Observer for Weighted Timed Event Graphs

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Abstract: This paper addresses observer design for Weighted Timed Event Graphs (WTEGs). WTEGs are a more general class of Timed Discrete Event Systems than standard Timed Event Graphs (TEGs). Hence our results represents a generalization of observer synthesis methods in Hardouin et al. (2007).

Keywords: Timed Event Graphs, Discrete Event Systems, synchronization, max-plus algebra, dioid.

1. INTRODUCTION AND MOTIVATION

Many Discrete Event System (DES), such as transportation networks, communication networks, and assembly lines can be modeled by timed Petri nets, see, e.g. Heidergott et al. (2006); Max Plus (1991); Cassandras and Lafortune (1999). TEGs are a subclass of Timed Petri nets, which can model synchronization and delay phenomena, but not conflicts. TEGs are timed Petri nets, where each place has exactly one upstream and one downstream transition and the weights of all arcs are one. TEGs admit a linear model in the (max,+) algebra (Baccelli et al. (1992); Heidergott et al. (2006)). These has lead to the development of a specific control theoretic framework where problems such as model predictive control van den Boom et al. (2005), optimal feedforward control (Cohen et al. (1989); Menguy et al. (1998)) and feedback control (Cottenceau et al. (2003); Maia et al. (2003); Lhommeau et al. (2003)) have been addressed. More recently, an estimation problem for TEGs has been considered in (Hardouin et al. (2007)). In this context, it is desired to generate an optimal estimate for the firing times of all transitions in the TEG where only the firing times of a subset of transitions can be directly observed. To solve this problem, a structure reminiscent of the classical Luenberger observer was proposed in the max-plus algebra. Moreover, observer-based control strategies have been proposed in (Hardouin et al. (2017, 2018)). In this paper, we study similar estimation techniques, but for a class of timed Petri nets that is more general than TEGs. These are Weighted Timed Event Graphs, i.e., TEGs where the weights of arcs may be different from one. In WTEGs two phenomena can be modeled which are not compatible with TEG models, namely batching of events and multiplication of events. The former describes scenarios where several occurrences of a certain event are needed before another event can happen. The latter describes a scenario where the single occurrence of a certain event enables the multiple occurrence of another event. WTEGs can no longer be described by linear equations in the (max,+) algebra. However, their behavior can be modeled in a specific dioid of operators, denoted by $\mathcal{E}[\![\delta]\!]$, which was introduced in (Cottenceau et al. (2014)). Moreover, in (Trunk et al. (2017)) it was shown how addition, multiplication, and residuation (approximate division) of element in $\mathcal{E}[\![\delta]\!]$ can be conveniently computed by reducing these operations to operations between matrices with entries in the well established dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ (Baccelli et al. (1992)). In this paper, observer design for WTEGs is addressed in the specific dioid $\mathcal{E}[\![\delta]\!]$. Hence, results for observer design for standard TEGs obtained in the (max,+) algebra are generalized to WTEGs in the dioid $\mathcal{E}[\![\delta]\!]$. The proposed observer records the number of firings of certain inputs and outputs transitions of the WTEG and generates ab optimal estimate for the number of firings of internal transitions up to the current instant of time. Optimality in this context means the following. The estimated number of firings should be as close as possible to the actual number, but should never be smaller than the latter. This is done in the presence of disturbances, i.e., events that can neither be controlled nor directly recorded. Practical applications provide ample motivation for addressing this kind of estimation problem. For example, in a production process, disturbances may describe machine breakdown or failure in component supply. The proposed observer can then be used for an early detection of such disturbances. The paper is organized as follows. Section II summarizes the required facts on timed Petri nets, TEGs, and WTEGs. Section III summarizes dioid and residuation theory and recalls the specific dioid $\mathcal{E}[\![\delta]\!]$. In Section IV, the modeling of WTEGs in the dioid $\mathcal{E}[\![\delta]\!]$ is discussed. In Section V, observer synthesis for WTEGs is addressed and it is shown that the optimal observer of a consistent WTEG is a consistent WTEG.

2. WEIGHTED TIMED EVENT GRAPHS

2.1 Petri nets and Timed Event Graphs

In the following we briefly summarize some facts on Petri nets and introduce the notation used in the paper. Matrices and vectors are indicated by bold letters. Moreover, $(\boldsymbol{A})_{i,j}$ (respectively $(\boldsymbol{A})_{:,j}$, $(\boldsymbol{A})_{i,:}$) denotes the $(i,j)^{th}$ entry (resp. j^{th} column, i^{th} row) of matrix $\boldsymbol{A}$.

Definition 1. A Petri net graph is a directed bipartite graph $\mathcal{N} = (P, T, w)$, where:

- $P = \{p_1, p_2, \ldots, p_n\}$ is the finite set of places.
- $T = \{t_1, t_2, \ldots, t_m\}$ is the finite set of transitions.
- $w : (P \times T) \cup (T \times P) \to \mathbb{N}_0$ is the weight function.

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Manuscript 73 submitted to 15th IFAC Workshop on Discrete Event Systems.
Received December 23, 2019.

$A := \{(p_i, t_j)|w(p_i, t_j) > 0\} \cup \{(t_j, p_i)|w(t_j, p_i) > 0\}$ is the arc set, and $\boldsymbol{W} \in \mathbb{Z}^{n\times m}$, where $(\boldsymbol{W})_{i,j} = w(t_j, p_i) - w(p_i, t_j)$, is the incidence matrix of the Petri net graph $\mathcal{N}$. Furthermore, the set of upstream and downstream transitions (respectively places) are defined as follows:

- ${}^\bullet p_i := \{t_j \in T|(t_j, p_i) \in A\}$ is the set of upstream transitions of p_i,
- $p_i^\bullet := \{t_j \in T|(p_i, t_j) \in A\}$ is the set of downstream transitions of place p_i,
- ${}^\bullet t_j := \{p_i \in P|(p_i, t_j) \in A\}$ is the set of upstream places of transition t_j,
- $t_j^\bullet := \{p_i \in P|(t_j, p_i) \in A\}$ is the set of downstream places of transition t_j.

Definition 2. A Petri net consists of a Petri net graph $\mathcal{N}$ and a vector of initial markings $\boldsymbol{\mathcal{M}}_0 \in \mathbb{N}_0^n$, i.e. an initial distribution of tokens over places in $\mathcal{N}$. ◁

A transition t_j can fire, iff $\forall p_i \in {}^\bullet t_j,\ (\boldsymbol{\mathcal{M}})_i \geq w(p_i, t_j)$. If a transition t_j fires the marking changes according to $\boldsymbol{\mathcal{M}}' = \boldsymbol{\mathcal{M}} + (\boldsymbol{W})_{:,j}$, where $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{M}}'$ are the markings before and after the firing of t_j. A potential firing sequence can be encoded by a vector $\boldsymbol{t} \in \mathbb{N}_0^m$ (called Parikh vector), where $(\boldsymbol{t})_i$ gives the number of firings of t_i in the sequence. If the encoded firing sequence can actually occur when the marking is $\boldsymbol{\mathcal{M}}$, the new marking is obtained as $\boldsymbol{\mathcal{M}}' = \boldsymbol{\mathcal{M}} + \boldsymbol{W}\boldsymbol{t}$. A Petri net is said to be bounded if the marking in all places is bounded. Moreover, a Petri net is said to be live if any transition can ultimately fire from any reachable marking (Teruel et al. (1992)). The structural properties of a Petri net can be analyzed by linear algebraic techniques. In particular, the right and left null spaces of the incidence matrix $\boldsymbol{W}$ reveal invariants of the net structure.

Definition 3. A vector $\boldsymbol{\xi}$ is called T-semiflow if $\boldsymbol{\xi} \in \mathbb{N}^{m\times 1}$ and $\boldsymbol{W}\boldsymbol{\xi} = \boldsymbol{0}$, where $\boldsymbol{0}$ is the zero vector. ◁

Note that a T-semiflow is a strictly positive integer vector. Therefore, it describes a firing sequence which involves all transitions in the Petri net and, if it can occur at marking $\boldsymbol{\mathcal{M}}$, leaves the latter invariant, i.e., $\boldsymbol{\mathcal{M}}' = \boldsymbol{\mathcal{M}}$. It can then of course be repeated indefinitely and is therefore also called repetitive vector.

Definition 4. A timed Petri net with holding times is a triple $(\mathcal{N}, \boldsymbol{\mathcal{M}}_0, \boldsymbol{\phi})$, where $(\mathcal{N}, \boldsymbol{\mathcal{M}}_0)$ is a Petri net and $\boldsymbol{\phi} \in \mathbb{N}_0^n$ represents the holding times of the places, i.e., $(\boldsymbol{\phi})_i$ is the time a token has to remain in place p_i before it contributes to the firing of a transition in $p_i^\bullet$. ◁

2.2 Weighted Timed Event Graphs

Definition 5. A timed Petri net $(\mathcal{N}, \boldsymbol{\mathcal{M}}_0, \boldsymbol{\phi})$ is called Weighted Timed Event Graph (WTEG), if every place has exactly one upstream and one downstream transition i.e., $\forall p_i \in P : |p_i^\bullet| = |{}^\bullet p_i| = 1$. ◁

Note that WTEGs, other than standard TEGs, do not require all arcs to have weight one.

Definition 6. A WTEG with a T-semiflow $\boldsymbol{\xi}$, is called consistent. ◁

In this paper we focus on consistent WTEGs, since a non-consistent WTEG is either not live or not bounded (Teruel et al. (1992)). The former mean that the WTEG may eventually not be able to fire any transition, while the latter means that the number of tokens may surpass all bounds. A basic directed path $t_{\underline{i}} \to p_i \to t_{\bar{i}}$ of a WTEG is such that $t_{\underline{i}} \in {}^\bullet p_i$ and $t_{\bar{i}} \in p_i^\bullet$. As $|p_i^\bullet| = |{}^\bullet p_i| = 1\ \forall p_i \in P$, each place appears in precisely one basic directed path, which we will denote π_i. The gain of π_i is defined by

$$\Gamma(\pi_i) = \Gamma(t_{\underline{i}}, p_i, t_{\bar{i}}) = \frac{w(t_{\underline{i}}, p_i)}{w(p_i, t_{\bar{i}})}.$$

A directed path is a sequence of basic directed paths, $\pi = \pi_{i_1} \cdots \pi_{i_q}$, where $p_{i_j}^\bullet = {}^\bullet p_{i_{j+1}}, j = \{1, \cdots, q-1\}$. The gain of a directed path $\pi = \pi_{i_1} \cdots \pi_{i_q}$ is then the product of all basic directed path gains, i.e., $\Gamma(\pi) = \prod_{j=1}^q \Gamma(\pi_{i_j})$.

Proposition 1. Let $(\mathcal{N}, \boldsymbol{\mathcal{M}}_0, \boldsymbol{\phi})$ be a consistent WTEG with T-semiflow $\boldsymbol{\xi}$. Then the directed path $\pi = \pi_{i_1} \cdots \pi_{i_q}$ has gain

$$\Gamma(\pi) = \frac{(\boldsymbol{\xi})_{\bar{i}_q}}{(\boldsymbol{\xi})_{\underline{i}_1}}.$$

Proof. According to the definition of T-semiflows, $\boldsymbol{\xi}$ is a positive integer vector such that

$$\boldsymbol{W}\boldsymbol{\xi} = \boldsymbol{0}, \quad (1)$$

where $\boldsymbol{W} \in \mathbb{Z}^{n\times m}$ is the incidence matrix of the WTEG. Lines $i_j,\ j \in \{1, \cdots, q\}$ of (1) read as follows:

$$w(t_{\underline{i}_j}, p_{i_j})(\boldsymbol{\xi})_{\underline{i}_j} - w(p_{i_j}, t_{\underline{i}_{j+1}})(\boldsymbol{\xi})_{\underline{i}_{j+1}} = 0, \quad \text{for } j = 1, \cdots, q-1$$
$$w(t_{\underline{i}_j}, p_{i_j})(\boldsymbol{\xi})_{\underline{i}_j} - w(p_{i_j}, t_{\bar{i}_j})(\boldsymbol{\xi})_{\bar{i}_j} = 0, \quad \text{for } j = q.$$

Equivalently,

$$\frac{(\boldsymbol{\xi})_{\underline{i}_{j+1}}}{(\boldsymbol{\xi})_{\underline{i}_j}} = \frac{w(t_{\underline{i}_j}, p_{i_j})}{w(p_{i_j}, t_{\underline{i}_{j+1}})} = \Gamma(\pi_{i_j}), \quad \text{for } j = 1, \cdots, q-1$$
$$\frac{(\boldsymbol{\xi})_{\bar{i}_q}}{(\boldsymbol{\xi})_{\underline{i}_q}} = \frac{w(t_{\underline{i}_q}, p_{i_q})}{w(p_{i_q}, t_{\bar{i}_q})} = \Gamma(\pi_{i_q}).$$

Therefore:

$$\Gamma(\pi) = \prod_{j=1}^q \Gamma(\pi_{i_j}) = \frac{(\boldsymbol{\xi})_{\bar{i}_q}}{(\boldsymbol{\xi})_{\underline{i}_1}}.$$

□

As for standard TEGs, set of transitions of a WTEG can be partitioned into internal, input and output transitions:

- input transition are transitions with only downstream places,
- internal transition are transitions with both upstream and downstream places,
- output transition are transitions with only upstream places.

We assume that each output transition t_o has precisely one upstream place p_i and the upstream transition of this place is an internal transition $t_{\underline{i}}$; the holding time of place p_i is zero and $w(t_{\underline{i}}, p_i) = w(p_i, t_o) = 1$. Conversely, we assume that each input transition t_j has precisely one downstream place p_i and the downstream transition of this place is an internal transition $t_{\bar{i}}$; the holding time of place p_i is zero and $w(t_j, p_i) = w(p_i, t_{\bar{i}}) = 1$. Note that these assumptions are not restrictive, since in case, they do not hold for an input (respectively output) transition, we can extend the set of internal transitions by this input (respectively output) transition and add a new input (respectively output) transitions satisfying the assumption.

Definition 7. (Earliest Functioning Rule). A WTEG is operating under the earliest functioning rule, if all internal and output transitions fire as soon as they are enabled. ◁

3. DIOID THEORY

3.1 Basics

In some dioids, the equations describing the evolution of TEGs operating under the earliest functioning rule are linear (Baccelli et al. (1992); Heidergott et al. (2006)). See also (Hardouin et al. (2018)) for a recent tutorial overview. Formally a dioid $\mathcal{D}$ is an algebraic structure with two binary operations, $\oplus$ (addition) and $\otimes$ (multiplication). Addition is commutative, associative and idempotent (i.e. $\forall a \in \mathcal{D},\ a \oplus a = a$). The neutral element for addition, denoted by ε, is absorbing for multiplication (i.e., $\forall a \in \mathcal{D},\ a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$). Multiplication is associative, distributive over addition and has a neutral element denoted by e. Note that, as in conventional algebra, the multiplication symbol $\otimes$ is often omitted. Both operations can be extended to the matrix case. For matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{D}^{m\times n}$, $\boldsymbol{C} \in \mathcal{D}^{n\times q}$, matrix addition and multiplication are defined by

$$(\boldsymbol{A} \oplus \boldsymbol{B})_{i,j} := (\boldsymbol{A})_{i,j} \oplus (\boldsymbol{B})_{i,j},$$
$$(\boldsymbol{A} \otimes \boldsymbol{C})_{i,j} := \bigoplus_{k=1}^{n} ((\boldsymbol{A})_{i,k} \otimes (\boldsymbol{C})_{k,j})\,. \tag{2}$$

The identity matrix, denoted by $\boldsymbol{I}$, is a square matrix with e on the diagonal and ε entries elsewhere. A dioid $\mathcal{D}$ is said to be complete if it is closed for infinite sums and if multiplication distributes over infinite sums. A complete dioid is a partially ordered set, with a canonical order $\preceq$ defined by $b \oplus a = b \Leftrightarrow a \preceq b$. The infimum operator can then be defined by $a \wedge b = \bigoplus\{x \in \mathcal{D}\,|x \preceq a, x \preceq b\}, \forall a, b \in \mathcal{D}$.

Moreover, in a complete dioid, the Kleene star of an element $a \in \mathcal{D}$ is defined by $a^* = \bigoplus_{i=0}^{\infty} a^i$ with $a^0 = \mathrm{e}$ and $a^{i+1} = a \otimes a^i$.

Theorem 1. (Baccelli et al. (1992)). In a complete dioid $\mathcal{D}$, $x = a^*b$ is the least solution of the implicit equation $x = ax \oplus b$. ◁

Residuation theory is a formalism to address the problem of approximate inversion of mappings over ordered sets, see Baccelli et al. (1992).

Definition 8. (Residuation). Let $\mathcal{D}$ and $\mathcal{L}$ be complete dioids and $f : \mathcal{D} \to \mathcal{L}$ an isotone mapping, i.e., $a \preceq b$ implies $f(a) \preceq f(b)$. The mapping f is said to be residuated if for all $y \in \mathcal{L}$, the least upper bound of the subset $\{x \in \mathcal{D} | f(x) \preceq y\}$ exists and lies in this subset. It is denoted $f^\sharp(y)$, and mapping $f^\sharp$ is called the residual of f. ◁

For instance, in a complete dioid, the mapping $R_a : x \mapsto xa$, ("right multiplication") respectively $L_a : x \mapsto ax$ ("left multiplication") are residuated. The residual mappings are denoted $R_a^\sharp(b) = b \phi a = \bigoplus\{x | xa \preceq b\}$ (right division by a) resp. $L_a^\sharp(b) = a \backslash\!\!\!\circ\, b = \bigoplus\{x | ax \preceq b\}$ (left division by a). In analogy to the extension of the product to the matrix case, we can extend left and right division to matrices with entries in a complete dioid. For matrices $\boldsymbol{A} \in \mathcal{D}^{m\times n}, \boldsymbol{B} \in \mathcal{D}^{m\times q}$, $\boldsymbol{C} \in \mathcal{D}^{n\times q}$

$$(\boldsymbol{A} \backslash\!\!\!\circ\, \boldsymbol{B})_{i,j} = \bigwedge_{k=1}^{m} \big((\boldsymbol{A})_{k,i} \backslash\!\!\!\circ\, (\boldsymbol{B})_{k,j}\big), \tag{3}$$

$$(\boldsymbol{B} \phi \boldsymbol{C})_{i,j} = \bigwedge_{k=1}^{q} \big((\boldsymbol{B})_{i,k} \phi (\boldsymbol{C})_{j,k}\big). \tag{4}$$

3.2 Dioid $\mathcal{E}[\![\delta]\!]$

Unlike standard TEGs, WTEGs may exhibit event variant behavior. E.g., if two consecutive events are needed to induce a following event. In this section, we develop the algebraic tools to describe the evolution of WTEGs under the earliest functioning rule in a dioid setting. We start by introducing a set of operators to model the event-variant behavior of WTEGs. Sum and composition of these operators satisfy a dioid structure and can be expressed as ultimately periodic series in a dioid denoted $\mathcal{E}[\![\delta]\!]$.

Dioids of Operators For the modeling process of WTEGs, a counter function $x_i : \mathbb{Z} \to \mathbb{Z}_{min}$, where $\mathbb{Z}_{min} = \mathbb{Z} \cup \{\infty, -\infty\}$, is associated to each transition t_i. $x_i(t)$ gives the accumulated number of firings up to time t. A counter function is naturally a non-decreasing function, i.e. $x_i(t+1) \geq x_i(t)$, and the set of counter functions is denoted by Σ. Two specific counter functions are defined as $\forall t \in \mathbb{Z},\ \tilde{\varepsilon}(t) = \infty$ and $\forall t \in \mathbb{Z},\ \tilde{\top}(t) = -\infty$. An operator is a map $\Sigma \mapsto \Sigma$, and the set of operators is denoted by $\mathcal{O}$. On this set, addition and multiplication are defined by $\forall x \in \Sigma,\ o_1, o_2 \in \mathcal{O}$,

$$((o_1 \oplus o_2)x)(t) = \min\big((o_1x)(t), (o_2x)(t)\big), \tag{5}$$
$$((o_1 \otimes o_2)x)(t) = \big(o_1(o_2x)\big)(t). \tag{6}$$

Dynamic phenomena arising in WTEGs can be described by the following operators:

$$\tau \in \mathbb{Z},\ \delta^\tau : \forall x \in \Sigma,\ (\delta^\tau x)(t) = x(t-\tau), \tag{7}$$
$$\nu \in \mathbb{Z},\ \gamma^\nu : \forall x \in \Sigma,\ (\gamma^\nu x)(t) = x(t) + \nu, \tag{8}$$
$$b \in \mathbb{N},\ \beta_b : \forall x \in \Sigma, (\beta_b x)(t) = \lfloor x(t)/b \rfloor, \tag{9}$$
$$m \in \mathbb{N},\ \mu_m : \forall x \in \Sigma, (\mu_m x)(t) = m \times x(t), \tag{10}$$

where $\lfloor a \rfloor$ is the greatest integer less than or equal to a. The γ and δ operators can be interpreted as event-shift and time-shift. The μ_m and β_b operators can be interpreted as event-multiplication and event-division. For the μ_m and β_b operator the following relations hold,

$$\gamma^{m\times n}\mu_m = \mu_m\gamma^n, \tag{11}$$
$$\beta_b\gamma^{b\times n} = \gamma^n\beta_b. \tag{12}$$

Example 1. Let us consider the simple WTEG shown in Fig. 1, for which the counter functions x_1 and x_2 are associated to transitions t_1 and t_2. The earliest firing relation between t_1 and t_2 is given as

$$x_2(t) = \Big\lfloor \frac{3x_1(t-1)+1}{2} \Big\rfloor.$$

This corresponds to an operator representation $x_2 = \beta_2\gamma^1\mu_3\delta^1x_1$. ◁

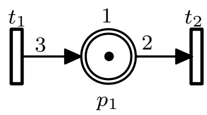

Fig. 1. Simple WTEG.

The three operators $\{\gamma^\nu, \mu_m, \beta_b\}$ are essential to describe the event-variant behavior of WTEGs. Therefore, in the following we discuss them in detail.

Definition 9. (Dioid of E-operators $\mathcal{E}$ (Cottenceau et al. (2014))). We denote by $\mathcal{E}$ the dioid of operators obtained by sums and compositions of operators in $\{\gamma^\nu, \beta_b, \mu_m, \varepsilon, \top, \mathrm{e}\}$ with $\nu \in \mathbb{Z}$, and $b, m \in \mathbb{N}$, equipped with addition and multiplication defined in (5) and (6). e, ε, $\top$, are unit, zero, and top element in this dioid, i.e.,

$$\forall x \in \Sigma,\ \varepsilon(x) = \tilde{\varepsilon},$$
$$\forall x \in \Sigma,\ \mathrm{e}(x) = x,$$
$$\forall x \in \Sigma \backslash \{\tilde{\varepsilon}\}, \top(x) = \tilde{\top} \text{ and } \top(\tilde{\varepsilon}) = \tilde{\varepsilon}.$$

◁

Note that the operator δ^{ς} is not in $\mathcal{E}$. $\mathcal{E}$ is a complete dioid, and an element $w \in \mathcal{E}$ is called E-operator hereafter. Moreover, the unit operator can be written as $\mathrm{e} = \gamma^0 = \mu_1 = \beta_1$. Since E-operators only affect event numbering, the effect of, an E-operator w can be described by a Counter-value to Counter-value (C/C) function $\mathcal{F}_w : \mathbb{Z}_{min} \to \mathbb{Z}_{min}$. If $y = w(x),\ x, y \in \Sigma$, then $\mathcal{F}_w$ maps the value $x(t) = k$ of the input counter function x to the value $y(t)$ of the output counter function y, i.e. $y(t) = \mathcal{F}_w(k)$. For instance, let $\mu_2\beta_3\gamma^1 \in \mathcal{E}$ then $(\mu_2\beta_3\gamma^1 x)(t) = \lfloor (x(t)+1)/3 \rfloor 2$ which leads to $\mathcal{F}_{\mu_2\beta_3\gamma^1}(k) = \lfloor (k+1)/3 \rfloor 2$. Thus, there is an isomorphism between the set of E-operators and the set of (C/C) functions. The order relation in the dioid $\mathcal{E}$ is given by the order in Σ. For $w_1, w_2 \in \mathcal{E}$,

$$\begin{aligned} & w_2 \preceq w_1 \Leftrightarrow w_1 \oplus w_2 = w_2, \\ & \Leftrightarrow w_1 x \oplus w_2 x = w_2 x,\ \forall x \in \Sigma, \\ & \Leftrightarrow (w_1 x)(t) \oplus (w_2 x)(t) = (w_2 x)(t),\ \forall x \in \Sigma, \forall t \in \mathbb{Z}, \\ & \Leftrightarrow \mathcal{F}_{w_1}(k) \geq \mathcal{F}_{w_2}(k),\ \forall k \in \mathbb{Z}_{min}. \end{aligned} \tag{13}$$

Definition 10. (Periodic E-operators (Cottenceau et al. (2014))). An E-operator $w \in \mathcal{E}$ is called (m, b)-periodic if $\forall k \in \mathbb{Z}_{min},\ \mathcal{F}_w(k + b) = m + \mathcal{F}_w(k)$, with $m, b \in \mathbb{N}$. The set of periodic E-operators is denoted by $\mathcal{E}_{per}$. The gain of an (m, b)-periodic E-operator $w \in \mathcal{E}_{per}$ is defined as $\Gamma(w) := m/b$. ◁

Remark 1. The basic E-operators $\{\mathrm{e}, \varepsilon, \top, \gamma^{\nu}, \beta_b, \mu_m\}$ are periodic, since their (C/C) functions satisfy $\forall k \in \mathbb{Z}_{min},\ m, b \in \mathbb{N}$:

$$\begin{aligned} & \mathcal{F}_{\mathrm{e}}(k + b) = b + \mathcal{F}_{\mathrm{e}}(k), \\ & \mathcal{F}_{\varepsilon}(k + b) = \mathcal{F}_{\varepsilon}(k) + \tilde{m},\ \ \tilde{m} \text{ arbitrary in } \mathbb{N}, \\ & \mathcal{F}_{\top}(k + b) = \mathcal{F}_{\top}(k) + \tilde{m},\ \ \tilde{m} \text{ arbitrary in } \mathbb{N}, \\ & \mathcal{F}_{\gamma^{\nu}}(k + b) = b + \mathcal{F}_{\gamma^{\nu}}(k), \\ & \mathcal{F}_{\beta_b}(k + b) = 1 + \mathcal{F}_{\beta_b}(k), \\ & \mathcal{F}_{\mu_m}(k + b) = mb + \mathcal{F}_{\mu_m}(k). \end{aligned}$$

Hence, the gains of the basic E-operators are $\Gamma(e) = 1$, $\Gamma(\gamma^{\nu}) = 1$, $\Gamma(\beta_b) = 1/b$, $\Gamma(\mu_m) = m$, while the gains of ε and $\top$ are arbitrary positive rational numbers. ◁

Definition 11. (Dioid $\mathcal{E}[\![\delta]\!]$ Cottenceau et al. (2014)). We denote by $\mathcal{E}[\![\delta]\!]$ the quotient dioid in the set of formal power series in one variable δ with exponents in $\mathbb{Z}$ and coefficients in $\mathcal{E}$ induced by the equivalence relation $\forall s \in \mathcal{E}[\![\delta]\!]$,

$$s = (\gamma^1)^* s = s(\gamma^1)^* = (\delta^{-1})^* s = s(\delta^{-1})^*.$$

The zero and unit element are $\varepsilon = \bigoplus_{\tau \in \mathbb{Z}} \varepsilon \delta^{\tau}$ and $\mathrm{e} = \mathrm{e}\,\delta^0$, respectively. ◁

E-operators commute with the time shift operator δ^{τ}, i.e., $\forall w \in \mathcal{E},\ \delta^{\tau} w = w \delta^{\tau}$. In addition, taking the quotient structure of $\mathcal{E}[\![\delta]\!]$ into account allows us to assimilate the time shift operator δ^{τ} with the variable delta δ of the dioid $\mathcal{E}[\![\delta]\!]$.

Remark 2. (Subset $\mathcal{E}_{per}[\![\delta]\!]$). The subset of $\mathcal{E}[\![\delta]\!]$ obtained by restricting the coefficients to $\mathcal{E}_{per}$, i.e. the set of periodic operators, is denoted by $\mathcal{E}_{per}[\![\delta]\!]$. ◁

A monomial in $\mathcal{E}_{per}[\![\delta]\!]$ is defined as $w\delta^{\varsigma}$ where $w \in \mathcal{E}_{per}$. A polynomial (respectively series) in $\mathcal{E}_{per}[\![\delta]\!]$ is a finite sum $p = \bigoplus_{i=1}^{I} w_i \delta^{\tau_i}$ (respectively infinite sum $s = \bigoplus_i w_i \delta^i$) of monomials such that $\Gamma(w_i) = \Gamma(w_j),\ \forall i, j$. Then the gain $\Gamma(p)$ (respectively $\Gamma(s)$) of a polynomial p (respectively series s) is defined to be the gain of its coefficient, i.e., $\Gamma(p) = \Gamma(w_i)$ (respectively $\Gamma(s) = \Gamma(w_i)$).

A series $s \in \mathcal{E}_{per}[\![\delta]\!]$ is said to be ultimately periodic (UP) if it can be written as

$$s = p \oplus q(\gamma^{\nu}\delta^{\tau})^* \tag{14}$$

where, $\nu, \tau \in \mathbb{N}$ and p, q are polynomials in $\mathcal{E}_{per}[\![\delta]\!]$. Note that all coefficients in every series in $\mathcal{E}_{per}[\![\delta]\!]$ are (m,b)-periodic. An UP series in $\mathcal{E}_{per}[\![\delta]\!]$ additionally satisfies the UP-definition (14).

Proposition 2. (Cottenceau et al. (2014)). Let $s_1, s_2 \in \mathcal{E}_{per}[\![\delta]\!]$ be two (UP) series then:

- $\Gamma(s_1) = \Gamma(s_2) \Rightarrow s_1 \oplus s_2$ (respectively $s_1 \wedge s_2$) is an (UP) series, with $\Gamma(s_1 \oplus s_2) = \Gamma(s_1)$ (respectively $\Gamma(s_1 \wedge s_2) = \Gamma(s_1)$).
- $s_1 \otimes s_2$ (respectively $s_2 \otimes s_1$) is an (UP) series, with $\Gamma(s_1 \otimes s_2) = \Gamma(s_2 \otimes s_1) = \Gamma(s_1) \times \Gamma(s_2)$.
- $\Gamma(s_1) = 1 \Rightarrow s_1^*$ is an (UP) series, with $\Gamma(s_1^*) = 1$.
- $(s_2 \backslash\!\!\circ s_1)$ (respectively $(s_1 /\!\!\circ s_2)$) is an (UP) series, with $\Gamma(s_2 \backslash\!\!\circ s_1) = \Gamma(s_1 /\!\!\circ s_2) = \Gamma(s_1)/\Gamma(s_2)$.

The gain of a matrix $\boldsymbol{A} \in \mathcal{E}_{per}[\![\delta]\!]^{p \times g}$ is defined to be a matrix with entries given by the gains of the entries of $\boldsymbol{A}$, i.e., $(\Gamma(\boldsymbol{A}))_{i,j} := \Gamma((\boldsymbol{A})_{i,j})$. Then from Prop. 2 the following can be inferred.

Corollary 1. (Trunk (2019)). Then for matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{E}_{per}[\![\delta]\!]^{m \times n}$, $\boldsymbol{C} \in \mathcal{E}_{per}[\![\delta]\!]^{n \times q}$, and $\boldsymbol{D} \in \mathcal{E}_{per}[\![\delta]\!]^{m \times q}$

- $\boldsymbol{A} \oplus \boldsymbol{B} \in \mathcal{E}_{per}[\![\delta]\!]^{m \times n}$ iff $\Gamma(\boldsymbol{A}) = \Gamma(\boldsymbol{B})$
- $(\boldsymbol{A} \otimes \boldsymbol{C})_{i,j} = \bigoplus_{k=1}^{n} ((\boldsymbol{A})_{i,k} \otimes (\boldsymbol{C})_{k,j}) \in \mathcal{E}_{per}[\![\delta]\!]$ iff $\forall k \in \{2, \cdots n\}$, $\Gamma((\boldsymbol{A})_{i,k} \otimes (\boldsymbol{C})_{k,j}) = \Gamma((\boldsymbol{A})_{1,k} \otimes (\boldsymbol{C})_{k,1})$
- $(\boldsymbol{A} \backslash\!\!\circ \boldsymbol{D})_{i,j} = \bigwedge_{k=1}^{m} \left((\boldsymbol{A})_{k,i} \backslash\!\!\circ (\boldsymbol{D})_{k,j}\right) \in \mathcal{E}_{per}[\![\delta]\!]$ iff $\forall k \in \{2, \cdots m\}$, $\Gamma((\boldsymbol{A})_{k,i} \backslash\!\!\circ (\boldsymbol{D})_{k,j}) = \Gamma((\boldsymbol{A})_{1,i} \backslash\!\!\circ (\boldsymbol{D})_{1,j})$
- $(\boldsymbol{D} /\!\!\circ \boldsymbol{C})_{i,j} = \bigwedge_{k=1}^{q} \left((\boldsymbol{D})_{i,k} /\!\!\circ (\boldsymbol{C})_{j,k}\right) \in \mathcal{E}_{per}[\![\delta]\!]$ iff $\forall k \in \{2, \cdots q\}$, $\Gamma((\boldsymbol{D})_{i,k} /\!\!\circ (\boldsymbol{C})_{j,k}) = \Gamma((\boldsymbol{D})_{i,1} /\!\!\circ (\boldsymbol{C})_{j,1})$

◁

4. DIOID MODEL OF WTEGs

As indicated in Example 1, the firing relation between transitions in a WTEG can be encoded by operators in $\mathcal{E}_{per}[\![\delta]\!]$. More generally, let us consider any basic path constituted by the arcs (t_j, p_i) and (p_i, t_o). The influence of transition t_i onto transition t_o is coded by the operator

$$\beta_{w(p_i,t_o)} \gamma^{(\boldsymbol{\mathcal{M}}_0)_i} \mu_{w(t_j,p_i)} \delta^{(\boldsymbol{\phi})_i}$$

where $w(p_i, t_o)$ and $w(t_j, p_i)$ are weights of the arcs (p_i, t_o) and (t_j, p_i), $(\boldsymbol{\phi})_i$ is the holding time of place p_i and $(\boldsymbol{\mathcal{M}}_0)_i$ is the initial marking of p_i. Let us note that the gain of a path is equal to the gain of the corresponding operator, i.e.,

$$\Gamma(t_o, p_i, t_j) = \Gamma(\beta_{w(p_i,t_o)} \gamma^{(\boldsymbol{\mathcal{M}}_0)_i} \mu_{w(t_j,p_i)} \delta^{(\boldsymbol{\phi})_i}). \tag{15}$$

This relation holds true for any path in a consistent WTEG.

Recall that we partitioned the set of transitions of a WTEG into internal, input and output transitions, Section 2.1. We consider the case where the firings of internal transitions cannot be directly observed. In contrast the firings of output transitions can be ”seen” by an external agent. Finally, the set of input transitions is further partitioned into controllable and uncontrollable inputs. The firings of controllable input transitions can be freely chosen and are therefore known. In contrast, the firing of uncontrollable input transitions can neither be influenced nor directly observed. The latter can be interpreted as (unknown) disturbances. To model the dynamic behavior of a WTEG in the dioid $\mathcal{E}[\![\delta]\!]$, we associate to each transition a counter function. Then the vector $\boldsymbol{x}$ refers to counter functions of internal transitions. Respectively, the vectors $\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{y}$ are associated to the counter functions of controllable input transitions, uncontrollable input transitions and output transitions. A WTEG operating under the earliest functioning rule admits a representation in $\mathcal{E}_{per}[\![\delta]\!]$, of the form,

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{B}\boldsymbol{u} \oplus \boldsymbol{R}\boldsymbol{w}, \qquad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}. \tag{16}$$

The matrix $\boldsymbol{A} \in \mathcal{E}_{per}[\![\delta]\!]^{n\times n}$ describes the influence of internal transitions on each other. Note that due to the assumption on WTEGs regarding their input and output transitions, see Section 2.2, the entries $\boldsymbol{B}, \boldsymbol{R}$ and $\boldsymbol{C}$ are either ε or e. Furthermore, the matrices $\boldsymbol{B}, \boldsymbol{R}$ are such that each column has precisely one entry equal to e. Similarly, the matrix $\boldsymbol{C}$ is such that each row has precisely one entry equal to e. We assume further more that each internal transition is connected (by a basic path) to at most one controllable input transition, one uncontrollable input transition and on output transition. This means that each row of $\boldsymbol{B}$, and $\boldsymbol{R}$ hat at most one e entry and each column of $\boldsymbol{C}$ has at most one e entry.

Proposition 3. (Cottenceau et al. (2014)). Let $(\mathcal{N}, \boldsymbol{\mathcal{M}}_0, \boldsymbol{\phi})$ be a consistent WTEG with g controllable input and p output transitions, then the entries of the $p \times g$ transfer function matrix $\boldsymbol{H} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}$ are UP series in $\mathcal{E}_{per}[\![\delta]\!]$.

Example 2. The earliest functioning of the consistent WTEG shown in Fig. 2 is modeled in the dioid $\mathcal{E}_{per}[\![\delta]\!]$ by

$$\boldsymbol{x} = \begin{bmatrix} \varepsilon & \varepsilon & \gamma^3\mu_3 \\ \varepsilon & \varepsilon & \beta_2\gamma^2 \\ \beta_3\delta^4 & \mu_2\delta^3 & \varepsilon \end{bmatrix} \boldsymbol{x} \oplus \begin{bmatrix} \mathrm{e} & \varepsilon \\ \varepsilon & \mathrm{e} \\ \varepsilon & \varepsilon \end{bmatrix} \boldsymbol{u} \oplus \begin{bmatrix} \mathrm{e} & \varepsilon & \varepsilon \\ \varepsilon & \mathrm{e} & \varepsilon \\ \varepsilon & \varepsilon & \mathrm{e} \end{bmatrix} \boldsymbol{w}$$

$$y = [\varepsilon \ \varepsilon \ \mathrm{e}]\, \boldsymbol{x}$$

where $\boldsymbol{x} = [x_1\ x_2\ x_3]^T$ is the vector of counter functions associated with internal transitions t_3, t_4, and t_5. $\boldsymbol{u} = [u_1\ u_2]^T$ is the vector of counter functions associated with the controllable input transitions t_1 and t_2, y is the counter function associated with output transition t_6, and $\boldsymbol{w} = [w_1\ w_2\ w_3]^T$ is the vector of counter functions associated with the uncontrollable input transitions (disturbances) t_7, t_8, and t_9. The transfer function matrix for the system from u to y is

$$\boldsymbol{H} = \left[(\gamma^1\delta^4)^*(\beta_3\delta^4)\ (\gamma^1\delta^4)^*(\mu_2\delta^3)\right].$$

◁

Proposition 4. Let $(\mathcal{N}, \boldsymbol{\mathcal{M}}_0, \boldsymbol{\phi})$ be a consistent WTEG with g (controllable) input and p output transitions and transfer matrix $\boldsymbol{H} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B} \in \mathcal{E}_{per}[\![\delta]\!]^{p\times g}$, then the gain matrix $\Gamma(\boldsymbol{H})$ has rank 1.

Proof. Recall (15), i.e., the gain of a path is equivalent to the gain of its operational representation. Moreover, consistency implies the existence of a T-semiflow $\boldsymbol{\xi} = [\boldsymbol{\xi}_{t_i}^T\ \boldsymbol{\xi}_{t_s}^T\ \boldsymbol{\xi}_{t_o}^T]$, with subvectors $\boldsymbol{\xi}_{t_i}^T = [\xi_{i_1}\ \cdots\ \xi_{i_g}]$ associated with (controllable) input transitions and $\boldsymbol{\xi}_{t_o}^T = [\xi_{o_1}\ \cdots\ \xi_{o_p}]$ associated with output transitions. Due to Prop. 1, the relation between gain and T-semiflow must hold for all paths in $\mathcal{N}$. Therefore, the gain matrix $\Gamma(\boldsymbol{H})$ is of rank 1 and is given by

$$\Gamma(\boldsymbol{H}) = \begin{bmatrix}\xi_{o_1} & \cdots & \xi_{o_p}\end{bmatrix}^T \begin{bmatrix}\frac{1}{\xi_{i_1}} & \cdots & \frac{1}{\xi_{i_g}}\end{bmatrix}.$$

□

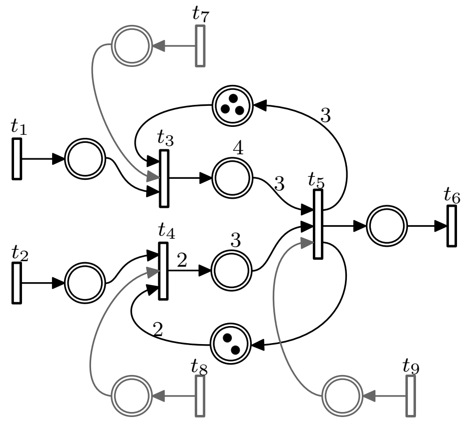

Fig. 2. Consistent WTEG

Clearly, Prop. 4 is also valid for the disturbance to output transfer function matrix $\boldsymbol{D} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{R}$, i.e., for a consistent WTEG, $\Gamma(\boldsymbol{D})$ has rank 1. In fact the rank condition also needs to hold for $\Gamma(\boldsymbol{A})$, $\Gamma(\boldsymbol{B})$, $\Gamma(\boldsymbol{C})$, $\Gamma(\boldsymbol{R})$. For any matrix $\boldsymbol{F} \in \mathcal{E}_{per}[\![\delta]\!]^{p\times g}$ where $\Gamma(\boldsymbol{F})$ has rank 1, we express $\Gamma(\boldsymbol{F})$ by a vector product $\Gamma(\boldsymbol{F}) = \boldsymbol{f}_c\boldsymbol{f}_r$, with $\boldsymbol{f}_c \in \mathbb{Q}^{p\times 1}$ and $\boldsymbol{f}_r \in \mathbb{Q}^{1\times g}$ are vectors with strictly positive entries.

Remark 3. We only consider consistent WTEGs and therefore both $\Gamma(\boldsymbol{H}) = \Gamma(\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})$ and $\Gamma(\boldsymbol{D}) = \Gamma(\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{R})$ have rank 1. This leads to the following necessary conditions on the gains for matrices $\boldsymbol{C}, \boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{R}$. First, as a consequence of Corollary 1, for $\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \boldsymbol{A}^2 \cdots$ we must satisfy $\Gamma(\boldsymbol{I}) = \Gamma(\boldsymbol{A}) = \Gamma(\boldsymbol{A}^2)\cdots$, with the rank of all gain matrices being 1. Recall that, $\Gamma(\mathrm{e}) = 1$, hence the diagonal elements of $\Gamma(\boldsymbol{A}) = \boldsymbol{a_c}\boldsymbol{a_r}$ must have gain 1, and this is only the case if $\forall i \in \{1, \cdots, n\}$, $(\boldsymbol{a_c})_i = ((\boldsymbol{a_r})_i)^{-1}$. Next, for the product $\boldsymbol{C}\boldsymbol{A}$, according to Corollary 1 we must satisfy, $\Gamma((\boldsymbol{C})_{1,1}(\boldsymbol{A})_{1,1}) = \Gamma((\boldsymbol{C})_{1,2}(\boldsymbol{A})_{2,1})\cdots$, etc. This leads to the following gain requirement for the matrix $\boldsymbol{C}$. Recall that $\boldsymbol{C} \in \{\mathrm{e}, \varepsilon\}^{p\times n}$ and that each row has precisely one entry equal to e. Recall that, $\Gamma(\mathrm{e}) = 1$ and that the gain of ε can be freely chosen to any positive value in $\mathbb{Q}$, see Remark 1. Then the gain matrix $\Gamma(\boldsymbol{C}) = \boldsymbol{c_c}\boldsymbol{c_r}$ is chosen such that $\forall i \in \{1, \cdots, n\}$, $(\boldsymbol{c_r})_i = ((\boldsymbol{a_c})_i)^{-1}$ and for $\boldsymbol{C}_{i,j} = \mathrm{e}$, $(\boldsymbol{c}_c)_j = (\boldsymbol{a}_c)_i$. As a result, the gain matrix of $\boldsymbol{C}\boldsymbol{A}$ is $\Gamma(\boldsymbol{C}\boldsymbol{A}) = \boldsymbol{c_c}\boldsymbol{a_r}$. Similarly, the gain matrix $\Gamma(\boldsymbol{B}) = \boldsymbol{b_c}\boldsymbol{b_r}$ is chosen such that $\forall i \in \{1, \cdots n\}$, $(\boldsymbol{b_c})_i = ((\boldsymbol{a_r})_i)^{-1}$, and for $(\boldsymbol{B})_{i,j} = \mathrm{e}$, $(\boldsymbol{b_r})_j = (\boldsymbol{a_r})_i$. Then the gain matrix of the transfer function matrix $\boldsymbol{H}$ is,

$$\Gamma(\boldsymbol{H}) = \Gamma(\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}) = \boldsymbol{c_c}\boldsymbol{b_r}.$$

Similar chooses are made for the disturbance to output transfer function matrix $\boldsymbol{D} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{R} \in \mathcal{E}_{per}[\![\delta]\!]^{p\times n_w}$. Hence, the gain matrix $\Gamma(\boldsymbol{R}) = \boldsymbol{r_c}\boldsymbol{r_r}$ is chosen such that $\forall i \in \{1, \cdots n\}$, $(\boldsymbol{r_c})_i = ((\boldsymbol{a_r})_i)^{-1}$, and for $(\boldsymbol{R})_{i,j} = \mathrm{e}$, $(\boldsymbol{r_r})_j = (\boldsymbol{a_r})_i$. Therefore,

$$\Gamma(\boldsymbol{D}) = \Gamma(\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{R}) = \boldsymbol{c}_c\boldsymbol{r}_r.$$

◁

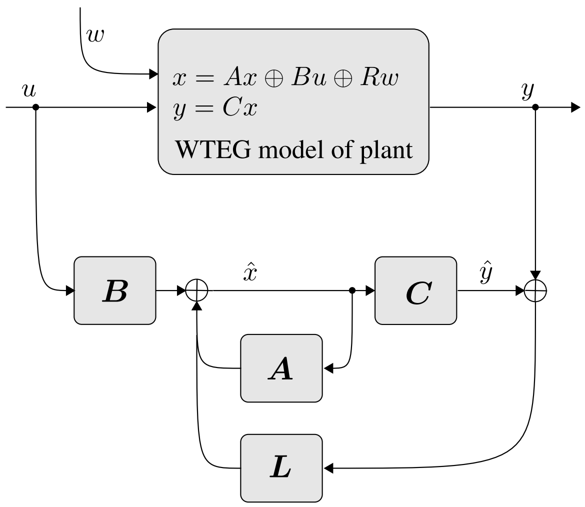

Fig. 3. Observer structure

5. OBSERVER FOR WTEGs

We now want to estimate, for any time unit t the number of firings of the internal transitions from the known number of firings of the controllable input transitions and the output transitions up to time unit t. For this we propose a structure reminiscent of the Luenberger observer in standard control theory. Note that this represents a generalization of the observer for TEGs that was proposed in Hardouin et al. (2007).

The postulated observer structure is shown in Fig. 3, where $\boldsymbol{x}$ is the vector of counter functions associated with internal transitions and $\hat{\boldsymbol{x}}$ denotes the estimate. The matrices $\boldsymbol{C}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{R}$ characterizing the WTEG are assumed to be known and $\boldsymbol{L} \in \mathcal{E}_{per}[\![\delta]\!]^{n\times p}$ is the observer matrix to be determined. This structure leads to the following observer equation,

$$\hat{\boldsymbol{x}} = \boldsymbol{A}\hat{\boldsymbol{x}} \oplus \boldsymbol{L}(\boldsymbol{y} \oplus \hat{\boldsymbol{y}}) \oplus \boldsymbol{B}\boldsymbol{u} \tag{17}$$

Using the equation $\hat{\boldsymbol{y}} = \boldsymbol{C}\hat{\boldsymbol{x}}$ and Theorem 1, the least solution of (17) in the dioid $\mathcal{E}[\![\delta]\!]$ is

$$\hat{\boldsymbol{x}} = (\boldsymbol{A} \oplus \boldsymbol{L}\boldsymbol{C})^*\boldsymbol{B}\boldsymbol{u} \oplus (\boldsymbol{A} \oplus \boldsymbol{L}\boldsymbol{C})^*\boldsymbol{L}\boldsymbol{y}, \tag{18}$$

Using the transfer function matrices $\boldsymbol{H}$ and $\boldsymbol{D}$, (18) can be rewritten as

$$\hat{\boldsymbol{x}} = (\boldsymbol{A} \oplus \boldsymbol{L}\boldsymbol{C})^*\boldsymbol{B}\boldsymbol{u} \oplus (\boldsymbol{A} \oplus \boldsymbol{L}\boldsymbol{C})^*\boldsymbol{L}\boldsymbol{C}(\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{u} \oplus \boldsymbol{A}^*\boldsymbol{R}\boldsymbol{w}). \tag{19}$$

Using the equation $(\boldsymbol{A} \oplus \boldsymbol{L}\boldsymbol{C})^* = \boldsymbol{A}^*(\boldsymbol{L}\boldsymbol{C}\boldsymbol{A}^*)^*$ which holds for any pair of square matrices $\boldsymbol{A}, \boldsymbol{L}\boldsymbol{C}$ with entries in a complete dioid (see e.g. Hardouin et al. (2018)), (19) can be rewritten as

$$\hat{\boldsymbol{x}} = (\boldsymbol{A} \oplus \boldsymbol{L}\boldsymbol{C})^*\boldsymbol{B}\boldsymbol{u} \oplus (\boldsymbol{A} \oplus \boldsymbol{L}\boldsymbol{C})^*\boldsymbol{L}\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{R}\boldsymbol{w}$$

We want to compute the greatest observer matrix $\boldsymbol{L}$ such that $\hat{\boldsymbol{x}} \preceq \boldsymbol{x}$, where both greatest and "$\preceq$" are in the sense of the dioid $\mathcal{E}[\![\delta]\!]$. The interpretation in standard algebra is as follows: we want to find the smallest estimates for the number of firings of the internal transitions under the restriction that the estimates may not be smaller than the actual number of firings. Recall that $\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{B}\boldsymbol{u} \oplus \boldsymbol{A}^*\boldsymbol{R}\boldsymbol{w}$ is the least solution of the WTEG's equation (16). Hence, we look for the greatest matrix $\boldsymbol{L}$ (in the dioid $\mathcal{E}[\![\delta]\!]$) such that,

$$(\boldsymbol{A} \oplus \boldsymbol{L}\boldsymbol{C})^*\boldsymbol{B} \preceq \boldsymbol{A}^*\boldsymbol{B} \tag{20}$$

and

$$(\boldsymbol{A} \oplus \boldsymbol{L}\boldsymbol{C})^*\boldsymbol{L}\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{R} \preceq \boldsymbol{A}^*\boldsymbol{R} \tag{21}$$

Adopting the argument in Hardouin et al. (2007) to the dioid $\mathcal{E}[\![\delta]\!]$ it is straight forward to show that the greatest solution of the inequalities above is given by the following proposition.

Proposition 5. The greatest observer matrix $\boldsymbol{L}$ such that (20) and (21) are satisfied is given by,

$$\boldsymbol{L}_{opt} = \boldsymbol{L}_1 \wedge \boldsymbol{L}_2,$$

where

$$\boldsymbol{L}_1 = (\boldsymbol{A}^*\boldsymbol{B}) \mathbin{/\!\!\!\circ} (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})$$

and

$$\boldsymbol{L}_2 = (\boldsymbol{A}^*\boldsymbol{R}) \mathbin{/\!\!\!\circ} (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{R}).$$

Proposition 6. Given a consistent WTEG described by matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{R}$. The observer, obtained in Prop. 5, for this WTEG is again a consistent WTEG.

Proof. In Trunk (2019) it was shown that any UP transfer function matrix $\boldsymbol{L} \in \mathcal{E}_{per}[\![\delta]\!]^{n\times p}$ can be realized by a consistent WTEG if $\Gamma(\boldsymbol{L})$ has rank 1. We therefore need to prove that $\Gamma(\boldsymbol{L}_{opt})$ has rank 1. Moreover, the gain of $\boldsymbol{L}_{opt}$ must satisfy $\Gamma(\boldsymbol{L}_{opt}\boldsymbol{C}) = \Gamma(\boldsymbol{A})$. Since, we only consider consistent WTEGs, it follows that $\Gamma(\boldsymbol{H}) = \Gamma(\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})$ and $\Gamma(\boldsymbol{D}) = \Gamma(\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{R})$ have rank 1, with gain matrices $\Gamma(\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}) = \boldsymbol{c}_c\boldsymbol{b}_r$ and $\Gamma(\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{R}) = \boldsymbol{c}_c\boldsymbol{r}_r$ (Remark 3). Moreover $\Gamma(\boldsymbol{A}^*\boldsymbol{B}) = \boldsymbol{a}_c\boldsymbol{b}_r$ and $\Gamma(\boldsymbol{A}^*\boldsymbol{R}) = \boldsymbol{a}_c\boldsymbol{r}_r$. Then according to (4) the first entry of quotient

$$\begin{aligned}((\boldsymbol{A}^*\boldsymbol{B}) \mathbin{/\!\!\!\circ} (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}))_{1,1} =& (\boldsymbol{A}^*\boldsymbol{B})_{1,1} \mathbin{/\!\!\!\circ} (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})_{1,1}\\ &\wedge (\boldsymbol{A}^*\boldsymbol{B})_{1,2} \mathbin{/\!\!\!\circ} (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})_{1,2}\\ &\wedge (\boldsymbol{A}^*\boldsymbol{B})_{1,3} \mathbin{/\!\!\!\circ} (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})_{1,3}\\ &\cdots\end{aligned}$$

and therefore

$$\begin{aligned}\Gamma((\boldsymbol{A}^*\boldsymbol{B})_{1,1} \mathbin{/\!\!\!\circ} (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})_{1,1}) &= \Gamma((\boldsymbol{A}^*\boldsymbol{B})_{1,2} \mathbin{/\!\!\!\circ} (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})_{1,2})\\ &= \Gamma((\boldsymbol{A}^*\boldsymbol{B})_{1,3} \mathbin{/\!\!\!\circ} (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})_{1,3})\\ &\cdots\end{aligned}$$

must be satisfied (see Corollary 1). Using $\Gamma(\boldsymbol{A}^*\boldsymbol{B}) = \boldsymbol{a}_c\boldsymbol{b}_r$ and $\Gamma(\boldsymbol{A}^*\boldsymbol{R}) = \boldsymbol{a}_c\boldsymbol{r}_r$ and Prop. 2 we can write,

$$\begin{aligned}&\frac{(\boldsymbol{a}_c)_1(\boldsymbol{b}_r)_1}{(\boldsymbol{c}_c)_1(\boldsymbol{b}_r)_1} = \frac{(\boldsymbol{a}_c)_1(\boldsymbol{b}_r)_2}{(\boldsymbol{c}_c)_1(\boldsymbol{b}_r)_2} = \frac{(\boldsymbol{a}_c)_1(\boldsymbol{b}_r)_3}{(\boldsymbol{c}_c)_1(\boldsymbol{b}_r)_3} = \cdots\\ \Leftrightarrow\ &\frac{(\boldsymbol{a}_c)_1}{(\boldsymbol{c}_c)_1} = \frac{(\boldsymbol{a}_c)_1}{(\boldsymbol{c}_c)_1} = \frac{(\boldsymbol{a}_c)_1}{(\boldsymbol{c}_c)_1} = \cdots\end{aligned}$$

Hence, Remark 3 is satisfied and the gain $\Gamma((\boldsymbol{A}^*\boldsymbol{B}) \mathbin{/\!\!\!\circ} (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}))_{1,1} = (\boldsymbol{a}_c)_1/(\boldsymbol{c}_c)_1$. Similarly, one can show that the quotient $(\boldsymbol{A}^*\boldsymbol{B}) \mathbin{/\!\!\!\circ} (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})$ satisfies Corollary 1 with gain matrix,

$$\Gamma(\boldsymbol{L}_1) = \Gamma((\boldsymbol{A}^*\boldsymbol{B}) \mathbin{/\!\!\!\circ} (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})) = \boldsymbol{a}_c\frac{(\boldsymbol{b}_r)_1}{(\boldsymbol{b}_r)_1}\bar{\boldsymbol{c}}_r = \boldsymbol{a}_c\bar{\boldsymbol{c}}_r,$$

where $\bar{\boldsymbol{c}}_r = [(\boldsymbol{c}_c)_1^{-1}\ (\boldsymbol{c}_c)_2^{-1}\ \cdots\ (\boldsymbol{c}_c)_p^{-1}]$. Similarly,

$$\Gamma(\boldsymbol{L}_2) = \Gamma((\boldsymbol{A}^*\boldsymbol{R}) \mathbin{/\!\!\!\circ} (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{R})) = \boldsymbol{a}_c\frac{(\boldsymbol{r}_r)_1}{(\boldsymbol{r}_r)_1}\bar{\boldsymbol{c}}_r = \boldsymbol{a}_c\bar{\boldsymbol{c}}_r.$$

Hence, $\Gamma(\boldsymbol{L}_1) = \Gamma(\boldsymbol{L}_2)$ and $\boldsymbol{L}_{opt} = \boldsymbol{L}_1 \wedge \boldsymbol{L}_2$ satisfy Corollary 1. Finally,

$$\begin{aligned}\Gamma(\boldsymbol{L}_{opt}\boldsymbol{C}) &= \boldsymbol{a}_c(\bar{\boldsymbol{c}}_r)_1(\boldsymbol{c}_c)_1\boldsymbol{c}_r,\\ &= \boldsymbol{a}_c\boldsymbol{c}_r, \text{because of } (\bar{\boldsymbol{c}}_r)_1 = (\boldsymbol{c}_c)_1)^{-1}\\ &= \boldsymbol{a}_c\boldsymbol{a}_r, \quad \text{because of Remark 3,}\end{aligned}$$

hence $\Gamma(\boldsymbol{L}_{opt}\boldsymbol{C}) = \Gamma(\boldsymbol{A})$. □

Example 3. The greatest observer for the system given in Example 2 is given by,

$$\boldsymbol{L}_{opt} = \begin{bmatrix} (\gamma^3\delta^4)^*(\gamma^3\mu_3) \\ (\gamma^1\delta^8)^*(\gamma^1\beta^2\gamma^1 \oplus \gamma^2\beta_2\delta^4) \\ (\gamma^1\delta^4)^* \end{bmatrix}$$

Recall the equation for estimate,

$$\begin{aligned} \hat{\boldsymbol{x}} &= \boldsymbol{A}\hat{\boldsymbol{x}} \oplus \boldsymbol{L}_{opt}(\boldsymbol{y} \oplus \hat{\boldsymbol{y}}), \\ &= \boldsymbol{A}\hat{\boldsymbol{x}} \oplus \boldsymbol{L}_{opt}\boldsymbol{y}. \end{aligned}$$

since $\hat{\boldsymbol{x}} \preceq \boldsymbol{x}$, therefore $\hat{\boldsymbol{y}} = \boldsymbol{C}\hat{\boldsymbol{x}} \preceq \boldsymbol{C}\boldsymbol{x} = \boldsymbol{y}$. Then

$$\begin{aligned} \boldsymbol{\xi} &= \boldsymbol{L}_{opt}\boldsymbol{y}, \\ &= \begin{bmatrix} (\gamma^3\delta^4)^*(\gamma^3\mu_3) \\ (\gamma^1\delta^8)^*(\gamma^1\beta^2\gamma^1 \oplus \gamma^2\beta_2\delta^4) \\ (\gamma^1\delta^4)^* \end{bmatrix} \boldsymbol{y}. \end{aligned}$$

The former equation is the solution of the following implicit equation

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} (\gamma^3\delta^4)^* & \varepsilon & \varepsilon \\ \varepsilon & (\gamma^1\delta^8)^* & \varepsilon \\ \varepsilon & \varepsilon & (\gamma^1\delta^4)^* \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \oplus \begin{bmatrix} (\gamma^3\mu_3) \\ (\gamma^1\beta_2\gamma^1 \oplus \gamma^2\beta_2\delta^4) \\ \mathrm{e} \end{bmatrix} \boldsymbol{y}.$$

Fig. 4 shows the WTEG together with the observer. Note that in (Trunk et al. (2017); Trunk (2019)), it was shown that, using the so-called core decomposition, all relevant operator on UP series in $\mathcal{E}_{per}[\![\delta]\!]$ can be reduced to operations on matrices with entries in the dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. Hence, the observer can be conveniently compute based on this core decomposition and the software tools MinMaxGD Hardouin et al. (2009). ◁

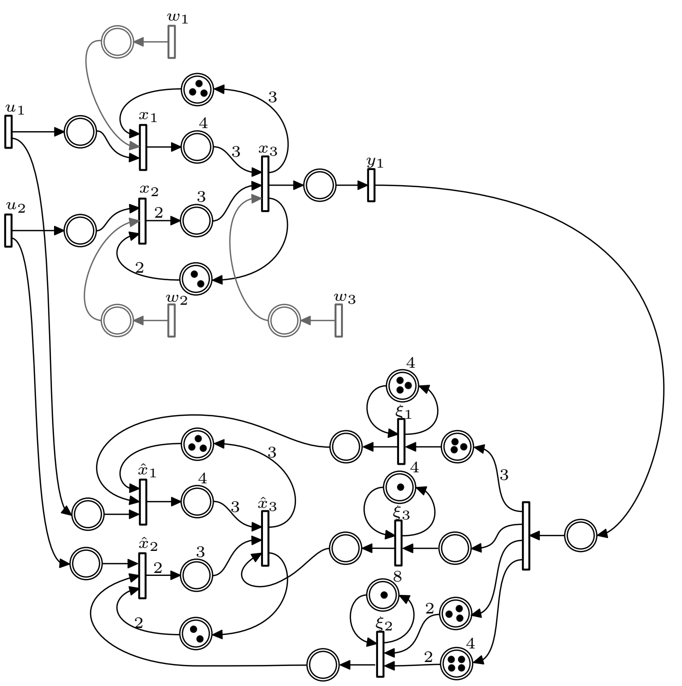

Fig. 4. System with observer

6. CONCLUSION

In this paper we proposed an observer for WTEGs. The observer yields an estimate of the number of firings of internal transitions. Our result generalizes the observer purposed for standard TEGs Hardouin et al. (2007) by using the dioid $\mathcal{E}[\![\delta]\!]$. It was shown that the optimal observer of a consistent WTEG can also be realized by a consistent WTEG. In future work we aim at observer based control for WTEGs.

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