influential in the development of the results presented in this chapter): the approach here is also based in concepts of geometric control, but while Katz (2007) is concerned with computing an entire semimodule (the maximal geometric invariant semimodule) - which can be a daunting task - the present one computes, with much less effort, only a subset of this semimodule. It is shown that for a subclass of problems this subset is sufficient to solve the control problem even if the initial condition lies outside of it (it is a globally attractive semimodule).

Forward, Maia et al. (2011a), treating a specific kind of semimodule $\mathscr{X}_{\texttt{cons}}$, considers that the initial condition is already inside the set. With this hypothesis, necessary and sufficient conditions were derived. Later, Maia et al. (2011b) treats the control problem for a generic (finitely generated) semimodule but assumes a specific initial condition (an eigenvector of the closed loop matrix).

The work presented in Amari et al. (2012) considers the problem for all initial conditions, but for a specific semimodule $\mathscr{S}$ (holding times in a place). Sufficient conditions were presented. Gonçalves et al. (2012) deals with a generic (finitely generated) semimodule of constraints and presents a sufficient condition given the desired semimodule of initial conditions (which is required to be inside the constraint set). This work generalizes the approaches in Maia et al. (2011a), Maia et al. (2011b). At last, Maia et al. (2013) presents a sufficient condition for any kind of initial condition and a specific kind of semimodule $\mathscr{X}_{\texttt{cons}}$, although the result can be easily extended for a generic (finitely generated) semimodule $\mathscr{X}_{\texttt{cons}}$. It will be shown that this result is a particular case of the one proposed in this chapter.

As it was shown in this review, there are very few papers dealing with the subject. Further, they are mainly concentrated in France. All the aforementioned authors in this section are either French or related to French universities.

It is also noteworthy that, while studied by few papers, the requirement that the initial condition is arbitrary is extremely useful. It induces a very desirable characteristic of robustness in the system: even if there is an arbitrary perturbation in the state (eventually driving $x[k]$ out of the constraint semimodule), the controller will eventually drive the system back again to the required specifications. This happens because, due to the fact that the matrices A, B, E, D do not depend on k (event invariance), one can consider the evolution from the perturbed state as a new evolution, of the same system, but with a new initial condition which is exactly this perturbed state. Since the convergence is guaranteed for any initial condition, eventually the system will converge again to the desired set.

4.2 Problem statement

The problem that is analyzed in this chapter will be stated formally.

Problem 4.2.1. (*Regulator problem for tropical linear event-invariant dynamical systems*) A *regulator problem for tropical linear event-invariant dynamical systems* denoted by $\mathscr{R}(A, B, E, D)$ (or simply by $\mathscr{R}$, when in the context the matrices are evident) is defined as follows.

Consider a tropical linear event-invariant dynamical system whose state evolution is given by the recursive equation

$$S : x[k+1] = Ax[k] \oplus Bu[k] \tag{4.2.1}$$

with $x[k] \in \mathbb{T}^{n}_{\max}$, $A \in \mathbb{T}^{n\times n}_{\max}$, $B \in \mathbb{T}^{n\times m}_{\max}$ and initial condition $x[0] \in \mathbb{T}^{n}_{\max}$. It is assumed, therefore, that all the states are measurable and can be used in the control.

The objective is to design a controller $u[k] \in \mathbb{T}^{m}_{\max}$ such that, *for all initial conditions* $x[0]$, the state $x[k]$ belongs to a particular set $\mathscr{X}_{\mathtt{cons}}(\mathscr{R})$ for all $k \geq k'$, for a given *finite* k'. This set is characterized by

$$\mathscr{X}_{\mathtt{cons}}(\mathscr{R}) = \{x \mid Ex = Dx\} \tag{4.2.2}$$

for given matrices $E, D \in \mathbb{T}^{q\times n}_{\max}$. □

Remark 4.2.1. The name *regulator problem* (which is a suggestion of the author of this thesis) comes from analogy to the traditional control theory. In the infinite horizon traditional linear quadratic regulator problem (or, at least, a particular version of it), one wishes to solve

$$\min_{u[k]} \sum_{i=0}^{\infty} x[i]^T Q x[i] \text{ such that } x[k+1] = Ax[k] + Bu[k] \tag{4.2.3}$$

(all the operators must be interpreted in the traditional algebra) for a positive semidefinite matrix Q and an initial condition $x[0]$. One sees that this problem is solved if and only if $x[k]$ approaches and stays at the null space of Q (if it is possible, otherwise the objective function value will be unbounded). Since this null space is a vector space, this implies the convergence to this specific class of sets. The proposed problem is similar in spirit: it is desired to drive, and maintain, the state $x[k]$ in a given semimodule (which is the analogue of vector spaces in semirings). □

This set $\mathscr{X}_{\mathtt{cons}}(\mathscr{R})$ is the most general form of tropical linear constraints (that is, a double-sided equation $Ex = Dx$), and was considered in Katz (2007); Maia et al. (2011b); Gonçalves et al. (2012). It is important to mention that any finitely generated semimodule $Im\{M\}$ can be written as in Equation (4.2.2) and that, conversely, $\mathscr{X}_{\mathtt{cons}}(\mathscr{R}) = Im\{M\}$ for a matrix M with a finite number of columns (both are consequences of the Duality Theorem and Finiteness Theorem, see Gaubert and M.Plus (1997)).

At last, it is important to stress that in some applications guaranteeing the desired constraints *only* in steady state is prohibitive, since the violation of these can imply inadmissible consequences. For instance, a manufacturing system for which one of the constraints is that a part cannot stay at the oven more than, say, 3 minutes: violation of this constraint may imply the loss of the part. In this case, the approaches referenced in Section 4.1 may be more appropriate. Nevertheless, the proposed approaches *can also* be used to this purpose provided that it is possible to choose a convenient initial condition $x[0]$ (see Remark 4.4.3).

4.3 (A,B) tropical geometrically invariant sets

4.3.1 Fundamental results and definitions

A key concept for solving $\mathscr{R}(A,B,E,D)$ is the one of *(A,B) tropical geometrical invariance* of sets.

Definition 4.3.1. *((A,B) tropical geometrically invariant sets, see Katz (2007))* A set $\mathcal{N} \subseteq \mathbb{T}^n_{\mathtt{max}}$ is said to be *(A,B) tropical geometrically invariant* (**(A,B)-TGI** from now on) if for every $x \in \mathcal{N}$ there exists $u \in \mathbb{T}^m_{\mathtt{max}}$ such that $Ax \oplus Bu \in \mathcal{N}$. □

Remark 4.3.1. Its important to note that if $\mathscr{R}(A,B,E,D)$ has a solution, an (A,B)-TGI set inside $\mathscr{X}_{\mathtt{cons}}(\mathscr{R})$ must exist.

To see this, suppose $\mathscr{R}(A,B,E,D)$ has a solution. Then, there must exist a natural k' such that for all $k \geq k'$ $x[k] \in \mathscr{X}_{\mathtt{cons}}(\mathscr{R})$. This implies that the set

$$\mathscr{X}_{\mathtt{geo}} = \{x[k] \mid k \geq k'\} \tag{4.3.1}$$

is an (A,B)-TGI set inside the set of constraints, since for all $x' \in \mathscr{X}_{\mathtt{geo}}$ there exists a $k \geq k'$ such that $x' = x[k]$ and thus there exists an u', namely $u[k]$, such that $Ax' \oplus Bu' = x[k+1] \in \mathscr{X}_{\mathtt{geo}}$. □

Note that the union of (A,B)-TGI sets is also (A,B)-TGI. By consequence, one can speak about the *maximal (A,B)-TGI set* (that is, the one created by the union of all (A,B)-TGI sets). Hence

Definition 4.3.2. (*Maximal (A,B)-TGI set inside the constraints, see Katz (2007)*) Given a problem $\mathscr{R}$, $\mathscr{X}^{\top}_{\mathtt{geo}}(\mathscr{R})$, the *maximal (A,B)-TGI set inside the constraints*, is the maximal (A,B)-TGI set inside $\mathscr{X}_{\mathtt{cons}}(\mathscr{R})$. Formally

$$\mathscr{X}^{\top}_{\mathtt{geo}}(\mathscr{R}) \equiv \{\bigcup \mathscr{X} \mid (\mathscr{X} \text{ is (A,B)-TGI }) \;\&\; (\mathscr{X} \subseteq \mathscr{X}_{\mathtt{cons}}(\mathscr{R}))\}. \tag{4.3.2}$$

□

Hence, an important definition can be made.

Definition 4.3.3. *(Controllable coupled problem)* A problem $\mathscr{R}(A, B, E, D)$ is said to be *controllable coupled* if it having a solution implies that

$$\exists M \in \mathbb{N} \mid \forall x \in \mathscr{X}^{\top}_{\mathtt{geo}}(\mathscr{R}) - \{\bot\}\ ,\ \ \forall\, i, j\ |x_i - x_j| \le M. \tag{4.3.3}$$

That is, controllable coupled problems have the property that in steady state the difference between the entries of the state will be bounded. □

Remark 4.3.2. This text is only interested in controllable coupled problems. Indeed, it could be argued that otherwise they are meaningless in practice or can be broken in independent subproblems (which then can be solved separately). If the system is not controllable coupled, in the steady state and under control there will be disjoint sets of transitions (those created from the quotient by the equivalence relation "is controllable coupled with" such that i is controllable coupled with j if and only if $|x_i[k] - x_j[k]|$ is bounded) that operate in different rates. This means that no interesting synchronization was imposed between these disjoint subsets.

For instance, Katz (2007) discusses specifications of the form $x[k] \succeq Qx[k]$. That paper argues that, frequently, in practical applications the entries Q_{ij} of this matrix can be chosen to be different than $\bot$ (by replacing it by a very large negative number, for instance). This alone would imply that $x_j[k] - x_i[k] \le -Q_{ij}$ and $x_i[k] - x_j[k] \le -Q_{ji}$ and thus $|x_i[k] - x_j[k]| \le \max(-Q_{ij}, -Q_{ji})$, which is finite under the consideration that $Q_{ij} \ne \bot$ for all i and j. Then, the system is controllable coupled.

An example of a non controllable coupled system is two completely independent machines in which, as a constraint, one is required to produce one piece at every 2 minutes and the other at every 1 minute. This problem is not controllable coupled because if $x_1[k]$ and $x_2[k]$ represent the time of completion of the k^{th} pieces for the first and second machines, respectively, then $|x_1[k] - x_2[k]|$ grows roughly with $2^k - 1^k = 1^k = k$, which is unbounded. There is no interesting additional requirement in the firing dates, at least in steady state, that can be imposed between them. □

Remark 4.3.3. If $\mathscr{R}(A, B, E, D)$ is controllable coupled and it has a solution, $\mathscr{X}^{\top}_{\mathtt{geo}}(\mathscr{R})$ is finitely generated: there exists a matrix X with a finite number of columns such that $\mathscr{X}^{\top}_{\mathtt{geo}}(\mathscr{R}) = Im\{X\}$.

For any $x \in \mathscr{X}^{\top}_{\mathtt{geo}}(\mathscr{R})$, $|x_i - x_j| \le M$ must hold since the problem is controllable coupled by hypothesis. Then, using the concept of *volume* presented in Katz (2007) and noting that $x \in \mathbb{T}^n_{\mathtt{max}}$ has only integers or $\bot$ entries, one can see that the number of *different* normalized vectors $x \not{/} (\oplus_i x_i)$ of $\mathscr{X}^{\top}_{\mathtt{geo}}(\mathscr{R})$ is less than or equal to $(2M + 1)^n$ (the operators in this expressions are in traditional algebra, see Katz (2007) for an in-depth discussion). It is clear that these normalized vectors, in which the number of them is finite, can be used to generate the entire semimodule. □

Remark 4.3.4. Another very important property to consider is if $\mathscr{X}_{\mathtt{geo}}^{\top}(\mathscr{R})$ is finitely generated, that is, $\mathscr{X}_{\mathtt{geo}}^{\top}(\mathscr{R}) = Im\{X\}$ for a matrix $X \in \mathbb{T}_{\mathtt{max}}^{n\times s}$ with a finite number of columns, then the affine tropical equation

$$AX \oplus BU = XV \tag{4.3.4}$$

for the unknowns $U \in \mathbb{T}_{\mathtt{max}}^{m\times s}$ and $V \in \mathbb{T}_{\mathtt{max}}^{s\times s}$ has a solution.

Indeed, for every column $x[i]$ of X, the equation

$$Ax[i] \oplus Bu[i] = Xv[i] \tag{4.3.5}$$

must have a solution for the unknowns $u[i]$ and $v[i]$, since $\mathscr{X}_{\mathtt{geo}}^{\top}(\mathscr{R})$ is a tropical (A,B)-TGI set and everything which is inside this set can be written as $Im\{X\}$. This readily implies the desired result (in which U and V are constructed by using the $u[i]$ and $v[i]$ as columns, respectively). □

Before continuing, a very important definition is necessary.

Definition 4.3.4. (*Spectral characteristic equations and characteristic spectrum*) The *spectral characteristic equation* for a problem $\mathscr{R}(A,B,E,D)$, $\mathfrak{S}(\mathscr{R})$, is defined as follows

$$\mathfrak{S}(\mathscr{R}) : \begin{cases} (i) : E\chi = D\chi; \\ (ii) : \lambda\chi = A\chi \oplus B\mu. \end{cases}$$

in which the members of the triple $\{\lambda, \chi, \mu\}$ are the unknowns. The *strong spectral characteristic equation*, $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$, is defined as

$$\mathfrak{S}_{\mathtt{str}}(\mathscr{R}) : \begin{cases} (i) : E\chi = D\chi; \\ (ii) : \lambda\chi = (\lambda^{-1}A)^{*}B\mu. \end{cases}$$

A triple $\{\lambda, \chi, \mu\}$ such that $\mathfrak{S}(\mathscr{R})$ (resp. $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$) holds is a *proper solution* to $\mathfrak{S}(\mathscr{R})$ (resp. $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$) if χ has no $\perp$ entries. The set of all λ such that $\{\lambda, \chi, \mu\}$ is a proper solution to $\mathfrak{S}(\mathscr{R})$ is the *characteristic spectrum* of the problem, and is denoted by $\Lambda(\mathscr{R})$. Formally

$$\Lambda(\mathscr{R}) \equiv \{\lambda \mid \{\lambda, \chi, \mu\} \text{ is a solution to } \mathfrak{S}(\mathscr{R}) \text{ and } \chi \text{ does not have a } \perp \text{ entry}\}. \tag{4.3.6}$$

□

Remark 4.3.5. The article Maia et al. (2011b) presents a particular case of $\mathfrak{S}(\mathscr{R})$: $(A \oplus BF)\chi = \lambda\chi$, which is (in principle) a stronger equation than $\mathfrak{S}(\mathscr{R})$. However, this stronger equation is tropical non-linear and thus bound to be difficult to solve. Maia et al. (2011b) presents a sufficient condition for solving it, that is, finding F and χ. As a byproduct of the results of this chapter, it will be shown that if $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ has a solution then so does $(A \oplus BF)\chi = \lambda\chi$. □

Remark 4.3.6. It is easy to see that $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ is indeed a stronger equation than $\mathfrak{S}(\mathscr{R})$. Both are equivalent when $\lambda > \rho(A)$, due to the equivalence $x = Cx \oplus d \iff x = C^*d$ if $\rho(C) < 0$ (see Baccelli et al. (1992)). Otherwise, only the $\Leftarrow$ part of this implication holds. □

Remark 4.3.7. As mentioned in Subsection 4.1.2, the work of Katz (2007) provides a method for computing $\mathscr{X}^{\top}_{\mathtt{geo}}(\mathscr{R})$. However, this can be quite expensive in terms of time and space. On the other hand, one can quickly note that given a solution $\{\lambda, \chi, \mu\}$ to $\mathfrak{S}(\mathscr{R})$, the set $\mathscr{X}_{\mathtt{gp}}(\chi) = \{\alpha\chi \mid \alpha \in \mathbb{T}_{\mathtt{max}}\}$ is (A,B)-TGI (indeed, it is sufficient to pick $u = \alpha\mu$ and then $A(\alpha\chi) \oplus B(\alpha\mu) = (\lambda\alpha)\chi \in \mathscr{X}_{\mathtt{gp}}(\chi)$). So, the characteristic spectral equation provides a method to compute a subsemimodule of $\mathscr{X}^{\top}_{\mathtt{geo}}(\mathscr{R})$. As it will be soon clear, this subsemimodule is very useful. □

Remark 4.3.8. It will soon be shown that it is of main interest to solve $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ instead of $\mathfrak{S}(\mathscr{R})$. However, keeping in mind Remark 4.3.6, $\mathfrak{S}(\mathscr{R})$ is equivalent to $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ if $\lambda > \rho(A)$. Hence, in this case, $\mathfrak{S}(\mathscr{R})$ (which has a more convenient form, as it will be clear below) can be solved instead of $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$. The case $\lambda = \rho(A)$ can be considered directly in $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ because, with this value of λ fixed, $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ is a tropical linear equation and hence it can be solved efficiently with a myriad of algorithms (see Chapter 2).

Equation $\mathfrak{S}(\mathscr{R})$, however, is tropical nonlinear for the parameters $\{\lambda, \chi, \mu\}$. The problem of finding a solution λ and $y = (\chi^T\ \mu^T)^T$ with this vector without $\perp$ entries (which implies that the solution is proper, by the previous discussion) can be transformed in a *two-sided eigenproblem* . This problem can be stated (Gaubert and Sergeev (2013)) as solving the equation $Uy = \lambda Vy$ for the unknowns λ and $y \neq \perp$. Note that $y = (\chi^T\ \mu^T)^T \neq \perp$ does not necessarily imply that χ has no $\perp$ entries. However, since the set $\mathscr{X}_{\mathtt{gp}}(\chi) = \{\alpha\chi \mid \alpha \in \mathbb{T}_{\mathtt{max}}\}$ is (A,B)-TGI (see Remark 4.3.7) and the problem is controllable coupled by hypothesis then, necessarily, if χ has at least one non-$\perp$ entry then it has no $\perp$ entries. Now, it is also clear that if y is not $\perp$, then either χ or μ (possibly both) has a non-$\perp$ entry. If it is the former case, the problem is solved. If it is the latter, that is, μ does have a non-$\perp$ entry, by the fact that $\lambda\chi \succeq B\mu$ and the assumption (without loss of generality) that B does not have a $\perp$ column (otherwise that control input plays no role in the system and then can be removed), one can readily conclude that χ also does have a non-$\perp$ entry. In either case, the fact that χ does not have any $\perp$ entry is then guaranteed.

A two-sided eigenproblem can be shown to be equivalent to find the zeroes λ of a *spectral function* $s_{\mathscr{R}}(\lambda)$, which is always non-positive, piecewise affine and Lipschitz (Gaubert and Sergeev (2013)). The algorithm proposed in Gaubert and Sergeev (2013) has a pseudo-polynomial complexity, and thus it is bound to be efficient in average cases.

To transform the problem in a two-sided eigenproblem, multiply both sides of Equation $\mathfrak{S}(\mathscr{R})$-(i) by λ. The resulting set of equations will be equivalent as long as λ is invertible ($\lambda \neq \perp$), which is a very weak assumption. Now, substitute the second equation in the first one, but only in the *left* side of the first equation (thus, creating a slight asymmetry). The resulting equation is

$$\begin{aligned} &(i) : \lambda\chi = A\chi \oplus B\mu \\ &(ii) : \lambda D\chi = EA\chi \oplus EB\mu; \end{aligned} \tag{4.3.7}$$

or

$$\lambda \begin{pmatrix} I & \perp \\ D & \perp \end{pmatrix} \begin{pmatrix} \chi \\ \mu \end{pmatrix} = \begin{pmatrix} A & B \\ EA & EB \end{pmatrix} \begin{pmatrix} \chi \\ \mu \end{pmatrix} \tag{4.3.8}$$

(the symmetric equation obtained by switching the roles of E and D can also be used) ergo, a two-sided eigenproblem.

It is also important to note that it is possible that the λ and χ, μ found have rational entries, thus not in $\mathbb{T}_{\mathtt{max}}$. This is not a practical problem, however. All these elements can be written as a fraction of integers (or $\perp$) u_i/v, in which the value v is the same for all of them. Redefining the time units in the system by multiplying all of them by v [1], the scaled elements $v \cdot \lambda$ and $v \cdot \chi$, $v \cdot \mu$ will all have integers of $\perp$ entries, and hence in $\mathbb{T}_{\mathtt{max}}$. $\square$

Example 4.3.1. The characteristic spectrum $\Lambda(\mathscr{R})$ can be quite complex, composed of an union of disjoint intervals of $\mathbb{R}$. To exemplify this, consider the problem $\mathscr{R}(A, B, E, D)$

[1] For example, if all the time units of the system are in minutes and $v = 60$, the new time unit is seconds. If $v = 30$, "double" seconds, and so on...

$$x[k+1] = \begin{pmatrix} 2 & \perp & \perp & \perp \\ 0 & \perp & \perp & \perp \\ \perp & 0 & \perp & \perp \\ 2 & 15 & \perp & \perp \end{pmatrix} x[k] \oplus \begin{pmatrix} 0 \\ \perp \\ \perp \\ 0 \end{pmatrix} u[k] \tag{4.3.9}$$

with the constraint $x_4[k] - x_2[k] \geq 10$ which can be written as $x_4[k] \oplus 10x_2[k] = x_4[k]$ (and thus as $Ex = Dx$). The characteristic spectrum of this problem can be shown to be $\Lambda(\mathscr{R}) = [2\ 5] \cup [10\ \infty)$. □

Then, as a consequence of the definition and remarks, the following important proposition can be stated.

Proposition 4.3.1. *(Necessity)*: Suppose that $\mathscr{R}(A, B, E, D)$ is controllable coupled, then it has a solution only if Equation $\mathfrak{S}(\mathscr{R})$ has a proper solution.

Proof. Remark 4.3.3 implies that $\mathscr{X}^{\top}_{\mathtt{geo}}(\mathscr{R})$ is finitely generated. Hence, $\mathscr{X}^{\top}_{\mathtt{geo}}(\mathscr{R}) = Im\{X\}$ for a matrix X with a finite number of columns.

Remark 4.3.4 then states that $AX \oplus BU = XV$ must have a solution for U and V. Since V is a square matrix, it has an eigenvector v so $Vv = \lambda v$. Thus, post-multiplying $AX \oplus BU = XV$ by v

$$A(Xv) \oplus B(Uv) = \lambda(Xv). \tag{4.3.10}$$

Further, since $\mathscr{X}^{\top}_{\mathtt{geo}}(\mathscr{R}) \subseteq \mathscr{X}_{\mathtt{cons}}(\mathscr{R})$, $EX = DX$ and thus $E(Xv) = D(Xv)$. Hence, it is clear that Equation $\mathfrak{S}(\mathscr{R})$ must have a solution. For instance, $\chi = Xv$ and $\mu = Uv$.

Assume without loss of generality that X has no $\perp$ columns. For showing that there is a solution for which all the χ_i are not $\perp$, it is sufficient to show that all entries of Xv are not $\perp$. Since the constraint $|x_i - x_j| \leq M$ must hold (the problem is controllable coupled and X is the generator of the maximal (A,B)-TGI set inside the constraints) and no column of x is $\perp$, then all the columns of X are free of $\perp$ entries. Finally, since at least one element of v is non-null (it is an eigenvector), this implies the desired result. □

It will be soon shown that under some wide circumstances, finding a proper solution to Equation $\mathfrak{S}(\mathscr{R})$ is also sufficient for solving $\mathscr{R}(A, B, E, D)$. Indeed, one can note that if $\mathfrak{S}(\mathscr{R})$ has a solution then $\lambda\chi \succeq A\chi$. If χ has no null entries (the solution is proper) this implies that $\lambda \succeq \rho(A)$. Hence, if $\lambda \in \Lambda(\mathscr{R})$ then $\lambda \succeq \rho(A)$. This motivates a very important definition.

Definition 4.3.5. *(Controllable critical and controllable non-critical problems)* A problem $\mathscr{R}(A, B, E, D)$ is said to be *controllable critical* if $\Lambda(\mathscr{R}) = \{\rho(A)\}$. Otherwise, it is said to be *controllable non-critical*. □

Controllable critical problems are very specific: their characteristic spectrum has only one possible value, which is exactly the smallest possible λ (since $\lambda \succeq \rho(A)$). Controllable non-critical problems will be, as mentioned in the introduction, the main subject of this chapter.

4.4 The Spectral Regulator

It will be shown that, for controllable coupled and controllable non-critical problems, there exists a $F_{\mathtt{SR}} \in \mathbb{T}_{max}^{m\times n}$ such that the controller $u[k] = F_{\mathtt{SR}}x[k]$ solves the proposed problem. Of course, in order to this controller to be realizable, it must be *causal*: that means that their entries are either non-negative or $\perp$. Otherwise, the controller would require the forecasting of events [2]. This problem will be disregarded for now, but will be discussed later in Section 4.6.

Before showing how to compute the feedback controller $F_{\mathtt{SR}}$, it is necessary a lemma.

Lemma 4.4.1. *(Necessary and sufficient condition for feedback controllers)* $\mathscr{R}(A, B, E, D)$ has a solution $u[k] = Fx[k]$ if and only if there exists a natural number l such that

$$E(A \oplus BF)^l = D(A \oplus BF)^l. \tag{4.4.1}$$

Proof.
Only if: With the controller $u[k] = Fx[k]$, one has that $x[k+1] = (A \oplus BF)x[k]$ or $x[k] = (A \oplus BF)^k x[0]$. Since it is necessary that there exists a l such that $\forall k \geq l\ x[k] \in \mathscr{X}_{\mathtt{cons}}(\mathscr{R})$, or , $Ex[k] = Dx[k]$, there must exist a l such that

$$E(A \oplus BF)^l x[0] = D(A \oplus BF)^l x[0]. \tag{4.4.2}$$

Since the same must hold for any $x[0]$, one obtains the necessity of Equation (4.4.1) (it is sufficient to choose $x[0]$ as the columns of the identity matrix).
If: The sufficiency comes easily after post-multiplying both sides of Equation (4.4.1) by $(A \oplus BF)^{k-l}x[0]$, $k \geq l$. Hence, one has that $x[k] \in \mathscr{X}_{\mathtt{cons}}(\mathscr{R})\ \forall\ k \geq l$, as desired. □

[2] Consider the control rule u[k]=(-2)x[k]. It reads as "fire u for the k^{th} time 2 time units before x fires for the k^{th} time", which would require a forecasting of $x[k]$.

Solving Equation (4.4.1) with $l > 1$ is a difficult task, since it is tropical non-linear (one can use the method of Schutter and Moor (1996), but it can be extremely time and space consuming). This equation will be solved indirectly by means of $\mathfrak{S}(\mathscr{R})$. To this end, one makes the following definition.

Definition 4.4.1. (*Convergence number*) For a given square matrix $M \in \mathbb{T}_{max}^{n\times n}$ with $\rho(M) \leq 0$, the *convergence number of* M, $\kappa(M)$, is the smallest r such that

$$M^* = \bigoplus_{i=0}^{r} M^i. \tag{4.4.3}$$

□

Remark 4.4.1. It can be shown that if $\rho(M) \leq 0$ and $M \in \mathbb{T}_{max}^{n\times n}$ then $\kappa(M) \leq n$ (see Baccelli et al. (1992)). □

Then, an important definition is necessary.

Definition 4.4.2. *(Feedback synthesis equation)* The *feedback synthesis equation* $\mathfrak{F}(\mathscr{R},\lambda,\chi,\mu)$ is the following (tropical affine) equation for the unknown ζ

$$\mathfrak{F}(\mathscr{R},\lambda,\chi,\mu): \begin{cases} (i): \ \zeta^T B\mu = \lambda; \\ (ii): \ \zeta^T A \preceq \lambda\zeta^T; \\ (iii): \ \chi\zeta^T \succeq (\lambda^{-1}A)^*. \end{cases}$$

□

Given a proper solution $\{\lambda,\chi,\mu\}$ to $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ and a solution ζ to Equation $\mathfrak{F}(\mathscr{R},\lambda,\chi,\mu)$, the controller will take the form $u[k] = F_{\text{SR}}x[k] = \mu\zeta^T x[k]$, that is, $F_{\text{SR}} = \mu\zeta^T$ solves Equation (4.4.1) for a l that will be specified soon. This kind of controller will receive the name *Spectral Regulator*. To show that this controller solves the problem, it is necessary to derive the following lemma.

Lemma 4.4.2. *(Left eigenvector)* Let $M \equiv A \oplus BF_{\text{SR}} = A \oplus B\mu\zeta^T$ be the closed loop matrix. If Equation $\mathfrak{F}(\mathscr{R},\lambda,\mu)$ - (i,ii) holds, then $\zeta^T M = \lambda\zeta^T$.

Proof. Straightforward: $\zeta^T M = \zeta^T A \oplus (\zeta^T B\mu)\zeta^T = \lambda\zeta^T$ by Equation $\mathfrak{F}(\mathscr{R},\lambda,\chi,\mu)$- (i,ii). □

Hence

Proposition 4.4.1. *(Spectral Regulator solves* $\mathscr{R}$*)* Let $\{\chi,\mu,\lambda\}$ be a solution of $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ and ζ of $\mathfrak{F}(\mathscr{R},\lambda,\chi,\mu)$. So $l = \kappa(\lambda^{-1}A)+1$ and $F_{\text{SR}} = \mu\zeta^T$ solve Equation (4.4.1) and hence, by Lemma 4.4.1 (specially the *if* part), $u[k] = F_{\text{SR}}x[k]$ solves $\mathscr{R}(A,B,E,D)$.

Proof. Let $M[k] = (A \oplus B\mu\zeta^T)^k$. By Lemma 4.4.2, one concludes that $\zeta^T M[k] = \lambda^k \zeta^T$. So

$$M[k+1] = (A \oplus B\mu\zeta^T)M[k] = \tag{4.4.4}$$
$$AM[k] \oplus B\mu(\zeta^T M[k]) = AM[k] \oplus \lambda^k B\mu\zeta^T.$$

Multiplying both members of Equation (4.4.4) by $\lambda^{-(k+1)}$, it is possible to work with the normalized matrix $M_{[\lambda]}[k] = \lambda^{-k} M[k]$:

$$M_{[\lambda]}[k+1] = (\lambda^{-1}A)M_{[\lambda]}[k] \oplus (\lambda^{-1}B)\mu\zeta^T. \tag{4.4.5}$$

Iterating Equation (4.4.5) (note that $M[0] = (A \oplus B\mu\zeta^T)^0 = I$)

$$M_{[\lambda]}[k+1] = (\lambda^{-1}A)^{k+1} \oplus \lambda^{-1} \left(\bigoplus_{i=0}^{k} (\lambda^{-1}A)^i \right) B\mu\zeta^T. \tag{4.4.6}$$

Choose $k = \kappa(\lambda^{-1}A)$. Hence

$$M_{[\lambda]}[l] = (\lambda^{-1}A)^l \oplus \lambda^{-1}(\lambda^{-1}A)^* B\mu\zeta^T. \tag{4.4.7}$$

Since $\{\lambda, \chi, \mu\}$ is a proper solution for $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$, $\lambda^{-1}(\lambda^{-1}A)^* B\mu = \chi$. Thus

$$M_{[\lambda]}[l] = (\lambda^{-1}A)^l \oplus \chi\zeta^T. \tag{4.4.8}$$

By $\mathfrak{F}(\mathscr{R}, \lambda, \chi, \mu)$-(iii), $\chi\zeta^T \succeq (\lambda^{-1}A)^* \succeq (\lambda^{-1}A)^k$ for any k. Then

$$M_{[\lambda]}[l] = \chi\zeta^T \tag{4.4.9}$$

and thus $M[l] = \lambda^l \chi\zeta^T$. Since $E\chi = D\chi$, after post-multiplication by $\lambda^l \zeta^T$ one concludes that $EM[l] = DM[l]$. And the proposition is proved.

□

Perhaps surprisingly, a solution ζ to the Equation $\mathfrak{F}(\mathscr{R}, \lambda, \chi, \mu)$ (when $\{\lambda, \chi, \mu\}$ is a solution to $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$) always exists when $\mathscr{R}$ is a controllable coupled problem. Further, the greatest one can be computed explicitly with little effort.

Proposition 4.4.2. *(Computing a solution to the feedback equation)* Given a proper solution $\{\lambda, \chi, \mu\}$ to $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$, if $\mathscr{R}$ is controllable coupled, there is always a solution ζ to Equation $\mathfrak{F}(\mathscr{R}, \lambda, \chi, \mu)$. In particular, $\zeta^T = (-\chi)^T (\lambda^{-1}A)^*$ is the greatest one.

Proof. Equation $\mathfrak{F}(\mathscr{R},\lambda,\chi,\mu)$-(ii) can be written equally as $\zeta^T = \zeta^T(\lambda^{-1}A)^*$. So, equivalently there exists g such that $\zeta^T = g^T(\lambda^{-1}A)^*$ and Equation $\mathfrak{F}(\mathscr{R},\lambda,\chi,\mu)$-(ii) will hold by default. Making these substitutions in Equation $\mathfrak{F}(\mathscr{R},\lambda,\chi,\mu)$-(i,iii), one concludes that $\mathfrak{F}(\mathscr{R},\lambda,\chi,\mu)$ reduces to simply

$$\begin{aligned}(i):&\ \ g^T\chi = 0; \\ (ii):&\ \ \chi g^T(\lambda^{-1}A)^* \succeq (\lambda^{-1}A)^*.\end{aligned} \tag{4.4.10}$$

Equation (4.4.10) always has a solution for g as long as χ has not $\perp$ entries (which can be assumed for controllable coupled systems, see Proposition 4.3.1). For instance, choose $g = -\chi$. It is clear that Equation (4.4.10)-(i) will hold, and so will Equation (4.4.10)-(ii), since $\chi g^T \succeq I$ (note that the diagonal entries of χg^T are $\chi_i + g_i = \chi_i + (-\chi_i) = 0$). It is also clear that this is the greatest g possible, since by Equation (4.4.10)-(i) $g \preceq -\chi$.

□

Hence, the most important result of this chapter can be stated.

Proposition 4.4.3. (*Necessary and sufficient condition for controllable coupled and controllable non-critical problems*) If $\mathscr{R}$ is controllable coupled and controllable non-critical, it has a solution if and only if $\mathfrak{S}(\mathscr{R})$ has a proper solution.

Proof.
Only if: comes directly from Proposition 4.3.1.
If: : If $\mathscr{R}$ is controllable non-critical, there exists a $\lambda \neq \rho(A)$ such that $\{\lambda,\chi,\mu\}$ is a solution to $\mathfrak{S}(\mathscr{R})$. In this case, a solution to $\mathfrak{S}(\mathscr{R})$ is also a solution to $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$. With such solution, Proposition 4.4.1 ensures that a feedback of the form $u[k] = F_{\mathtt{SR}}x[k]$ solves the problem as long as there is a solution to $\mathfrak{F}(\mathscr{R},\lambda,\chi,\mu)$. Proposition 4.4.2 guarantees the existence of at least one solution to this equation. And the proposition is proved. □

Remark 4.4.2. As mentioned in Subsection 4.1.2, the results presented in this section are more general then the one presented in Maia et al. (2013). In that paper, it is shown that if there exists a vector p without $\perp$ entries such that $EBp = DBp$ (the original paper considers that $D = I$, but this assumption is unnecessary) and B has no row full of $\perp$ entries, then $\mathscr{R}(A,B,E,D)$ has a solution with $l = 1$.

It can be shown that, under these conditions, $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ has a solution. Indeed, let p be such that $EBp = DBp$ and that p has no $\perp$ entries. Choose $\lambda \geq \rho(A)$ huge enough so

$$Bp \succeq (\lambda^{-1}A)^k Bp \ \forall \ k = 1, 2, ..., n. \tag{4.4.11}$$

Such thing is possible with a finite λ because B has no row full of $\perp$ entries and p has no $\perp$ entries, so the vector Bp has no $\perp$ entries. Then, due to Equation (4.4.11), it is clear that $Bp = (\lambda^{-1}A)^* Bp$, and therefore $\chi = \lambda^{-1}(\lambda^{-1}A)^* Bp$, $\mu = p$ and this λ form a solution triple to $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$. Hence, by Proposition 4.4.1 and Proposition 4.4.2 (see the *If* part of Proposition 4.4.3), $\mathscr{R}(A, B, E, D)$ has a solution. □

Remark 4.4.3. As a final note, if the proposed regulator problem is weakened so the initial condition $x[0]$ can be chosen, then this weakened problem is solvable if and only if $\mathfrak{S}(\mathscr{R})$ has a solution, no matter if the problem is controllable non-critical or not. This is because $\mathfrak{S}(\mathscr{R})$ is also necessary for this kind of problem (Proposition 4.3.1). It is also sufficient because one can simple choose $x[0] = \alpha\chi \in \mathscr{X}_{\mathtt{cons}}(\mathscr{R})$ for any scalar α. The (open-loop) controller $u[k]$ can be taken as $u[k] = \alpha\lambda^k\mu$. If a closed-loop controller is desired, one can take $u[k] = Fx[k] = \mu v^T$ in which $v^T\chi = 0$ (which obviously exists as long as $\chi \neq \perp$). □

Then

Algorithm 4.4.1. *Spectral Regulator for solving* $\mathscr{R}$

1. Find a proper solution $\{\lambda, \chi, \mu\}$ to $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ (see Remark 4.3.8);
2. Solve $\mathfrak{F}(\mathscr{R}, \lambda, \chi, \mu)$ for ζ (suggestion: $\zeta^T = (-\chi)^T(\lambda^{-1}A)^*$);
3. Compute $F_{\mathtt{SR}} = \mu\zeta^T$;
4. Use as the controller $u[k] = F_{\mathtt{SR}}x[k]$.

4.5 The Feedback Accelerator

4.5.1 Motivation

The convergence speed of the proposed controller in terms of number of events is related to the number $\kappa(\lambda^{-1}A)$, as shown in Proposition 4.4.1. The convergence number plays then a role similar to the eigenvalues of the matrix A in the traditional system theory. An enticing idea would be improving this convergence number, in order to hasten the system. This section will pursue this endeavour.

One simple idea to try to improve the convergence number is to "pre-close" the loop with a linear feedback, while still keeping a free term on the control input: $u[k] = F_{\mathtt{FA}}x[k] \oplus g[k]$. This free term can be used to implement the Spectral Regulator, for example. Thus, using this input on the system given by Equation (4.2.1):

$$S_{\mathtt{FA}}: \; x[k+1] = (A \oplus BF_{\mathtt{FA}})x[k] \oplus Bg[k]. \tag{4.5.1}$$

Let $A_{\mathtt{FA}} \equiv A \oplus BF_{\mathtt{FA}}$. The objective is thus design $F_{\mathtt{FA}}$ in a way that the convergence number $\kappa(\lambda^{-1}A_{\mathtt{FA}})$ is as smaller as possible.

Ideally, $A \oplus BF_{\mathtt{FA}} = \alpha Q^*$ for a matrix Q and a scalar α. Indeed, if $\lambda \geq \alpha$, the convergence number $\kappa(\lambda^{-1}(A \oplus BF_{\mathtt{FA}}))$ would be 1 (the smallest possible, except for the very particular case of a diagonal matrix with non-positive entries, which has convergence number of 0), and the Spectral Regulator would be very efficient. However, this equation for the unknowns $F_{\mathtt{FA}}, Q, \alpha$ is highly (tropical) non-linear and thus hard to solve.

One can proceed by using $\alpha = \lambda$ and $Q = (\lambda^{-1}A)$, with λ being the same value used in the design of the desired Spectral Regulator $F_{\mathtt{SR}}$. The resulting equation is now tropical affine and easily solvable, if it has a solution. However, frequently such solution does not exists. Thus, one possibility is to weaken the problem by finding the greatest $F_{\mathtt{FA}}$ solution to the inequation

$$A_{\mathtt{FA}} = A \oplus BF_{\mathtt{FA}} \preceq \lambda(\lambda^{-1}A)^* \tag{4.5.2}$$

(and if a solution to the non-weakened inequation exists, such $F_{\mathtt{FA}}$ will also be found by solving this weakened inequation. See Baccelli et al. (1992)). Since naturally $\lambda(\lambda^{-1}A)^* \succeq A$, Equation (4.5.2) is equivalent to $BF_{\mathtt{FA}} \preceq \lambda(\lambda^{-1}A)^*$.

However, another possible concern is that $F_{\mathtt{FA}}$ must be a *causal* matrix: with only non-negative or $\perp$ entries. Let $\mathscr{F}$ be the set of all such causal matrices. Therefore, the problem asks for

$$\max_{F_{\mathtt{FA}} \in \mathscr{F}} BF_{\mathtt{FA}} \preceq \lambda(\lambda^{-1}A)^{*}. \tag{4.5.3}$$

Since the set of causal matrices is closed under (tropical) addition (if $F[1], F[2] \in \mathscr{F}$, so is $F[1] \oplus F[2]$), Problem 4.5.3 has a solution. To show this, it is necessary to define the concept of *causal projection*.

Definition 4.5.1. *(Causal projection)* The causal projection of a matrix F, denoted by $C_{\mathrm{cp}}(F)$, is obtained from F by exchanging all negative entries to $\perp$. □

Then, one proceeds by computing $B \backslash (\lambda(\lambda^{-1}A)^{*})$ (that is, solve by residuation Problem 4.5.3 disregarding the causality constraint) and applying the causal projection to this matrix. Thus

$$F_{\mathtt{FA}} = C_{\mathrm{cp}}(B \backslash (\lambda(\lambda^{-1}A)^{*})). \tag{4.5.4}$$

4.5.2 Effects on the system

A natural question is how this linear feedback (Equation (4.5.4)) affects the system, that is, if it is really capable of improving the system performance. It can be proved that this approach at least maintains the convergence number. This can be shown using the following result.

Lemma 4.5.1. *(Inequality in the convergence number)* Let $X \succeq Y$, $X \preceq Y^{*}$, with $\rho(X), \rho(Y) \leq 0$. Then $\kappa(X) \preceq \kappa(Y)$.

Proof. Let $t \equiv \kappa(Y)$. Then $Y^{*} = \bigoplus_{i=0}^{t} Y^{i}$. Since $X \succeq Y$, $\bigoplus_{i=0}^{t} X^{i} \succeq \bigoplus_{i=0}^{t} Y^{i}$. By $Y^{*} \succeq X$, one can conclude that [3] $(Y^{*})^{k} = Y^{*} \succeq X^{k}$. Finally, one has that $\bigoplus_{i=0}^{t} X^{i} \succeq \bigoplus_{i=0}^{t} Y^{i} = Y^{*} \succeq X^{k}$, for any k. This final conclusion, $\bigoplus_{i=0}^{t} X^{i} \succeq X^{k}$, implies that $X^{*} = \bigoplus_{i=0}^{t} X^{i}$. Then, $\kappa(X)$ is at most t and the proof is complete. □

Using $Y = \lambda^{-1}A$, $X = \lambda^{-1}A_{\mathtt{FA}}$, one can see by using Proposition 4.5.1 that, for any $A_{\mathtt{FA}}$ such that Equation (4.5.2) holds, $\kappa(\lambda^{-1}A_{\mathtt{FA}}) \preceq \kappa(\lambda^{-1}A)$. Thus, as claimed, the feedback approach, in the worst case, maintains the convergence number. Hence

Corollary 4.5.1. *(Feedback Accelerator is never deleterious to the convergence number)* For any Feedback Accelerator $F_{\mathtt{FA}}$ such that Equation (4.5.2) holds (in special, the one in Equation (4.5.4)), the convergence number of the closed loop $A \oplus BF_{\mathtt{FA}}$ is always less than or equal to the original one A. □

[3] Recall a basic property of Kleene Closures (see Baccelli et al. (1992)): for any natural $k > 0$, $(Y^{*})^{k} = Y^{*}$.

4.5.3 Spectral Regulator and Feedback Accelerator

Given the Spectral Regulator matrix $F_{\mathtt{SR}}$ designed for the system given by Equation (4.2.1), a natural question would be if it works for the accelerated system given by Equation (4.5.1), with $F_{\mathtt{FA}}$ given as in Equation (4.5.4). The answer is, yes. This is true because of the following lemma.

Lemma 4.5.2. *(Parameters also solve other equations)*
Let $\{\lambda, \chi, \mu\}$ be a solution of $\mathfrak{S}_{\mathtt{str}}(\mathscr{R}(A,B,E,D))$ and ζ of $\mathfrak{F}(\mathscr{R}(A,B,E,D),\lambda,\chi,\mu)$. Let $A_{\mathtt{FA}} = A \oplus BF_{\mathtt{FA}}$, with

$$F_{\mathtt{FA}} = C_{\mathtt{cp}}(B \diagdown (\lambda(\lambda^{-1}A)^*)). \tag{4.5.5}$$

Then, $\{\lambda, \chi, \mu\}$ also solves $\mathfrak{S}_{\mathtt{str}}(\mathscr{R}(A_{\mathtt{FA}},B,E,D))$ and ζ also solves $\mathfrak{F}(\mathscr{R}(A_{\mathtt{FA}},B,E,D),\lambda,\chi,\mu)$.

Proof. It is sufficient to show that $(\lambda^{-1}A)^* = (\lambda^{-1}A_{\mathtt{FA}})$ and also that, if $\zeta^T A \preceq \lambda\zeta$ then $\zeta^T A_{\mathtt{FA}} \preceq \lambda\zeta$. By inspection, the former fact implies the solution for $\mathfrak{S}_{\mathtt{str}}(\mathscr{R}(A_{\mathtt{FA}},B,E,D))$ and both the former and the latter the solution for $\mathfrak{F}(\mathscr{R}(A_{\mathtt{FA}},B,E,D),\lambda,\chi,\mu)$.

To show the former affirmation, let $Y = \lambda^{-1}A$, $X = \lambda^{-1}A_{\mathtt{FA}}$. So as in Proposition 4.5.1, $X \succeq Y$, $X \preceq Y^*$. Due to the monotonicity of the Kleene Closure, applying it to both sides one concludes from the first Equation that [4] $X^* \succeq Y^*$ and from the second $X^* \preceq Y^*$. Thus $X^* = Y^*$ and then $(\lambda^{-1}A)^* = (\lambda^{-1}A_{\mathtt{FA}})^*$.

To show the latter, one notes that $\zeta^T A \preceq \lambda\zeta$ is equivalent to $\zeta^T = \zeta^T(\lambda^{-1}A)^*$. By the previous result, this implies $\zeta^T = \zeta^T(\lambda^{-1}A_{\mathtt{FA}})^*$, and hence that $\zeta^T A_{\mathtt{FA}} \preceq \lambda\zeta$. And the lemma is proved.

□

Lemma 4.5.2 together with Proposition 4.4.1 imply the following corollary.

Corollary 4.5.2. *(Spectral Regulator and Feedback Accelerator)* Let $\{\lambda, \chi, \mu\}$ be a solution of $\mathfrak{S}_{\mathtt{str}}(\mathscr{R}(A,B,E,D))$ and ζ of $\mathfrak{F}(\mathscr{R}(A,B,E,D),\lambda,\chi,\mu)$. Let $A_{\mathtt{FA}} = A \oplus B_{\mathtt{FA}}$, with

$$F_{\mathtt{FA}} = C_{\mathtt{cp}}(B \diagdown (\lambda(\lambda^{-1}A)^*)). \tag{4.5.6}$$

Then $F_{\mathtt{SR}} = \mu\zeta^T$ solves both $\mathscr{R}(A,B,E,D)$ and $\mathscr{R}(A_{\mathtt{FA}},B,E,D)$. By consequence, the action of both feedbacks can be implemented simply by using $F = F_{\mathtt{SR}} \oplus F_{\mathtt{FA}}$. □

[4] Another basic property of Kleene Closures (see Baccelli et al. (1992)): $(Y^*)^* = Y^*$.

Then

Algorithm 4.5.1. *Spectral Regulator with Feedback Accelerator for solving $\mathscr{R}$*

1. Find a proper solution $\{\lambda, \chi, \mu\}$ to $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ (see Remark 4.3.8);
2. Compute $F_{\mathtt{SR}}$ (See Algorithm 4.4.1);
3. Compute $F_{\mathtt{FA}} = C_{\mathtt{cp}}(B \backslash\!\!\!\circ (\lambda(\lambda^{-1}A)^*))$;
4. Use as the controller $u[k] = (F_{\mathtt{SR}} \oplus F_{\mathtt{FA}})x[k]$.

4.6 Causality

Causality is a concern for the applicability of any controller. A control rule as $u_1[k] = -4 + x_2[k]$ would mean that u_1 must fire at the k^{th} time 4 time units before x_2 fire at the k^{th} time. Then, in order to decide when to fire, it is necessary to foresee when x_2 fires (contrast with, for example, $u_1[k] = 1 + x_2[k]$, in which u_1 fires 1 time unit *after* x_2 fired, and thus it is possible to wait when the firing of x_2 happened to decide when fire u_1).

Thus, solving $\mathscr{R}$ may not be sufficient to solve the practical problem because the feedback $F_{\mathtt{SR}}$ may be non-causal, rendering it unapplicable. However, it turns out that this non-causal feedback $F_{\mathtt{SR}}$ may be used for finding a causal one.

In closed loop, with the feedback $F_{\mathtt{SR}} = \mu\zeta^T$, one has that

$$x[k+1] = Ax[k] \oplus B\mu(\zeta^T x[k]). \tag{4.6.1}$$

Consider the reduced system $S_{\mathtt{red}}$ obtained from S:

$$S_{\mathtt{red}}(S, \mu) \; : \; x[k+1] = Ax[k] \oplus B\mu v[k]. \tag{4.6.2}$$

Note that the matrix B was replaced by the matrix $B\mu$ and now the control action $v[k]$ is a scalar. Before continuing, a definition is necessary.

Definition 4.6.1. *(Extended controllability matrices)* Given a system S, the r^{th} *extended controllability matrix* $K_r(S)$ is defined recursively as

$$K_{r+1}(S) \equiv (AK_r(S)\ B) \tag{4.6.3}$$

with $K_0(S) \equiv I$. □

Remark 4.6.1. Note that the extended controllability matrices are just the traditional controllability matrices (in the tropical setting) with the addition of the terms A^n. □

Then

Lemma 4.6.1. *(Order for the image of the extended controllability matrices)* If $s \geq r$ then $Im\{K_s(S)\} \subseteq Im\{K_r(S)\}$.

Proof. It suffices to prove the case in which $s = r + 1$. The full proof then follows by repeated application of this result.

Indeed, let

$$x \in Im\{K_{r+1}(S)\} \iff x = A^{r+1}x[0] \oplus \bigoplus_{i=0}^{r} A^{r-i}Bu[i]. \tag{4.6.4}$$

for a given $x[0]$ and control sequence $u[i]$. Then

$$x = A^r(Ax[0] \oplus Bu[0]) \oplus \bigoplus_{i=0}^{r-1} A^{r-1-i}Bu[i+1] = A^r x'[0] \oplus \bigoplus_{i=0}^{r-1} A^{r-1-i}Bu'[i] \in Im\{K_r(S)\}. \tag{4.6.5}$$

And the proof is complete.

□

Also

Lemma 4.6.2. *(Law of sum of exponents)* Let

$$\zeta^T_{\texttt{alt}} K_t(S_{\texttt{red}}(S,\mu)) = \zeta^T K_t(S_{\texttt{red}}(S,\mu)). \tag{4.6.6}$$

Then

$$(A \oplus B\mu\zeta^T)(A \oplus B\mu\zeta^T_{\texttt{alt}})^t = (A \oplus B\mu\zeta^T_{\texttt{alt}})^{t+1}. \tag{4.6.7}$$

Proof. Then

$$(A \oplus B\mu\zeta^T)(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^t = A(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^t \oplus B\mu(\zeta^T(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^t). \tag{4.6.8}$$

Clearly, $(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^t \in Im\{K_t(S_{\mathtt{red}}(S,\mu))\}$ (this can be seen by expanding $(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^t$). Hence, by Equation (4.6.6)

$$\zeta^T(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^t = \zeta_{\mathtt{alt}}^T(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^t. \tag{4.6.9}$$

And therefore

$$A(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^t \oplus B\mu(\zeta^T(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^t) = A(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^t \oplus B\mu(\zeta_{\mathtt{alt}}^T(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^t) = (A \oplus B\mu\zeta_{\mathtt{alt}}^T)(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^t = (A \oplus B\mu\zeta_{\mathtt{alt}}^T)^{t+1}. \tag{4.6.10}$$

Thus

$$(A \oplus B\mu\zeta^T)(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^t = (A \oplus B\mu\zeta_{\mathtt{alt}}^T)^{t+1}. \tag{4.6.11}$$

And the lemma is proved.

□

And, finally, the main result of this section can be stated.

Proposition 4.6.1. *(Another solution)* Let $F_{\mathtt{SR}} = \mu\zeta^T$ and $l = s$ be a solution to Equation (4.4.1). Let $\zeta_{\mathtt{alt}}$ be such that

$$\zeta_{\mathtt{alt}}^T K_r(S_{\mathtt{red}}(S,\mu)) = \zeta^T K_r(S_{\mathtt{red}}(S,\mu)) \tag{4.6.12}$$

for a r. Then $F_{\mathtt{SRalt}} = \mu\zeta_{\mathtt{alt}}^T$ and $l = s + r$ also solve Equation (4.4.1), implying that the feedback law $u[k] = F_{\mathtt{SRalt}}x[k]$ solves $\mathscr{R}$.

Proof. By hypothesis

$$E(A \oplus B\mu\zeta^T)^s = D(A \oplus B\mu\zeta^T)^s. \tag{4.6.13}$$

Post-multiply both sides by $(A \oplus B\mu\zeta_{\mathtt{alt}}^T)^r$

$$E(A\oplus B\mu\zeta^T)^s(A\oplus B\mu\zeta_{\mathtt{alt}}^T)^r = D(A\oplus B\mu\zeta^T)^s(A\oplus B\mu\zeta_{\mathtt{alt}}^T)^r. \tag{4.6.14}$$

Now, it will be proved that, if Equation (4.6.12) holds, then

$$(A\oplus B\mu\zeta^T)^s(A\oplus B\mu\zeta_{\mathtt{alt}}^T)^r = (A\oplus B\mu\zeta_{\mathtt{alt}}^T)^{s+r} \tag{4.6.15}$$

which is clearly sufficient to complete the proof.

First, by consequence of Lemma 4.6.1, $\zeta_{\mathtt{alt}}^T K_r(S_{\mathtt{red}}(S,\mu)) = \zeta^T K_r(S_{\mathtt{red}}(S,\mu))$ implies $\zeta_{\mathtt{alt}}^T K_t(S_{\mathtt{red}}(S,\mu)) = \zeta^T K_t(S_{\mathtt{red}}(S,\mu))$ for any $t \geq r$. Equation (4.6.15) can then be proved by a repeated application of Lemma 4.6.2. Suppose $s \geq 1$ (otherwise the proof is trivial). Then $(A\oplus B\mu\zeta^T)^s = (A\oplus B\mu\zeta^T)^{s-1}(A\oplus B\mu\zeta^T)$ and thus

$$(A\oplus B\mu\zeta^T)^s(A\oplus B\mu\zeta_{\mathtt{alt}}^T)^r = (A\oplus B\mu\zeta^T)^{s-1}((A\oplus B\mu\zeta^T)(A\oplus B\mu\zeta_{\mathtt{alt}}^T)^r). \tag{4.6.16}$$

Using Lemma 4.6.2, $(A\oplus B\mu\zeta^T)(A\oplus B\mu\zeta_{\mathtt{alt}}^T)^r = (A\oplus B\mu\zeta_{\mathtt{alt}}^T)^{r+1}$ and hence

$$(A\oplus B\mu\zeta^T)^s(A\oplus B\mu\zeta_{\mathtt{alt}}^T)^r = (A\oplus B\mu\zeta^T)^{s-1}(A\oplus B\mu\zeta_{\mathtt{alt}}^T)^{r+1}. \tag{4.6.17}$$

If $s-1 \geq 1$ (otherwise the proof is complete), one can apply the same procedure again (remember that by Lemma 4.6.2, $\zeta_{\mathtt{alt}}^T K_r(S_{\mathtt{red}}(S,\mu)) = \zeta^T K_r(S_{\mathtt{red}}(S,\mu))$ implies $\zeta_{\mathtt{alt}}^T K_t(S_{\mathtt{red}}(S,\mu)) = \zeta^T K_t(S_{\mathtt{red}}(S,\mu))$ for any $t \geq r$ and thus in particular for $t = r+1$) to conclude that

$$(A\oplus B\mu\zeta^T)^{s-1}(A\oplus B\mu\zeta_{\mathtt{alt}}^T)^{r+1} = (A\oplus B\mu\zeta^T)^{s-2}(A\oplus B\mu\zeta_{\mathtt{alt}}^T)^{r+2} \tag{4.6.18}$$

and so on. Then one can conclude that Equation (4.6.15) holds and the proof is complete

□

Proposition 4.6.1 is a powerful result for finding, from a non-causal feedback, a causal solution. Suppose, without loss of generality, that the vector μ solution to $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ has the smallest non-$\perp$ entry as 0 (otherwise, just take $\mu' = \mu_i^{-1}\mu$, in which i is the index with the smallest non-$\perp$ entry). Then, in order to the feedback $F_{\mathtt{SRalt}} = \mu\zeta_{\mathtt{alt}}^T$ be causal, it suffices that $\zeta_{\mathtt{alt}}$ is causal. Thus, given a fixed r and a ζ, one can find a solution to the following tropical affine equation $\zeta_{\mathtt{alt}}^T$

$$\zeta_{\mathtt{alt}}^T K_r(S_{\mathtt{red}}(S,\mu)) = \zeta^T K_r(S_{\mathtt{red}}(S,\mu)) \tag{4.6.19}$$

with the constraint that $\zeta^T_{\mathtt{alt}}$ is causal, that is $\zeta^T_{\mathtt{alt}} = C_{\mathtt{cp}}(\zeta^T_{\mathtt{alt}})$. The following result is then auspicious.

Lemma 4.6.3. *(Causal solution to one-sided tropical affine equations)* The system of equations $Ux = v$ and $x = C_{\mathtt{cp}}(x)$ has a solution if and only if $UC_{\mathtt{cp}}(U \circ\!\!\!\backslash v) = v$. Further, $x = C_{\mathtt{cp}}(U \circ\!\!\!\backslash v)$ is the greatest solution.

Proof. The *if* part along with the conclusion that it is the greatest one is trivial, since $x \preceq U \circ\!\!\!\backslash v$ and hence, by applying the causal projection (which is non-decreasing), $x = C_{\mathtt{cp}}(x) \preceq C_{\mathtt{cp}}(U \circ\!\!\!\backslash v)$. The *only if* comes as follows. Suppose there is a solution y, that is, $Uy = v$ and $C_{\mathtt{cp}}(y) = y$. Hence, $y \preceq U \circ\!\!\!\backslash v$ and, by applying the causal projection (which is non-decreasing) and using the fact that $C_{\mathtt{cp}}(y) = y$, one concludes that $y \preceq C_{\mathtt{cp}}(U \circ\!\!\!\backslash v)$. Then, multiplying by U one has that $Uy = v \preceq UC_{\mathtt{cp}}(U \circ\!\!\!\backslash v)$.

Now, $C_{\mathtt{cp}}(U \circ\!\!\!\backslash v) \preceq U \circ\!\!\!\backslash v$ (because $C_{\mathtt{cp}}(x) \preceq x$ holds for any x). Multiplying by U in both sides $UC_{\mathtt{cp}}(U \circ\!\!\!\backslash v) \preceq U(U \circ\!\!\!\backslash v) \preceq v$. And hence $UC_{\mathtt{cp}}(U \circ\!\!\!\backslash v) = v$, which implies that $C_{\mathtt{cp}}(U \circ\!\!\!\backslash v)$ is also a solution. And the proof is complete. □

Hence

Algorithm 4.6.1. *Causalisation of feedbacks*

1. Find a feedback $F_{\mathtt{SR}} = \mu\zeta^T$ solution to Equation (4.4.1). Ensure that the smallest non-$\perp$ entry of μ is 0 and hence μ is causal;

2. Find r such that $\zeta^T_{\mathtt{alt}} K_r(S_{\mathtt{red}}(S,\mu)) = \zeta^T K_r(S_{\mathtt{red}}(S,\mu))$ in which $\zeta^T_{\mathtt{alt}} = C_{\mathtt{cp}}((\zeta^T K_r(S_{\mathtt{red}}(S,\mu))) \circ\!\!\!/ K_r(S_{\mathtt{red}}(S,\mu)))$;

3. Use as a causal feedback $F_{\mathtt{SRalt}} = \mu\zeta^T_{\mathtt{alt}}$.

Note that the causalisation may interfere in the closed-loop system behaviour, since it increases the parameter l in Equation (4.4.1) (and thus it may increase the number of steps taken for convergence). Such is the price one needs to pay to have causality.

Finally, it may be the case that the causalisation procedure is unnecessary. Indeed, in the next chapter the implementation of *observers* will be discussed. Frequently, in practice one does not have all the states $x[k]$ measured, but just a combination of them in variables $y[k]$ (the outputs). The next chapter will discuss how to implement the feedback law $u[k] = Fx[k]$ with just the measured variables, and it will be shown that there are cases in which the observer implementation of this feedback law is causal even if the one of F is not. This will be observed in the practical implementation in Chapter 6.

4.7 Critical problems and faux-criticality

Proposition 4.4.3 gives a sufficient and necessary condition for the so-called controllable non-critical problem. The question then is how broad this class of problems is. Frequently, the system designer requires that the system works as fast as possible, and this requirement is imbued in the constraint set $\mathscr{X}_{\mathtt{cons}}$. Faster as possible may imply, if possible, $\lambda = \rho(A)$, and then the problem is controllable critical. In this case, in principle, the proposed methodology is not applicable. However, problems $\mathscr{R}$ in which $\Lambda(\mathscr{R}) = \{\rho(A)\}$ is a consequence of tight control constraints are only *controllable faux-critical*: one can construct a controllable non-critical problem $\mathscr{R}_\delta$ arbitrarily close to the original problem by relaxing the constraints as much as required. In this case, an infinitesimal relaxation in the constraints causes the controllable characteristic spectrum $\Lambda(\mathscr{R}_\delta)$ to contain something different from $\rho(A)$ and hence the problem is controllable non-critical. By taking the limit when $\mathscr{R}_\delta \to \mathscr{R}$, one concludes that this class of controllable critical problems is, in practice, controllable non-critical (hence the adjective *faux*) and the proposed methodology can be applied.

A simple example of such problem is the following.

$$x[k+1] = \begin{pmatrix} 1 & \perp \\ 0 & \perp \end{pmatrix} x[k] \oplus \begin{pmatrix} 0 \\ \perp \end{pmatrix} u[k] \tag{4.7.1}$$

with the constraint $(0 \ \perp)x[k] = (\perp \ \ 1)x[k]$ $(x_1[k]-x_2[k] = x_1[k]-x_1[k-1] = 1)$. In this case, the only possible choice of λ in $\mathfrak{S}(\mathscr{R})$ is $\lambda = \rho(A) = 1$ and the problem is controllable critical. However, the problem is controllable faux-critical. Indeed, the weakened constraints $\delta(0 \ \perp)x[k] \succeq (\perp \ 1)x[k]$ and $\delta(\perp \ \ 1)x[k] \succeq (0 \ \ \perp)x[k]$ imply that any $1 \leq \lambda \leq 1+\delta$ is possible for solving $\mathfrak{S}(\mathscr{R}_\delta)$, and for any $\delta > 0$ (no matter how small) the problem is controllable non-critical and hence solvable by the proposed methodology.

For controllable faux-critical problems, one must solve $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ instead of $\mathfrak{S}(\mathscr{R})$ in order to obtain χ and μ. In this case, criticality is mainly due to very tight constraints.

Definition 4.7.1. *(Controllable faux-critical and controllable structurally critical problems)*: A problem

$\mathscr{R}$, is said to be *controllable faux-critical* if it is controllable critical and $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ has a proper solution. If it is controllable critical but not controllable faux-critical it will be called *controllable structurally critical*. □

Controllable structurally critical problems do exist. For instance, consider the problem

$$x[k+1] = \begin{pmatrix} 1 & \perp \\ \perp & 2 \end{pmatrix} x[k] \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} u[k] \tag{4.7.2}$$

with the constraint $x_2[k] - x_1[k] = 1$. This problem is controllable critical ($\Lambda(\mathscr{R}) = \{\rho(A)\} = \{2\}$) and solvable, but no slight relaxation of the constraint will make the characteristic spectrum change. Controllable structurally critical problems require something more general than the Spectral Regulator. This issue will be discussed in a future work.

4.8 An illustrative problem

In order to better illustrate the methodology, the control problem proposed (and solved, with another methodology) Attia et al. (2010) will be used as example. It represents a cluster tool operating the manufacture of semiconductor wafers (details can be found in Attia et al. (2010)).

The original dynamical equations in Attia et al. (2010) are (see Figure 4.1)

$$\begin{aligned}
x_1[k+1] &= w x_8[k] \oplus \tau_1 x_2[k] \oplus u_1[k]; \\
x_2[k+1] &= s x_1[k+1]; \\
x_3[k+1] &= v x_2[k+1] \oplus \tau_2 x_8[k]; \\
x_4[k+1] &= s x_3[k+1]; \\
x_5[k+1] &= w x_4[k+1] \oplus \tau_1 x_6[k] \oplus u_3[k]; \\
x_6[k+1] &= s x_5[k+1]; \\
x_7[k+1] &= v x_6[k+1] \oplus \tau_2 x_4[k+1] \oplus u_4[k]; \\
x_8[k+1] &= s x_7[k+1].
\end{aligned} \tag{4.8.1}$$

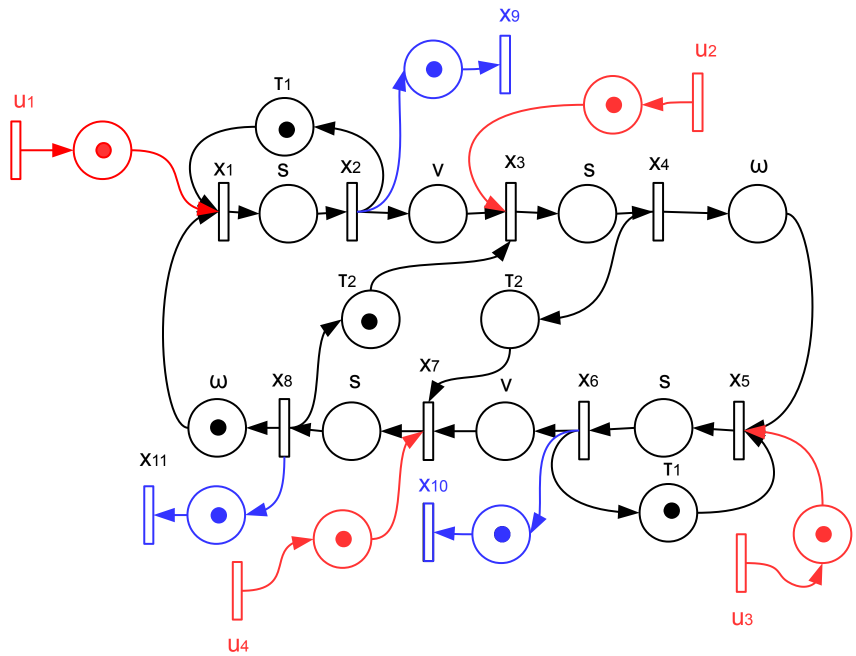

Figure 4.1: TEG for the model. The numbers above places represents delays (no number means no delay). The red transitions/places represents control inputs, while the blue ones delayed variables.

The constraints as described in Attia et al. (2010) are

$$x_1[k] \leq \tau_1 d_1 x_2[k-1]; \tag{4.8.2}$$
$$x_5[k] \leq \tau_1 d_1 x_6[k-1];$$
$$x_3[k] \leq \tau_2 d_2 x_8[k-1];$$
$$x_7[k] \leq \tau_2 d_2 x_4[k].$$

In order to write the constraints in Equation (4.8.4) in the form $Ex = Dx$, it is necessary to introduce in Equation (4.8.3) three new states and three new dynamical equations:

$$x_9[k+1] = x_2[k]; \tag{4.8.3}$$
$$x_{10}[k+1] = x_6[k];$$
$$x_{11}[k+1] = x_8[k]$$

and hence

$$x_1[k] \leq \tau_1 d_1 x_9[k]; \tag{4.8.4}$$
$$x_5[k] \leq \tau_1 d_1 x_{10}[k];$$
$$x_3[k] \leq \tau_2 d_2 x_{11}[k];$$
$$x_7[k] \leq \tau_2 d_2 x_4[k].$$

Further, from Equation (4.8.3), one can also infer the following structural constraints

$$x_2[k] \geq s x_1[k]; \tag{4.8.5}$$
$$x_3[k] \geq v x_2[k];$$
$$x_4[k] \geq s x_3[k];$$
$$x_5[k] \geq w x_4[k];$$
$$x_6[k] \geq s x_5[k];$$
$$x_7[k] \geq v x_6[k] \oplus \tau_2 x_4[k];$$
$$x_8[k] \geq s x_7[k].$$

The nominal values being $w = 4, v = 1, s = 2, \tau_1 = 22, \tau_2 = 9, d_1 = 1, d_2 = 1$.

In order to make the problem controllable coupled, one more constraint (absent from Attia et al. (2010)) will be posed in the system: $x_1[k] \geq (-100)x_8[k]$. Note that this constraint is mostly innocuous to the system, and will only be used to guarantee that the problem is controllable coupled.

Then, the matrices $A \in \mathbb{T}_{max}^{11\times 11}$ and $B \in \mathbb{T}_{max}^{11\times 4}$, as in Equation (4.2.1), can be obtained

$$A = \begin{pmatrix} \perp & 22 & \perp & \perp & \perp & \perp & \perp & 4 & \perp & \perp & \perp \\ \perp & 24 & \perp & \perp & \perp & \perp & \perp & 6 & \perp & \perp & \perp \\ \perp & 25 & \perp & \perp & \perp & \perp & \perp & 9 & \perp & \perp & \perp \\ \perp & 27 & \perp & \perp & \perp & \perp & \perp & 11 & \perp & \perp & \perp \\ \perp & 31 & \perp & \perp & \perp & 22 & \perp & 15 & \perp & \perp & \perp \\ \perp & 33 & \perp & \perp & \perp & 24 & \perp & 17 & \perp & \perp & \perp \\ \perp & 36 & \perp & \perp & \perp & 25 & \perp & 20 & \perp & \perp & \perp \\ \perp & 38 & \perp & \perp & \perp & 27 & \perp & 22 & \perp & \perp & \perp \\ \perp & 0 & \perp & \perp & \perp & \perp & \perp & \perp & \perp & \perp & \perp \\ \perp & \perp & \perp & \perp & \perp & 0 & \perp & \perp & \perp & \perp & \perp \\ \perp & \perp & \perp & \perp & \perp & \perp & \perp & 0 & \perp & \perp & \perp \end{pmatrix}; \tag{4.8.6}$$

$$B^T = \begin{pmatrix} 0 & 2 & 3 & 5 & 9 & 11 & 14 & 16 & \perp & \perp & \perp \\ \perp & \perp & 0 & 2 & 6 & 8 & 11 & 13 & \perp & \perp & \perp \\ \perp & \perp & \perp & \perp & 0 & 2 & 3 & 5 & \perp & \perp & \perp \\ \perp & \perp & \perp & \perp & \perp & \perp & 0 & 2 & \perp & \perp & \perp \end{pmatrix}. \tag{4.8.7}$$

Further, the constraints can be written as $x[k] \succeq Qx[k]$, and hence as $x[k] = Q^* x[k]$. Thus, $E = I$ and $D = Q^*$:

$$Q^* = \begin{pmatrix} 0 & -86 & -87 & -89 & -95 & -97 & -98 & -100 & 22 & -73 & -78 \\ 2 & 0 & -85 & -87 & -93 & -95 & -96 & -98 & 24 & -71 & -76 \\ 3 & 1 & 0 & -86 & -92 & -94 & -95 & -97 & 25 & -70 & 9 \\ 5 & 3 & 2 & 0 & -7 & -9 & -10 & -95 & 27 & 15 & 11 \\ 9 & 7 & 6 & 4 & 0 & -5 & -6 & -91 & 31 & 22 & 15 \\ 11 & 9 & 8 & 6 & 2 & 0 & -4 & -89 & 33 & 24 & 17 \\ 14 & 12 & 11 & 9 & 3 & 1 & 0 & -86 & 36 & 25 & 20 \\ 16 & 14 & 13 & 11 & 5 & 3 & 2 & 0 & 38 & 27 & 22 \\ -23 & -109 & -110 & -112 & -118 & -120 & -121 & -123 & 0 & -96 & -101 \\ -14 & -16 & -17 & -19 & -23 & -28 & -29 & -114 & 8 & 0 & -8 \\ -7 & -9 & -10 & -96 & -102 & -104 & -105 & -107 & 15 & -80 & 0 \end{pmatrix} \tag{4.8.8}$$

and then the problem is clearly controllable coupled (see Remark 4.3.2).

The first step in Algorithm 4.6.1 is solving $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$. For that, one assumes first that $\lambda > \rho(A) = 24$ (see Remark 4.3.8), and then $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ is equivalent to $\mathfrak{S}(\mathscr{R})$, which in turn can be transformed in Equation (4.3.7) in order to write it as the two sided eigenproblem $Uy = \lambda V y$. Using the method described in Gaubert and Sergeev (2013), it is possible to find that (see Remark 4.3.8)

$$s_{\mathscr{R}}(\lambda) = \begin{cases} \frac{24-\lambda}{5} & \text{for } 24 \leq \lambda \leq 26.5; \\ \frac{25-\lambda}{3} & \text{for } 26.5 \leq \lambda. \end{cases} \tag{4.8.9}$$

Thus, the only possible λ to choose ($s_{\mathscr{R}}(\lambda) = 0$) is $\lambda = \rho(A) = 24$, and the problem is controllable critical. In this case, Equation $\mathfrak{S}(\mathscr{R})$ is only necessary, not sufficient. Equation $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ must have a solution (with $\lambda = 24$) to the problem be solvable with the proposed methodology.

Then, it is possible (with $\lambda = 24$) to solve the tropical linear equation $\mathfrak{S}_{\mathtt{str}}(\mathscr{R})$ (see Chapter 2), implying that the proposed problem is faux-critical, to find as a solution

$$\mu^T = (0\ 3\ 12\ 12); \tag{4.8.10}$$

$$\chi^T = (-24\ -22\ -21\ -19\ -12\ -10\ -9\ -7\ -46\ -34\ -31). \tag{4.8.11}$$

Solving $\mathfrak{F}(\mathscr{R}, \lambda, \chi, \mu)$ with the suggestion $\zeta^T = (-\chi)^T(\lambda^{-1}A)^*$ (see Algorithm 4.6.1) it is possible to obtain

$$\zeta^T = (24\ 22\ 21\ 19\ 12\ 10\ 9\ 7\ 46\ 34\ 31). \tag{4.8.12}$$

Both μ and ζ are causal vectors, so the controller is causal. Indeed

$$F_{\mathtt{SR}} = \mu\zeta^T = \begin{pmatrix} 24 & 22 & 21 & 19 & 12 & 10 & 9 & 7 & 46 & 34 & 31 \\ 27 & 25 & 24 & 22 & 15 & 13 & 12 & 10 & 49 & 37 & 34 \\ 36 & 34 & 33 & 31 & 24 & 22 & 21 & 19 & 58 & 46 & 43 \\ 36 & 34 & 33 & 31 & 24 & 22 & 21 & 19 & 58 & 46 & 43 \end{pmatrix}. \tag{4.8.13}$$

The Feedback Accelerator can then be computed (see Algorithm 4.6.1). The original convergence number is $\kappa((-24)A) = 3$. Computing $F_{\mathtt{FA}}$ one has

$$F_{\mathtt{FA}} = \begin{pmatrix} \perp & 22 & \perp & \perp & \perp & 7 & \perp & 4 & \perp & \perp & \perp \\ \perp & 25 & \perp & \perp & \perp & 12 & \perp & 9 & \perp & \perp & \perp \\ \perp & 31 & \perp & \perp & \perp & 22 & \perp & 15 & \perp & \perp & \perp \\ \perp & 36 & \perp & \perp & \perp & 25 & \perp & 20 & \perp & \perp & \perp \end{pmatrix} \tag{4.8.14}$$

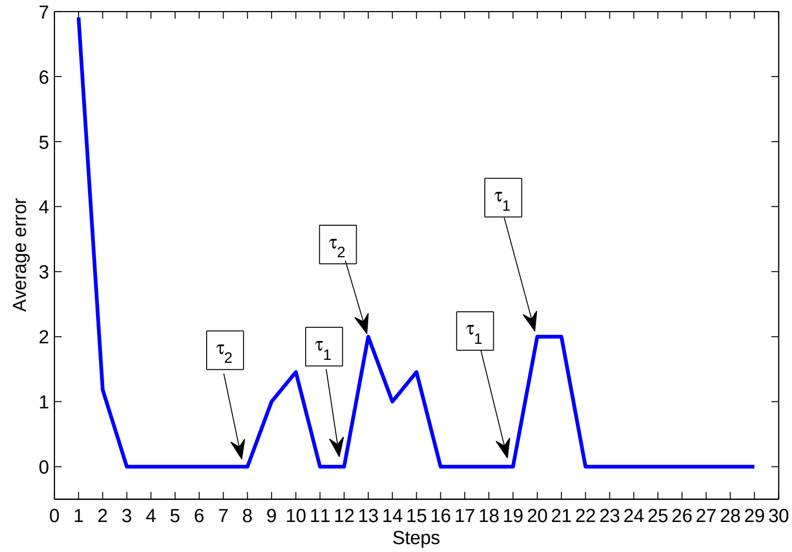

Figure 4.2: Evolution of the average error $e_{\mathtt{avg}}[k]$. Note that the feedback controller is capable of rejecting the occasional perturbations.

which is already causal. The new closed loop $A_{\mathtt{FA}} = A \oplus BF_{\mathtt{FA}}$ is such that $\kappa((-24)A_{\mathtt{FA}}) = 2$, so the convergency number was improved by 1.

Thus, the final control action is

$$u[k] = F_{\mathtt{FA}}x[k] \oplus F_{\mathtt{SR}}x[k] = (F_{\mathtt{FA}} \oplus F_{\mathtt{SR}})x[k]. \tag{4.8.15}$$

To illustrate the robustness of the controller, a simulation was done with the randomly generated initial condition $x[0] = (52\ 54\ 55\ 57\ 61\ 63\ 73\ 75\ 17\ 18\ 7)^T$, which does not complies with the requirement in Equation (4.8.4). At each step, a perturbation will afflict the system: with 10% of chance τ_1 is delayed by 8 time units, while with also 10% of chance τ_2 gets delayed by 6 time units (the faults are independent and can in principle happen concomitantly). The τ's represent processing times. Figure 4.2 shows the evolution of the average error according to the equations

$$\begin{aligned} e[k] &= Ex[k] - Dx[k]; \\ e_{\mathtt{avg}}[k] &= \frac{1}{n}\sum_{i=1}^{n} |e_i[k]|; \end{aligned} \tag{4.8.16}$$

pointing when a perturbation happened (and in which τ) at each step.

4.9 Conclusion

This chapter presented the definition of (controllable coupled) controllable critical and controllable non-critical problems. A necessary and sufficient condition was proposed to the latter problem and also for some class of controllable critical problems (controllable faux-critical), which "functionally" are non-critical. The important issue of causality was also discussed.

The reader may wonder about the scope of the proposed result: are controllable non-critical (or controllable faux-critical) problems frequent in practice? The Definition 4.3.5 itself suggests that the answer is "yes": for the problem to be critical, only one possible value of λ for the spectral characteristic equation is possible. Even in this case, some problems (faux-critical) are covered by the proposed result.

Experimental results also point to this direction. *All* the regulation problems that the author faced throughout his PHD research were successfully solved by the proposed methodology (some were non-critical, others faux-critical). This include the train scheduling problem presented in Katz (2007), the processor problem presented in Gonçalves et al. (2012), the medium-sized train scheduling problem proposed in Gonçalves et al. (2013a), both (small and large scale) High-Throughput Screening Systems problems presented in Brunsch (2014), the cluster tool problem presented Attia et al. (2010) (see Section 4.8 above), the small traffic problem presented in Maia et al. (2013), the small workshop problem presented in Maia et al. (2011a), the illustrative problem presented in Amari et al. (2012) and the example presented in Section 6.1 of Atto et al. (2011).

Chapter 5

On the Observer Problem for Tropical Linear Event-Invariant Dynamical Systems: Non-Critical Case

This chapter deals with an observer problem, which in some ways is the dual of the regulator problem described in Chapter 4. Concepts as “controllable coupled” and “controllable critical/non-critical” will receive counterparts in the setting of the observer problem. A necessary and sufficient condition for the so-called *(observable coupled) observable non-critical problems* will then be presented.

A preliminary version of the results developed in this chapter will be presented in a conference (see Gonçalves et al. (2014b)).

5.1 Introduction

The regulator problem proposed and solved in Chapter 4 assumes the measurement of *all* states. Frequently, such hypothesis is unfeasible. In this case, as in the traditional system theory, one alternative is to construct an *observer* of the state variables x using the known outputs y and inputs u. This way, the state feedback can be implemented using the observed state.

In the tropical setting, very few research was done in the subject. Indeed, to the authors’ knowledge, only two papers studied problems related to observability in tropical setting. Loreto et al. (2010) proposes an observer for a descriptor system (which can model uncertainties in the parameters). Hence, given that the initial condition of the system $x[0]$ is known, it is possible to discover at a given step k all the possible values of the state $x[k]$ that could be reached by the system with uncertainties. Although interesting (and indeed, the concepts presented in that paper were fundamental for the developments of this chapter), the proposed observation problem is not fitting for the main objective of this chapter (implementing a state feedback control using only inputs). Also, the approach has a (possible) double exponential complexity in the number of states, which can hinder the application in more complex systems. The second one, Hardouin et al. (2010b), used transfer series methods to devise a Luenberger-like observer that reconstructs the greatest state estimation $\hat{x}[k]$ that is less than or equal to the real state $x[k]$ (bounds for the error between the real and reconstructed state can be seen in Hardouin et al. (2010a)). It is also important to mention that the output feedback strategy (that is, controlling using an observer) was also considered (in the setting of transfer series) in Hardouin (2004),

This chapter is interested, specifically, in the following problem (it will be posed formally later): using only the system outputs $y[k]$ and inputs $u[k]$, construct a sequence $\hat{x}[k]$ that converges in a finite number of steps to a linear functional $Wx[k]$, for a given matrix W, no matter what the initial condition $x[0]$ of the system be. In principle, the approach described in Hardouin et al. (2010b) could be used: observing $x[k]$ and then computing $Wx[k]$ (in that case, in 0 steps the observed state will already match the real one, with no transients). However, the conditions that ensure strict equality can be quite restrictive. Computing the form $Wx[k]$ directly (in opposition of computing $x[k]$ and then $Wx[k]$), as it will be proposed in this chapter, can be handy. It is clear that if W is chosen as a feedback matrix F, then the approach can be used to solve the regulator problem using only the measured outputs. Of course, the results are not only limited to this particular application: they can be used, for instance, to make diagnostics in the system (see Loreto et al. (2010); Hardouin et al. (2010b)).

5.2 The problem

5.2.1 Problem statement

Problem 5.2.1. *(Generalized steady state observation problem for tropical linear event-invariant dynamical systems)* The *generalized steady state observation problem for tropical linear event-invariant dynamical systems*, denoted by $\mathcal{O}(S, W)$, is defined as follows.

Consider the tropical linear event-invariant dynamical system

$$S: \begin{cases} x[k+1] = Ax[k] \oplus Bu[k]; \\ y[k] = Cx[k] \oplus Gu[k]; \end{cases} \tag{5.2.1}$$

for $A \in \mathbb{T}_{\max}^{n \times n}$, $B \in \mathbb{T}_{\max}^{n \times m}$, $C \in \mathbb{T}_{\max}^{d \times n}$ and $G \in \mathbb{T}_{\max}^{d \times m}$. For a given matrix $W \in \mathbb{T}_{\max}^{g \times n}$, using the inputs $u[k]$ and outputs $y[k]$, construct a sequence $s[k]$ such that there exists a finite l in which $s[k] = Wx[k]$ $\forall k \geq l$. □

Assumption 5.2.1. *(Assumptions in A, B, $x[0]$)* Since the firing dates are non-decreasing ($x[k+1] \succeq x[k]$), one can, without loss of generality, always assume that $A \succeq I$. Further, one can also always assume that no column of B is $\perp$, because otherwise the corresponding action plays no role in the system and can be removed. Finally, it is also always possible to assume that $x[0]$ has no $\perp$ entries. This is very reasonable because $x[0]$ represents firing dates. □

Assumption 5.2.2. *(Common growth rate of $Wx[k]$)* For all i, j

$$\lim_{k \to \infty} \frac{\{Wx[k]\}_i}{k} = \lim_{k \to \infty} \frac{\{Wx[k]\}_j}{k}. \tag{5.2.2}$$

□

Remark 5.2.1. Assumption 5.2.2 may seem restrictive but, in fact, it is not. If one wishes to observe a linear functional $Wx[k]$ with different rates, it is possible to split this observation problems in many different problems in which in each problem this assumption holds. For instance, let $W = (W[0]^T \; W[1]^T)^T$, in which all the entries of $W[0]x[k]$ have the same rate $\lambda[0]$ and all the entries of $W[1]x[k]$ have the same rate $\lambda[1]$ with $\lambda[0] \neq \lambda[1]$. One can then consider two different observation problems: one for $W[0]x[k]$ and other for $W[1]x[k]$. In this case, for each observation problem Assumption 5.2.2 will hold. □

And also another very important assumption.

Assumption 5.2.3. *(System is strongly connected)* Suppose Assumption 5.2.2 and that the common rate for $Wx[k]$ is $\lambda_{\mathtt{rate}}$. It is then assumed that for all i, j

$$\lim_{k \to \infty} \frac{x_i[k]}{k} = \lim_{k \to \infty} \frac{y_j[k]}{k} = \lambda_{\mathtt{rate}}. \tag{5.2.3}$$

□

Remark 5.2.2. Assumption 5.2.3 may seem to be very restrictive at a first glance. It will be argued that this is not the case.

Consider then the following partition of the states x

$$\begin{aligned}\mathcal{Q} &\equiv \{i \mid \lim_{k\to\infty} \frac{x_i[k]}{k} > \lambda_{\mathtt{rate}}\};\\ \mathcal{W} &\equiv \{i \mid \lim_{k\to\infty} \frac{x_i[k]}{k} = \lambda_{\mathtt{rate}}\};\\ \mathcal{E} &\equiv \{i \mid \lim_{k\to\infty} \frac{x_i[k]}{k} < \lambda_{\mathtt{rate}}\}\end{aligned} \tag{5.2.4}$$

and also an analogous partition for the outputs y.

Suppose, without loss of generality, that the indexes of the state x are ordered so the first indexes are the ones of $\mathcal{Q}$, then of $\mathcal{W}$ and then of $\mathcal{E}$. Suppose the same ordering is made in the outputs y . System S can then be written in the following form

$$S : \begin{cases} \begin{pmatrix} x_{\mathcal{Q}}[k+1] \\ x_{\mathcal{W}}[k+1] \\ x_{\mathcal{E}}[k+1] \end{pmatrix} = \begin{pmatrix} A_{\mathcal{Q}\mathcal{Q}} & A_{\mathcal{Q}\mathcal{W}} & A_{\mathcal{Q}\mathcal{E}} \\ \perp & A_{\mathcal{W}\mathcal{W}} & A_{\mathcal{W}\mathcal{E}} \\ \perp & \perp & A_{\mathcal{E}\mathcal{E}} \end{pmatrix} \begin{pmatrix} x_{\mathcal{Q}}[k] \\ x_{\mathcal{W}}[k] \\ x_{\mathcal{E}}[k] \end{pmatrix} \oplus \begin{pmatrix} B_{\mathcal{Q}} \\ B_{\mathcal{W}} \\ B_{\mathcal{E}} \end{pmatrix} u[k]; \\ \begin{pmatrix} y_{\mathcal{Q}}[k] \\ y_{\mathcal{W}}[k] \\ y_{\mathcal{E}}[k] \end{pmatrix} = \begin{pmatrix} C_{\mathcal{Q}\mathcal{Q}} & C_{\mathcal{Q}\mathcal{W}} & C_{\mathcal{Q}\mathcal{E}} \\ \perp & C_{\mathcal{W}\mathcal{W}} & C_{\mathcal{W}\mathcal{E}} \\ \perp & \perp & C_{\mathcal{E}\mathcal{E}} \end{pmatrix} \begin{pmatrix} x_{\mathcal{Q}}[k] \\ x_{\mathcal{W}}[k] \\ x_{\mathcal{E}}[k] \end{pmatrix} \oplus \begin{pmatrix} G_{\mathcal{Q}} \\ G_{\mathcal{W}} \\ G_{\mathcal{E}} \end{pmatrix} u[k]. \end{cases} \tag{5.2.5}$$

and $Wx[k] = W_{\mathcal{W}}x_{\mathcal{W}}[k] \oplus W_{\mathcal{E}}x_{\mathcal{E}}[k]$ (there is no contribution of $x_{\mathcal{Q}}[k]$ because, otherwise, $Wx[k]$ would have an entry with rate $> \lambda_{\mathtt{rate}}$).

Note that, necessarily, $A_{\mathcal{W}\mathcal{Q}} = \perp$, $A_{\mathcal{E}\mathcal{Q}} = \perp$ and $A_{\mathcal{E}\mathcal{W}} = \perp$, because otherwise there would be a contradiction with the definition of the sets $\mathcal{W}$ and $\mathcal{E}$. An analogous observation can be applied to the matrices $C_{\mathcal{W}\mathcal{Q}} = \perp$, $C_{\mathcal{E}\mathcal{Q}} = \perp$ and $C_{\mathcal{E}\mathcal{W}} = \perp$.

Note that, in terms of the observation problem, the states and outputs in $\mathcal{Q}$ are not relevant, since the growth rate of all the entries $\{Wx[k]\}_i$ is $\lambda_{\mathtt{rate}}$, which is strictly smaller than the growth rate of the states in $\mathcal{Q}$ (hence, they will not be used in a tropical linear observer because, otherwise, a rate greater than $\lambda_{\mathtt{rate}}$ would be imposed in at least one entry of $Wx[k]$). Therefore, the system S could be simplified by the removing these states and outputs. Further, in steady state one can say that $A_{\mathcal{W}\mathcal{W}}x_{\mathcal{W}}[k] \succ A_{\mathcal{W}\mathcal{E}}x_{\mathcal{E}}[k]$ and $C_{\mathcal{W}\mathcal{W}}x_{\mathcal{W}}[k] \succ C_{\mathcal{W}\mathcal{E}}x_{\mathcal{E}}[k]$, eventually (note that this must hold true, because otherwise there would be a contradiction with the definition of $\mathcal{W}$). Finally, eventually $W_{\mathcal{W}}x_{\mathcal{W}}[k] \succ W_{\mathcal{E}}x_{\mathcal{E}}[k]$ (because, otherwise, Assumption 5.2.2 would not hold). With these considerations, one can create a simplified system $S_{\mathtt{ss}}$ by swapping $A_{\mathcal{W}\mathcal{E}}$ for $\perp$, $C_{\mathcal{W}\mathcal{E}}$ for $\perp$, $W_{\mathcal{E}}$ for $\perp$ and removing the states in $\mathcal{Q}$. The resulting system will be equivalent in steady state to the original one:

$$S_{\mathrm{ss}}\ :\ \begin{cases} \begin{pmatrix} x_{\mathscr{W}}[k+1] \\ x_{\mathscr{E}}[k+1] \end{pmatrix} = \begin{pmatrix} A_{\mathscr{W}\mathscr{W}} & \perp \\ \perp & A_{\mathscr{E}\mathscr{E}} \end{pmatrix} \begin{pmatrix} x_{\mathscr{W}}[k] \\ x_{\mathscr{E}}[k] \end{pmatrix} \oplus \begin{pmatrix} B_{\mathscr{W}} \\ B_{\mathscr{E}} \end{pmatrix} u[k]; \\ \begin{pmatrix} y_{\mathscr{W}}[k] \\ y_{\mathscr{E}}[k] \end{pmatrix} = \begin{pmatrix} C_{\mathscr{W}\mathscr{W}} & \perp \\ \perp & C_{\mathscr{E}\mathscr{E}} \end{pmatrix} \begin{pmatrix} x_{\mathscr{W}}[k] \\ x_{\mathscr{E}}[k] \end{pmatrix} \oplus \begin{pmatrix} G_{\mathscr{W}} \\ G_{\mathscr{E}} \end{pmatrix} u[k]. \end{cases} \tag{5.2.6}$$

in which is desirable to observe $W_{\mathscr{W}} x_{\mathscr{W}}[k]$.

Note that, for this problem, the states and outputs in $\mathscr{E}$ are innocuous to the system S_{ss}. Indeed, using them in a tropical linear observer would make no difference in steady state (since these variables grow in a rate less than the rate of the variable $Wx[k]$ that it is desired to observe). Therefore, this system can be further simplified by dropping these variables.

$$S_{\mathrm{sss}}\ :\ \begin{cases} x_{\mathscr{W}}[k+1] = A_{\mathscr{W}\mathscr{W}} x[k] \oplus B_{\mathscr{W}} u[k]; \\ y_{\mathscr{W}}[k] = C_{\mathscr{W}\mathscr{W}} x_{\mathscr{W}}[k] \oplus G_{\mathscr{W}} u[k]. \end{cases} \tag{5.2.7}$$

Clearly, Assumption 5.2.3 holds for S_{sss}. This system is equivalent, in terms of the observation problem, in steady state to the original one S. Assuming that this pre-processing is done, Assumption 5.2.3 can be assumed without loss of generality. In practice, $u[k]$ frequently ensures a common rate in the system, and hence this hypothesis holds by default without no further pre-processing (that is, the sets $\mathscr{Q}$ and $\mathscr{E}$ are empty in both x and y). □

It is noteworthy that the usual observation problem is obtained by taking $W = I$. However, the choice of another matrix can substantially weaken the problem. As it will be shown in the application section, sometimes it is sufficient to observe only a linear functional of the states, not every one of them (the aforementioned choice $W = I$).

5.2.2 Manipulating semimodules and congruences

The next section will make intense use of properties of both semimodules and congruences. Hence, some properties and definitions need to be presented.

Definition 5.2.1. *(Operations in semimodules, see Loreto et al. (2010))* Given semimodules $\mathscr{S}, \mathscr{T}$ and a matrix M of adequate dimensions.

$$M\mathscr{S} \equiv \{Ms \mid s \in \mathscr{S}\}. \tag{5.2.8}$$

$$M^{-1}\mathscr{S} \equiv \{s \mid Ms \in \mathscr{S}\}. \tag{5.2.9}$$

$$\mathscr{S} \oplus \mathscr{T} \equiv \{s \oplus t \mid s \in \mathscr{S} \ , \ t \in \mathscr{T}\}. \tag{5.2.10}$$

□

Definition 5.2.2. *(Operations in congruences, see Loreto et al. (2010))* Given congruences $\mathscr{C}, \mathscr{D}$ and a matrix M of adequate dimensions.

$$M\mathscr{C} \equiv \{\{Mc, M\overline{c}\} \mid \{c, \overline{c}\} \in \mathscr{C}\}. \tag{5.2.11}$$

$$M^{-1}\mathscr{C} \equiv \{\{c, \overline{c}\} \mid \{Mc, M\overline{c}\} \in \mathscr{C}\}. \tag{5.2.12}$$

$$\mathscr{C} \cap \mathscr{D} \equiv \{\{e, \overline{e}\} \mid \{e, \overline{e}\} \in \mathscr{C} \ , \ \{e, \overline{e}\} \in \mathscr{D}\}. \tag{5.2.13}$$

□

Property 5.2.1. *(Sum and intersection, see Loreto et al. (2010))* It is straightforward to see that if $\mathscr{S} = Im\{S\}$ and $\mathscr{T} = Im\{T\}$ then

$$\mathscr{S} \oplus \mathscr{T} = Im\{(S \ T)\}. \tag{5.2.14}$$

Further, if $\mathscr{C} = Ker\{C\}$ and $\mathscr{D} = Ker\{D\}$

$$\mathscr{C} \cap \mathscr{D} = Ker\left\{\begin{pmatrix} C \\ D \end{pmatrix}\right\}. \tag{5.2.15}$$

□

Definition 5.2.3. *(Orthogonal operators, see Loreto et al. (2010))* Given a semimodule $\mathscr{S} \in \mathbb{T}^n_{\mathtt{max}}$, the *orthogonal* of this semimodule, $\mathscr{S}^\perp$, is the congruence

$$\mathscr{S}^\perp \equiv \{\{c, \overline{c}\} \mid s^T c = s^T \overline{c} \ , \ \forall s \in \mathscr{S}\}. \tag{5.2.16}$$

Dually, given a congruence $\mathscr{C} \in \mathbb{T}^n_{\mathtt{max}} \times \mathbb{T}^n_{\mathtt{max}}$, the *orthogonal* of this congruence, $\mathscr{C}^\top$, is the semimodule

$$\mathscr{C}^\top \equiv \{s \mid s^T c = s^T \overline{c} \ , \ \forall \{c, \overline{c}\} \in \mathscr{C}\}. \tag{5.2.17}$$

□

Property 5.2.2. *(Properties of congruences and semimodules, see Loreto et al. (2010))* Let $\mathscr{S} = Im\{S\}$ and $\mathscr{T} = Im\{T\}$ be semimodules and $\mathscr{C} = Ker\{C\}$, $\mathscr{D} = Ker\{D\}$ congruences, in which the matrices S, T, C, D have finite dimension (so the semimodules and congruences are closed, see Loreto et al. (2010)). Let M be a matrix of appropriate dimensions. Then

$$(\mathscr{S}^{\perp})^{\top} = \mathscr{S} \ , \ (\mathscr{C}^{\top})^{\perp} = \mathscr{C}. \tag{5.2.18}$$

$$(\mathscr{S} \oplus \mathscr{T})^{\perp} = \mathscr{S}^{\perp} \cap \mathscr{T}^{\perp} \ , \ (\mathscr{C} \cap \mathscr{D})^{\top} = \mathscr{C}^{\top} \oplus \mathscr{D}^{\top}. \tag{5.2.19}$$

$$(M\mathscr{S})^{\perp} = (M^T)^{-1}\mathscr{S}^{\perp} \ , \ (M\mathscr{C})^{\top} = (M^T)^{-1}\mathscr{C}^{\top}. \tag{5.2.20}$$

$$Im\{S\}^{\perp} = Ker\{S^T\} \ , \ Ker\{C\}^{\top} = Im\{C^T\}. \tag{5.2.21}$$

$$M\,Im\{S\} = Im\{MS\} \ , \ M^{-1}Ker\{C\} = Im\{CM\}. \tag{5.2.22}$$

$$\mathscr{S} \supseteq \mathscr{T} \implies \mathscr{S}^{\perp} \subseteq \mathscr{T}^{\perp} \ , \ \mathscr{C} \subseteq \mathscr{D} \implies \mathscr{C}^{\top} \supseteq \mathscr{D}^{\top}. \tag{5.2.23}$$

□

And

Lemma 5.2.1. (*Additional property*) For a matrix M, a semimodule $\mathscr{S}$ and a congruence $\mathscr{C}$ of adequate dimensions:

$$(M^{-1}\mathscr{S})^{\perp} \supseteq M^T\mathscr{S}^{\perp} \ , \ (M^{-1}\mathscr{C})^{\top} \supseteq M^T\mathscr{C}^{\top}. \tag{5.2.24}$$

Proof. Only Equation (5.2.24)-(left) will be proved. The other follows with a similar reasoning (and will not be used in this text).

Hence

$$(M^{-1}\mathscr{S})^{\perp} = \{\{h, \overline{h}\} \mid h^T v = \overline{h}^T v \ \forall \ \{v \mid Mv \in \mathscr{S}\}\}. \tag{5.2.25}$$

Now, if $\{h, \overline{h}\} \in M^T\mathscr{S}^{\perp}$, $h = M^T u$ and $\overline{h} = M^T\overline{u}$ in which $u^T s = \overline{u}^T s$ for all $s \in \mathscr{S}$. In particular, consider all $s \in \mathscr{S}$ such that $s = Mv$ for a v. Hence, $u^T Mv = \overline{u}^T Mv$ or $(M^T u)^T v = (M^T\overline{u})^T v$ or $h^T v = \overline{h}^T v$, which implies that $\{h, \overline{h}\} \in (M^{-1}\mathscr{S})^{\perp}$. And the proof is complete.

□

Then, two useful definitions will be introduced by the author of this thesis:

Definition 5.2.4. *(Relaxed semimodules and constrained congruences)* Given a semimodule $\mathscr{S}$ and a congruence $\mathscr{C}$ of appropriate finite dimensions:

The *semimodule* $\mathscr{S}$ *relaxed by the congruence* $\mathscr{C}$, $\mathscr{S} \uparrow \mathscr{C}$, is the following semimodule

$$\mathscr{S} \uparrow \mathscr{C} \equiv \{\bar{s} \mid \{s, \bar{s}\} \in \mathscr{C} \ , \ s \in \mathscr{S}\}. \tag{5.2.26}$$

Dually, the *congruence* $\mathscr{C}$ *constrained by the semimodule* $\mathscr{S}$, $\mathscr{C} \downarrow \mathscr{S}$, is the following set [1]

$$\mathscr{C} \downarrow \mathscr{S} \equiv \{\{c, \bar{c}\} \mid \{c, \bar{c}\} \in \mathscr{C} \ , \ c \in \mathscr{S} \ , \ \bar{c} \in \mathscr{S}\}. \tag{5.2.27}$$

Further, if $\mathscr{S} = Im\{S\}$ and $\mathscr{C} = Ker\{C\}$ for matrices C, S, then the notations $Im\{S \uparrow C\} \equiv \mathscr{S} \uparrow \mathscr{C}$ and $Ker\{C \downarrow S\} \equiv \mathscr{C} \downarrow \mathscr{S}$ will be used. □

The following properties are straightforward using the definition of relaxed semimodules and constrained congruences and the properties in Property 5.2.2 and Lemma 5.2.1

Property 5.2.3. *(Properties of relaxed semimodules and constrained congruences)* Let $\mathscr{S} = Im\{S\}$ and $\mathscr{C} = Ker\{C\}$ for matrices C, S of appropriate finite dimensions. Then

$$\mathscr{S} \uparrow \mathscr{C} \supseteq \mathscr{S} \ , \ \mathscr{C} \downarrow \mathscr{S} \subseteq \mathscr{C}. \tag{5.2.28}$$

$$Im\{S \uparrow C\} = C^{-1} Im\{CS\} \ , \ Ker\{C \downarrow S\} = S Ker\{CS\}. \tag{5.2.29}$$

$$(\mathscr{S} \uparrow \mathscr{C})^{\perp} \supseteq \mathscr{S}^{\perp} \downarrow \mathscr{C}^{\top} \ , \ (\mathscr{C} \downarrow \mathscr{S})^{\top} = \mathscr{C}^{\top} \uparrow \mathscr{S}^{\perp}. \tag{5.2.30}$$

□

Equation (5.2.28), in particular, justifies the names *relaxed semimodules* and *constrained congruences*.

The following lemma relating constrained congruences and relaxed semimodules will be very important later.

Proposition 5.2.1. *(Equivalence)* Let $\mathscr{Z} = Ker\{Z\}$, $\mathscr{W} = Ker\{W\}$, $\mathscr{H} = Im\{H\}$ and R be matrices of appropriate dimensions. Then

$$R(\mathscr{Z} \downarrow \mathscr{H}) \subseteq \mathscr{W} \tag{5.2.31}$$

if and only if the equation

$$LZH = WRH \tag{5.2.32}$$

has a solution L.

[1] It is not, in general, a congruence. For example, not for every x the pair $\{x, x\}$ is in this set, which contradicts with the fact that a congruence is an equivalence relation, and thus, in particular, reflexive.

Proof. Apply the orthogonal operator $(\cdot)^\top$ in both sides of Equation (5.2.31). Using the properties of congruences, specifically Equation (5.2.19) and Equation (5.2.23) in Property 5.2.2 and Equation (5.2.30) in Property 5.2.3, Equation (5.2.31) holds only if

$$(R^T)^{-1}(\mathscr{Z}^\top \uparrow \mathscr{H}^\perp) \supseteq \mathscr{W}^\top. \tag{5.2.33}$$

Using Equation (5.2.29) in Property 5.2.3 and Equation (5.2.21) in Property 5.2.2, one has equivalently

$$(R^T)^{-1}((H^T)^{-1} Im\{H^T Z^T\}) \supseteq Im\{W^T\}. \tag{5.2.34}$$

This equation holds *if and only if* (since $M^{-1}\mathscr{X} \supseteq \mathscr{Y} \longleftrightarrow \mathscr{X} \supseteq M\mathscr{Y}$)

$$Im\{H^T Z^T\} \supseteq H^T R^T Im\{W^T\}. \tag{5.2.35}$$

And then, using Equation (5.2.22) in Property 5.2.2

$$Im\{H^T Z^T\} \supseteq Im\{H^T R^T W^T\}. \tag{5.2.36}$$

For two given matrices X, Y, $Im\{X\} \supseteq Im\{Y\}$ if and only if there exists a matrix M such that $XM = Y$. Hence, there exists a matrix L such that

$$H^T Z^T L^T = H^T R^T W^T. \tag{5.2.37}$$

After transposing, it is clear that the *only if* part holds. The *if* part comes after reversing the steps. All of them are straightfoward, with the exception of the first one because Equation (5.2.30)-(left) does not hold with equality and thus one cannot claim, in general, that $((R^T)^{-1}(\mathscr{Z}^\top \uparrow \mathscr{H}^\perp))^\perp = R(\mathscr{Z} \downarrow \mathscr{H})$. This is not necessary, however. Indeed, if

$$(R^T)^{-1}((H^T)^{-1} Im\{H^T Z^T\}) \supseteq Im\{W^T\}. \tag{5.2.38}$$

applying the $(\cdot)^\perp$ operator, Equation (5.2.23) and Equation (5.2.21) in Property 5.2.2, Lemma 5.2.1 and Equation (5.2.30)-(left):

$$R(\mathscr{Z} \downarrow \mathscr{H}) \subseteq R(\mathscr{Z}^\top \uparrow \mathscr{H}^\perp)^\perp \subseteq ((R^T)^{-1}(\mathscr{Z}^\top \uparrow \mathscr{H}^\perp))^\perp \subseteq \mathscr{W} \tag{5.2.39}$$

and the result holds.

□

5.2.3 Generalized t-observation problem

In order to solve $\mathscr{O}(S,W)$ another kind of problem will be introduced.

Problem 5.2.2. *(Generalized t-observation problem for tropical linear event-invariant dynamical systems with information of initial value)*

The *generalized t-observation problem for tropical linear event-invariant dynamical systems with information of initial value*, denoted by $\mathscr{O}^t_{\mathtt{str}}(S,W)$, will be defined as follows.

Consider the tropical linear event-invariant dynamical system as in Equation (5.2.1). Given a natural number t, using the inputs $u[k]$, the outputs $y[k]$ *and the values* $Wx[k]$, $k \leq t$, construct a sequence $s[k]$ such that $s[k] = Wx[k]\ \forall k > t$.

□

The main differences between $\mathscr{O}(S,W)$ and $\mathscr{O}^t_{\mathtt{str}}(S,W)$ are that the latter requires convergence in at most t steps and it also assumes the knowledge of $Wx[k]$, $k \leq t$. Further, the reader may consider, from a first glance, the problem innocuous since in this case the information of initial conditions is given, which contradicts with the role that one would expect from an observer. It will be shown later that under some conditions this information can be dispensed with.

The following lemma is clear.

Lemma 5.2.2. *(Stronger problem)*: There is a solution for $\mathscr{O}(S,W)$ *only if* $\mathscr{O}^t_{\mathtt{str}}(S,W)$ has a solution for a natural t. □

Indeed, if for all t $\mathscr{O}^t_{\mathtt{str}}(S,W)$ is not solvable, then $\mathscr{O}(S,W)$ cannot be solvable, since the equality between $s[k]$ and $Wx[k]$ cannot be achieved with *more information* (the values $Wx[k]$, $k \leq t$).

For now, the effort will be focused in solving $\mathscr{O}^t_{\mathtt{str}}(S,W)$. Further, some conditions will be posed so that solving $\mathscr{O}^t_{\mathtt{str}}(S,W)$ for a t is sufficient for solving $\mathscr{O}(S,W)$.

5.2.4 Solving $\mathscr{O}^t_{\mathtt{str}}(S,W)$

The following definition is important.

Definition 5.2.5. ($\mathscr{H}_S[k]$ *semimodule*) The k^{th} $\mathscr{H}_S$ semimodule, $\mathscr{H}_S[k] \in \mathbb{T}_{\mathtt{max}}^{(n+m+d)\cdot(k+1)}$, is defined in the following way. A vector $z[k]$, in which $z[k] \equiv (u[0{:}k]^T\ y[0{:}k]^T\ x[0{:}k]^T)^T$ is in $\mathscr{H}_S[k]$ if and only if

$$\begin{aligned}&\text{For } i = 0,1,2,...,k-1:\\ (i)\ &: x[i+1] = Ax[i] \oplus Bu[i];\\ &\text{For } i = 0,1,2,...,k:\\ (ii)\ &: y[i] = Cx[i] \oplus Gu[i]. \end{aligned} \tag{5.2.40}$$

□

One notices that if $z[t] \in \mathscr{H}_S[t]$, then $z[t]$ can be interpreted as a sequence of inputs, outputs and states generated by a dynamical system S from $k = 0$ to $k = t$, since its components comply with the dynamical equations of S.

Since Equation (5.2.40) is tropical linear in $z[k]$, the following lemma is clear.

Lemma 5.2.3. *(Semimodule $\mathscr{H}_S[k]$ is generated by a matrix)* There exists a matrix $H_S[k]$ such that Equation (5.2.40) can be written as $z[k] = H_S[k]v$ for a vector v, and hence $\mathscr{H}_S[k] = Im\{H_S[k]\}$. □

There is a systematic procedure for finding the matrices $H_S[k]$. Note that the variables $u[0{:}k]$ are free (there are no dynamic constraints in them). This is also true for $x[0]$. All the other variables are constrained.

Then, let

$$u[i] = v[i]\ ,\ i = 0,1,2,...,k \text{ and } x[0] = w. \tag{5.2.41}$$

Iterating the equation $x[k+1] = Ax[k] \oplus Bu[k]$, one can easily see that

$$x[k] = A^k x[0] \oplus \bigoplus_{i=0}^{k-1} A^{k-i} Bu[i]. \tag{5.2.42}$$

Or, using the equations for $v[i]$ and w

$$x[k] = A^k w \oplus \bigoplus_{i=0}^{k} A^{k-i} Bv[i]\ \ i = 0,1,2,...,k. \tag{5.2.43}$$

Hence

$$y[k] = CA^k w \oplus \bigoplus_{i=0}^{k} CA^{k-i} Bv[i] \oplus Gv[k]. \tag{5.2.44}$$

Using Equation (5.2.41), Equation (5.2.43), and Equation (5.2.44), one can write these equations in a matricial form as

$$\begin{pmatrix} u[0] \\ u[1] \\ u[2] \\ \ldots \\ u[k] \\ y[0] \\ y[1] \\ y[2] \\ \ldots \\ y[k] \\ x[0] \\ x[1] \\ x[2] \\ \ldots \\ x[k] \end{pmatrix} = H_S[k] \begin{pmatrix} v[0] \\ v[1] \\ v[2] \\ \ldots \\ v[k] \\ w \end{pmatrix} \tag{5.2.45}$$

in which the matrix $H_S[k]$ is

$$\begin{array}{c} \\ u[0] \\ u[1] \\ u[2] \\ \cdots \\ u[k] \\ y[0] \\ y[1] \\ y[2] \\ \cdots \\ y[k] \\ x[0] \\ x[1] \\ x[2] \\ \cdots \\ x[k] \end{array}
\begin{array}{c} \begin{array}{cccccc} v[0] & v[1] & v[3] & \cdots & v[k] & w \end{array} \\
\left(\begin{array}{ccccc|c}
I & \perp & \perp & \cdots & \perp & \perp \\
\perp & I & \perp & \cdots & \perp & \perp \\
\perp & \perp & I & \cdots & \perp & \perp \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\perp & \perp & \perp & \cdots & I & \perp \\ \hline
G & \perp & \perp & \cdots & \perp & C \\
CB & G & \perp & \cdots & \perp & CA \\
CAB & CB & G & \cdots & \perp & CA^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
CA^{k-1}B & CA^{k-2}B & CA^{k-3}B & \cdots & G & CA^k \\ \hline
\perp & \perp & \perp & \cdots & \perp & I \\
B & \perp & \perp & \cdots & \perp & A \\
AB & B & \perp & \cdots & \perp & A^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
A^{k-1}B & A^{k-2}B & A^{k-3}B & \cdots & \perp & A^k
\end{array}\right) \end{array}. \tag{5.2.46}$$

Further, the following definition is necessary.

Definition 5.2.6. ($\mathscr{Z}_S[k]$ *congruence*) Given a system S, the k^{th} $\mathscr{Z}_S$ congruence, $\mathscr{Z}_S[k] \in \mathbb{T}_{\mathtt{max}}^{(n+m+g)\cdot(k+1)} \times \mathbb{T}_{\mathtt{max}}^{(n+m+g)\cdot(k+1)}$, is defined in the following way. A pair $\{z[k], \overline{z}[k]\}$ in which $z[k] \equiv (u[0{:}k]^T \; y[0{:}k]^T \; x[0{:}k]^T)^T$ and $\overline{z}[k] = (\overline{u}[0{:}k]^T \; \overline{y}[0{:}k]^T \; \overline{x}[0{:}k]^T)^T$ is in $\mathscr{Z}_S[k]$ if and only if

$$\begin{aligned} &\text{For } \; i = 0, 1, 2, \ldots, k: \\ (i) \; &: u[i] = \overline{u}[i]; \\ (ii) \; &: y[i] = \overline{y}[i]; \\ (iii) \; &: Wx[i] = W\overline{x}[i]. \end{aligned} \tag{5.2.47}$$

□

If one considers the vectors $z[t]$ and $\overline{z}[t]$ as a history of inputs, outputs and states of a system S, then $\{z[t], \overline{z}[t]\} \in \mathscr{Z}_S[t]$ implies that they have the same inputs (Equation (5.2.47)-(i)), same outputs (Equation (5.2.47)-(ii)) and the same initial values $Wx[k] = W\overline{x}[k]$ (Equation (5.2.47)-(iii)) from $k = 0$ to $k = t$.

Since Equation (5.2.47) is tropical linear in $z[k], \overline{z}[k]$ and the components of $z[k]$ only appear on the left side while the ones of $\overline{z}[k]$ appear on the right, with the same matrices, the following lemma is clear.

Lemma 5.2.4. *(Congruence $\mathscr{Z}_S[k]$ is generated by a matrix)* There exists a matrix $Z_S[k]$ such that Equation (5.2.47) can be written as $Z_S[k]z[k] = Z_S[k]\overline{z}[k]$ and hence $\mathscr{Z}_S[k] = Ker\{Z_S[k]\}$. □

Then, if

$$Z_S[k] \begin{pmatrix} u[0] \\ u[1] \\ \ldots \\ u[k] \\ y[0] \\ y[1] \\ \ldots \\ y[k] \\ x[0] \\ x[1] \\ \ldots \\ x[k] \end{pmatrix} = Z_S[k] \begin{pmatrix} \overline{u}[0] \\ \overline{u}[1] \\ \ldots \\ \overline{u}[k] \\ \overline{y}[0] \\ \overline{y}[1] \\ \ldots \\ \overline{y}[k] \\ \overline{x}[0] \\ \overline{x}[1] \\ \ldots \\ \overline{x}[k] \end{pmatrix} \tag{5.2.48}$$

one can show that $Z_S[k]$ is equal to

$$
\begin{array}{c}
 \\ u[0] \\ u[1] \\ \dots \\ u[k] \\ y[0] \\ y[1] \\ \dots \\ y[k] \\ x[0] \\ x[1] \\ \dots \\ x[k]
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
u[0] & u[1] & \dots & u[k] & y[0] & y[1] & \dots & y[k] & x[0] & x[1] & \dots & x[k]
\end{array} \\
\left(\begin{array}{cccc|cccc|cccc}
I & \bot & \dots & \bot & \bot & \bot & \dots & \bot & \bot & \bot & \dots & \bot \\
\bot & I & \dots & \bot & \bot & \bot & \dots & \bot & \bot & \bot & \dots & \bot \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\bot & \bot & \dots & I & \bot & \bot & \dots & \bot & \bot & \bot & \dots & \bot \\
\hline
\bot & \bot & \dots & \bot & I & \bot & \dots & \bot & \bot & \bot & \dots & \bot \\
\bot & \bot & \dots & \bot & \bot & I & \dots & \bot & \bot & \bot & \dots & \bot \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\bot & \bot & \dots & \bot & \bot & \bot & \dots & I & \bot & \bot & \dots & \bot \\
\hline
\bot & \bot & \dots & \bot & \bot & \bot & \dots & \bot & W & \bot & \dots & \bot \\
\bot & \bot & \dots & \bot & \bot & \bot & \dots & \bot & \bot & W & \dots & \bot \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\bot & \bot & \dots & \bot & \bot & \bot & \dots & \bot & \bot & \bot & \dots & W
\end{array}\right)
\end{array}. \tag{5.2.49}
$$

And also

Definition 5.2.7. ($R_S[k]$ *matrix*) Given a system S and a natural k, the matrix $R_S[t]z[k] \in \mathbb{T}_{\mathtt{max}}^{n\times(n+m+g)\cdot k}$ is defined in a way that, given $z[k] \equiv (u[0{:}k]^T\ y[0{:}k]^T\ x[0{:}k]^T)^T$, $R_S[k]z[k] = Ax[k] \oplus Bu[k]$, $\forall z[k]$. □

It is clear that the matrix $R_S[k]$ is equal to

$$
\begin{array}{c}
\begin{array}{cccccccccccc}
u[0] & u[1] & \dots & u[k] & y[0] & y[1] & \dots & y[k] & x[0] & x[1] & \dots & x[k]
\end{array} \\
\left(\begin{array}{cccc|cccc|cccc}
\bot & \bot & \dots & B & \bot & \bot & \dots & \bot & \bot & \bot & \dots & A
\end{array}\right)
\end{array}. \tag{5.2.50}
$$

With these definitions, an important result can be stated.

Proposition 5.2.2. (*Necessary and sufficient condition for* $\mathscr{O}^t_{\mathtt{str}}(S,W)$) Let $z[t] \equiv (u[0{:}t]^T\ y[0{:}t]^T\ x[0{:}t]^T)^T$. Then $\mathscr{O}^t_{\mathtt{str}}(S,W)$ has solution if and only if the equation

$$\mathfrak{M}^t(S,W)\ :\ L[t]Z_S[t]H_S[t] = WR_S[t]H_S[t] \tag{5.2.51}$$

(with $H_S[k]$, $Z_S[k]$ defined as in Lemma 5.2.3 and 5.2.4, respectively) has a solution $L[t]$.

Proof. *Only if*: $\mathcal{O}^t_{\texttt{str}}(S,W)$ having a solution implies that it is possible to, using *only* the information of $Wx[k]\ k \leq t$, of the inputs $u[k]\ k \leq t$ and of the outputs $y[k]\ k \leq t$, to obtain $Wx[t+1] = W(Ax[t] \oplus Bu[t])$. Let f be the function that uses this set of values to retrieve $Wx[t+1]$.

Now, let $\{z[t], \overline{z}[t]\} \in \mathscr{Z}_S[t] \downarrow \mathscr{H}_S[t]$, then the vectors $z[t], \overline{z}[t]$ can be interpreted as the inputs, outputs and states of a sequence generated by the system S (since each of them is inside the semimodule $\mathscr{H}[t]$). Further, $z[t], \overline{z}[t]$ have the same outputs, inputs and initial values $Wx[k] = W\overline{x}[k]$, $k \leq t$ (since they are in $\mathscr{Z}_S[t]$). Since they have the same outputs and same inputs for $k \leq t$ and have the same $Wx[k] = W\overline{x}[k]\ k \leq t$, this implies that they must generate the same $Wx[t+1] = W(Ax[t] \oplus Bu[t]) = WR_S[t]z[t]$ (due to the definition of $R_S[t]$) under the action of f. This implies that any member $\{R_S[t]z[t], R_S[t]\overline{z}[t]\}$ of the set $R_S[t](\mathscr{Z}_S[t] \uparrow \mathscr{H}_S[t])$ is inside the congruence $\mathscr{W} = Ker\{W\}$, that is, $R_S[t](\mathscr{Z}_S[t] \downarrow \mathscr{H}_S[t]) \subseteq \mathscr{W}$. Proposition 5.2.1 then implies that this equation has a solution if and only if $\mathfrak{M}^t(S,W)$ has a solution. This proves the *only if* part.

If: Define the vector $z[k,t] \equiv (u[k{:}k+t]^T\ y[k{:}k+t]^T\ x[k{:}k+t]^T)^T$. If these values are obtained from a system S, the definition of $H_S[k]$ guarantees that there exists a vector h such that $H_S[k]h = z[k,t]$. Suppose that, indeed, $z[k,t]$ represents inputs, outputs and states generated from a system S. Then, post multiplying $\mathfrak{M}^t(S,W)$ by h

$$L[t]Z_S[t]z[k,t] = WR_S[t]z[k,t]. \tag{5.2.52}$$

According to the definition of $R_S[t]$, $R_S[t]z[k,t] = Ax[k+t] \oplus Bu[k+t] = x[k+t+1]$. Hence

$$Wx[k+t+1] = L[t]Z_S[t]z[k,t]. \tag{5.2.53}$$

According to the definition of the matrix $Z_S[t]$

$$Z_S[t]z[k,t] = \begin{pmatrix} u[k{:}k+t] \\ y[k{:}k+t] \\ (Wx^T[k{:}k+t])^T \end{pmatrix}. \tag{5.2.54}$$

Let $L[t] = (L_u^T[0{:}t]^T\ L_y^T[0{:}t]^T\ L_s^T[0{:}t]^T)$. Then, using this decomposition and Equation 5.2.54, Equation 5.2.53 reduces to

$$Wx[k+t+1] = \bigoplus_{i=0}^{t} L_s[i]Wx[k+i] \oplus \bigoplus_{i=0}^{t} L_u[i]u[k+i] \oplus \bigoplus_{i=0}^{t} L_y[i]y[k+i]. \quad (5.2.55)$$

Define $Wx[k] = s[k]$

$$s[k+t+1] = \bigoplus_{i=0}^{t} L_s[i]s[k+i] \oplus \bigoplus_{i=0}^{t} L_u[i]u[k+i] \oplus \bigoplus_{i=0}^{t} L_y[i]y[k+i]. \quad (5.2.56)$$

Then it is clear that the recursive equation given in Equation (5.2.56) solves $\mathcal{O}^t_{\mathtt{str}}(S,W)$ if $s[k] = Wx[k]$ for $k \leq t$. And the proposition is proved.

□

$\mathfrak{M}^t(S,W)$ is an equation of the form $L[t]U[t] = V[t]$, in which $U[t] = Z_S[t]H_S[t]$, $V[t] = R_S[t]H_S[t]$ is known and $L[t]$ is unknown. It is a well known fact that this kind of equation has solution if and only if $(V[t] \phi U[t])U[t] = V[t]$ and that in this case $L[t] = V[t] \phi U[t]$ is the greatest solution (see Baccelli et al. (1992)). Thus, the sufficient condition can be checked (and the parameters of L computed) very easily in polynomial time.

By inspection of the matrices $H_S[k]$ and $Z_S[k]$ in Equation (5.2.46) and Equation (5.2.49) respectively, one can see that $\mathfrak{M}^t(S,W)$ can also be written in the following form

$$\mathfrak{M}^t(S,W): \begin{cases} (i): WA^{t+1} = \\ \bigoplus_{i=0}^{t} \left(L_y[i]C \oplus L_s[i]W\right)A^i; \\ \text{For } \ j = 0,1,2,...,t: \\ (ii): WA^{t-j}B = L_u[j] \oplus L_y[j]G \oplus \\ \bigoplus_{i=j+1}^{t} \left(L_y[i]C \oplus L_s[i]W\right)A^{i-j-1}B. \end{cases}$$

In Gonçalves et al. (2014b), a sufficient condition was presented to solve $\mathcal{O}(S,W)$. It turns out that the sufficient equation presented is a special form of $\mathfrak{M}^t(S,W)$, which is a necessary and sufficient condition for solving $\mathcal{O}^t_{\mathtt{str}}(S,W)$. Indeed, by setting $G = \perp$ (in the previous work, only the dynamic $y[k] = Cx[k]$ was considered) and enforcing that $L_s[0{:}k] = \perp$ and $L_u[0{:}k] = \perp$, one obtains the condition presented in the aforementioned paper.

This outlines the fact that solving problem $\mathcal{O}^t_{\mathtt{str}}(S, W)$ can be useful to solve $\mathcal{O}(S, W)$. The results presented in Gonçalves et al. (2014b) show that if $G = \perp$, $L_s[0{:}k] = \perp$ and $L_u[0{:}k] = \perp$, solving $\mathcal{O}^t_{\mathtt{str}}(S, W)$ can also provide a solution to $\mathcal{O}(S, W)$. This condition can be weakened, and this will be the subject of discussion of the next subsection.

5.2.5 Solving $\mathcal{O}(S, W)$

In order to continue, two important definitions are necessary.

Definition 5.2.8. (*Observable coupled problem*) A problem $\mathcal{O}(S, W)$ is said to be *observable coupled* if W has no $\perp$ entries. □

Apparently, a problem being observable coupled seems to be a restrictive assumption. Indeed, if W has no $\perp$ entries, all the entries $s_j[k]$ grow with the same rate, which is the maximum rate in which an entry $x_i[k]$ grows. This alone implies that no problem in which it is desirable to observe variables with different rates cannot be observable coupled. However, by Assumption 5.2.3, the system in closed loop is strongly connected and thus it has only one rate. Hence, given a problem in which all the entries of $s_j[k]$ grow with the same rate, one can use very negative entries in the matrix W in order to make it without $\perp$ entries. For instance, in a system with 3 states $x_1[k], x_2[k]$ and $x_3[k]$, if one wishes to observe $s[k] = x_1[k]$, which is not a coupled problem, one can instead observe $\bar{s}[k] = x_1[k] \oplus (-N)x_2[k] \oplus (-N)x_3[k]$ for $N \neq \perp$. This problem is coupled because $W = (0\ -N\ -N)$ has no $\perp$ entries. Although the two problems are not exactly the same, $s[k] = \bar{s}[k]$ as long as $x_1[k] \succeq (-N)x_2[k]$ and $x_1[k] \succeq (-N)x_3[k]$, and these inequations hold provided that N is chosen judiciously. Then in practice, due to these observations, the problem being observable coupled is not a very strict assumption.

And

Definition 5.2.9. (*Structurally observable systems/problems, see Baccelli et al. (1992)*) A system S is said to be *structurally observable* if, in the respective TEG, for any state x_i there is at least one path to at least one output y_j. If, in the problem $\mathcal{O}(S, W)$, S is structurally observable, then the problem is also said to be structurally observable. □

The system being structurally observable is also not a restrictive assumption.

Definition 5.2.10. *(Extended matrices)* The following matrices are defined [2]

$$L_s^{\text{ex}}[t] \equiv \begin{pmatrix} \bot & I & \bot & \bot & \dots & \bot \\ \bot & \bot & I & \bot & \dots & \bot \\ \bot & \bot & \bot & I & \dots & \bot \\ \dots & \dots & \dots & \dots & \dots & \dots \\ L_s[0] & L_s[1] & L_s[2] & L_s[3] & \dots & L_s[t] \end{pmatrix} \tag{5.2.57}$$

$$L_u^{\text{ex}}[t] \equiv \begin{pmatrix} \bot & \bot & \bot & \bot & \dots & \bot \\ \bot & \bot & \bot & \bot & \dots & \bot \\ \bot & \bot & \bot & \bot & \dots & \bot \\ \dots & \dots & \dots & \dots & \dots & \dots \\ L_u[0] & L_u[1] & L_u[2] & L_u[3] & \dots & L_u[t] \end{pmatrix} \tag{5.2.58}$$

and

$$L_y^{\text{ex}}[t] \equiv \begin{pmatrix} \bot & \bot & \bot & \bot & \dots & \bot \\ \bot & \bot & \bot & \bot & \dots & \bot \\ \bot & \bot & \bot & \bot & \dots & \bot \\ \dots & \dots & \dots & \dots & \dots & \dots \\ L_y[0] & L_y[1] & L_y[2] & L_y[3] & \dots & L_y[t] \end{pmatrix}. \tag{5.2.59}$$

Further, let $W^{\text{ex}}[t] \equiv (W^T\ (WA)^T\ \dots\ (WA^t)^T)^T$ and $C^{\text{ex}}[t] \equiv (C^T\ (CA)^T\ \dots\ (CA^t)^T)^T$. □

With these definitions, Equation $\mathfrak{M}^t(S,W)-(i)$ can be written as

$$W^{\text{ex}}[t]A = L_s^{\text{ex}}[t]W^{\text{ex}}[t] \oplus L_y^{\text{ex}}[t]C^{\text{ex}}[t]. \tag{5.2.60}$$

Then, the following lemma can be stated.

Lemma 5.2.5. *(Bound on the spectral radius)* If the problem $\mathcal{O}(S,W)$ is observable coupled, then all the solutions $L[t]$ of $\mathfrak{M}^t(S,W)$ are such that $\rho(L_s^{\text{ex}}[t]) \preceq \rho(A)$.

Proof. Considering Equation (5.2.60), one can conclude that

$$W^{\text{ex}}[t]A \succeq L_s^{\text{ex}}[t]W^{\text{ex}}[t]. \tag{5.2.61}$$

Let $l[t]^T$ be a left eigenvector of $L_s^{\text{ex}}[t]$ associated to the greatest eigenvalue, and a the right eigenvector of A associated with the greatest eigenvalue. Pre-multiplying Equation (5.2.61) by $l[t]^T$ and

[2] Note that $L_s^{\text{ex}}[t]$ is a *companion matrix*.

post-multiplying by a one can conclude that

$$(l[t]^T W^{\mathsf{ex}}[t]a)\rho(A) \succeq (l[t]^T W^{\mathsf{ex}}[t]a)\rho(L_s^{\mathsf{ex}}[t]). \tag{5.2.62}$$

Since $A \succeq I$ (see Assumption 5.2.1), it is easy to see that for any k $WA^k \succeq W$. Further, by the fact that $\mathscr{O}(S,W)$ is observable coupled, by hypothesis, this implies that $W^{\mathsf{ex}}[t]$ has no $\perp$ entries. Hence, since $l[t] \neq \perp$ and $a \neq \perp$, one can easily see that $l[t]^T W^{\mathsf{ex}}[t]a \neq \perp$. By Equation (5.2.62), this implies $\rho(A) \succeq \rho(L_s^{\mathsf{ex}}[t])$ and the lemma is proved. □

Keeping in mind this lemma, another definition can be made.

Definition 5.2.11. (*Observable non-critical problem*) A problem $\mathscr{O}(S,W)$ is said to be *observable non-critical* if there exists a t such that $\mathfrak{M}^t(S,W)$ has a solution with $\rho(L_s^{\mathsf{ex}}[t]) \prec \rho(A)$ (note the strict inequality). Otherwise it is said to be *observable critical*. □

Before stating the main result of this chapter, two lemmas are necessary.

Lemma 5.2.6. *(Constructing a solution with no $\perp$ entries)* Consider a tropical affine equation $Up = v$ for the unknown p. Suppose v has no $\perp$ entries. Let $\mathscr{D}(p)$ be the set of non-$\perp$ entries of p (that is, $i \in \mathscr{D}(p)$ if and only if $p_i \neq \perp$).

Let p be a particular solution of this equation. Then, there always exists another solution $\overline{p}$ with no $\perp$ entries such that $\overline{p}_i = p_i$, $\forall i \in \mathscr{D}(p)$.

Proof. Let $u[j]$ be the j^{th} column of U. Then

$$\bigoplus_{j\in\mathscr{D}(p)} u[j]p_j = v. \tag{5.2.63}$$

Consider also the greatest solution q_j of

$$u[j]q_j \preceq v \tag{5.2.64}$$

for all $j \notin \mathscr{D}(p)$. Since v has no $\perp$ entries, it is clear that this greatest solution q_j is non $\perp$ for all $j \notin \mathscr{D}(p)$. It is also clear that $\overline{p}$ such that $\overline{p}_j = p_j$ if $j \in \mathscr{D}(p)$ and q_j otherwise is such that $U\overline{p} = v$. Further, $\overline{p}$ has no $\perp$ entries. And the lemma is proved.

□

The next lemma is quite technical, but it is necessary to continue the development.

Lemma 5.2.7. *(Consensus of sequences)* Consider two sequences

$$\begin{aligned} s_{\mathtt{A}}[k+1] &= L s_{\mathtt{A}}[k] \oplus d_{\mathtt{A}}[k], \\ s_{\mathtt{B}}[k+1] &= L s_{\mathtt{B}}[k] \oplus d_{\mathtt{B}}[k], \end{aligned} \tag{5.2.65}$$

with initial conditions $s_{\mathtt{A}}[0]$ and $s_{\mathtt{B}}[0]$, respectively. Consider that

1. There exists a finite $\overline{k}$ such that $d_{\mathtt{A}}[k] = d_{\mathtt{B}}[k]\ \forall k \geq \overline{k}$;

2. There exists a non-empty set of indexes $\mathscr{J}$ of entries of $d_{\mathtt{A}}[k]$ ($d_{\mathtt{B}}[k]$ too, since they are eventually equal by the first hypothesis) such that

$$\lim_{k\to\infty} \frac{\{d_{\mathtt{A}}[k]\}_j}{k} = \lim_{k\to\infty} \frac{\{d_{\mathtt{B}}[k]\}_j}{k} > \rho(L)\ \ \forall\ j \in \mathscr{J}; \tag{5.2.66}$$

3. In the precedence graph of L, generated by setting an arc from node $i[1]$ to node $i[2]$ if and only if $L_{i[2]i[1]} \neq \perp$, for all $i \notin \mathscr{J}$, there is a path from at least one node $j \in \mathscr{J}$ to i.

Then, there exists a finite $\hat{k}$ such that $s_{\mathtt{A}}[k] = s_{\mathtt{B}}[k]\ \forall k \geq \hat{k}$.

Proof. By Hypothesis 3 in the statement, for any node $i \notin \mathscr{J}$ there is a path from at least one $j \in \mathscr{J}$ to this node i. Let $f(i)$ denote the minimal number of nodes one must traverse to go from a node in $\mathscr{J}$ to a node i (so $f(i) < \infty$ for any $i \notin \mathscr{J}$). This means that $\{L^{f(i)}\}_{ij} \neq \perp\ \forall i \notin \mathscr{J}$.

Then, iterating the sequences in Equation (5.2.65)

$$\begin{aligned} s_{\mathtt{A}}[l+r+1] &= L^{r+1} s_{\mathtt{A}}[l] \oplus \bigoplus_{n=0}^{r} L^{r-n} d_{\mathtt{A}}[l+n], \\ s_{\mathtt{B}}[l+r+1] &= L^{r+1} s_{\mathtt{B}}[l] \oplus \bigoplus_{n=0}^{r} L^{r-n} d_{\mathtt{B}}[l+n]. \end{aligned} \tag{5.2.67}$$

Let $r = \max_{i\notin\mathscr{J}} f(i)$ if there is an $i \notin \mathscr{J}$ or $r = 0$ otherwise. Either way, this value is finite, by hypothesis. Then, it is necessary to see that *all* the entries of both

$$\begin{aligned} e_{\mathtt{A}}[k] &\equiv \bigoplus_{n=0}^{r} L^{r-n} d_{\mathtt{A}}[k+n], \\ e_{\mathtt{B}}[k] &\equiv \bigoplus_{n=0}^{r} L^{r-n} d_{\mathtt{B}}[k+n] \end{aligned} \tag{5.2.68}$$

grow with a rate strictly greater than $\rho(L)$. Indeed, for $e_{\mathtt{A}}$, for any entry $i \in \mathscr{J}$ this is clear because the summand has the term $d_{\mathtt{A}}[k+r]$, in which the i^{th} entry grows with a rate greater than $\rho(L)$ by the Hypothesis 2 presented in the statement. This is also true for all other entries $i \notin \mathscr{J}$, because $e_{\mathtt{A}}$ has the term $L^{f(i)} d_{\mathtt{A}}[k+r-f(i)]$ (recall that $r = \max_{i \notin \mathscr{J}} f(i)$). In this case $\{L^{f(i)}\}_{ij} \neq \perp$ for at least one $j \in \mathscr{J}$, and hence the i^{th} entry grows at least as much as $\{L^{f(i)}\}_{ij}\{d_{\mathtt{A}}[k+r-f(i)]\}_j$, which in turn grows with a rate greater than $\rho(L)$. An analogous statement holds for $e_{\mathtt{B}}[k]$.

Hence

$$
\begin{aligned}
s_{\mathtt{A}}[l+r+1] &= L^{r+1} s_{\mathtt{A}}[l] \oplus e_{\mathtt{A}}[l], \\
s_{\mathtt{B}}[l+r+1] &= L^{r+1} s_{\mathtt{B}}[l] \oplus e_{\mathtt{B}}[l]
\end{aligned}
\tag{5.2.69}
$$

in which the fact that $e_{\mathtt{A}}[k]$ (respectively $e_{\mathtt{B}}[k]$) has all their entries growing strictly faster than $\rho(L)$ was already established. Then, iterating again the sequences in Equation (5.2.69)

$$
\begin{aligned}
& s_{\mathtt{A}}[l+(m+1)\cdot(r+1)] = \\
L^{(m+1)\cdot(r+1)} s_{\mathtt{A}}[l] \oplus & \bigoplus_{n=0}^{m} L^{r\cdot(m-n)} e_{\mathtt{A}}[l+n\cdot(r+1)], \\
& s_{\mathtt{B}}[l+(m+1)\cdot(r+1)] = \\
L^{(m+1)\cdot(r+1)} s_{\mathtt{B}}[l] \oplus & \bigoplus_{n=0}^{m} L^{r\cdot(m-n)} e_{\mathtt{B}}[l+n\cdot(r+1)].
\end{aligned}
\tag{5.2.70}
$$

Choose l large enough so $l \geq \overline{k}$. So $\bigoplus_{n=0}^{m} L^{r\cdot(m-n)} e_{\mathtt{A}}[l+n\cdot(r+1)] = \bigoplus_{n=0}^{m} L^{r\cdot(m-n)} e_{\mathtt{B}}[l+n\cdot(r+1)]$ (see Hypothesis 1 in the statement of the lemma). Let this common value be denominated simply as $\overline{d}[m]$. Hence

$$
\begin{aligned}
s_{\mathtt{A}}[l+(m+1)\cdot(r+1)] &= L^{(m+1)\cdot(r+1)} s_{\mathtt{A}}[l] \oplus \overline{d}[m], \\
s_{\mathtt{B}}[l+(m+1)\cdot(r+1)] &= L^{(m+1)\cdot(r+1)} s_{\mathtt{B}}[l] \oplus \overline{d}[m].
\end{aligned}
\tag{5.2.71}
$$

Since $\overline{d}[m] \succeq e_{\mathtt{A}}[l+m\cdot(r+1)]$, this means that as m grows, all the entries of $\overline{d}[m]$ grow with a rate greater than $\rho(L)^{r+1}$, since $e_{\mathtt{A}}[k]$ grows with a rate greater than $\rho(L)$. This implies that m can be chosen large enough (in function of $s_{\mathtt{A}}[l]$ and $s_{\mathtt{B}}[l]$) so

$$
\begin{aligned}
\overline{d}[m] &\succeq L^{(m+1)\cdot(r+1)} s_{\mathtt{A}}[l], \\
\overline{d}[m] &\succeq L^{(m+1)\cdot(r+1)} s_{\mathtt{B}}[l],
\end{aligned}
\tag{5.2.72}
$$

since $L^{(m+1)\cdot(r+1)}$ grows with a rate $\rho(L)^{r+1}$ with m. This establishes that, for such l, m and r

$$s_{\mathtt{A}}[l+(m+1)\cdot(r+1)] = \overline{d}[m] = s_{\mathtt{B}}[l+(m+1)\cdot(r+1)]. \tag{5.2.73}$$

It should be clear then that $\hat{k} = l + (m+1)\cdot(r+1)$ can be taken. Indeed, since $\hat{k} \geq l \geq \overline{k}$, $d_{\mathtt{A}}[k] = d_{\mathtt{B}}[k]$ for all $k \geq \hat{k}$ and one can easily prove by induction in Equation (5.2.65), with the basis case $s_{\mathtt{A}}[\hat{k}] = s_{\mathtt{B}}[\hat{k}]$ that was already proved, that $s_{\mathtt{A}}[k] = s_{\mathtt{B}}[k]$ $\forall k \geq \hat{k}$. And the lemma is proved.

□

And then, the principal result of this chapter can be stated

Proposition 5.2.3. (*Necessary and sufficient condition for observable coupled, structurally observable non-critical problems*) A problem $\mathscr{O}(S, W)$ which is observable coupled, structurally observable and observable non-critical has a solution if and only if there exists a t such that $\mathfrak{M}^t(S, W)$ has a solution. Further, provided a specific solution $L[t]$ to $\mathfrak{M}^t(S, W)$, the observer can be implemented by the recursive equation

$$s[k+t+1] = \bigoplus_{i=0}^{t} L_s[i]s[k+i] \oplus \bigoplus_{i=0}^{t} L_u[i]u[k+i] \oplus \bigoplus_{i=0}^{t} L_y[i]y[k+i] \quad k \geq 0. \tag{5.2.74}$$

for any set of initial conditions $s[0{:}t]$, $y[0{:}t]$ and $u[0{:}t]$.

Proof. Only if: by Lemma 5.2.2, $\mathscr{O}(S, W)$ has a solution only if there is a t such that $\mathscr{O}^t_{\mathtt{str}}(S, W)$ is solvable. According to Proposition 5.2.2, the latter is solvable if and only if, and particularly only if, $\mathfrak{M}^t(S, W)$ has a solution. This proves the necessity (indeed, this holds true even if the problem is not observable coupled or observable non-critical).

If: Due to Assumption 5.2.1 and the fact that W has no $\perp$ entries (it is observable coupled, by hypothesis), it is easy to see that neither WA^k nor WA^kB have $\perp$ entries for all k. This implies that for any solution $L[t]$ obtained for $\mathfrak{M}^t(S, W)$, one can derive another solution $\overline{L}[t]$ such that $\overline{L}_s^T[0{:}t]^T = L_s^T[0{:}t]^T$ and also $\overline{L}_y^T[0{:}t]^T$ has no $\perp$ entries (see Lemma 5.2.6, noting that $\mathfrak{M}^t(S, W)$ is a tropical affine equation).

Due to the fact that the problem is observable non-critical, by hypothesis, there exists a t and a solution $L[t]$ to $\mathscr{O}^t_{\mathtt{str}}(S, W)$ such that $\rho(L_s^{\mathtt{ex}}[t]) \prec \rho(A)$. Further, due to the previous discussion, this solution can be chosen so $L_y^T[0{:}t]^T$ has no $\perp$ entries (the overline "$\overline{L}$" will be dropped in L, for simplicity of notation). Now, it will be argued that, in this condition, the long term behavior of the recursive sequence presented in Equation 5.2.74 is independent of the initial conditions $s[0{:}t]$, $y[0{:}t]$ and $u[0{:}t]$.

Equation (5.2.74) can be written as

$$s[k+t+1{:}k+1] = L_s^{\mathtt{ex}}[t]s[k+t{:}k] \oplus L_u^{\mathtt{ex}}[t]u[k+t{:}k] \oplus L_y^{\mathtt{ex}}[t]y[k+t{:}k]. \tag{5.2.75}$$

Let

$$d[k] \equiv L_u^{\mathtt{ex}}[t]u[k+t{:}k] \oplus L_y^{\mathtt{ex}}[t]y[k+t{:}k]. \tag{5.2.76}$$

Then, Lemma 5.2.7 will be applied. Note that

1. Given two pairs of initial conditions $\{y_{\mathtt{A}}[0{:}t], u_{\mathtt{A}}[0{:}t]\}$ and $\{y_{\mathtt{B}}[0{:}t], u_{\mathtt{B}}[0{:}t]\}$, there exists a finite k such that their respective sequences $d_{\mathtt{A}}[k], d_{\mathtt{B}}[k]$ satisfies $d_{\mathtt{A}}[k] = d_{\mathtt{B}}[k]\ \forall k \geq \overline{k}$. Indeed, $\overline{k} = t+1$ can be chosen;

2. There exists a non-empty set of indexes $\mathscr{J}$ of entries of $d[k]$ such that

$$\lim_{k\to\infty} \frac{\{d[k]\}_j}{k} > \rho(L_s^{\mathtt{ex}}[t])\ \ \forall\, j \in \mathscr{J}, \tag{5.2.77}$$

   This is because the matrix $L_y^{\mathtt{ex}}[t]$ has no $\perp$ entries, which in turn, together with the fact that the problem is structurally observable, imply that the last g entries (remember that $W \in \mathbb{T}_{\mathtt{max}}^{g\times n}$) of $d[k]$ grow with a rate of at least $\rho(A)$. Indeed, this happens because, since S is structurally observable, there is a path from any state $x_i[k]$ which grows with the rate $\rho(A)$ (such state exists, because by Assumption 5.2.1 $x[0]$ has no $\perp$ entries) to at least one output $y_q[k]$. Since $L_y^{\mathtt{ex}}[t]$ has no $\perp$ entries, this means that all entries of $L_y[t]y[k{:}k-t]$ grow with the rate of at least $\rho(A)$. Since $\rho(L_s^{\mathtt{ex}}[t]) < \rho(A)$ by hypothesis, this means that the growth rate of all the g entries of $d[k]$, which is minored by $L_y[t]y[k{:}k-t]$, are strictly greater than $\rho(L_s^{\mathtt{ex}}[t])$. Hence, $\mathscr{J}$ can be chosen as the last g entries of $L_s^{\mathtt{ex}}[t]$;

3. In the precedence graph of $L_y^{\mathtt{ex}}[t]$, there is always a path from the g last nodes to all the other ones. This can be seen by inspecting Equation (5.2.57). See also Figure 5.1 for a pictoric explanation.

Hence, the application of Lemma 5.2.7 permits one to conclude that the initial conditions $s[0{:}t]$, $y[0{:}t]$ and $u[0{:}t]$ are irrelevant for $s[k]$ in long terms. Since there exists at least one particular choice of $s[0{:}t]$, $y[0{:}t]$ and $u[0{:}t]$ such that the sequence $s[k]$ converges to $Wx[k]$ (pick $s[k] = Wx[k]$

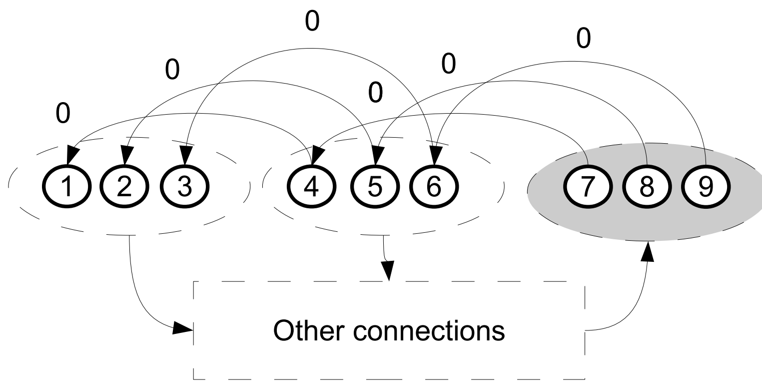

Figure 5.1: Precedence graph for the companion matrix $L_s^{\mathsf{ex}}[2]$ when $g = 3$ (see Equation (5.2.57)). Only the connections relative to the first six rows of the matrix $L_s^{\mathsf{ex}}[2]$ are explicitly shown (in the upper connections), while the ones relative to the third final rows are only represented by the "Other connections" rectangle. If $\mathscr{J}$ is the grayed set of nodes ($\{7,8,9\}$), then there is always a path from at least one node $j \in \mathscr{J}$ to a node $i \notin \mathscr{J}$.

for $k = 0,...,t$ and $y[0{:}t]$, $u[0{:}t]$ as the real system outputs and inputs, respectively, see Proposition 5.2.2), this implies that for any choice eventually the sequence will converge to the desired value.

And the proposition is proved.

□

Remark 5.2.3. Note that, in the light of Proposition 5.2.3, in order to solve $\mathscr{O}(S,W)$ with the proposed methodology, it is necessary to solve $\mathfrak{M}^t(S,W)$ under the constraint $\rho(L_s^{\mathsf{ex}}[t]) \prec \rho(A)$. The constraint $\rho(L_s^{\mathsf{ex}}[t]) \prec \rho(A)$ can be quite difficult to deal with, but it can be weakened to the constraint $\{L_s^{\mathsf{ex}}[t]\}_{ij} \preceq \rho(A) - \delta$, in which $\delta > 0$ is a given tolerance. It is clear that in this case, $\{L_s[0:t]\}_{ij} \preceq \rho(A) - \delta$ implies $\rho(L_s^{\mathsf{ex}}[0:t]) \prec \rho(A)$ (but the converse is not, in general, true). This weakened constraint is much easier to deal with.

Hence, one can choose a $\delta > 0$ and proceed to solve the following tropical affine equation for $L[t]$:

$$L[t]Z_S[t]H_S[t] = WR_S[t]H_S[t];$$
$$\{L_s[0:t]\}_{ij} \preceq \rho(A) - \delta \ \forall\ i,j. \tag{5.2.78}$$

However, it is also a concern that the matrix $L[t]$ is *causal*, because otherwise the implementation may be impossible (it would require the foretelling of events). In that case, Lemma 4.6.3 may be used. Disregarding for a while the constraint $\{L_s[0:t]\}_{ij} \preceq \rho(A)-\delta$, since the equation $L[t]Z_S[t]H_S[t] = WR_S[t]H_S[t]$ is an one-sided tropical affine equation, one concludes by using Lemma 4.6.3 that a causal solution to this equation exists if and only if

$$\hat{L}[t] = C_{\mathrm{cp}}((WR_S[t]H_S[t]) \not\phi (Z_S[t]H_S[t])) \tag{5.2.79}$$

is a solution (and further, it is the greatest one). After this solution is computed, one can impose the constraint $\{L_s[0:t]\}_{ij} \preceq \rho(A)-\delta$ by taking the infimum between $\rho(A)-\delta$ and the entries of $\hat{L}[t]$ respective to $L_s[0:t]$ (remember that $\hat{L}[t]$ can be decomposed as $\hat{L}[t] = (\hat{L}_u^T[0{:}t]^T \ \hat{L}_y^T[0{:}t]^T \ \hat{L}_s^T[0{:}t]^T)$). Then, one just needs to check if this modified $\hat{L}[t]$ is a solution to $L[t]Z_S[t]H_S[t] = WR_S[t]H_S[t]$, because it clearly will be causal, as long as $\rho(A)-\delta \geq 0$ or $\rho(A)-\delta = \perp$, and will respect the constraint $\{L_s[0:t]\}_{ij} \preceq \rho(A)-\delta$.□

Hence

Algorithm 5.2.1. *Observer for solving $\mathcal{O}$*

1. Find a solution $L[t]$ to $\mathfrak{M}^t(S,W)$ for a t with the constraint $\rho(L_s^{\mathrm{ex}}[t]) \prec \rho(A)$ and $L[t]$ causal (see Remark 5.2.3);

2. Use as observer

$$s[k+1] = \bigoplus_{i=0}^{t} L_s[i]s[k-t+i] \oplus \bigoplus_{i=0}^{t} L_u[i]u[k-t+i] \oplus \bigoplus_{i=0}^{t} L_y[i]y[k-t+i]. \tag{5.2.80}$$

   with any initial condition $s[-t{:}0]$, $y[-t{:}0]$ and $u[-t{:}0]$.

5.3 An illustrative problem

As mentioned, one of the possible (and perhaps the main) applications of the proposed methodology is in the implementation of a state feedback control law of the form $u[k] = Fx[k]$ in situations in which the matrix controller F is known (previously designed) but the state $x[k]$ is not. In this case, the problem can be overcome by estimating the value $Fx[k]$. So, this problem can be easily recast as the (generalized) observation problem presented in this chapter, by choosing $W = F$. However, the observer takes some iterations to achieve the correct value and this implies perturbations in the system. Hence, in order to the proposed methodology to be useful, the feedback controller must be robust in the sense that it can reject any kind of eventual perturbations, a good example being the Spectral Regulator proposed in Chapter 4. In principle, the implementation of the observer could degrade the controller performance, and this is much as true as larger is the value of t in $\mathfrak{M}^t(S, W)$. This will be, indeed, observed in simulations.

In order to illustrate the methodology, consider the problem considered in Maia et al. (2013) which models a small traffic light. The matrices A and B are (see Figure 5.2)

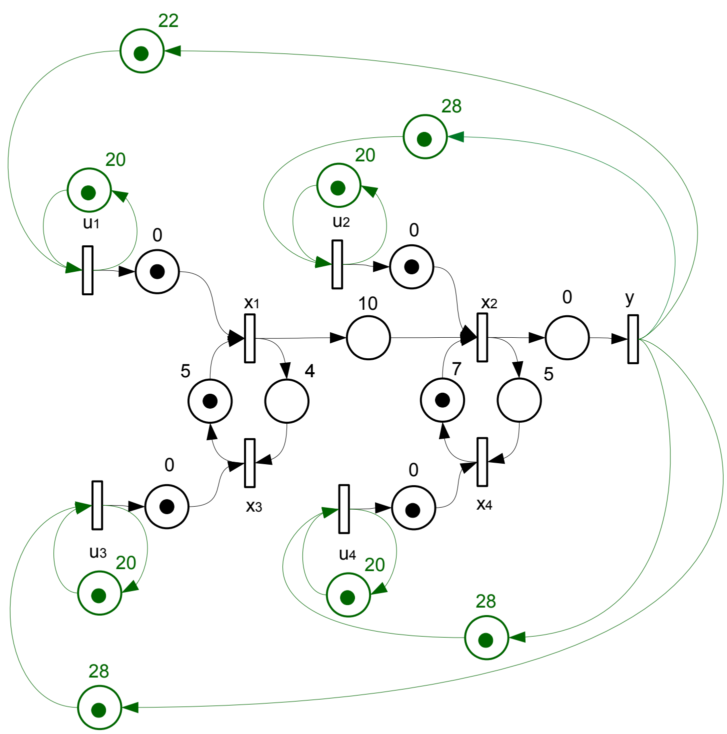

Figure 5.2: The TEG for a small traffic light. In green, the implementation of the output feedback controller.

$$A = \begin{pmatrix} 0 & \perp & 5 & \perp \\ 10 & 0 & 15 & 7 \\ 4 & \perp & 9 & \perp \\ 15 & 5 & 20 & 12 \end{pmatrix}, \; B = \begin{pmatrix} 0 & \perp & \perp & \perp \\ 10 & 0 & \perp & \perp \\ 4 & \perp & 0 & \perp \\ 15 & 5 & \perp & 0 \end{pmatrix}. \qquad (5.3.1)$$

The constraint is

$$x[k] = \begin{pmatrix} 0 & -15 & -15 & -30 \\ 10 & 0 & -5 & -15 \\ 6 & -11 & 0 & -26 \\ 15 & 5 & 0 & 0 \end{pmatrix} x[k] \tag{5.3.2}$$

which implies that the problem is controllable coupled. Using the methodology proposed in Chapter 4, it can be shown that the controller $u[k] = F_{\mathrm{SR}} x[k] = \mu \zeta^T x[k]$ with

$$\begin{aligned} \mu &= (5\ 11\ 11\ 11)^T; \\ \zeta &= (0\ 0\ 0\ 0)^T \end{aligned} \tag{5.3.3}$$

solves the problem.

The Spectral Regulator proposed in Chapter 4 can always be written in the form $F_{\mathrm{SR}} = \mu\zeta^T$ for vectors μ and ζ. This factorization is highly proficuous for the proposed methodology. Indeed, using this feedback

$$x[k+1] = Ax[k] \oplus (B\mu)(\zeta^T x[k]). \tag{5.3.4}$$

In light of this, consider the reduced system

$$S_{\mathtt{red}}(S,\mu)\ :\ x[k+1] = Ax[k] \oplus B\mu v[k]. \tag{5.3.5}$$

Then, in order to implement this controller one can consider this new system in which $v[k]$ is scalar, and the original controller can be recovered if $v[k] = \zeta^T x[k]$. This implies that it is only necessary to observe a scalar functional $\zeta^T x[k]$ in this new system.

Suppose one only observes $x_2[k]$, that is $C = (\perp\ \ 0\ \ \perp\ \perp)$ and $y[k] = x_2[k]$ (see Figure 5.2). Then, the task is to implement the state feedback controller using only this output and the inputs. To this end, it is possible to solve $\mathfrak{M}^t(S_{\mathtt{red}}(S,\mu),\zeta^T)$ for the reduced system with $t = 1$ and the constraint $\rho(L_s^{\mathtt{ex}}[t]) \preceq \perp$ (it is desirable that the observer is as fast as possible) and the causality condition (see Remark 5.2.3). Then, it is possible to obtain $L_s[0] = L_s[1] = \perp$ and $L_y[0] = L_y[1] = 17$, $L_u[0] = \perp$ and $L_u[1] = 20$ (note that, despite the notation "u" in the matrices $L_u[0], L_u[1]$, the control action of the reduced system is v). Hence, Equation (5.2.80) reduces to

$$s[k+1] = 20v[k] \oplus 17y[k] \oplus 17y[k-1] = 20v[k] \oplus 17y[k] \tag{5.3.6}$$

since $y[k] \succeq y[k-1]$. As the control input will be chosen as $\zeta^T x[k]$, $v[k] = s[k]$ and hence one has the dynamical equation for the control action of the reduced system

$$v[k+1] = 20v[k] \oplus 17y[k] \tag{5.3.7}$$

in which the initial conditions $v[-1]$ and $y[-1]$ can be chosen in an arbitrary manner. Therefore, post-multiplying both sides of Equation (5.3.7) by μ, it is easy to see that the control input $u[k] = \mu v[k]$ of the original system can be computed according to the dynamical equation

$$u[k+1] = 20u[k] \oplus \begin{pmatrix} 22 \\ 28 \\ 28 \\ 28 \end{pmatrix} y[k] \tag{5.3.8}$$

which is a dynamical (in relation to the outputs) controller.

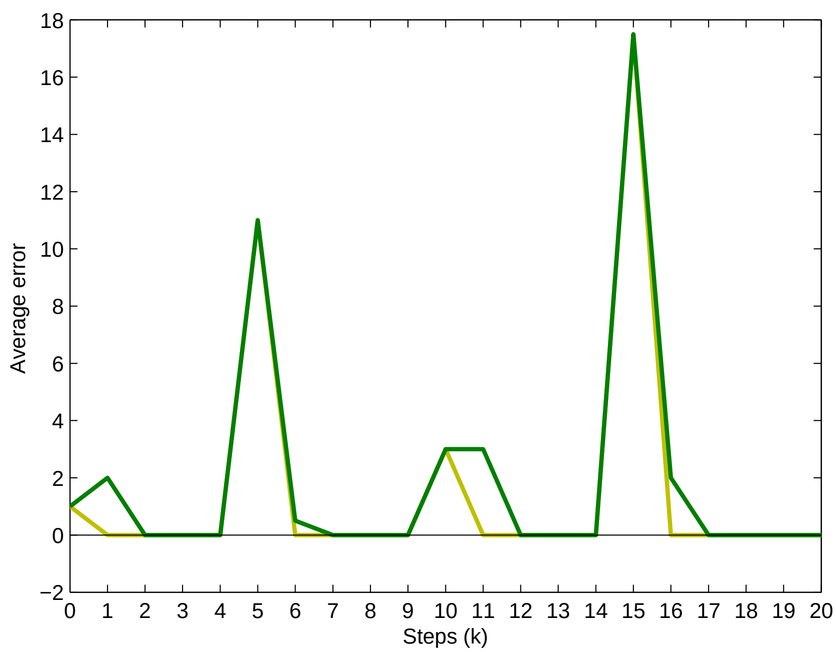

Figure 5.3: Average error at each step. Yellow for the state feedback and green for the output feedback.

in which the initial conditions $u[-1]$ and $y[-1]$ can be chosen in an arbitrary manner. See Figure 5.2 for the implementation.

The performance of both the state feedback controller and output controller will now be tested and compared. For fairness, in both cases the same initial condition was considered and the same perturbations were inflicted in them. 21 steps were simulated (from $k = 0$ to $k = 20$).

The perturbations are

- At $k = 4$, a delay of 20 time units was added at x_1 and 15 time units at x_3;
- At $k = 9$, a delay of 12 time units was added at x_2 and 20 time units at x_4;
- At $k = 14$, a delay of 8 time units was added at x_1 and 30 time units at x_2.

Consider the initial condition

$$x[0] = (11\ 27\ 15\ 32)^T \tag{5.3.9}$$

chosen at random. The initial conditions $u[-1], y[-1]$ were chosen as $\perp$. In Figure 5.3, it is possible to see the average error from the constraint set ($\hat{e}[k] = \frac{1}{4}\sum_{i=1}^{4} e_i[k]$ in which $e[k] = Dx[k] - x[k]$). One can see that, clearly, the insertion of the observer degrades slightly the performance of the controller, since instead of only one step it takes two steps to totally reject the perturbation.

5.4 Conclusion

In this chapter, a condition was proposed for handling the observer problem in tropical setting, which received attention from researchers only very recently. The condition is necessary and sufficient for a specific class of problems, the so-called *(observable coupled) observable non-critical systems*, mirroring the results obtained in Chapter 4.

As discussed in Chapter 4, one natural question is if this class of problems ((observable coupled) observable non-critical systems) covers a wide range of applications. Unfortunately, there is a scarcity of observation problems in the literature so the proposed methodology was tested in only few cases. Nevertheless, it was successful in all instances. Despite this, an investigation is necessary before claiming that the method is widely applicable (as it was done in the case of the Spectral Regulator proposed in Chapter 4). In the next chapter, one will show a practical application, using a plant of an assembly line.

Chapter 6

On the Practical Implementation of the Regulator and Observer

In order to show the applicability of the approaches proposed in Chapter 4 and Chapter 5, this chapter will show the implementation of the Spectral Regulator and the observer in a real plant installed in the *Université d'Angers*.

Although the proposed problem is not very complex, it still serves as a convincing "proof of concept" that the approaches are practical and can be used in real situations. It is worthy noting that, to the author's knowledge, this is the first (at least published) real implementation of feedback controllers and observers for TEGs for this class of problems (regulation).

6.1 Overview

The system located at the *Laboratoire Angevin de Recherche en Ingénierie des Systèmes (LARIS)* at the *Université d'Angers* is a conveyor belt system that moves pallets through circuits (see Figure 6.1). In its more general configuration, it allows by external signals the blocking of pallets using *buttons* located in different parts of the system (see Figure 6.2) and also the dynamic change of the paths that the pallets can follow through. The time in which is desirable to turn on or off the buttons or to make the path modifications are the *inputs* of the system. Further, there are many proximity sensors (see Figure 6.2) along the circuits which can detect the presence of the pallet. With this information, it is possible to have as *outputs* the times in which a pallet (any pallet, there is not, in principle, any distinction between them) passed through a given point. The entire system receives commands from a programable logic controller, which in turn receives actions either directly from its user interface or through a C++ program at a computer.

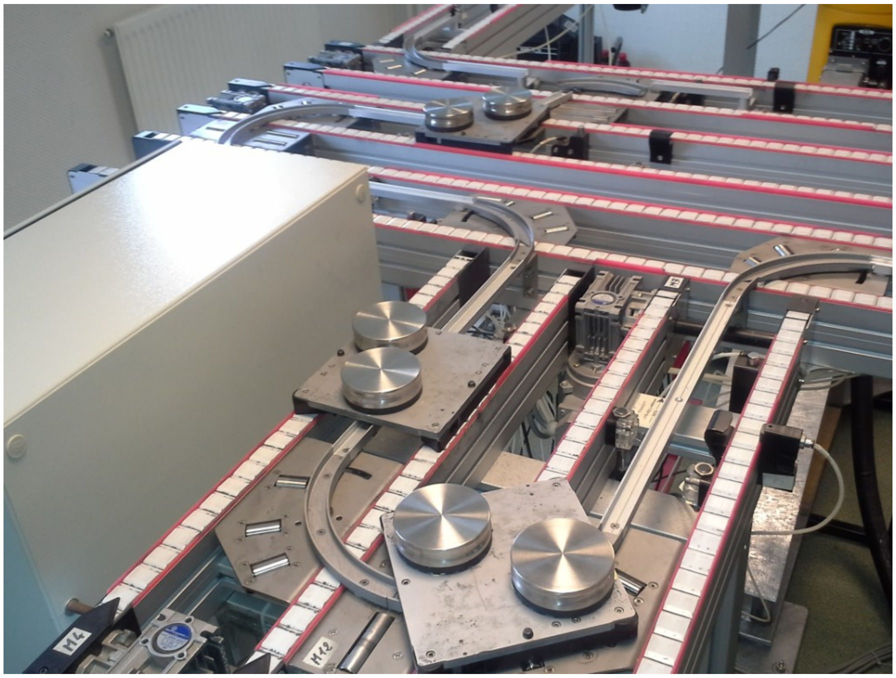

Figure 6.1: Photo of the system in the *Laboratoire Angevin de Recherche en Ingénierie des Systèmes (LARIS)* at the *Université d'Angers*.

In order for the system to be modeled by a TEG, the paths need to be static through all the experiment (so, no changing of paths are allowed). Further, the buttons are programmed so, when they are turned off (that is, the button is down and the pallet is allowed to move), they automatically go on (up) again in 2 seconds. This guarantees that if more than one pallet is waiting in line, when the button fires one time, only one pallet continues (see Figure 6.2).

Figure 6.3 presents the schematics of the system, viewed from the top. There are two independent circuits, ten buttons ($B1$ to $B10$) and ten proximity sensors (one sensor just before each button). The upper circuit has three pallets, all of them located initially just before $B1$. The lower circuit has also three pallets, but two of them located just before $B5$ and another just before $B6$. The pallets in the upper circuit move clockwise while the ones in the lower circuit move counter-clockwise. For each *stretch* between two successive buttons (for instance, the stretch $B1 \to B2$), there is an associated timing and also an associated capacity of pallets. The timing gives the time that a pallet needs to go from just before the initial button to the successive one when there is nothing in the path (thus, it is the *minimal* time). These timings were obtained through multiple experiments, and the average

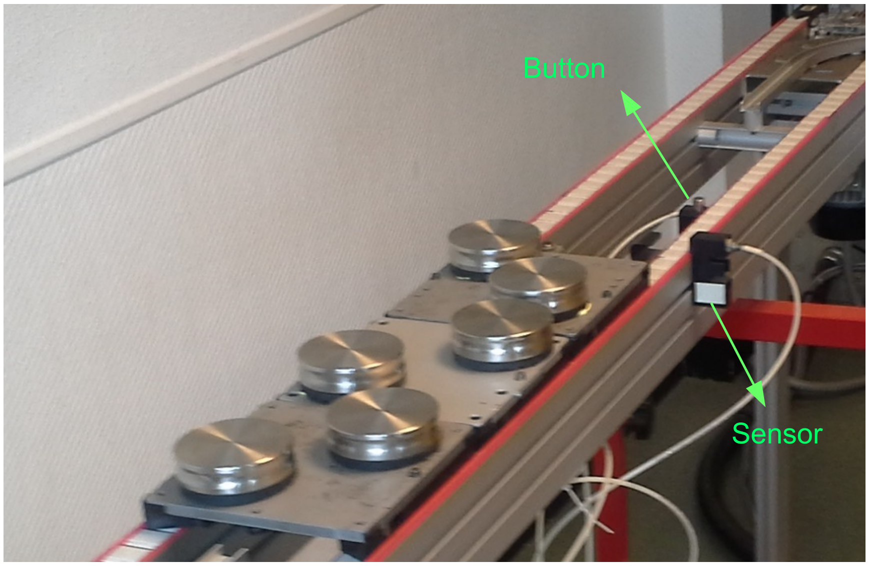

Figure 6.2: Three pallets in a line, approaching a button. When the button "fires" once, only one pallet continues moving and the other two continue waiting for another firing of the button.

of the results was taken as the value. Thus, from Figure 6.3 it is possible to conclude that, without anything in the path, a pallet takes in average 8 seconds [1] to go from just before $B1$ to just before $B2$. The capacity tells the maximum number of tokens that this stretch can hold. Thus, the stretch $B1 \rightarrow B2$ can hold at most 3 pallets. The system is programmed so, if the following path is full, the button will not fire (go down). This capacity constraint is inconsequential for the upper loop, since all the capacities are three and there is only three pallets in the upper loop, but it is important for the lower one.

As programmed, in order for a button to fire it is necessary that three conditions hold: (i) there must exist one *presence token* for that button, that is, there must be a pallet waiting just before a button, (ii) there must exist at least one *capacity token* for that button, that is, there must exist at least one free space for a pallet in the following stretch (so, for example, $B5$ is only allowed to fire if there is at most 1 pallet in the stretch $B5 \rightarrow B6$) and finally (iii) there must exists at least one *control token*, which represents an external action in the system. The first two kind of tokens obviously represent a "physical" constraint of the system, while the third one represents a logical constraint in which the engineer can act to obtain a desired behaviour. When the button fires, one of each token is consumed.

[1] Of course, the average of the values was not exactly an integer amount of seconds. Rounded numbers were used.

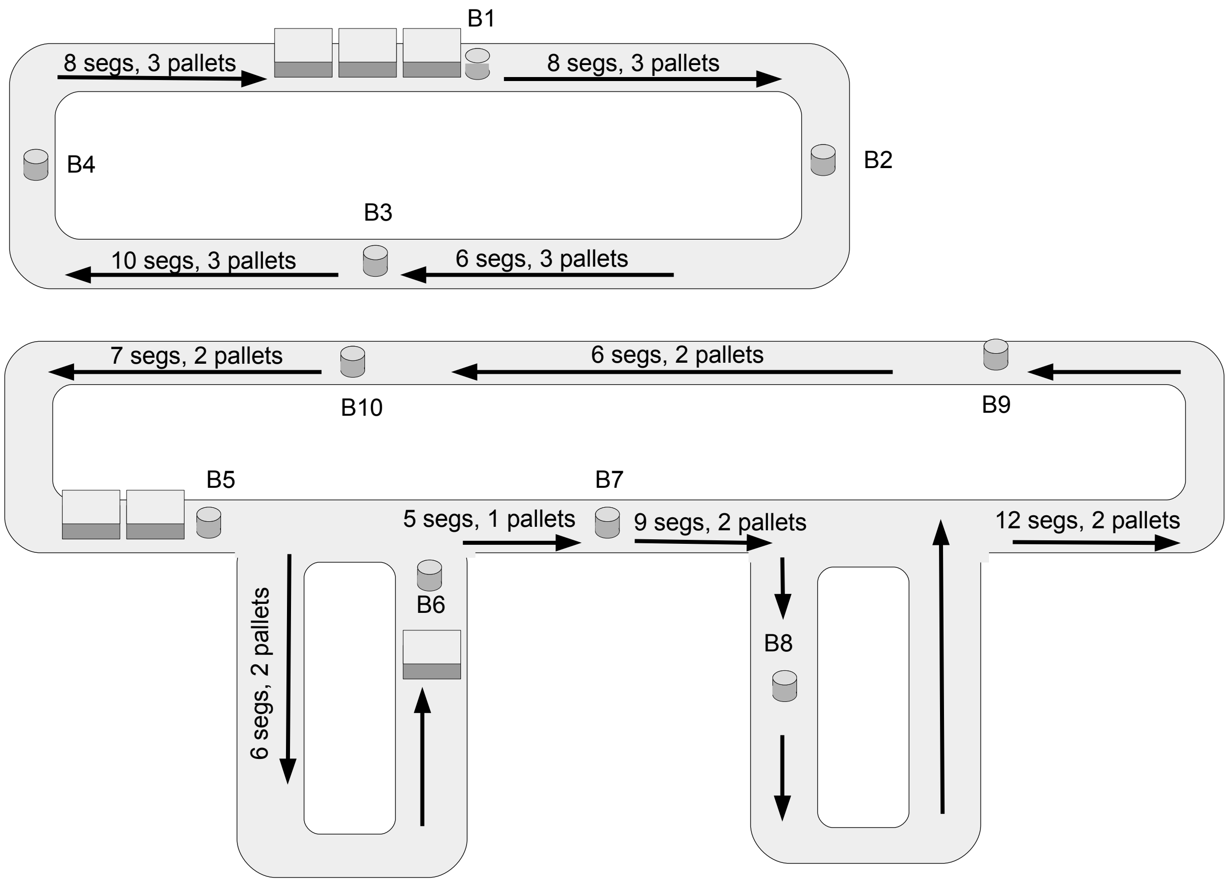

Figure 6.3: Schematic picture of the transportation system.

The system is also programmed to make a forced synchronization between the upper and lower circuits using the buttons $B3$ and $B10$. This means that, for these two buttons in particular, there is an additional *fourth* token necessary for firing. $B3$ fires only if (but not if) there is a presence token in $B10$ and $B10$ fires only if (but not if) there is a presence token in $B3$. This way, their firings are always synchronized.

6.2 System modelling

Now, a TEG model for this system will be derived. Before continuing, it is important to define the *inputs* $u[k]$ and *outputs* $y[k]$ of the system. Hence, it will be defined as $u_i[k]$, $i = 1, 2, ..., 10$ the time in which the k^{th} *control token* is available for the i^{th} button. Further, $y_i[k]$, $i = 1, 2, ..., 10$ the time in which the k^{th} palled arrived just before the i^{th} button. Note that these definitions comply with what one can act and observe in the system.

The modelling begins by analyzing each stretch. Suppose, to begin with, stretch $B1 \to B2$. For this stretch, which holds three pallets at most, one can think in three possible status for a pallet (respective of one place in a TEG):

• $P1$: Stopped, in the first place in line before $B2$;

• $P2$: Stopped, in the second place in line before $B2$;

• $P3$: Stopped, in the third place in line before $B2$;

Also, one needs four actions (respective of one transition in a TEG), labeled as x_1, x_2, x_3 and x_4:

• x_1: Button $B1$ fired, began moving to $B2$;

• x_2: Began moving from the third place to the second place in line;

• x_3: Began moving from the second place to the first place in line;

• x_4: Button $B2$ fired, began moving to $B3$.

Note that

• x_1 can only fire if there is at least one capacity token (that is, a free space for a pallet) in the stretch $B1 \to B2$, there is a presence token in the previous place (there is one, initially, see Figure 6.3) and at least one control token is available. Further, at every firing of x_4, one capacity token is restored to the stretch because one pallet is leaving. Since the stretch begins free of pallets (see Figure 6.3), initially there are three capacity tokens;

• A pallet can only begin to move from the third place in line to the second one if there are no pallets in the second place. Thus, there must not exist a token/pallet in $P2$. Further, only one token/pallet can be at $P2$ at a given time (because only one pallet can be at the second place) and every time x_3 fires the space becomes free to a new pallet to go to the second position;

• A pallet can only begin to move from the second place in line to the first one if there are no pallets in the first place. Thus, there must not exist a token/pallet in $P1$. Further, only one token/pallet can be at $P1$ at a given time (because only one pallet can be at the first place) and every time x_4 fires the space becomes free to a new pallet to go to the first position;

• Of course, x_4 can only fire if there is at least one capacity token in the stretch $B2 \to B3$, which initially is devoid of pallets and thus $B2$ has initially three capacity tokens. Further, there must exist a presence token in $P1$ and also at least one control token for $B2$.

A final concern is that $P3$ can only have one token/pallet at a given time. The above constraints naturally ensure this, and thus there is no need to force it artificially. Indeed, suppose there are two token/pallets at $P3$ at a given time. Then, there must be one token/pallet in $P2$ because, otherwise, x_2 would have fired and there would be just one token/pallet in $P1$. This, in turn, implies that there must be at least one token/pallet in $P1$ because, otherwise, x_3 would have fired and there would be no token/pallet in $P2$. Hence, there would be four tokens in $P1, P2$ and $P3$. This is impossible, because one of the constraints above ensures at most three tokens in the stretch.

Then, it remains to discuss the timings of $P1, P2, P3$. Clearly, the sum of them needs to be 8 seconds (see Figure 6.3). One then needs to discover how much time it takes to a pallet to move from the second to the first position and from the third to the second position. Experiments show that this timing is of 2 seconds for both movements, and indeed this is true for all other stretches (it is simply the time the belt takes to move a pallet a distance of one length of a pallet, and thus the length of the pallet divided by the belt speed). Hence, the timing of $P1$ and $P2$ are 2 seconds and the timing of $P3$ is $8-2-2=4$ seconds.

Taking all of this in consideration, the model of the stretch $B1 \rightarrow B2$ can be seen in Figure 6.4.

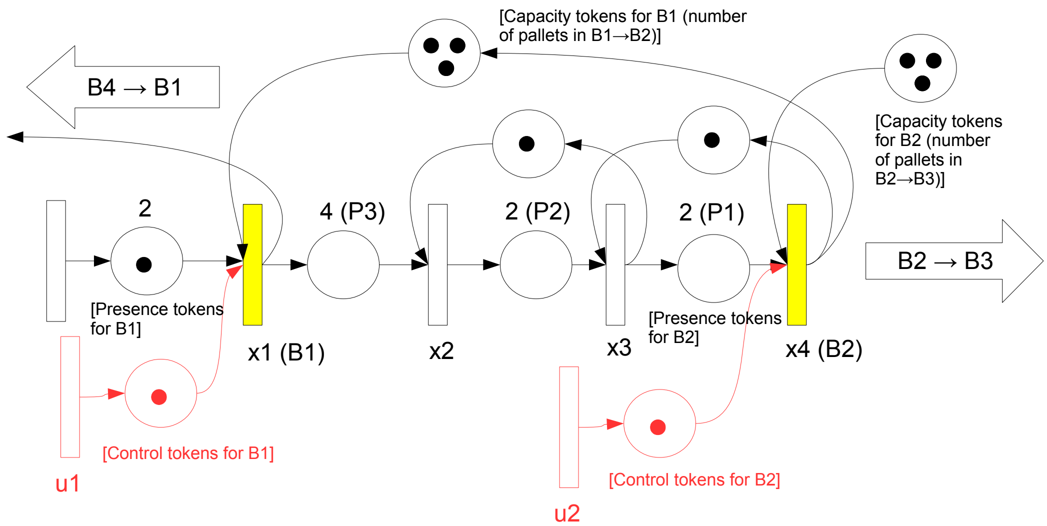

Figure 6.4: TEG for the stretch $B1 \rightarrow B2$. The number over the places represent the holding times and when absent it is considered to be 0. More details concerning system modelling can be found in Baccelli et al. (1992).

In an analogous way, models for all the stretches can be derived. It is necessary, though, to be careful about initial conditions (number of pallets initially in the stretch) and maximum number of pallets, according to Figure 6.3. After that, all these models can be connected (connecting the model of the stretch $B1 \rightarrow B2$ with the one of the stretch $B2 \rightarrow B3$ and so on). The resulting TEG can be seen in Figure 6.5. Note that the output transitions, drawn in green, are constructed in a way that they represents the arrival time of a pallet in each button, which is what is measured by the proximity sensor. For instance, $y_2[k] = 2x_3[k]$ which is the time that the k^{th} pallet arrives at $B2$. Some of these output transitions have an associated place with tokens ($B1$, $B5$ and $B6$) because, initially, there is already a pallet close to these buttons (see Figure 6.3). Note, also, the aforementioned synchronization between $B3$ and $B10$ in transition x_7.

By creating the delayed variables

$$\begin{aligned}
x_{23}[k+1] &= x_4[k]; \\
x_{24}[k+1] &= x_{23}[k]; \\
x_{25}[k+1] &= x_7[k]; \\
x_{26}[k+1] &= x_{25}[k]; \\
x_{27}[k+1] &= x_{10}[k]; \\
x_{28}[k+1] &= x_{27}[k]; \\
x_{29}[k+1] &= x_{19}[k]; \\
x_{30}[k+1] &= x_{21}[k]; \\
x_{31}[k+1] &= x_{12}[k]; \\
x_{32}[k+1] &= x_{13}[k]; \\
x_{33}[k+1] &= x_{15}[k] \qquad (6.2.1)
\end{aligned}$$

one can write the respective dynamic equations of the TEG in Figure 6.5, as in Equation (5.2.1), with

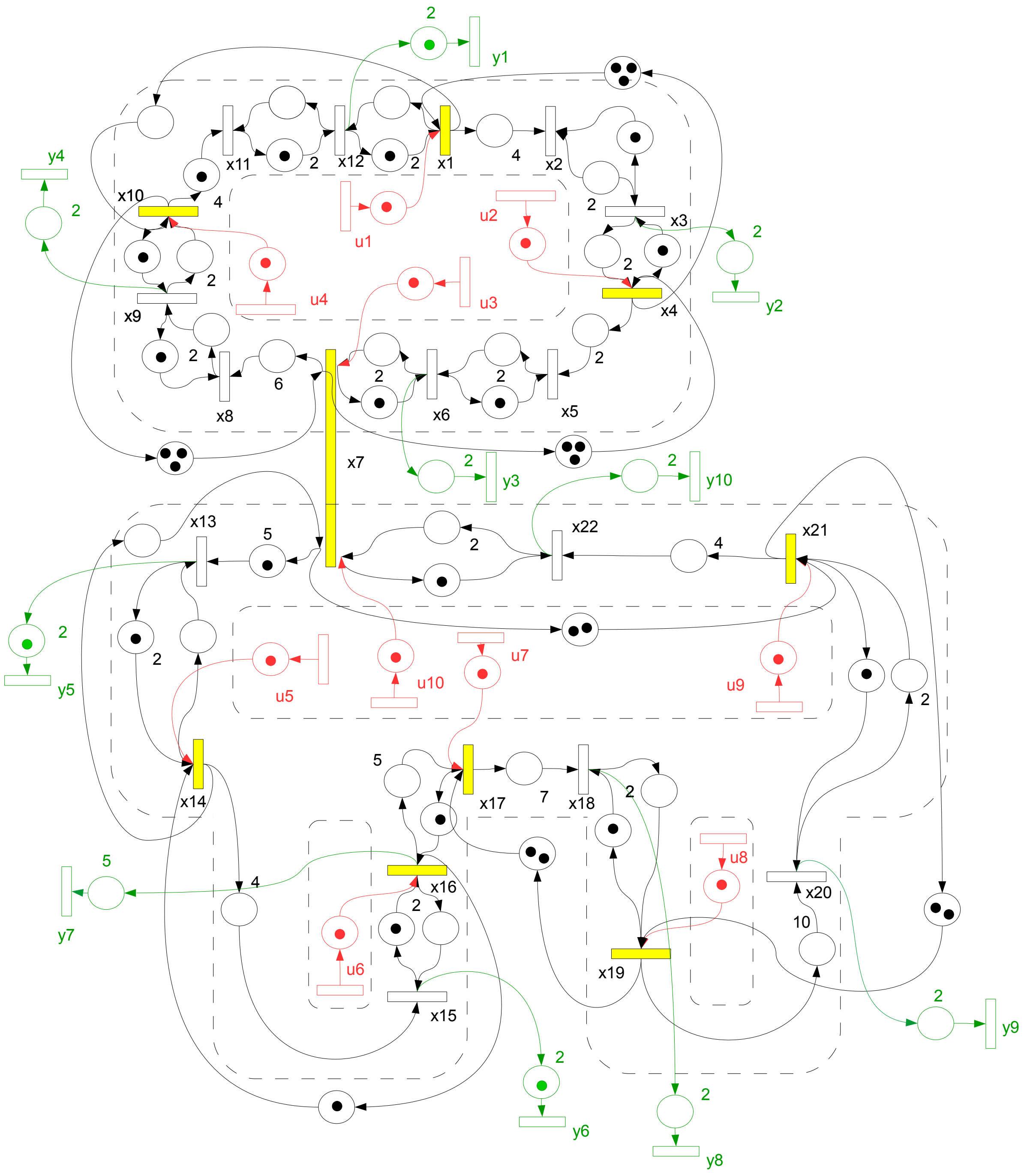

Figure 6.5: Full TEG for the system. The yellow transitions represent buttons, the red parts concern inputs and the green parts the outputs. The numbers above each place represent timings and when absent it is considered to be 0.

$$
A=\left(\begin{array}{ccccccccccccccccccccccccccccccccc}
\bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 2 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 0 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & 0 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 6 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 4 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & 2 & 0 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 8 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 6 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & 4 & 2 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 10 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 8 & \bot & 0 & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & 6 & 4 & \bot & 0 & \bot & \bot & \bot & \bot & \bot & 12 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 10 & \bot & 2 & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & 8 & 6 & \bot & 2 & 0 & \bot & \bot & \bot & \bot & 14 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 12 & \bot & 4 & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & 10 & 8 & \bot & 4 & 2 & \bot & \bot & \bot & \bot & 16 & 2 & \bot & 36 & 0 & 34 & \bot & 22 & \bot & 8 & \bot & 6 & 14 & \bot & 6 & \bot & 0 & 30 & 20 & \bot & \bot & \bot \\
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\bot & \bot & 18 & 16 & \bot & 12 & 10 & \bot & 2 & 0 & \bot & 24 & 10 & \bot & 44 & 8 & 42 & \bot & 30 & \bot & 16 & \bot & 14 & 22 & \bot & 14 & \bot & 8 & 38 & 28 & \bot & \bot & \bot \\
\bot & \bot & 20 & 18 & \bot & 14 & 12 & \bot & 4 & 2 & \bot & 26 & 12 & \bot & 46 & 10 & 44 & \bot & 32 & \bot & 18 & \bot & 16 & 24 & \bot & 16 & \bot & 10 & 40 & 30 & \bot & \bot & \bot \\
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\bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 0 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot
\end{array}\right); \tag{6.2.2}
$$

$$B^T = \begin{pmatrix}
0 & 4 & 6 & 8 & 10 & 12 & 14 & 20 & 22 & 24 & 0 & 0 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & \bot & 0 & 2 & 4 & 6 & 12 & 14 & 16 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & \bot & \bot & \bot & \bot & 0 & 6 & 8 & 10 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 0 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & \bot & \bot & \bot & \bot & 0 & 6 & 8 & 10 & \bot & \bot & 0 & 0 & 4 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & \bot & \bot & \bot & \bot & 34 & 40 & 42 & 44 & \bot & \bot & \bot & \bot & 0 & 0 & 5 & 12 & 14 & 26 & 28 & 32 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & \bot & \bot & \bot & \bot & 29 & 35 & 37 & 39 & \bot & \bot & \bot & \bot & \bot & \bot & 0 & 7 & 9 & 21 & 23 & 27 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & \bot & \bot & \bot & \bot & 20 & 26 & 28 & 30 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 0 & 12 & 14 & 18 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & \bot & \bot & \bot & \bot & 6 & 12 & 14 & 16 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 0 & 4 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & \bot & \bot & \bot & \bot & 0 & 6 & 8 & 10 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot
\end{pmatrix}; \tag{6.2.3}$$

$$C = \begin{pmatrix}
\bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 2 & \bot & \bot \\
\bot & \bot & 2 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & \bot & \bot & \bot & 2 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
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\bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 2 & \bot \\
\bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 2 \\
\bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 5 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 2 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 2 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot \\
\bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & 2 & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot & \bot
\end{pmatrix} \tag{6.2.4}$$

and $G = \bot$.

6.3 The problem

Buttons $B3$ and $B10$ are synchronized, but if the buttons are left uncontrolled, it may be the case that one pallet has to wait a long time behind one of the buttons waiting for the other to arrive.

This is the case for the impulse response $u[k] = \bot\ \forall k$. In this case, there are plenty of control tokens and the buttons fire as soon as they can (when there are pallets behind of them and there are enough free spaces in the next stretches, and for $B3$ and $B10$ in particular when there are pallets behind both buttons). Figure 6.6 shows the difference between the arrival times at $B10$ ($y_{10}[k]$) and $B3$ ($y_3[k]$) measured from the real system when this policy is applied. One can see that $y_{10}[k]$ is always greater than $y_3[k]$ and that, as a consequence, a pallet always waits behind $B3$ until a pallet arrives at $B10$. Further, this sojourn time oscillates, in steady state, between 1 and 18 seconds.

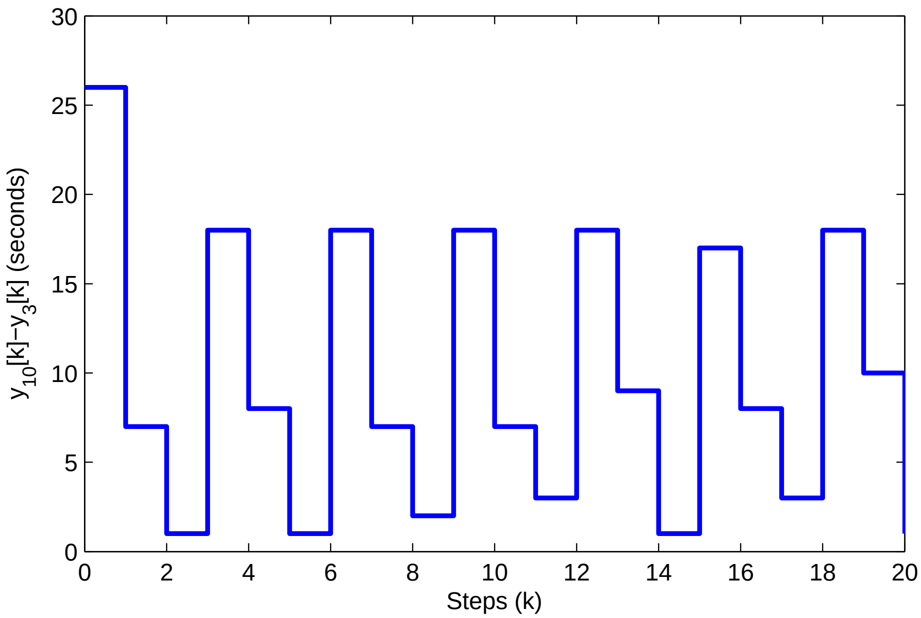

Figure 6.6: Difference between the arrival times at $B10$ ($y_{10}[k]$) and $B3$ ($y_3[k]$) measured from the system when the policy $u[k] = \perp$ is used.

It will be imposed that both pallets do not wait much time at the synchronization buttons. The maximum sojourn time that will be tolerated is 3 seconds. This can be done in the following way: note that, by Figure 6.3, the minimum time between $B3$ and $B4$ is 10 seconds. Thus, if it is imposed that

$$\begin{aligned} y_4[k] - y_3[k] = x_9[k] - x_6[k] \leq 13 \\ |y_{10}[k] - y_3[k]| = |x_{22}[k] - x_6[k]| \leq 3 \leftrightarrow -3 \leq x_{22}[k] - x_6[k] \leq 3 \end{aligned} \tag{6.3.1}$$

then it is guaranteed that the sojourn time for both pallets is at most 3 seconds (note that just $|y_{10}[k] - y_3[k]| \leq 3$ alone is not enough, since the two pallets can arrive with at most 3 apart but still wait together a long time behind their respective buttons before continuing).

In order to ensure that the problem is controllable coupled, an innocuous set of constraints $-100 \leq x_i[k] - x_j[k] \leq 100$ $i, j = 1, 2, ..., 33$ will be posed to the system. It is clear, by the timings in Figure 6.3, that these constraints are very loose.

6.4 Regulator and observer synthesis

The system in open loop is strongly connected (which means that there is a connection, even if indirect, between two given states, see Baccelli et al. (1992)) and the spectral radius of A is $\rho(A) = 15+\frac{2}{3}$. Equation $\mathfrak{S}(\mathscr{R})$ can be solved with $\lambda = 16 > \rho(A)$ (by limitations of the system, only natural values of seconds are possible) so one can obtain

$$\begin{gathered}\mu = (17\ 17\ 17\ 17\ 10\ 0\ 5\ 14\ 17\ 17)^T;\\ \chi = (1\ 5\ 7\ 9\ 11\ 13\ 18\ 24\ 26\ 28\ 16\ 2\ 7\ -6\ -2\ -16\ \ldots\\ -11\ -4\ -2\ 10\ 12\ 16\ -7\ -23\ 2\ -14\ 12\ -4\ -18\ -4\ -14\ -9\ -18)^T\\ \zeta^T = (-1\ -5\ -7\ -9\ -11\ -13\ -18\ -24\ -26\ -28\ -16\ -2\ -7\ 6\ 2\ \ldots\\ 16\ 11\ 4\ 2\ -10\ -12\ -16\ 7\ 23\ -2\ 14\ -12\ 4\ 18\ 4\ 14\ 9\ 18)^T.\end{gathered} \tag{6.4.1}$$

The convergence number $\kappa((-16)A)$ is equal to 8. Note that the feedback causal law $u[k] = \mu\zeta^T x[k]$ is not causal, because the matrix $F_{\mathtt{SR}} = \mu\zeta^T$ has non-$\perp$ negative entries. However, as mentioned at the end of Section 4.6, it will be soon clear that the observer implementation of this feedback is causal, and hence no causalisation procedure is necessary.

As it was done in Section 5.3, a reduced system will be used to simplify the calculations. Consider the system

$$S_{\mathtt{red}}(S,\mu)\ :\ x[k+1] = Ax[k] \oplus B\mu v[k] \tag{6.4.2}$$

in which the matrix B was replaced by $B\mu$ and $v[k]$ is a scalar. The real plant will also be controlled using the reduced system. This implies that only the scalar variable $v[k]$ will be controlled, and the control input will be fixed to $u[k] = \mu v[k]$.

Then, one wishes to observe the linear functional $\zeta^T x[k]$, which is an observable coupled problem since ζ has no $\perp$ entries. Solving equation $\mathfrak{M}^t(S_{\mathtt{red}}(S,\mu), \zeta^T)$ with $t = 3$, the constraint $\rho(L_s^{\mathtt{ex}}[t]) \preceq \perp$ (it is desirable that the observer is as fast as possible) and the causality condition (see Remark 5.2.3), one obtains

$$L_u^T[0{:}3]^T = (64\ 48\ 32\ 16)^T$$
$$L_y^T[0{:}3]^T = (\perp\ \ 38\ 32\ 22\ \ \perp\ \perp\ \ 49\ \ \perp\ \perp\ \perp\ \ 44\ 36\ 30\ 20\ 54\ 64\ \ldots$$
$$59\ 50\ 36\ 30\ 31\ 23\ 17\ 4\ 38\ 48\ 43\ 34\ 20\ 14\ 15\ 7\ 1\ \ \perp\ \ 23\ 32\ 27\ 18\ 4\ \ \perp)^T \tag{6.4.3}$$

and $L_s^T[0{:}3]^T = \perp$. Hence, the observer implementation for $\zeta^T x[k]$ is causal. The (rather long) equation for the observer is, according to Equation (5.2.80)

$$s[k+1] = 64v[k-3] \oplus 48v[k-2] \oplus 32v[k-1] \oplus 16v[k] \oplus$$
$$38y_2[k-3] \oplus 32y_3[k-3] \oplus 22y_4[k-3] \oplus 49y_7[k-3] \oplus$$
$$44y_1[k-2] \oplus 36y_2[k-2] \oplus 30y_3[k-2] \oplus 20y_4[k-2] \oplus$$
$$54y_5[k-2] \oplus 64y_6[k-2] \oplus 59y_7[k-2] \oplus 50y_8[k-2] \oplus 36y_9[k-2] \oplus 30y_{10}[k-2] \oplus$$
$$31y_1[k-1] \oplus 23y_2[k-1] \oplus 17y_3[k-1] \oplus 4y_4[k-1] \oplus 38y_5[k-1] \oplus$$
$$48y_6[k-1] \oplus 43y_7[k-1] \oplus 34y_8[k-1] \oplus 20y_9[k-1] \oplus 14y_{10}[k-1] \oplus$$
$$15y_1[k] \oplus 7y_2[k] \oplus 1y_3[k] \oplus 23y_5[k] \oplus 32y_6[k] \oplus$$
$$27y_7[k] \oplus 18y_8[k] \oplus 4y_9[k]. \tag{6.4.4}$$

Since it is desirable to have $v[k] = s[k]$, one has a recursive equation for the control input $v[k]$ in this reduced system. The complete input $u[k]$ can be recovered from the equation $u[k] = \mu v[k]$.

6.5 Results

The observer implementation of the Spectral Regulator was applied to the real plant using a `C++` code. The signals of the two imposed constraints can be seen in Figure 6.7 and Figure 6.8. In these figures, one can observe that initially the constraints were not respected. After a while, they began to hold. Then, around the 5^{th} event, a disturbance (a man-made disturbance in which the author held some pallets with his hands) happened in the system and the two constraints were transgressed. After that, however, the controller was able to drive the system again to the desired set. Note that, even in steady state, there are some small fluctuations (inside the desired limit, however). These are consequences of natural perturbations in the system, the most common being a pallet being stuck for a little while when it is moving through a corner (this is because there are some lubrication problems in the wheels that help the pallets to move in that area).

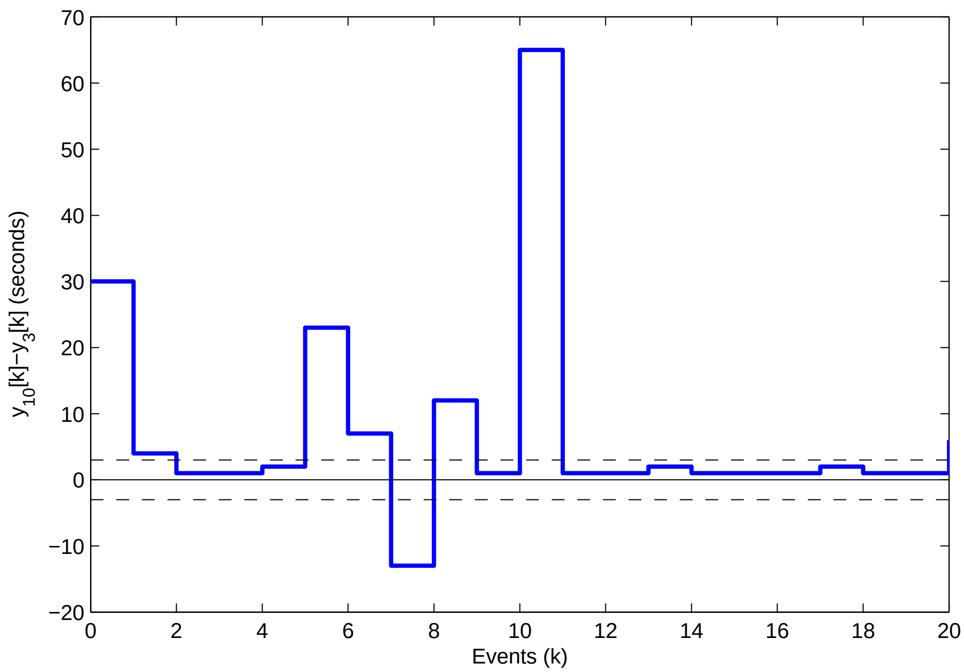

Figure 6.7: Constraint $|y_{10}[k] - y_3[k]| \leq 3$.

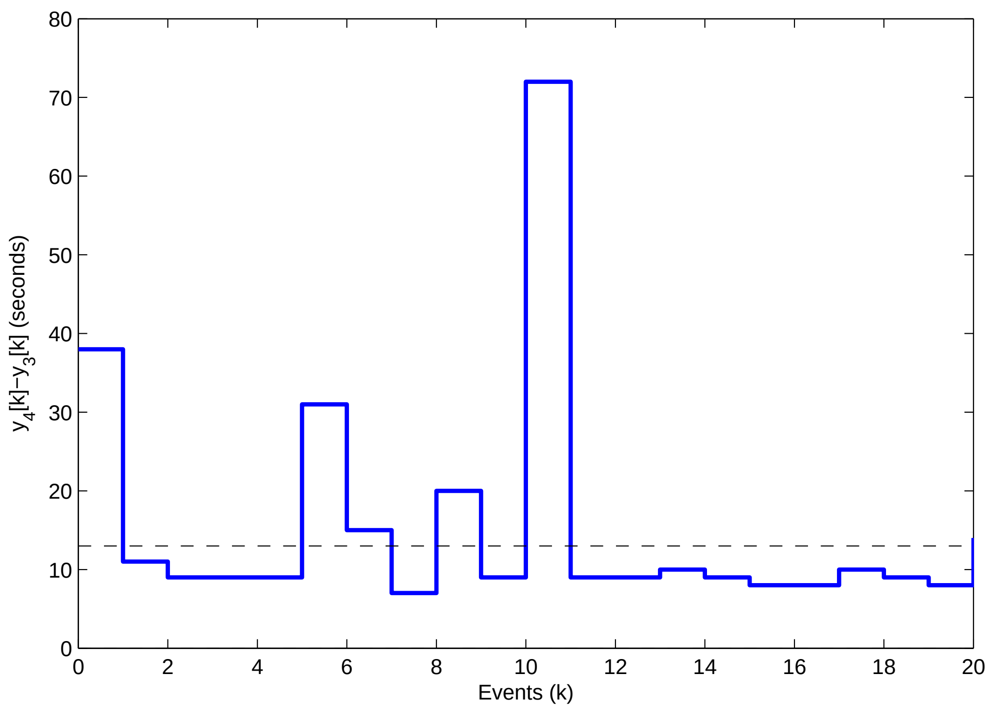

Figure 6.8: Constraint $y_4[k] - y_3[k] \leq 13$.

Finally, the conclusion is that it is clear that the proposed regulator+observer strategy was successful in controlling the system behaviour.

6.6 Conclusion

In this chapter, the real implementation of the observer and the regulator proposed in this thesis were presented. To the author's knowledge, this is the first published report of such work. Although the example presented in this chapter is less complex compared to the ones that can appear in real industrial applications, it still serves quite well as a "proof of concept" of the proposed methodology.

Chapter 7

Future Research

Future topics which the author considers important to pursue are:

- **Make a rigorous comparison between the algorithms for solving TFLPs**

As mentioned in Chapter 3, a comparison between the proposed algorithms and the already published ones would be extremely important to verify the efficiency of the proposed methods.

- **Deal with controllable structurally critical problems.**

Chapter 4 presents a necessary and sufficient condition for (controllable coupled) controllable non-critical problems and also controllable faux-critical problems. In order to give a complete solution for the proposed problem, it is necessary to devise a methodology for the so-called controllable structurally critical problems.

The author already has some sufficient conditions for this class of problems, but he believes that they are not yet in an appropriate form, requiring some polishing. Further, he also has some clues about which direction he should pursue for the final answer (necessary and sufficient condition).

- **Deal with observable critical problems.**

Dually, Chapter 5 presents a necessary and sufficient condition for (observable coupled) observable non-critical problems. In order to give a complete solution for the proposed problem, it is necessary to devise a methodology for the observable critical problems.

- **Extend the methodology to other class of problems**

The author believes that a more general class of systems, other than Tropical Linear Event-Invariant Systems, can have a regulator-like problem solved with a controller inspired by the Spectral Regulator (Chapter 4). The author already has some results in that regard, but they are still in an initial stage.

Appendix A

Mathematical Tools

In this chapter, a (brief) background of the mathematical tools used in this thesis will be given.

A.1 Tropical Algebra Basics

A.1.1 Dioids and semimodules

Definition A.1.1. *(Dioid, see Baccelli et al. (1992))* A *dioid* $\mathbb{D} = \{\mathscr{S}, \oplus, \otimes\}$, alternatively *idempotent semiring*, is a set $\mathscr{S}$, together with two operations $\{\oplus, \otimes\}$ (the "sum" and the "product", respectively) such that

$\{\mathscr{S}, \oplus\}$ is a *commutative idempotent monoid*, meaning that:

- $\oplus$ is closed in $\mathscr{S}$: $\forall a, b \in \mathscr{S}$, $a \oplus b \in \mathscr{S}$;
- $\oplus$ is commutative : $\forall a, b \in \mathscr{S}$, $a \oplus b = b \oplus a$;
- $\oplus$ is associative : $\forall a, b, c \in \mathscr{S}$, $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;
- $\oplus$ has a neutral element : $\exists \perp \in \mathscr{S}$ such that, $\forall a \in \mathscr{S}$, $a \oplus \perp = \perp \oplus a = a$;
- $\oplus$ is idempotent: $\forall a \in \mathscr{S}$, $a \oplus a = a$;

$\{\mathscr{S}, \otimes\}$ is a *monoid*, meaning that:

- $\otimes$ is closed in $\mathscr{S}$: $\forall a, b \in \mathscr{S}$, $a \otimes b \in \mathscr{S}$;
- $\oplus$ is associative : $\forall a, b, c \in \mathscr{S}$, $a \otimes (b \otimes c) = (a \otimes b) \otimes c$;
- $\oplus$ has a neutral element : $\exists e \in \mathscr{S}$ such that, $\forall a \in \mathscr{S}$, $a \otimes e = e \otimes a = a$;

Further

- $\otimes$ distributes over $\oplus$: $\forall a, b, c \in \mathscr{S}, a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$;

- $\perp$ is absorbing for $\otimes$: $\forall a \in \mathscr{S}$, $a \otimes \perp = \perp \otimes a = \perp$;

□

Example A.1.1. The *Tropical Algebra*, $\mathbb{T}_{max}$ in which $\mathscr{S} = \mathbb{Z} \cup \{-\infty\}$, $\oplus = \max$, $\otimes = +$, $\perp = -\infty$ and $e = 0$, is a dioid. □

Definition A.1.2. *(Complete dioid, see Baccelli et al. (1992))* A *dioid* $\mathbb{D} = \{\mathscr{S}, \oplus, \otimes\}$ is said to be complete if it is closed under infinite "sums" and if the "product" distributes over any infinite "sum". Formally, if for any $c \in \mathscr{S}$ and $\forall \mathscr{X} \subseteq \mathscr{S}$

$$c \otimes \left(\bigoplus_{x \in \mathscr{X}} x \right) = \left(\bigoplus_{x \in \mathscr{X}} c \otimes x \right) \in \mathscr{S}. \tag{A.1.1}$$

□

Example A.1.2. The *Complete Tropical Algebra*, $\mathbb{T}_{max}$ in which $\mathscr{S} = \mathbb{Z} \cup \{-\infty, \infty\}$, $\oplus = \max$, $\otimes = +$, $\perp = -\infty$ and $e = 0$, is a complete dioid. The inclusion of the element $+\infty$, denominated here as $\top$, is necessary because otherwise $\bigoplus_{x \in \mathbb{Z}} x$, which is equal to $+\infty$, would not be in $\mathbb{Z} \cup \{-\infty\}$. □

Definition A.1.3. *(Natural order, see Baccelli et al. (1992))* Given a dioid $\{\mathscr{S}, \oplus, \otimes\}$, then the induced order $\succeq$ is a partial order in $\mathscr{S}$ such that

$$a \succeq b \iff a \oplus b = a. \tag{A.1.2}$$

Further, if $a \succeq b$, one also writes that $b \preceq a$. □

Example A.1.3. In the *Tropical Algebra*, $\mathbb{T}_{max}$, the natural order is simply the traditional one, that is, $a \succeq b$ if and only if $a \geq b$. In this case, it is also a total order. □

Remark A.1.1. Given a positive natural number n, the n^{th} *order matricial Tropical Algebra*, $\mathbb{T}_{max}^{n \times n}$, is a dioid in which $\mathscr{S} = (\mathbb{Z} \cup \{-\infty, \infty\})^{n \times n}$, that is, the square matrices of order n in which all the entries are either integers or $-\infty$.

In this case, $\oplus$ is defined in a way that, given $A, B \in \mathscr{S}$

$$\{A \oplus B\}_{ij} = A_{ij} \oplus B_{ij}; \tag{A.1.3}$$

that is, in complete analogy with the traditional matricial sum (only swapping $+$ by max). Further, $\otimes$ is defined in a way that

$$\{A \otimes B\}_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}; \tag{A.1.4}$$

again in complete analogy with the traditional matricial product (only swapping + by max and × by +). In this case, $\perp$ is the matrix in which $\{\perp\}_{ij} = -\infty \ \forall i, j$ and e is the diagonal matrix in which $e_{ii} = 0$ and $e_{ij} = \perp$ if $i \neq j$. Instead of using the symbol e for the neutral element of the product, the symbol I (of *identity*, following the traditional matricial algebra convention) will be used. Also, $\perp$ will be used for both the neutral element of the sum (the matrix of the n^{th} order) and as the element $-\infty$, without any confusion.

One could also define, in complete analogy, the n^{th} *order matricial Complete Tropical Algebra*. □

Example A.1.4. Let $n = 2$ and

$$A = \begin{pmatrix} 1 & 5 \\ 3 & 8 \end{pmatrix}, \ B = \begin{pmatrix} 4 & \perp \\ 7 & 2 \end{pmatrix}. \tag{A.1.5}$$

Then in the n^{th} *order matricial Tropical Algebra*

$$A \oplus B = \begin{pmatrix} \max(1,4) & \max(5,-\infty) \\ \max(3,7) & \max(8,2) \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}. \tag{A.1.6}$$

And

$$A \otimes B = \begin{pmatrix} \max(1+4, 5+7) & \max(1+(-\infty), 5+2) \\ \max(3+4, 8+7) & \max(3+(-\infty), 8+2) \end{pmatrix} = \begin{pmatrix} 12 & 7 \\ 15 & 10 \end{pmatrix}. \tag{A.1.7}$$

Further, the symbol $\otimes$, just like in the traditional algebra, is usually omitted. For instance, instead of writing $A \otimes B$ one writes just AB.

□

Remark A.1.2. In the n^{th} *order matricial Tropical Algebra*, $A \succeq B$ means simply that $A_{ij} \succeq B_{ij}$ for all $i, j = 1, 2, .., n$. For $n > 1$, this order is clearly not total, but just partial. For instance, let A and B as in Equation (A.1.5), then it is clear that neither $A \succeq B$ nor $B \succeq A$ holds. □

Remark A.1.3. One could, using an "abuse of notation", define a matricial "Tropical Algebra" in which the set $\mathscr{S}$ has matrices of different sizes. In this case, the sum $A \oplus B$ in (A.1.3) will be defined only if A and B have the same dimension and $A \otimes B$ in (A.1.4) if the number of columns of A is equal to the number of rows of B. Formally, this is not a dioid because $A \oplus B$ and $A \otimes B$ are not defined for all pairs of members of $\mathscr{S}$.

But it is also clear that this is just an abuse of notation, because given an upper bound h of the number of rows and columns of the matrices in $\mathscr{S}$ (such bound can be always assumed in practice), the matricial sum or product can be "interpreted" as a matricial sum of product in the h^{th} *order matricial Tropical Algebra*, but removing the unnecessary information. For instance

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \tag{A.1.8}$$

(remember the convention of omitting $\otimes$) with matrices of different (but compatible for the product) sizes can be "interpreted" as the following product in the 3^{rd} *order matricial Tropical Algebra*

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 & \perp & \perp \\ 1 & \perp & \perp \\ 2 & \perp & \perp \end{pmatrix} \tag{A.1.9}$$

of matrices of same size. In this case, the last two columns of resulting matrix in Equation A.1.9 can be dispensed with (they will be composed solely of $\perp$ entries, and hence has no interesting information), and then this result reduces to the product in Equation A.1.8.

This "abuse of notation" will be used through this thesis, and the set of all matrices with n rows and m columns with elements in $\mathbb{Z} \cup \{-\infty\}$ will be denoted by $\mathbb{T}^{n \times m}_{max}$. If $m = 1$, the simpler notation $\mathbb{T}^{n}_{max}$ will be used. If $n = m = 1$, simply $\mathbb{T}_{max}$ will be used. $\square$

Definition A.1.4. *(Semimodules and sub-semimodules, see Cohen et al. (2004))* A (left) semimodule $\mathbb{M} = \{\mathscr{M}, \mathbb{D}, \oplus_{\mathscr{M}}, \otimes_{\mathscr{M}}\}$ over a semiring $\mathbb{D} = \{\mathscr{S}, \oplus, \otimes\}$ is a set of elements $\mathscr{M}$, together with an operation $\oplus_{\mathscr{M}} : \mathscr{M} \times \mathscr{M} \mapsto \mathscr{M}$ such that

$\{\mathscr{M}, \oplus_{\mathscr{M}}\}$ is a *commutative additive semigroup with zero element*, meaning that:

- $\oplus_{\mathscr{M}}$ is commutative : $\forall a, b \in \mathscr{M}$, $a \oplus_{\mathscr{M}} b = b \oplus_{\mathscr{M}} a$;
- $\oplus_{\mathscr{M}}$ is associative : $\forall a, b, c \in \mathscr{M}$, $a \oplus_{\mathscr{M}} (b \oplus_{\mathscr{M}} c) = (a \oplus_{\mathscr{M}} b) \oplus c$;
- $\oplus_{\mathscr{M}}$ has a neutral element : $\exists \perp_{\mathscr{M}} \in \mathscr{M}$ such that, $\forall a \in \mathscr{S}$, $a \oplus_{\mathscr{M}} \perp_{\mathscr{M}} = \perp_{\mathscr{M}} \oplus_{\mathscr{M}} a = a$;

And further, such that there is a map $\mathscr{S} \times \mathscr{M} \mapsto \mathscr{M}$, the *left action*, denoted by $f_a(x) = a \otimes_{\mathscr{M}} x$ such that

- $\otimes_{\mathscr{M}}$ right distributes over $\oplus$, that is : $\forall a, b \in \mathscr{S}$, $x \in \mathscr{M}$, $(a \oplus b) \otimes_{\mathscr{M}} x = a \otimes_{\mathscr{M}} x \oplus_{\mathscr{M}} b \otimes_{\mathscr{M}} x$;

- $\otimes$ left distributes over $\oplus_{\mathscr{M}}$, that is : $\forall a \in \mathscr{S}$, $x, y \in \mathscr{M}$, $a \otimes_{\mathscr{M}} (x \oplus_{\mathscr{M}} y) = a \otimes_{\mathscr{M}} x \oplus_{\mathscr{M}} a \otimes_{\mathscr{M}} y$;
- Law of neutral element for the sum : $\forall a \in \mathscr{S}$, $x \in \mathscr{M}$, $a \otimes_{\mathscr{M}} \perp_{\mathscr{M}} = \perp \otimes_{\mathscr{M}} x = \perp_{\mathscr{M}}$;
- Law of neutral element for the product : $\forall x \in \mathscr{M}$, $e \otimes_{\mathscr{M}} x = x$;

One could also define, in a similar way, a (right)-semimodule, in which instead of left actions $f_a(x) = a \otimes_{\mathscr{M}} x$, right actions $g_a(x) = x \otimes_{\mathscr{M}} a$ are considered. If the semiring $\mathbb{S}$ is commutative, both semimodules are isomorphic.

A member of $\mathscr{M}$ is said to be a *vector*. A semimodule $\mathbb{M}' = \{\mathscr{M}', \mathbb{D}, \oplus_{\mathscr{M}}, \otimes_{\mathscr{M}}\}$ is said to be a *sub-semimodule* of $\mathbb{M} = \{\mathscr{M}, \mathbb{D}, \oplus_{\mathscr{M}}, \otimes_{\mathscr{M}}\}$ if $\mathscr{M}' \subseteq \mathscr{M}$. □

Remark A.1.4. A (left or right) semimodule over a semiring is the counterpart of vector space over a field (there is no notion of *left and right* in a vector space because the sum, in a field, is necessarily commutative). □

Definition A.1.5. *(Upper and lower bounded vectors)* A vector x is said to be *lower bounded* if no entry is $-\infty$ ($\perp$). Dually, it is said to be *upper bounded* if no entry is $+\infty$ ($\top$). □

Example A.1.5. An useful example of semimodule is, for a given n, the dioid $\mathbb{D}$ being the Tropical Algebra and $\mathscr{M}$ the set of all columns vectors with entries in $\mathbb{Z} \cup \{-\infty\}$, that is, $\mathbb{T}^n_{max}$.

In this case, $\oplus_{\mathscr{M}}$ is defined so $\{x \oplus_{\mathscr{M}} y\}_i = x_i \oplus y_i$ (the tropical sum of both vectors), and $a \otimes_{\mathscr{M}} x$ as $\{a \otimes_{\mathscr{M}} x\}_i = a \otimes x_i$. $\perp_{\mathscr{M}}$ is the column vector of order n in which all the entries are $\perp$.

If $x = (1\ 2\ 3)^T$, $y = (4\ \perp\ -1)^T$ and $a = 5$, then $x \oplus_{\mathscr{M}} y = (4\ 2\ 3)^T$ and $a \otimes_{\mathscr{M}} x = (6\ 7\ 8)^T$.

□

Remark A.1.5. Usually, the symbol $\otimes_{\mathscr{M}}$ is omitted, and $\oplus_{\mathscr{M}}$ is written simply as $\oplus$. Further, $\perp_{\mathscr{M}}$ is written simply as $\perp$. □

Definition A.1.6. (*Tropical linear combination*) Let $\mathbb{M} = \{\mathscr{M}, \mathbb{D}, \oplus, \otimes\}$, $\mathbb{D} = \{\mathscr{S}, \oplus, \otimes\}$ be a semimodule. In this context, $x \in \mathscr{M}$ is said to be a *tropical linear combination* of members $y_i \in \mathscr{M}$, $i = 1, 2, ..., t$ if there exists $\alpha_i \in \mathscr{S}$ such that

$$x = \bigoplus_{i=1}^{t} \alpha_i x_i \tag{A.1.10}$$

□

Definition A.1.7. (*Tropical linear maps over a semimodule*) Let $\mathbb{M}_1 = \{\mathscr{M}_1, \mathbb{D}, \oplus_1, \otimes_1\}$, $\mathbb{M}_2 = \{\mathscr{M}_2, \mathbb{D}, \oplus_2, \otimes_2\}$ and $\mathbb{D} = \{\mathscr{S}, \oplus, \otimes\}$. A map $f : \mathscr{M}_1 \mapsto \mathscr{M}_2$ is said to be *tropical linear* if, for any $x, y \in \mathscr{M}_1$, $\alpha, \beta \in \mathscr{S}$

$$f(\alpha x \oplus \beta y) = \alpha f(x) \oplus \beta f(y). \tag{A.1.11}$$

□

Example A.1.6. Let $A \in \mathbb{T}_{max}^{n \times m}$, then it induces a tropical linear map over semimodules: $f(x) = Ax$ (provided that the members of the set in the domain are column vectors of dimension m). □

Definition A.1.8. *(Image of matrices)* Given a matrix $A \in \mathbb{T}_{max}^{n \times m}$, the *Image* of a matrix *A*, *Im*$\{A\}$, is the image of the tropical linear application $x \mapsto Ax$. □

Remark A.1.6. The image of a matrix is the set of all tropical linear combinations of the columns of *A*, and hence is a semimodule. □

Definition A.1.9. (*Finitely generated semimodule, see Gaubert and M.Plus (1997)*) Let $\mathbb{M}$ be a semimodule. This semimodule is said to be *finitely generated* if there is a matrix $A \in \mathbb{T}_{max}^{n \times m}$, n and m finite, such that $\mathscr{M} = Im\{A\}$. □

Remark A.1.7. Vector spaces lying in a dimension n (that is, that the number of components of the vectors is n) are finitely generated. Indeed, it can be said that the number of vectors is at most n. This is not the case of semimodules: they can lie in a dimension n and the minimal number of generators can be more than n, or even infinite. In fact, let

$$A[k] = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 2 & 3 & \dots & k \\ 0 & -1 & -2 & -3 & \dots & -k \end{pmatrix} \tag{A.1.12}$$

then the semimodule $\lim_{k \to \infty} Im\{A[k]\}$ lies in a three-dimensional space but is not finitely generated. This happens mainly because no column of $A[k]$ can be written as a tropical linear combination of other ones (see Cuninghame-Green (1979) and also Gaubert and M.Plus (1997)). □

Definition A.1.10. (*Congruence, see Loreto et al. (2010)*) A congruence $\mathbb{G}$ over a semimodule $\mathbb{M} = \{\mathscr{M}, \mathbb{D}, \oplus, \otimes\}$, with $\mathbb{D} = \{\mathscr{S}, \oplus, \otimes\}$, is an equivalence relation (that is, a relation which is symmetric, reflexive and transitive) on $\mathscr{M} \times \mathscr{M}$ such that

- $\forall\ x, y \in \mathscr{M},\ \alpha \in \mathscr{S},\ \{x, y\} \in \mathbb{G} \ \rightarrow\ \{\alpha x, \alpha y\} \in \mathbb{G}$;
- $\forall\ x, y, z \in \mathscr{M},\ \{x, y\} \in \mathbb{G} \ \rightarrow\ \{x \oplus z, y \oplus z\} \in \mathbb{G}$;

□

Remark A.1.8. A semimodule can be thought as an equivalence relation with a semimodule-like structure. □

Definition A.1.11. *(Kernel of matrices, see Loreto et al. (2010))* Given a matrix $A \in \mathbb{T}_{max}^{n\times m}$, the *Kernel* of a matrix A, *Ker*$\{A\}$, is the set of all pairs $\{x, y\}$ such that they are equal under $z \mapsto Az$, that is, $Ax = Ay$. □

Remark A.1.9. One can easily prove that *Ker*$\{A\}$ is a congruence. □

Definition A.1.12. (*Finitely generated congruence*) Let $\mathbb{G}$ be a congruence. This congruence is said to be *finitely generated* if there is a matrix $A \in \mathbb{T}_{max}^{n\times m}$, n and m finite, such that $\mathscr{M} = Ker\{A\}$. □

Remark A.1.10. Kernels and semimodules are duals, similarly how vector spaces and null spaces are duals (orthogonals of each other). The difference is that in the vector space/null space setting, a strong duality holds in the sense that any null space can be written as a vector space and *vice-e-versa*. Due to the absence of a "subtraction" in tropical algebra, this strong duality does not hold: they do not even have the same dimension (the dual of a semimodule embedded in dimension n is a congruence of dimension $2n$). However, some duality results still hold. See Loreto et al. (2010). □

A.1.2 Spectral theory

Definition A.1.13. *(Eigenvector and eigenvalue, see Baccelli et al. (1992))* Let $A \in \mathbb{T}_{max}^{n\times n}$. An *eigenvector* of A with associated eigenvalue $\lambda \in \mathbb{Q} \cup \{-\infty\}$ is any vector $v \in (\mathbb{Q} \cup \{-\infty\})^n \neq \perp$ (that is, that has at least one non-$\perp$ entry such that) $Av = \lambda v$. □

Remark A.1.11. One may wonder why the vector v and λ have rational entries (in $\mathbb{Q}$) even though the elements of A are in $\mathbb{T}_{max}$, that is, are either integers or $\perp$. This is because the eigenvalue problem is not "closed" in the set $\mathbb{T}_{max}$: there exist matrices $A \in \mathbb{T}_{max}^{n\times n}$ such that the (non-null) solution v and λ to $Av = \lambda v$ necessarily has a non-integer rational entry (only rational, though, reals are not necessary as it will be clear further).

To exemplify, let

$$A = \begin{pmatrix} \perp & 1 \\ 2 & \perp \end{pmatrix} \tag{A.1.13}$$

which clearly has all their entries in $\mathbb{T}_{max}$. The equations for the eigenproblem $Av = \lambda v$ reads as, in the traditional algebra

$$1 + v_2 = \lambda + v_1; \tag{A.1.14}$$

$$2 + v_1 = \lambda + v_2. \tag{A.1.15}$$

This equation is linear in the traditional sense, and has as a general solution $v_1 = \alpha, v_2 = \alpha + 1/2$ and $\lambda = 3/2$ for any α. Hence, it is clear that λ is necessary a non-integer rational number (even though α can be chosen so v has only integers entries, for instance, $\alpha = 1/2$).

In practice, however, it can be assumed *without loss of generality* that λ and v have entries in $\mathbb{T}_{max}$. This is due to the fact that one can redefine the dimensions of the problem to ensure that the eigenvectors and eigenvalues have only integer or $\perp$ entries.

For instance, frequently the entries of the matrices are times, which are measured in, for example, minutes. Suppose the eigenproblem has as $\lambda = 1/2$ minute $= 0.5$ minute. One could use seconds instead of minutes in the data of the problem. This will imply that all the entries of the matrix are multiplied by 60 (since 1 minute = 60 seconds), and the eigenvalue in this new entry will be 30 seconds, which is an integer value. This always possible because λ and v have only rationals and $\perp$ entries, so all numbers x_i in the λ and v can be written as $x_i = h_i/m$, in which $h_i \in \mathbb{T}_{max}$ and m is a positive natural number (a *common base*). Hence, multiplying all the data by m, a simple matter of change of dimension, ensure that v and λ will be integers.

□

Definition A.1.14. *(Spectral radius, see Baccelli et al. (1992))* The spectral radius of A, $\rho(A)$, is the greatest eigenvalue of A. □

Definition A.1.15. *(Precedence graph and incidence matrix, see Baccelli et al. (1992))* Let $A \in \mathbb{T}_{max}^{n\times n}$. The *precedence graph* of A is a weighted graph with n nodes and an arc going to node i to j, with weight A_{ij}, if and only if $A_{ij} \neq \perp$. Dually, one says that A is the *incidence matrix* of this graph. □

Example A.1.7. Consider the graph in Figure A.1.

The incidence matrix is

$$A = \begin{pmatrix} 1 & \perp & -4 & 8 \\ -1 & \perp & \perp & \perp \\ \perp & 2 & \perp & \perp \\ \perp & 5 & \perp & \perp \end{pmatrix}. \tag{A.1.16}$$

□

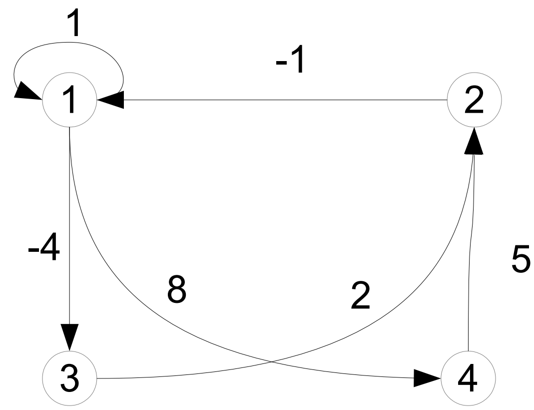

Figure A.1: A graph.

Definition A.1.16. *(Path, cycle, weight and length, see Baccelli et al. (1992))* Consider the incidence matrix A of a graph $\mathcal{G}$. A *path* in a graph is a sequence of nodes $i[0] \to i[1] \to i[2] \to \ldots \to i[k]$. k is said to be the *length* of this path, while $A_{i[0]i[1]} + A_{i[1]i[2]} + \ldots + A_{i[k-1]i[k]}$ is its *weight*. A path is a *cycle* if $i[k] = i[0]$. □

Example A.1.8. In the graph in Figure A.1, $1 \to 4 \to 2 \to 1$ is a cycle with length 3 and weight $8 + 5 + (-1) = 12$. □

Definition A.1.17. *(Strongly connected and irreducible matrix, see Baccelli et al. (1992))* A graph $\mathcal{G}$ is said to be *strongly connected* if, for any two given nodes i and j, there is a path from i to j. The incidence matrix of a strongly connected graph is said to be *irreducible*. □

Example A.1.9. The graph in Figure A.1 is strongly connected and hence the matrix A in Equation A.1.16 is irreducible. □

Theorem A.1.1. *(Spectral theorem in Tropical Algebra, see Baccelli et al. (1992))* Let $\mathcal{C}$ be the set of all cycles of the precedence graph of a matrix $A \in \mathbb{T}_{max}^{n \times n}$. Let $L(P)$ be the length of the path P and $W(P)$ its weight. Then $\rho(A)$ is the *maximum cycle mean*, that is

$$\rho(A) = \bigoplus_{P \in \mathcal{C}} \frac{W(P)}{L(P)} \tag{A.1.17}$$

with the convention of the empty sum if $\mathcal{C} = \emptyset$ (that is, if there are no cycles in the precedence graph of A, $\rho(A) = \perp$). Further, if A is irreducible, there is only one eigenvalue, which is exactly $\rho(A)$. □

Remark A.1.12. Equation (A.1.17) corroborates with the claim made before that the eigenvalue (and hence the eigenvectors) of a matrix with integer or $\perp$ data is a member of $\mathbb{T}_{max}$. For the general result, one needs to prove that all the other eigenvalues are also mean cycles (of subgraphs). See Butkovic et al. (2009) for details of how to compute the entire spectra. □

Example A.1.10. The matrix in Equation (A.1.16) has spectral radius 4 (greatest cycle being $1 \to 4 \to 2 \to 1$, with length 3 and weight $8+5+(-1)=12$, thus $\rho(A)=12/3=4$), and it is its only eigenvalue since the precedence graph is strongly connected. □

Definition A.1.18. (*Power of a matrix, see Baccelli et al. (1992)*) Let $A \in \mathbb{T}_{max}^{n\times n}$. The k^{th} power of a matrix, A^k, is defined recursively for any natural number $k \geq 0$ as $A^k = A^{k-1}A$ with $A^0 = I$. □

Definition A.1.19. *(Kleene Closure, see Baccelli et al. (1992))* Let $A \in \mathbb{T}_{max}^{n\times n}$ and $\rho(A) \leq 0$. Then

$$A^* = \bigoplus_{i=0}^{\infty} A^i. \tag{A.1.18}$$

□

Remark A.1.13. A^* can be also defined in the case that $\rho(A) > 0$, but this needs to be done in a complete dioid, since in this case at least one entry of A^* will be $+\infty = \top$. □

Property A.1.1. *(Of Kleene Closures, see Baccelli et al. (1992)*) Consider $\rho(A), \rho(B) \leq 0$. Then:

1. $A \succeq B \;\Rightarrow\; A^* \succeq B^*$;
2. $(A^*)^* = A^*$;
3. $A^* \succeq A^k$ for any natural number k;
4. $X \succeq AX \iff X = A^*X$;
5. If $A \in \mathbb{T}_{max}^{n\times n}$, $A^* = \bigoplus_{i=0}^{n} A^i$;
6. $\rho(A^*) = 0$;
7. $X = A^*B$ is the smallest solution of $X = AX \oplus B$. Further, it is the only one if $\rho(A) < 0$;

Theorem A.1.2. *(Relation between graphs, power of matrices and Kleene Closures, see Baccelli et al. (1992))* Let $\mathcal{G}$ be the precedence graph of A. Then $\{A^k\}_{ij}$ is the maximum weight among all paths that go from node i to j taking exactly k nodes, with the convention that with 0 nodes one can only go from a node to itself with the weight 0, and that if there is not a path from i to j, the weight is $\perp$. Hence, $\{A^*\}_{ij}$ is the maximum weight among all paths going from node i to node j (no matter how much nodes it takes). □

Remark A.1.14. Theorem A.1.2 suggests that computing A^* is tantamount to solving an all-to-all maximum path problem in graphs, which is a classic in computer science. There are many (strongly) polynomial algorithms for solving it, see for instance, Robert (1962); Stephen (1962). □

Example A.1.11. Consider the matrix

$$B = \begin{pmatrix} -3 & \perp & -8 & 4 \\ -5 & \perp & \perp & \perp \\ \perp & -2 & \perp & \perp \\ \perp & 1 & \perp & \perp \end{pmatrix}. \tag{A.1.19}$$

which is simply the matrix in Equation (A.1.16) with all their entries decreased by $\rho(A) = 4$. It is clear then that $\rho(B) = 0$.

One has that

$$B^2 = BB = \begin{pmatrix} -6 & 5 & -11 & 1 \\ -8 & \perp & -13 & -1 \\ -7 & \perp & \perp & \perp \\ -4 & \perp & \perp & \perp \end{pmatrix}. \tag{A.1.20}$$

and

$$B^* = I \oplus B \oplus B^2 \oplus B^3 \oplus B^4 = \begin{pmatrix} 0 & 5 & -8 & 4 \\ -5 & 0 & -13 & -1 \\ -7 & -2 & 0 & -3 \\ -4 & 1 & -12 & 0 \end{pmatrix}. \tag{A.1.21}$$

□

A.1.3 Residuation Theory

Definition A.1.20. *(Poset, see Schröder (2003))* A *partially ordered set*, or *poset*, $\mathbb{P} = \{\mathscr{P}, \succeq\}$ is a set $\mathscr{P}$ together with a partial order $\succeq$. □

Remark A.1.15. Every dioid $\mathbb{D} = \{\mathscr{S}, \oplus, \otimes\}$ induces a partial order $\succeq$, and hence a poset $\mathbb{P} = \{\mathscr{S}, \succeq\}$ as well. The same with semimodules $\mathbb{M} = \{\mathscr{M}, \mathbb{D}, \oplus_{\mathscr{M}}, \otimes_{\mathscr{M}}\}$, in which the order $\succeq_{\mathscr{M}}$ in $\mathscr{M}$ is as $x \succeq_{\mathscr{M}} y \longleftrightarrow x = x \oplus_{\mathscr{M}} y$. Hence, in this case $\mathbb{P} = \{\mathscr{M}, \succeq_{\mathscr{M}}\}$. □

Definition A.1.21. *(Non-decreasing mapping, see Baccelli et al. (1992))* Given two posets $\mathbb{P}_1 = \{\mathscr{P}_1, \succeq_1\}$, $\mathbb{P}_2 = \{\mathscr{P}_2, \succeq_2\}$, a function $f : \mathscr{P}_1 \mapsto \mathscr{P}_2$ is said to be *non-decreasing* (or *isotone*) if, for any $x, y \in \mathscr{P}_1$, $x \succeq_1 y \rightarrow f(x) \succeq_2 f(y)$. □

Remark A.1.16. Given two semimodules $\mathbb{M}_1 = \{\mathscr{M}_1, \mathbb{D}, \oplus_1, \otimes_1\}$, $\mathbb{M}_2 = \{\mathscr{M}_2, \mathbb{D}, \oplus_2, \otimes_2\}$, their respective induced posets $\mathbb{P}_1$ and $\mathbb{P}_2$ and the map $f : \mathscr{M}_1 \mapsto \mathscr{M}_2$ given by $f(x) = Ax$ for a matrix A, it is non-decreasing. □

Definition A.1.22. *(Residuated mapping, see Baccelli et al. (1992))* Given two posets $\mathbb{P}_1 = \{\mathscr{P}_1, \succeq_1\}$, $\mathbb{P}_2 = \{\mathscr{P}_2, \succeq_2\}$, a non-decreasing function $f : \mathscr{P}_1 \mapsto \mathscr{P}_2$ is said to be *residuated* if, for all $y \in \mathscr{P}_2$, the maximal solution $f^\sharp(y)$ to the inequality $y \succeq_2 f(x)$ exists, that is, $y \succeq_2 f(x) \longleftrightarrow f^\sharp(y) \succeq_1 x$. The function $f^\sharp : \mathscr{P}_2 \mapsto \mathscr{P}_1$ is said to be the *residuated mapping* of f.

Dually, if the minimal solution $f^\natural(y)$ to the inequality $f(x) \succeq_2 y$ exists, that is, $f(x) \succeq_2 y \longleftrightarrow x \succeq_1 f^\natural(y)$, f is said to *dually residuated*. The function $f^\natural : \mathscr{P}_2 \mapsto \mathscr{P}_1$ is said to be the *dual residuated mapping* of f. □

Property A.1.2. *(Of the residuated mapping, see Baccelli et al. (1992))* Let f be a function and $f^\sharp$ its residuated mapping. Then

1. $f^\sharp$ is non-decreasing;
2. $x \succeq f(f^\sharp(x))$;
3. $f^\sharp(f(x)) \succeq x$;

Dual properties hold for the dual residuated mapping $f^\natural$ if the order is swapped. □

Definition A.1.23. (*Product residuation operator for a matrix, see Baccelli et al. (1992)*) Let $A \in \mathbb{T}_{max}^{n\times m}$ and a column vector $x \in \mathbb{T}_{max}^{n}$. Then the operation $A \backslash x$ is defined as

$$\{A \backslash x\}_i \equiv \bigwedge_{j=1}^{n} (-A_{ji}) x_j \ , \ i = 1, 2, ..., m \tag{A.1.22}$$

in which $a \wedge b = \min(a, b)$, and the convention that $\bot - \bot = \top$ is used. □

Remark A.1.17. Note that the vector $A \backslash\!\!\!\circ\, x$ can have $+\infty = \top$ entries, and thus it is in general in a complete dioid. □

Example A.1.12. Let

$$A = \begin{pmatrix} 1 & 2 \\ -3 & 5 \\ \bot & 8 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}. \tag{A.1.23}$$

Then

$$A \backslash\!\!\!\circ\, b = \begin{pmatrix} \min(-1+0, 3+2, +\infty+6) \\ \min(-2+0, -5+2, -8+6) \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}. \tag{A.1.24}$$

□

Theorem A.1.3. (*Tropical product can be residuated, see Baccelli et al. (1992)*) Given two semimodules $\mathbb{M}_1 = \{\mathscr{M}_1, \mathbb{D}, \oplus_1, \otimes_1\}$, $\mathbb{M}_2 = \{\mathscr{M}_2, \mathbb{D}, \oplus_2, \otimes_2\}$, their respective induced posets $\mathbb{P}_1$ and $\mathbb{P}_2$ and the map $f : \mathscr{M}_1 \mapsto \mathscr{M}_2$ given by $f(x) = Ax$ for a matrix A, then f is residuated and $f^{\sharp}(y) = A \backslash\!\!\!\circ\, y$. □

Remark A.1.18. Given A and b as in Equation (A.1.23), the greatest solution to the inequality $b \succeq Ax$ is $x_{max} = A \backslash\!\!\!\circ\, b$. Further, the solution set to $b \succeq Ax$ can be characterized as $x_{max} \succeq x$. □

Definition A.1.24. (*Sum residuation operator for a matrix, see Baccelli et al. (1992)*) Let $a \in \mathbb{T}^n_{max}$ be a column vector $x \in \mathbb{T}^n_{max}$. Then the operation $x \circ\!\!\!\!-\, a$ is defined as

$$\{x \circ\!\!\!\!-\, a\}_i \equiv \begin{cases} x_i & \text{if } x_i > a_i \\ \bot & \text{if } x_i \leq a_i \end{cases} \tag{A.1.25}$$

□

Theorem A.1.4. (*Tropical sum can be dually residuated, see Baccelli et al. (1992)*) Given two semimodules $\mathbb{M}_1 = \{\mathscr{M}_1, \mathbb{D}, \oplus_1, \otimes_1\}$, $\mathbb{M}_2 = \{\mathscr{M}_2, \mathbb{D}, \oplus_2, \otimes_2\}$, their respective induced posets $\mathbb{P}_1$ and $\mathbb{P}_2$ and the map $f : \mathscr{M}_1 \mapsto \mathscr{M}_2$ given by $f(x) = x \oplus a$ for a column vector a, then f is dually residuated and $f^{\natural}(y) = y \circ\!\!\!\!-\, a$. □

Example A.1.13. Let $x = (1\ 2\ 10)^T$ and $y = (5\ 2\ 0)^T$, then $x \circ\!\!\!\!-\, y = (\bot\ \bot\ 10)^T$. □.

Theorem A.1.5. (*Solving equations, see Baccelli et al. (1992)*) Given two semimodules $\mathbb{M}_1 = \{\mathscr{M}_1, \mathbb{D}, \oplus_1, \otimes_1\}$, $\mathbb{M}_2 = \{\mathscr{M}_2, \mathbb{D}, \oplus_2, \otimes_2\}$, their respective induced posets $\mathbb{P}_1$ and $\mathbb{P}_2$ and the map $f : \mathscr{M}_1 \mapsto \mathscr{M}_2$.

If f is residuated with residuated mapping $f^{\sharp}$, then the equation $f(x) = y$ has a solution for x if and only if $f(f^{\sharp}(y)) = y$, and $x = f^{\sharp}(y)$ is the greatest solution. Dually, if f is dually residuated with dual residuated mapping $f^{\natural}$, then the equation $f(x) = y$ has a solution for x if and only if $f(f^{\natural}(y)) = y$, and $x = f^{\natural}(y)$ is the smallest solution. □

Example A.1.14. Let A be as in Equation (A.1.23), and

$$c = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}. \tag{A.1.26}$$

Then

$$A \circ\!\!\!\backslash c = \begin{pmatrix} \min(-1+0, 3+2, +\infty+5) \\ \min(-2+0, -5+2, -8+5) \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}. \tag{A.1.27}$$

One has that $A(A \circ\!\!\!\backslash c) = c$, and hence $Ax = c$ has a solution, with $A \circ\!\!\!\backslash c$ being the greatest one. Compare with the equation $Ax = b$, with b being as in Equation (A.1.23). This time, $A(A \circ\!\!\!\backslash b) \neq b$ and hence the equation does not have a solution. □

A.2 Timed Event Graph Basics

Definition A.2.1. *(Timed Event Graphs and P-Timed Event Graphs, see Baccelli et al. (1992))* A *Timed Event Graph* is a *Timed Petri Net* (see the definition in Baccelli et al. (1992)) such that all places are connected to at most one transition, no more than one transition is connected to the same place and all the arcs have unitary weight.

A *P-Timed Event Graph* ("P" stands for "Place") is a Timed Event Graph in which the timings are in the places. □

Remark A.2.1. The timing in the places mentioned in the previous definition stands for the minimal amount of time that a token must be held, after it is delivered to the place, till it can be used to enable a transition. See Example A.2.2 for details. □

Example A.2.1. Consider the three Petri Nets in Figure A.2.

PNA is a Timed Event Graph. PNB is not because there is a place connected to more than one transition. PNC is also not because there are more than one transition connected to the same place.

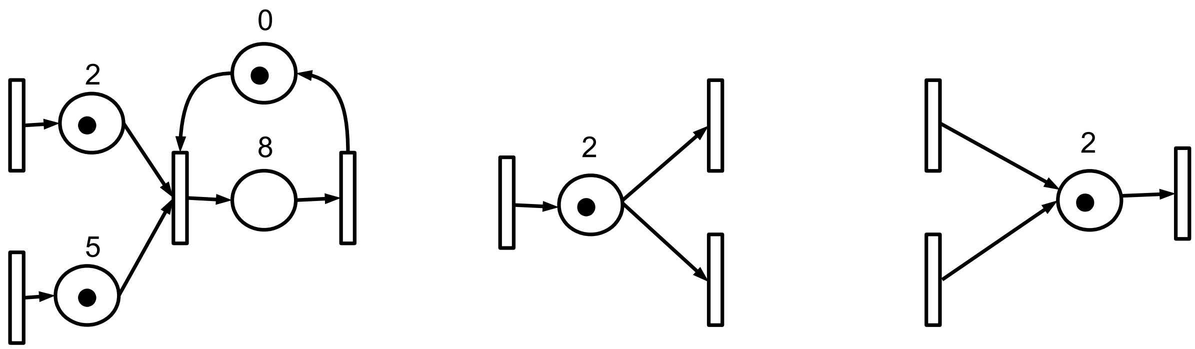

Figure A.2: Three Petri Nets.

Note also the number above each place: it is their respective *timing*, as mentioned in Definition A.2.1.□

Definition A.2.2. (*Tropical Linear Event-Invariant Dynamical Systems*) A *Tropical linear event-invariant dynamical system* is a recursive equation of the form

$$x[k+1] = Ax[k] \oplus Bu[k]; \tag{A.2.1}$$

$$y[k] = Cx[k] \oplus Gu[k]; \tag{A.2.2}$$

with a given $x[0] \in \mathbb{T}^n_{max}$, in which $x[k] \in \mathbb{T}^n_{max}$, $u[k] \in \mathbb{T}^m_{max}$, $y[k] \in \mathbb{T}^d_{max}$, $A \in \mathbb{T}^{n\times n}_{max}$, $B \in \mathbb{T}^{n\times m}_{max}$, $C \in \mathbb{T}^{d\times n}_{max}$ and $G \in \mathbb{T}^{d\times m}_{max}$. □

Remark A.2.2. Sometimes in this thesis, a reduced form of Equation (A.2.1), without the equation $y[k] = Cx[k] \oplus Gu[k]$, will be also be referred as Tropical linear event-invariant dynamical system. □

Remark A.2.3. The name "Tropical linear event-invariant dynamical system" comes in analogy with the linear time-invariant dynamical systems:

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k]; \\ y[k] &= Cx[k] + Gu[k]; \end{aligned} \tag{A.2.3}$$

in which all products and sums are interpreted in the traditional algebra. Note that in the context of Equation (A.2.3) k represents a *time* while in Equation (A.2.1) an *event*. $\square$

Theorem A.2.1. (*Dater dynamics of non-autonomous timed event graph, see Baccelli et al. (1992)*) Let $\mathcal{T}$ be a Timed Event Graph with n state transitions, m input transitions (such that no place are connected to them) and d output transitions (such that they are connected to no place). Let $x_i[k]$ be the earliest date that the i^{th}, $i = 1, 2, ..., n$, state transition can fire for the k^{th} time, $u_j[k]$ the date in which the j^{th}, $j = 1, 2, .., m$, input transition fires for the k^{th} time and $y_l[k]$ be the earliest date that the l^{th}, $l = 1, 2, ..., n$, output transition can fire for the k^{th} time. Then there exist a Tropical linear event-invariant dynamical system, as in Equation (A.2.1), that relates the firing dates $x_i[k]$ and $y_l[k]$ with the ones of $u_j[k]$. $\square$

Example A.2.2. Consider the Timed Event Graph in Figure A.3

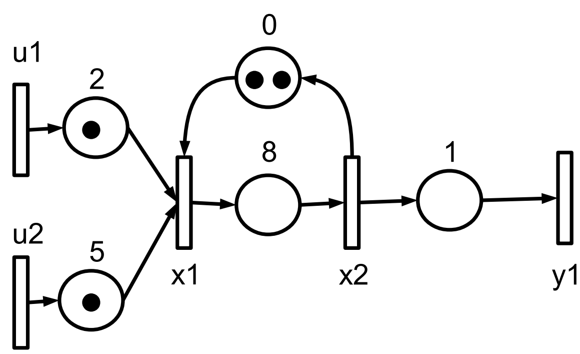

Figure A.3: A timed event graph.

The first thing one needs to do is to classify the transitions as states, inputs or outputs. In the given image, this was already done (the labels x, u and y given to the transitions), but this could be easily done by finding the transitions such that no place are connected to them (inputs), the ones such that they are connected to no place (outputs) and the remainder (states). Note that a transition that no place is connected to them *and* it is connected to no place is innocuous to the system and can be removed.

Now, consider transition x_1. One needs to discover, first, all the places that are connected to them, and the respective transitions to each one of these places (there will be only one transition assigned to each place, due to the fact that the Petri Net is a Timed Event Graph). In the case of x_1, there are three, namely:

- u_1 with a place with timing 2 time units and one token;

• u_2 with a place with timing 5 time units and one token;

• x_2 with a place with timing 0 time units and two tokens;

In order to x_1 to fire for the $(k+1)^{th}$ time, u_1 needs to have fired for the k^{th} time (since there is already one token) and waited at least 2 time units (the *minimum* holding time of the token there). *Also*, u_2 needs to have fired for the k^{th} time (since there is already one token) and waited at least 5 time units, the minimum holding time there. Finally, it is *also* necessary that x_2 fired for the $(k-1)^{th}$ time (since there is already two tokens) and waited 0 (nothing) time units (there is no minimum holding time in this case). Hence, one can write that

$$x_1[k+1] \geq \max(x_2[k-1], u_1[k]+2, u_2[k]+5). \tag{A.2.4}$$

With a similar analysis, one can derive the following equation for x_2

$$x_2[k+1] \geq 8 + x_1[k]; \tag{A.2.5}$$

and

$$y_1[k] \geq 1 + x_2[k]. \tag{A.2.6}$$

Then, the equations for the earliest firing time are obtained by swapping the $\geq$ with the equality

$$\begin{aligned} x_1[k+1] &= \max(x_2[k-1], 2+u_1[k], 5+u_2[k]); \\ x_2[k+1] &= 8 + x_1[k+1]; \\ y_1[k] &= 1 + x_2[k]. \end{aligned} \tag{A.2.7}$$

Note that in the equation for $x_1[k+1]$, a term in the event $k-1$, namely $x_2[k-1]$, appears at the right. Further, in the equation for $x_2[k+1]$ a term in the event $k+1$, namely $x_1[k+1]$, appears at the right. In order to write these equations as in Equation (A.2.1), only delays of order k in the states can be at the right side of the Equation.

The problem for $x_2[k-1]$ can be solved by creating a new state $x_3[k]$ and a new equation $x_3[k+1] = x_2[k]$. In this way, one can write that

$$
\begin{aligned}
x_1[k+1] &= \max(x_3[k], 2+u_1[k], 5+u_2[k]);\\
x_2[k+1] &= 8+x_1[k+1];\\
x_3[k+1] &= x_2[k];\\
y_1[k] &= 1+x_2[k].
\end{aligned}
\tag{A.2.8}
$$

To solve the problem for $x_1[k+1]$, one simply substitutes the first equation in Equation (A.2.7), that is, use the fact that $x_1[k+1] = \max(x_3[k], u_1[k]+2, u_2[k]+5)$, to conclude that

$$
\begin{aligned}
x_1[k+1] &= \max(x_3[k], 2+u_1[k], 5+u_2[k]);\\
x_2[k+1] &= \max(8+x_3[k], 10+u_1[k], 13+u_2[k]);\\
x_3[k+1] &= x_2[k];\\
y_1[k] &= 1+x_2[k].
\end{aligned}
\tag{A.2.9}
$$

Using the Tropical Algebra notation.

$$
\begin{aligned}
x_1[k+1] &= x_3[k] \oplus 2u_1[k] \oplus 5u_2[k];\\
x_2[k+1] &= 8x_3[k] \oplus 10u_1[k] \oplus 13u_2[k];\\
x_3[k+1] &= x_2[k];\\
y_1[k] &= 1x_2[k].
\end{aligned}
\tag{A.2.10}
$$

Equation (A.2.10) can then be written as Equation (A.2.1):

$$
\begin{pmatrix} x_1[k+1] \\ x_2[k+1] \\ x_3[k+1] \end{pmatrix} = \begin{pmatrix} \perp & \perp & 0 \\ \perp & \perp & 8 \\ \perp & 0 & \perp \end{pmatrix} \begin{pmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \end{pmatrix} \oplus \begin{pmatrix} 2 & 5 \\ 10 & 13 \\ \perp & \perp \end{pmatrix} \begin{pmatrix} u_1[k] \\ u_2[k] \end{pmatrix},
$$

$$
y[k] = \begin{pmatrix} \perp & 1 & \perp \end{pmatrix} \begin{pmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \end{pmatrix} \oplus \begin{pmatrix} \perp & \perp & \perp \end{pmatrix} \begin{pmatrix} u_1[k] \\ u_2[k] \end{pmatrix}. \tag{A.2.11}
$$

□

Appendix B

Report on the Thesis

Palaiseau, le 20 Novembre 2014

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Report on the PhD manuscript of Vinicius Mariano Gonçalves

The PhD manusript of Vinicius Mariano Gonçalves is entitled "Tropical algorithms for linear algebra and linear event-invariant dynamical systems".

This is a significant contribution to the theory of max-plus linear discrete event systems, and in particular to the "geometric approach", in which one formulates control problems as fixed point problems in spaces of modules and solutions of max-plus linear equations. The author presents several results, some of which have been already been published or accepted in articles with Maia and Hardouin.

The introduction summarizes the main results of the thesis. Instead of a cold presentation to be often found in theses, it rather gives a personal perspective, explaining how the author came to the different results, and the logic of their development.

Chapter 2 is a contribution of the theory of tropical linear equations. Vinicius Mariano Gonçalves studies tropical linear systems of the form $Ex = Dx$, i.e.,

$$\max_{1\leq j\leq n}(E_{ij}+x_j) = \max_{1\leq j\leq n}(D_{ij}+x_j),\ 1\leq i\leq m$$

where the E_{ij}, D_{ij} are given. It is known that the set of solutions of such systems are precisely the tropical analogues of polyhedral finitely generated convex cones. An important problem is to find a solution if there is one, or to verify that there is none. This problem has been previously been shown to be (Karp) polynomial time equivalent to mean payoff game (a well known problem in NP $\cap$ coNP for which no polynomial time algorithm is still known). However, algorithms often practically efficient have been previously developed : value or Kleene iteration; analogues of the cyclic projection algorithm of Von Neumann; policy iteration; pivoting algorithms, including tropical analogues of the simplex algorithm.

This chapter is concerned specifically with the following :

Problem Find a *minimal* x satisfying $Ex = Dx$ and $x \geq x_0$, where x_0 is given.

In the litterature, it is rather the dual problem, with $x \leq x_0$, and max instead of min, that has been considered. The present problem is harder, as there are in general several (exponentially many) minimal solutions. This is related to the fact that a tropically linear map is generally residuated but not dually residuated, meaning that, given y, there is a unique maximal x such that $Ax \leq y$, but not a unique mininal x such that $Ax \geq y$. To solve this difficulty, Vinicius Mariano GONÇALVES considers a notion of dominance matrices, which essentially represent *policies* for the associated mean payoff games. Geometrically, the set $\{x \mid Ex = Fx\}$ is a polyhedral complex and every cell of this complex corresponds to a pair of dominance matrices. Then, Vinicius Mariano GONÇALVES shows that the Problem above becomes solvable if $\{x \mid Ex = Dx\}$ is replaced by one of its cells, since the minimum element of a cell such that $x \geq x_0$ is given by a Kleene star operation. The proofs of this chapter rely on the introduction of a modified notion of residuated map, which is taylored to find minimal elements greater than a given vector. This result will be useful in practice to refine solutions found by other means. Although the consideration of cells and polytropes defined by Kleene stars is not entirely new or suprising, this result is not without theoretical interest, as it shows that the minimal solutions x of $Ex = Dx$ such that $x \geq x_0$ are canonically attached to the cell decomposition. This may have further interpretations in terms of duality, it would be interesting to see whether this can be related to the theory of minimal solutions of classical cover inequalities, $Ax \geq b$, with A a nonnegative matrix, and x, a Boolean or integer vectors, by Boros, Fredman, Elbassioni, Gurvich, Khachyan, and Makino (work on quasi-polynomial incremental algorithms to generate minimal elements).

Chapter 3 deals with the tropical analogue of fractional linear programming, i.e., the minimization of the ratio of two affine forms over the set of points of a tropical polyhedron. The fractional linear programming problem arises in applications to discrete event systems, a typical example of function to be minimized being $x_j - x_i$, the difference of two variables, representing generally a delay, a sojourn time, etc. (The difference $x_j - x_i$ is a non linear but fractional object, in the tropical structure.) The author introduces Algorithm 3.2.2 to solve non-fractional tropical linear programs, by reduction to a series of feasibility problems for tropical equations. One step of this algorithm exploits the result of the previous chapter by finding a minimal constrained element in a certain cell of a tropical polyhedron, determined by dominance matrices. The other step consists in looking for an nonoptimality certificate, and using it to improve the current point. As the authors notes it, this has some similarities with a policy iteration algorithm of Katz, Sergeev, and the reviewer, although the nonoptimality certificate is a different one, and leads to a somehow more direct method when the data are integer valued. Then, the author makes the relatively simple but very interesting observation that tropical fractional linear programming reduce to tropical linear programming by a homographic transformation, similar to the Charnes-Cooper transformation in classical fractional programming. By exploiting the symmetry properties of fractional linear programs, this leads to a direct equivalence between minimization and maximization problems for tropical linear programs, which was not noticed before.

Chapter 4 is in my opinion the heart of the thesis. It addresses an original regulator problem for tropically linear discrete event systems, in which one looks for a control such that the state *ultimately* belongs to a target space (representing a specification). This is to be contrasted with earlier works on the geometric approach of discrete event systems, by Katz, in which one looked for a subset of initial conditions for which there is a control making the state stay in the target space forever. The contribution of the author is to reduce, under appropriate technical assumptions, the regulation problem to the solution of a generalized eigenproblem over the tropical semiring, that the authors reduces to the solution of a parametric mean payoff game by extending an argument of Sergeev and the reviewer, developed in a more special case. This appears to be more tractable than the original approach of Katz, which required as an expensive step the solution of a fixed point problem in the set of finitely generated tropical modules. This is dispensed with here by considering a special family of one dimensional invariant sets, leading to a more tractable formulation. The conditions under which this formulation is valid are then carefully analysed. The author is showing in particular that the spectral formulation is necessary and sufficient under a “noncriticality” condition, which in particular, is valid for generic data. He also observes that certain nongeneric instances, but not all, can be reduced to the generic case by perturbation. He finally addresses the question of minimizing the convergence time to the target, and making the feedback matrix nonnegative, which corresponds to a causality assumption for timed event graphs.

Chapter 5 developed a dual theory, concerning this time an observation problem, in which one wishes to reconstruct ultimately (exactly after a finite time t) a linear function of the state. In the tropical setting, the duality is not so straightforward as in classical linear algebra : whereas primal spaces are the tropical analogues of modules, dual spaces (representing informations and observations) are represented by congruences, following earlier works of Cohen et al. and Di Loreto et al. The main result of this chapter shows that the observability problem, for a fixed time t, is equivalent to the solution of a tropical linear system of size $O(t)$. It uses a number of results of tropical spectral theory.

Finally, Chapter 6 presents the practical implementation of controller and observer on a realistic example, a conveyor belt system deployed at Laboratoire Angevin de Recherche en Ingénierie des Systèms (LARIS) of Angers. This confirms that the approach can be applied in practice, and indicates that it should be scalable, allowing one to solve larger industrial instances, out of reach of Katz's earlier method.

In conclusion, I found original and non trivial results in this thesis. The regulation and observability problems which are considered are original in themselves, and very relevant to practical applications. I was specially impressed by the developments of Chapter 4, the reduction of the regulation problem to a generalized eigenproblem is a clear progress by comparison with earlier results of several researchers in the field. The author showed a very good command of methods of linear systems theory and of tropical algebra, including spectral theory. He showed the ability to take a perspective, to formulate good questions and to solve them, which shows some scientific maturity. The manuscript is probably not so accessible to researchers not working in the field : the action starts quite immediately, with little backround results (the reader is assumed for instance to be familiar with residuation). However, when it comes to technical developments, the work is complete, so that it will be understood by experts in the area. I appreciated that the work is relatively wide (there are several distinct results), and that the author showed both the ability to develop theory and to apply his results to a realistic problem. The academic standard of the work is evidenced by the publication file, with, besides conference papers in refereed control or discrete event systems conferences, comprises two articles in Linear Algebra and Applications (there are not so many articles motivated by discrete event systems published in mathematical journals). The materials of Chapter 4, currently under review at IEEE-TAC, deserve in my opinion to be published in such a high standard control journal. I think this is a significant work in the field of max-plus linear discrete event systems, inovative in terms of modelling of control problems and in terms of solution methods. I am very happy to recommend the defense to take place.

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Bibliography

Akian, M., Gaubert, S., Guterman, A. E., 2010. The correspondence between tropical convexity and mean payoff games. In: Mathematical Theory of Networks and Systems. pp. 1295–1302.

Allamigeon, X., Benchimol, P., Gaubert, S., Joswig, M., 2013. Tropicalizing the simplex algorithm. 18th Conference of the International Linear Algebra Society, Rhode Island, USA.

Allamigeon, X., Gaubert, S., Goubault, E., 2010. The tropical double description method. In: 27th International Symposium on Theoretical Aspects of Computer Science (STACS 2010). Vol. 5. pp. 47–58.

Amari, S., Demongodin, I., Loiseau, J., 2005. Control of linear min-plus systems under temporal constraints. 44th IEEE Conference on Decision and Control and European Control Conference, Seville, Spain.

Amari, S., Demongodin, I., Loiseau, J., Jean-Jacques, J., Martinez, C., 2012. Max-plus control design for temporal constraints meeting in timed event graphs. IEEE Transactions on Automatic Control 57 (2), 462 – 467.

Amari, S., Demongodin, I., Loiseau, J.-J., 2004. Sizing and cycle time of an industrial plant using dioid algebra. Chapter 6 in Supply Chain Optimization, A. Dolgui, J. Soldek and O. Zaikin (Eds.), Series Applied Optimization, Springer-Verlag,, 71–85.

Attia, R., Amari, S., Martinez, C., 2010. Control of time-constrained dual-armed cluster tools using (max, +) algebra. In: 2010 Conference on Control and Fault Tolerant Systems, Nice, France. pp. 412–417.

Atto, A. M., Martinez, C., Amari, S., 2011. Control of discrete event systems with respect to strict duration: Supervision of an industrial manufacturing plant. Computers and Industrial Engineering 61 (4), 1149–1159.

Baccelli, F., Cohen, G., Olsder, G., Quadrat, J., 1992. Synchronization and Linearity. Wiley, New York.

Brackley, C. A., Broomhead, D., Romano, M. C., Thiel, M., 2011. A max-plus model of ribosome dynamics during mRNA translation. Journal of Theoretical Biology 302, 128–140.

Brunsch, T., 2014. Modeling and control of complex systems in a dioid framework. Ph.D. thesis, Technische Universität, Berlin.

Brunsch, T., Hardouin, L., Maia, C. A., Raisch, J., 2012. Duality and interval analysis over idempotent semirings. Linear Algebra and its Applications 437, 2436–2454.

Butkovic, P., Aminu, A., 2008. Introduction to max-linear programming. IMA Journal of Management Mathematics 20 (3), 233–249.

Butkovic, P., Cuninghame-Green, R. A., Gaubert, S., 2009. Reducible Spectral Theory with Applications to the Robustness of Matrices in Max-Algebra. Siam Journal on Matrix Analysis and Applications 31, 1412–1431.

Butkovic, P., Hegedus, G., 1984. An elimination method for finding all solutions of the system of linear equations over an extremal algebra. Ekonom. mat. Obzor 20 (2), 203–215.

Butkovic, P., MacCaig, M., 2013. On the integer max-linear programming problem. Discrete Applied Mathematics 162, 128–141.

Butkovic, P., Zimmermann, K., 2006. A strongly polynomial algorithm for solving two-sided linear systems in max-algebra. Theoretical Computer Science. 154 (3), 437–446.

Charnes, A., Cooper, W. W., 1962. Programming with linear fractional functionals. Naval Research Logistics Quarterly, 181 – 196.

Cohen, G., Gaubert, S., Quadrat., J., 2004. Duality and separation theorems in idempotent semi-modules. Linear Algebra and Applications 379, 395–422.

Cuninghame-Green, R., 1979. Minimax Algebra. Number 166 in Lecture notes in Economics and Mathematical Systems. Springer.

Cuninghame-Green, R. A., Butkovic, P., 2003. The equation $A \otimes x = B \otimes y$ over (max; +). Theoretical Computer Science 293 (1), 3 – 12.

Cuninghame-Green, R. A., Zimmermann, K., 2001. Equation with residuated functions. Commentationes Mathematicae Universitatis Carolinae 42 (4), 729–740.

Develin, M., Sturmfels, B., 2004. Tropical convexity. Documenta Math 9, 1–27.

Dhingra, V., Gaubert, S., 2006. How to solve large scale deterministic games with mean payoff by policy iteration. In Proceedings of the 1st international conference on Performance evaluation methodologies and tools, Pisa, Italy.

Gaubert, S., 1992. Théorie des systèmes linéaires dans les dioïdes (Theory of linear systems in dioids). Ph.D. thesis, École Nationale Supérieure des Mines de Paris, Paris, France.

Gaubert, S., Katz, R., 2009. The tropical analogue of polar cones. Linear Algebra and its Applications 431, 608–625.

Gaubert, S., Katz, R., Sergeev, S., 2012. Tropical linear-fractional programming and parametric mean payoff games. Journal of Symbolic Computation 47 (12), 1447–1478.

Gaubert, S., M.Plus, 1997. Methods and applications of (max,+) linear algebra. In: Proceedings of the 14th Annual Symposium on Theoretical Aspects of Computer Science. p. 261Ű282.

Gaubert, S., Sergeev, S., 2013. The level set method for the two-sided eigenproblem. Discrete Event Dynamic Systems 23 (2), 105–134.

Gonçalves, V. M., Maia, C. A., Hardouin, L., 2012. On the solution of max-plus linear equations with application on the control of timed event graphs. Workshop on Discrete Event Systems (WODES 2012), 91–97.

Gonçalves, V. M., Maia, C. A., Hardouin, L., 2013a. On the steady state control of timed event graphs with firing date constraints. IEEE Transactions on Automatic Control (Submitted).

Gonçalves, V. M., Maia, C. A., Hardouin, L., 2013b. On tropical fractional linear programming. Linear Algebra and its Applications 459 (0), 384 – 396.

Gonçalves, V. M., Maia, C. A., Hardouin, L., 2013c. Solving tropical linear equations with weak dual residuations. Linear Algebra and its Applications 445, 69–84.

Gonçalves, V. M., Maia, C. A., Hardouin, L., 2014a. Avanços na teoria de controle para sistemas lineares na álgebra tropical. In: Congresso Brasileiro de Automática 2014.

Gonçalves, V. M., Maia, C. A., Hardouin, L., Shang, Y., 2014b. An observer for tropical linear event-invariant dynamical systems. IEEE CDC-ECC, Los Angeles.

Hardouin, L., 2004. Habilitation à diriger des recherches: Sur la commande linéaire de systèmes á evenements discrets dans l'algèbre (max,+) (in french). `http://perso-laris.univ-angers.fr/~hardouin/hdr_hardouin.pdf`.

Hardouin, L., Cottenceau, B., Lhommeau, M., Le Corronc, E., 2009. Interval systems over idempotent semiring. Linear Algebra and its Applications 431 (5-7), 855–862.

Hardouin, L., Maia, C., Cottenceau, B., Lhommeau, M., September 2010a. Max-plus linear observer: Application to manufacturing systems. In: Workshop on Discrete Event Systems, WODES 2010. pp. 171–176.

Hardouin, L., Maia, C. A., Cottenceau, B., Lhommeau, M., 2010b. Observer design for (max,plus) linear systems. IEEE Trans. on Automatic Control 55-2, 538–543.

Heidergott, B., Olsder, G. J., van der Woude, J., 2006. Max Plus At Work. Princeton University Press.

Katz, R. D., 2007. Max-plus (A; B)-invariant spaces and control of timed discrete-event systems. IEEE Transactions on Automatic Control 52 (2), 229 – 241.

Lorenzo, E., de la Puente, M. J., 2011. An algorithm to describe the solution set of any tropical linear system $Ax = Bx$. Linear Algebra and its Applications 435 (4), 884–901.

Loreto, M. D., Gaubert, S., Katz, R. D., Loiseau, J.-J., 2010. Duality between invariant spaces for max-plus linear discrete event systems. SIAM J. Control Optim 48, 5606–5628.

Maia, C. A., Andrade, C. R., Hardouin, L., 2011a. On the control of maxplus linear system subject to state restriction. Automatica 47 (5), 988–992.

Maia, C. A., Cottenceau, B., Hardouin, L., Santos-Mendes, R., 2003. Optimal closed-loop control of timed event graphs in dioids. IEEE Transactions on Automatic Control 48 (12), 2284 – 2287.

Maia, C. A., Hardouin, L., R, C. J. E., 2013. Some results on the feedback control of max-plus linear systems under state constraints. IEEE CDC, Firenze, Italy.

Maia, C. A., Hardouin, L., Santos-Mendes, R., Loiseau, J. J., 2011b. A super-eigenvector approach to control constrained max-plus linear systems. IEEE CDC-ECC, Orlando, Florida.

Osborne, M. J., Rubinstein, A., 1994. A Course in Game Theory. The MIT Press.

Robert, W. F., 1962. Algorithm 97: Shortest path. Communications of the ACM 5 (6), 345.

Schröder, B. S. W., 2003. Ordered Sets: An Introduction. Birkhäuser, Boston.

Schutter, B. D., Moor, B. D., 1996. A method to find all solutions of a system of multivariate polynomial equalities and inequalities in the max algebra. Discrete Event Dynamic Systems: Theory and Applications.

Speyer, D., Sturmfels, B., 2004. Tropical mathematics. Idempotency, J. Gunawardena.

Stephen, W., 1962. A theorem on boolean matrices. Journal of the ACM 9 (1), 11–12.

T. Brunsch, J. R., Hardouin, L., 2012. Modeling and control of high-throughput screening systems. Control Engineering Practice 20, 14–23.

Truffet, L., 2010. A decomposition formula of idempotent polyhedral cones based on idempotent superharmonic spaces. Contributions to algebra and geometry 51 (2), 313–336.