On Tropical Fractional Linear Programming

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Abstract

Very recently, tropical counterparts of fractional linear programs have been studied. Some algorithms were proposed for solving them, with techniques ranging from bisection methods to homeomorphisms to formal power series. In this paper, some algorithms are also proposed. They mainly rely in the ability of finding the greatest and smallest solutions of tropical equations, a subject that was discussed in a previous work of the authors (Gonçalves et al. (2013)).

Keywords: Tropical Algebra, Tropical Linear Programming, Tropical Fractional Linear Programming, Optimization

1. Introduction

Tropical Fractional Linear Programs (denoted as **TFLP** hereafter) are problems of the form (Gaubert et al. (2012)) (see Section 2 for the notations)

$$\begin{aligned} \max / \min \ \ (\mathbf{w}^T\mathbf{p} \oplus \alpha) \phi (\mathbf{f}^T\mathbf{p} \oplus \beta) \ \text{ such that} \\ R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s}. \end{aligned} \tag{1}$$

This formulation can be used to solve optimization problems for multi-processor systems (see Butkovic and Aminu (2008)) and to compute the tightest inequality of the form $p_i - p_j \geq K$ if $\mathbf{p}$ is inside a tropical polyhedra, which finds applications in static analysis (see Gaubert et al. (2012)). It can also be used to check if a set of equalities $R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s}$ implies another equality $\mathbf{w}^T\mathbf{p} \oplus \alpha = \mathbf{f}^T\mathbf{p} \oplus \beta$ without the burden of finding all solutions explicitly [1]. This is true if and only if both max and min versions of Problem 1 have optimal value 0. This is worthy of mention because, in contrast with the traditional algebra, in the tropical setting there are equalities that can be logically deduced from a set of other equalities, but *cannot* be obtained by taking tropical linear combinations of these (see Gaubert and R.Katz (2009)). Hence, it is not always possible to expect to claim that $R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s}$ does not imply $\mathbf{w}^T\mathbf{p} \oplus \alpha = \mathbf{f}^T\mathbf{p} \oplus \beta$ by verifying the solvability of the equations $\mathbf{z}^T R = \mathbf{w}$, $\mathbf{z}^T\mathbf{r} = \alpha$, $\mathbf{z}^T S = \mathbf{f}^T$ and $\mathbf{z}^T\mathbf{s} = \beta$ for $\mathbf{z}$ [2]. With an efficient algorithm for solving TFLPs, one can check the validity of this proposition in an easy manner.

These kind of optimizations problems have began receiving attention from scientific community not a long time ago. By the authors knowledge, the first published work that has solved a particular case of Problem 1 (save the very particular cases in which it can be solved by a direct application of residuation theory, such as $A\mathbf{x} \preceq \mathbf{b}$) of Problem 1 was (Butkovic and Aminu (2008)). This special case is when $\mathbf{f} = \perp$ and $\beta = 0$ (Tropical Linear Programs, denoted as **TLP** hereafter)

$$\begin{aligned} \max / \min \quad & \mathbf{w}^T\mathbf{p} \oplus \alpha \;\text{ such that} \\ & R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s} \end{aligned} \tag{2}$$

and an algorithm was presented to solve them. The idea is that it is possible to check whether a value of objective function in Problem 1 is achievable by solving a tropical affine equation. Thus, if a lower and upper bound for the objective function is derived, it is possible to use a bisection method searching for the optimal value. The recent paper (Butkovic and MacCaig (2013)) pursues an integer solution to the problem when the entries are real numbers, also using a similar bisection approach.

(Gaubert et al. (2012)) studied the complete problem, and derived a Newton-like algorithm which works by solving a sequence of mean-payoff games. More recently, (Allamigeon et al. (2013)) used the field of *generalized Puiseux series* over $\mathbb{R}$, $\mathbb{K}$ (formal power series in one variable in which

[1] Finding all solutions of a tropical linear equation can be a very time consuming task, so, it is advantageous to avoid it whenever possible.

[2] The analogue of this affirmation for traditional algebra, i.e $A\mathbf{x} = \mathbf{b}$ implies $\mathbf{c}^T\mathbf{x} = d$ only if (the "if" part is trivial) there exists $\mathbf{y}$ such that $\mathbf{y}^T A = \mathbf{c}^T$ and $\mathbf{y}^T\mathbf{b} = d$ is a consequence of the Farkas Lemma.

the exponents can be any real number) to develop an alternative approach to the problem. It explores the idea of *valuation*, a function which maps each Puiseaux series to the opposite of its smallest exponent with a non 0 coefficient. In a special subset of $\mathbb{K}$, $\mathbb{K}_+$ (the set of "non-negative" Puiseux series), this valuation function is a homeomorphism between $\mathbb{K}_+$ and the tropical semiring $\mathbb{T}_{\texttt{max}}$. Since many of the results used in conventional linear programming rely only in axioms of ordered fields, such as $\mathbb{K}$, the classical Simplex algorithm can be adapted to solve linear programs over $\mathbb{K}$ (instead of the conventional $\mathbb{R}$) and hence, thanks to the valuation homeomorphism, tropical lineal programs as well.

In this paper, some algorithms will be proposed to solve the general Problem 1. The first algorithm, to solve max type TLPs, comes directly from a remarkable result about tropical affine equations: they do have a greatest solution. For min type TLPs, a more sophisticated approach using the recent developments in (Gonçalves et al. (2013)) is presented. The connection between TLPs and TFLPs is made by using a tropical version of the classical Charnes-Cooper method (Charnes and Cooper (1962)) for converting (traditional) fractional linear programs to (traditional) linear programs. As a byproduct of the derivation of those methods, some interesting conclusions can be obtained. Mainly that, as far as the solution $\mathbf{p}$ is concerned, the objective function do not matter for max TLPs and hence that the numerator and denominator of the objective functions in max and min TFLPs, respectively, also do not.

2. Tropical Algebra and Definitions

Tropical Algebra (also known as the Max-Plus Algebra), is the semiring[3]

$$\mathbb{T}_{\texttt{max}} = \{\mathbb{Z} \cup \{-\infty\}, \oplus, \otimes\} \tag{3}$$

in which $\oplus$ is the maximum and $\otimes$ is the traditional sum. It is usual, as well, to denote the neutral element of the sum, $-\infty$, as $\perp$. It is also usual to define the complete dioid $\overline{\mathbb{T}}_{\texttt{max}}$ augmented with the element ∞, here denoted as $\top$. It is also defined $\top \otimes \perp = \perp \otimes \top = \perp$. As in the traditional algebra, the symbol $\otimes$ is usually omitted.

[3]Usually, the name is used to denote the isomorphic dioid Min-Plus.

It is assumed from now on that the reader is familiar with this algebra basics and also with the concepts of residuation and Kleene Closure (Baccelli et al. (1992)). The pointwise infimum is denoted by $\wedge$. $\diagdown\!\!\!\circ$, ϕ are used to denote the left and right residuation of the product, respectively (more details in Blyth and Janowitz (1972)):

$$
\begin{aligned}
A \phi B &\equiv \max_{XB \preceq A} X, \\
A \diagdown\!\!\!\circ B &\equiv \max_{AX \preceq B} X.
\end{aligned}
\tag{4}
$$

The dual residuation of the sum, written as $\ominus$, is defined as:

$$
A \ominus B \equiv \min_{X \oplus B \succeq A} X. \tag{5}
$$

By analogy with the traditional algebra, if A is a matrix and α a scalar, $A \phi \alpha$ will be the pointwise scalar residuation of the entries of A by α. $\perp$ is a matrix of appropriate dimensions in which all the entries are $\perp$, while $\top$ is a matrix in which all the entries are $\top$.

An equation can be referred as *Problems* throughout this paper if it represents an optimization problem. For example, concerning the very first equation of this paper, it will be written "Problem 1" instead of "Equation (1)".

3. Dual and Primal Methods

In this section, a short review of two "cousins" algorithms, the Dual and Primal Methods, will be given (an in depth discussion can be found in Gonçalves et al. (2013)). Both methods solve linear tropical equations of the form

$$
E\mathbf{x} = D\mathbf{x}. \tag{6}
$$

3.1. Dual Method

The *Dual Method* (this is a denomination given in the paper (Gonçalves et al. (2013)), since as far as one is concerned there is no official denomination) is an iterative method described in literature. For example, it appears in (Dhingra and Gaubert (2006); Gaubert and Sergeev (2013)) and also as a particular case of a slight modification of the general algorithm of (Cuninghame-Green and Zimmermann (2001)). It is also related to the

Alternating Method presented in (Cuninghame-Green and Butkovic (2003)). The main appeal of this algorithm for this paper is the fact that it is able to find the *greatest solution* of a tropical linear equation [4].

It solves Equation (6) by iterating the sequence

$$\mathbf{x}[k+1] = E \diagdown (D\mathbf{x}[k]) \wedge D \diagdown (E\mathbf{x}[k]) \wedge \mathbf{x}[k] \tag{7}$$

on an initial $\mathbf{x}[0]$. It can also solve (Cuninghame-Green and Butkovic (2003))

$$R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s} \tag{8}$$

by solving the modified equation

$$R\mathbf{p} \oplus \mathbf{r}y = S\mathbf{p} \oplus \mathbf{s}y \tag{9}$$

which is linear and reduces to Equation (6) if one sets $\mathbf{x} = (\mathbf{p}^T \mid y)^T$, $E = (R \mid \mathbf{r})$ and $D = (S \mid \mathbf{s})$.

If a finite (no $\perp$ entries) solution exists, the Dual Method is able to find in a finite number of steps the greatest solution to Equation (8) smaller than or equal to a desired upper bound $\mathbf{p}_{\texttt{UB}}$. If one solves the linear Equation (9) with the initial condition $\mathbf{p}[0] = \mathbf{p}_{\texttt{UB}}$ and $y[0] = 0$, a solution smaller than or equal to $\mathbf{p}_{\texttt{UB}}$ will exist if and only if y remain 0 throughout the entire sequence of iterations (see Gonçalves et al. (2013)). Thus, the greatest solution can be found using the initial condition $\mathbf{x}[0] = (\top \mid 0)^T$.

Being closely related to the Alternating Method of (Cuninghame-Green and Butkovic (2003)) (see Gonçalves et al. (2013)), the Dual Method has a pseudo-polynomial complexity. (Cuninghame-Green and Butkovic (2003)) also presents conditions which ensure that the method converges in finite time (it is possible that one or more of the entries of $\mathbf{x}$ decrease indefinitely to $\perp$ thus implying an infinite number of steps for convergence). In practice, however, the algorithm seems to be efficient (convergence with finite time and with a low number of steps) for handling typical problems.

[4] The remarkable observation is that the greatest solution to any tropical affine equation does exist, thanks to the idempotence property of the sum in the tropical semiring. This can be concluded by the following simple observation. Let $\mathcal{X}$ be the set of all solutions of the affine equation, then $\bigoplus_{\mathbf{x}\in\mathcal{X}} \mathbf{x}$ is clearly the greatest member of $\mathcal{X}$, and thus it is itself a solution to the equation.

3.2. Primal Method

The *Primal Method* works, in some ways, in a dual manner to the Dual Method. Since the *smallest solution* of a tropical affine equation does not exists in general[5] , the algorithm is able to find the smallest solution in a particular subset. The method provides other solutions to Equation (6) by using a previously found one, denoted by $\mathbf{z}$. It works by finding the *dominances* of $\mathbf{z}$ on E and D. For instance, the dominance of $\mathbf{z}$ on E maps to each row i of E a column j, $j = \Upsilon(i)$, such that for the i^{th} row the j^{th} column dominates in the product $E\mathbf{z}$. Thus

$$\bigoplus_j E_{ij} z_j = E_{i\Upsilon(i)} z_{\Upsilon(i)} \quad \forall i. \tag{10}$$

For example, if the *first* row of E is $(1\ 2\ \ -3)$ and $\mathbf{z} = (0\ 3\ 10)^T$, then the product is $(1 \cdot 0) \oplus (2 \cdot 3) \oplus ((-3) \cdot 10) = 1 \oplus 5 \oplus 7 = 7$. Note that the *third* term $(-3) \cdot 10 = 7$ is the dominating term, so in this case $\Upsilon^{\mathbf{z}}_E(1) = 3$. A dominance may not be unique, as if $\mathbf{z}' = (0\ 5\ 10)^T$ is used instead of $\mathbf{z}$, in which the second and third terms achieves the maximum. In this case, any dominance can be chosen.

With these dominances $\Upsilon^{\mathbf{z}}_E$ and $\Upsilon^{\mathbf{z}}_D$ for $E \in \mathbb{T}^{n\times m}_{\mathtt{max}}$ and $D \in \mathbb{T}^{n\times m}_{\mathtt{max}}$ respectively, one can create matrices $\mathsf{W}^\flat(\Upsilon^{\mathbf{z}}_E, E) \in \mathbb{T}^{m\times n}_{\mathtt{max}}$ and $\mathsf{W}^\flat(\Upsilon^{\mathbf{z}}_D, D) \in \mathbb{T}^{m\times n}_{\mathtt{max}}$ such that

$$\begin{aligned} \{\mathsf{W}^\flat(\Upsilon^{\mathbf{z}}_E, E)\}_{ij} &\equiv -E_{ji} \text{ if } j = \Upsilon^{\mathbf{z}}_E(i); \\ \{\mathsf{W}^\flat(\Upsilon^{\mathbf{z}}_E, E)\}_{ij} &\equiv \perp \ \text{ otherwise,} \end{aligned} \tag{11}$$

and respectively for $\mathsf{W}^\flat(\Upsilon^{\mathbf{z}}_D, D)$ using D.

Thus, if one sets

$$\mathsf{H}(E, D, \Upsilon^{\mathbf{z}}_E, \Upsilon^{\mathbf{z}}_D) \equiv \mathsf{W}^\flat(\Upsilon^{\mathbf{z}}_E, E) D \oplus \mathsf{W}^\flat(\Upsilon^{\mathbf{z}}_D, D) E \tag{12}$$

then $\mathbf{x} = \mathsf{H}(E, D, \Upsilon^{\mathbf{z}}_E, \Upsilon^{\mathbf{z}}_D)^* \mathbf{y}$ is a solution for any $\mathbf{y}$.

As the Dual Method, the Primal is also able to solve Equation (8) by transforming it into the Equation (9) and using, for a desired $\mathbf{p}_{\mathtt{LB}}$, $\mathbf{y} = (\mathbf{p}^T_{\mathtt{LB}} \mid 0)^T$. The solution found will be the smallest one inside the so called

[5] A simple example: $x_1 \oplus x_2 = 0$. The solution set is obviously $((x_1 = 0)\ AND\ (x_2 \leq 0))\ OR\ ((x_1 \leq 0)\ AND\ (x_2 = 0))$, for which the smallest element does not exists.

dominance space induced by $\mathbf{z}$ on E and D that is greater than or equal to the lower bound $\mathbf{p}_{\text{LB}}$ (see Gonçalves et al. (2013)). Thus, the smallest solution inside this dominance space can be found by using $\mathbf{y} = (\perp \;\vdots\; 0)^T$ (that is, $\mathbf{p}$ being the last column of $\mathsf{H}(E, D, \Upsilon^{\mathbf{z}}_E, \Upsilon^{\mathbf{z}}_D)^*$ *without* the last entry).

The Primal Method relies in the ability of finding a solution to Equation (6), which can be done (for example) with the Dual Method. Disregarding the time for computing this solution, the Primal Method is essentially a problem of computing a Kleene Closure, which is tantamount to a problem of all-to-all maximum path in a graph. There exist strongly polynomial algorithms for solving this problem such as, for example, the *Floyd-Warshall Algorithm* (which is $\mathcal{O}(n^3)$ for a $n \times n$ matrix, see Robert (1962); Stephen (1962)). In some applications (for example, for solving TLPs as will be presented further in this text) it is necessary to compute only one specific solution (a column of $\mathsf{H}(E, D, \Upsilon^{\mathbf{z}}_E, \Upsilon^{\mathbf{z}}_D)^*$). In this case, a more efficient one-to-all maximum path algorithm can be used.

4. Solving TLPs and TFLPs

4.1. Max type programs

Max type TLPs are problems of the form

$$\begin{aligned} \max \quad & \mathbf{w}^T\mathbf{p} \oplus \alpha \;\; \text{such that} \\ & R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s}. \end{aligned} \tag{13}$$

As discussed in Subsection 3.1, the greatest solution to Equation (8) always exists, that is, there exists a solution $\mathbf{p}_{\mathtt{max}}$ such that for all other solutions $\mathbf{p}$ of this affine equation $\mathbf{p}_{\mathtt{max}} \succeq \mathbf{p}$.

Thus, Max-type TLP can be solved disregarding $\mathbf{w}$ and α, by finding this greatest solution of Equation (8). The Dual Method can be used for this purpose. Thus, the algorithm is very straightforward.

Algorithm 4.1. Solving max type TLP

1. Find the greatest solution of Equation (8) using the Dual Method.

4.2. Min type programs

Min type TLPs are problems of the form

$$\begin{aligned} \min \quad & \mathbf{f}^T\mathbf{p} \oplus \beta \;\; \text{such that} \\ & R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s}. \end{aligned} \tag{14}$$

The method for solving Min type TLPs is not as straightforward as solving Max type ones since, in general, the smallest solution of Equation (8) does not exists. The Primal Method addresses this problem partially by finding the smallest solution inside a special set, the *dominance space*. For that, it is necessary to find a solution to the equation first (see Subsection 3.2).

Given a solution $\mathbf{p}_{\mathtt{sol}}$ to Equation (8), the optimality of it can be ensured if and only if there is no solution to the following Equation

$$\begin{aligned} & \mathbf{f}^T\mathbf{p} \oplus \beta \preceq \gamma \\ & \text{with } \gamma = (-1)(\mathbf{f}^T\mathbf{p}_{\mathtt{sol}} \oplus \beta) \end{aligned} \tag{15}$$

inside the solution set of Equation (8). This is due to the fact that all the numbers used in the TLP are integers and thus an integer solution $\mathbf{p}$ exists (see Corollary 3.1 in Butkovic and Aminu (2008)). Hence the value of the objective function is also an integer. Since Equation (15) can be written as an equality

$$\mathbf{f}^T\mathbf{p} \oplus (\beta \oplus \gamma) = \gamma \tag{16}$$

it is possible to check the optimality of $\mathbf{p}_{\mathtt{sol}}$ by checking if there is a solution to the augmented equation

$$\begin{aligned} R\mathbf{p} \oplus \mathbf{r} &= S\mathbf{p} \oplus \mathbf{s} \\ \mathbf{f}^T\mathbf{p} \oplus (\beta \oplus \gamma) &= \gamma. \end{aligned} \tag{17}$$

Further, if a solution is found, it is guaranteed that this solution improves (decreases) the objective function value by at least one unit. Thus, one can apply the following procedure.

Algorithm 4.2. Solving min type TLP

1. Set k=0;
2. Find a solution $\mathbf{p}_{\mathtt{dl}}[0]$ to Equation (8), use as initial condition of the Dual Method $\mathbf{p}[0] = \top$, $y = 0$;
3. Use the Primal Method to solve Equation (8), using the solution $\mathbf{p}_{\mathtt{dl}}[k]$ to induce a dominance, to reduce this solution to a smallest solution in this dominance space, $\mathbf{p}_{\mathtt{pr}}[k]$ (see Subsection 3.2);
4. Check for optimality of $\mathbf{p}_{\mathtt{pr}}[k]$ by solving Equation (17), using as initial condition of the Dual Method $\mathbf{p}[0] = \top$, $y = 0$;
5. If optimality is found (there is no solution for Equation (17), see Subsection 3.1), end the algorithm with the solution $\mathbf{p}_{\mathtt{sol}} = \mathbf{p}_{\mathtt{pr}}[k]$;
6. Else, obtain the solution $\mathbf{p}_{\mathtt{dl}}[k+1]$ of Step 4, set k to $k+1$ and go to Step 3.

One can note that Step 3 can be avoided by replacing $\mathbf{p}_{\mathtt{pr}}[k]$ with $\mathbf{p}_{\mathtt{dl}}[k]$. However, the fact that the Primal Method finds a "small" solution (even if it is not the smallest) can substantially reduce the number of steps taken for convergence (the algorithm can naively just reduce the objective function one unit at each step, see Subsection 5.2).

It is important to Remark that there is a deep connection between Algorithm 4.2 and Algorithm 2 presented in (Gaubert et al. (2012)). In that paper the problem of solving Min type Problem 1 is shown to be equivalent to finding the smallest $\lambda \in \mathbb{R}$ such that $\phi(\lambda) \geq 0$ for a function $\phi : \mathbb{R} \mapsto \mathbb{R}$ constructed from all the parameters of the problem.

At each step, the authors find a so-called *left optimal strategy* σ, constructing a simplified function from this strategy, $\phi^{\sigma}(\lambda)$, and finding the smallest λ such that $\phi^{\sigma}(\lambda) \geq 0$. The latter problem can be solved by rather straightforward means using Kleene Closures. Due to the properties of left strategies, it is guaranteed that at each step $\lambda[k+1] \leq \lambda[k]$.

The concept of (max player) strategies is very close to the concept of dominances defined in Subsection 3.2. Then, the problem of finding the smallest λ such that $\phi^{\sigma}(\lambda) \geq 0$ can be interpreted as the problem of finding the smallest solution inside the dominance space, which is exactly Step 3 in Algorithm 4.2. Thus, the essential difference between the algorithms lies in the way that the monotonic convergence is guaranteed (left optimal strategies for Algorithm 2 of (Gaubert et al. (2012)) and extra constraint Equation (16) for Algorithm 4.2). Also, Algorithm 4.2 requires that the inputs are integers, while Algorithm 2 of (Gaubert et al. (2012)) do not (albeit one which requires this property is also presented).

Finally, Algorithm 4.2 solves Min-type TLPs, while Algorithm 2 of (Gaubert et al. (2012)) solves the more general Min type TFLPs. It will be shown in Subsection 4.3 that Algorithm 4.2 can also be used to solve this more general kind of problem.

It is also very important to remark that the method *may* take an infinite number of steps to converge if the optimal objective function value is $\perp$. If $\beta \neq \perp$ this bound is evident. If this does not hold, it is helpful to introduce such bound as a constraint or adding it a new β' to ensure that the number of steps will be finite. In practice such bound can be inherent of the structure of the problem. Nevertheless, (Butkovic and Aminu (2008)) shows how to compute lower and upper bounds which are finite in some situations. Basically, assuming that $\mathbf{r} \succeq \mathbf{s}$ (this can be always assumed, since the equation can be reordered in a way that this holds true) and $\beta = \perp$ (otherwise, β itself is a bound) this bound can be written as

$$V_{\text{LB}} = (\mathbf{f}^T \oslash S)(\mathbf{r} \ominus \mathbf{s}). \tag{18}$$

Indeed, with a very similar argument of those present in (Butkovic and Aminu (2008)), a symmetric version of Equation (18) (without the assumption $\mathbf{r} \succeq \mathbf{s}$) can be found

$$V_{\text{LB}} = (\mathbf{f}^T \oslash S)(\mathbf{r} \ominus \mathbf{s}) \oplus (\mathbf{f}^T \oslash R)(\mathbf{s} \ominus \mathbf{r}). \tag{19}$$

Finally, one can note that (as opposed to Max type programs) the solution of Min type programs depends on $\mathbf{f}$ and β.

4.3. Solving TFLPs

The complete problem (TFLPs) is defined as

$$\begin{aligned} \max / \min \quad & (\mathbf{w}^T \mathbf{p} \oplus \alpha) \oslash (\mathbf{f}^T \mathbf{p} \oplus \beta) \text{ such that} \\ & R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s}. \end{aligned} \tag{20}$$

It is straightforward to adapt the Charnes-Cooper transformation (Charnes and Cooper (1962)) to the tropical setting. Set

$$\begin{aligned} \mathbf{q} &= \mathbf{p} \oslash (\mathbf{f}^T \mathbf{p} \oplus \beta); \\ \hat{q} &= 0 \oslash (\mathbf{f}^T \mathbf{p} \oplus \beta). \end{aligned} \tag{21}$$

Then, by dividing (in Tropical Algebra) both sides of the affine equation on Equation (20) by $\mathbf{f}^T \mathbf{p} \oplus \beta$, it can be rewritten as

$$\begin{aligned} \max / \min \ \mathbf{w}^T \mathbf{q} \oplus \alpha \hat{q} \text{ such that} \\ R\mathbf{q} \oplus \mathbf{r}\hat{q} = S\mathbf{q} \oplus \mathbf{s}\hat{q}; \\ \mathbf{f}^T \mathbf{q} \oplus \beta \hat{q} = 0 \end{aligned} \tag{22}$$

in which the equation $\mathbf{f}^T \mathbf{q} \oplus \beta \hat{q} = 0$ condenses Equation (21). Thus, this problem in the transformed variables is a Max / Min type TLP. Once $\mathbf{q}$ and $\hat{q}$ are obtained, in order to come back to the original variable one just needs to revert Equation (21)

$$\mathbf{p} = (\mathbf{f}^T \mathbf{p} \oplus \beta)\mathbf{q} = \mathbf{q} \oslash \hat{q}. \tag{23}$$

It is necessary, however, to consider that *not all* the feasible space of Problem 22 (transformed problem) can be mapped back to a member of the feasible space of Problem 20 (original problem). This is the case for some problems in which the transformed problem has $\hat{q} = \top$. For instance, suppose that $\mathbf{r}$ and $\mathbf{s}$ are finite (no $\perp$ entries), $\mathbf{r} \neq \mathbf{s}$ and $\beta = \perp$. In this case, $\hat{q} = \top$ and any vector $\mathbf{q}$ with no $\top$ entries such that $\mathbf{f}^T \mathbf{q} = 0$ are in the feasible space of the transformed problem (since $\mathbf{r}\hat{q} = \mathbf{s}\hat{q} = \top$ and hence the equation $R\mathbf{q} \oplus \mathbf{r}\hat{q} = S\mathbf{q} \oplus \mathbf{s}\hat{q}$ clearly holds, regardless of $\mathbf{q}$). Transforming back to the original variables using Equation (23) yields to $\mathbf{p} = \perp$ (since $\hat{q} = \perp$ and no entry of $\mathbf{q}$ is $\top$), which is not in the original feasible space since by hypothesis $\mathbf{r} \neq \mathbf{s}$.

For Max type TFLPs, if one uses the method presented in Subsection 4.1 this consideration is crucial. This is due to the fact that the Dual Method will initialize the vector $(\mathbf{q}^T \,\vdots\, \hat{q}^T)^T$ in $\top$, and hence $\hat{q}$ will begin - and may stay - in $\top$. Thus, it is *necessary* to give an upper bound $\hat{q}$ (by any finite amount). This can be done by lower bounding $\mathbf{f}^T \mathbf{p} \oplus \beta$ by a finite amount. As mentioned previously, (Butkovic and Aminu (2008)) shows how to compute lower and upper bounds which are finite in some situations (Equation (18)).

Indeed, a cursory observation of Equation (18) is sufficient to conclude that as long as $\mathbf{f}$ has no $\perp$ entries and $\mathbf{r} \neq \mathbf{s}$ then the bound is finite.

Thus

Algorithm 4.3. Solving Max type TFLPs using Max type TLPs

1. Use Charnes-Cooper transformation to transform a Max type TFLP to a Max type TLP;

2. Find a finite lower bound V_{LB} for $\mathbf{f}^T\mathbf{p} \oplus \beta$ in the feasible space of Equation (8). That is, find $V_{\text{LB}} \succ \perp$ such that for all $\mathbf{p}$ such that $R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s}$ one has $\mathbf{f}^T\mathbf{p} \oplus \beta \succeq V_{\text{LB}}$. See (Butkovic and Aminu (2008)) for how to obtain this lower bound for certain problems;

3. Solve the Max type TLP using Algorithm 4.1 with the additional constraint $\hat{q} \preceq -V_{\text{LB}}$. This can be done by either setting the constraint $\hat{q} \oplus (-V_{\text{LB}}) = (-V_{\text{LB}})$ explicitly for the transformed problem or initializing the Dual Method with $\hat{q}[0] = -V_{\text{LB}}$ instead of $\top$;

4. Return to the original variables using Equation (23).

Also, one can solve Min type TFLPs using Algorithm 4.2. Hence

Algorithm 4.4. Solving Min type TFLPs using Min type TLPs

1. Use Charnes-Cooper transformation to transform a Min type TFLP to a Min type TLP;

2. Solve the Min type TLP using Algorithm 4.2 ;

3. Return to the original variables.

Now, a Min type TFLP is simply a Max type TFLP with the inverse of the objective function

$$\begin{aligned} &\min \ \ (\mathbf{w}^T\mathbf{p} \oplus \alpha) \not{o} (\mathbf{f}^T\mathbf{p} \oplus \beta) = \\ &- \max \ \ (\mathbf{f}^T\mathbf{p} \oplus \beta) \not{o} (\mathbf{w}^T\mathbf{p} \oplus \alpha). \end{aligned} \tag{24}$$

Thus, Min type programs can be transformed in conventional Max type programs and *vice-versa*. In summary, any of the Algorithms 4.3 or 4.4 can be used to solve either Max or Min type TFLPs. See Figure 1.

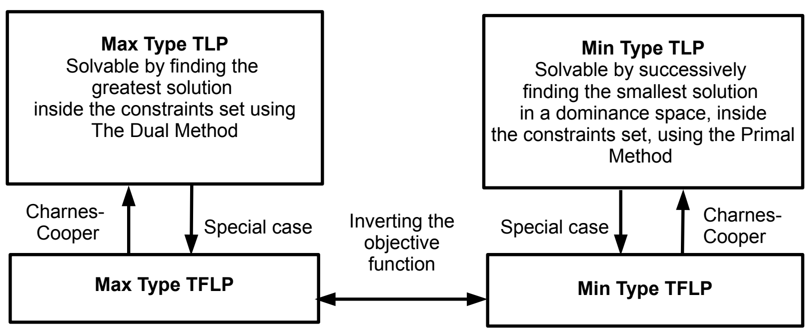

Figure 1: Connection between the problems.

It is important to note that, due to the discussion presented in this Subsection, Max type TFLPs solutions (that is, the value of $\mathbf{p}$) are independent of $\mathbf{f}$ and β. Dually, Min type programs are independent of $\mathbf{w}$ and α.

5. Example

The following example was taken from (Butkovic and Aminu (2008)).

$$\begin{gathered}
\min \quad 3p_1 \oplus 1p_2 \oplus 4p_3 \oplus (-2)p_4 \oplus p_5 \text{ such that} \\
\begin{pmatrix} 17 & 12 & 9 & 4 & 9 \\ 9 & 0 & 7 & 9 & 10 \\ 19 & 4 & 3 & 7 & 11 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} \oplus \begin{pmatrix} 12 \\ 15 \\ 13 \end{pmatrix} = \\
\begin{pmatrix} 2 & 11 & 8 & 10 & 9 \\ 11 & 0 & 12 & 20 & 3 \\ 2 & 13 & 5 & 16 & 4 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} \oplus \begin{pmatrix} 12 \\ 12 \\ 3 \end{pmatrix}.
\end{gathered} \tag{25}$$

The solution using Algorithms 4.3 and 4.2 will be presented.

5.1. Solving using Algorithm 4.3

For Algorithm 4.3, one must transform the original Min type TLP in a Max type TFLP, that is

$$\begin{gathered}
\min \quad 3p_1 \oplus 1p_2 \oplus 4p_3 \oplus (-2)p_4 \oplus p_5 = \\
\max 0 \not{o} (3p_1 \oplus 1p_2 \oplus 4p_3 \oplus (-2)p_4 \oplus p_5).
\end{gathered} \tag{26}$$

Then, it is necessary to compute a lower bound for $\mathbf{f}^T\mathbf{p} \oplus \beta$. Using Lemma 3.2 of (Butkovic and Aminu (2008)) (see Equation (18), note that $\mathbf{r} \succeq \mathbf{s}$), one can obtain $V_{\mathtt{LB}} = -5$. Thus, $\hat{q} \preceq 5$. The program in the modified variables is

$$\max \hat{q} \text{ such that}$$

$$\begin{pmatrix} 17 & 12 & 9 & 4 & 9 & 12 \\ 9 & 0 & 7 & 9 & 10 & 15 \\ 19 & 4 & 3 & 7 & 11 & 13 \\ 3 & 1 & 4 & -2 & 0 & \perp \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ \hat{q} \end{pmatrix} \oplus \begin{pmatrix} \perp \\ \perp \\ \perp \\ \perp \end{pmatrix} =$$

$$\begin{pmatrix} 2 & 11 & 8 & 10 & 9 & 12 \\ 11 & 0 & 12 & 20 & 3 & 12 \\ 2 & 13 & 5 & 16 & 4 & 3 \\ \perp & \perp & \perp & \perp & \perp & \perp \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ \hat{q} \end{pmatrix} \oplus \begin{pmatrix} \perp \\ \perp \\ \perp \\ 0 \end{pmatrix}. \tag{27}$$

Now, in order to solve Problem 5.1, it is sufficient to find the greatest solution to the affine constraint equation. Using the Dual Method with $(q_1[0]\ q_2[0]\ q_3[0]\ q_4[0]\ q_5[0]\ \hat{q}[0])^T = (\top\ \top\ \top\ \top\ \top\ 5)^T$ (note that $\hat{q}$ was initialized at 5 instead of $\top$. Another possibility is initializing in $\top$ and adding the constraint $\hat{q} \oplus 5 = 5$), it is possible to find, after 8 iterations of the Equation (7)

$$(q_1\ q_2\ q_3\ q_4\ q_5\ \hat{q}) = (-7\ \ -1\ \ -4\ \ -6\ 0\ \ -1). \tag{28}$$

Thus, the objective function value is $-\hat{q} = 1$, and $(p_1\ \ p_2\ \ p_3\ \ p_4\ \ p_5)^T = (-6\ 0\ \ -3\ 5\ 1)^T$ is a possible $\mathbf{p}$. The value obtained for the objective function is, obviously, the same as the one obtained in (Butkovic and Aminu (2008)). Also, the resulting solution $\mathbf{p}$ is exactly the same.

5.2. Solving using Algorithm 4.2

Solving Equation (5) with the Dual Method and initial condition

$$(p_1[0]\ p_2[0]\ p_3[0]\ p_4[0]\ p_5[0])^T = (0\ 0\ 0\ 0\ 0)^T, \tag{29}$$

it is possible to find after 2 iterations

$$\mathbf{p}_{\mathtt{d1}}[0] = (-6\ 0\ 0\ \ -5\ 0)^T \tag{30}$$

which has objective function value 4. For notational simplicity, let $U = (R \,\vdots\, \mathbf{r})$ and $V = (R \,\vdots\, \mathbf{s})$ and $\mathbf{z}[k] = (\mathbf{p}_{\mathtt{d1}}[k]^T \,\vdots\, 0)^T$ for any k.

Using the Primal Method with the dominances

$$\Upsilon_U^{\mathbf{z}[0]}(i) = \begin{cases} 6, & \text{if } i = 1 \\ 4, & \text{if } i = 2 \\ 2, & \text{if } i = 3 \end{cases}, \quad \Upsilon_V^{\mathbf{z}[0]}(i) = \begin{cases} 2, & \text{if } i = 1 \\ 6, & \text{if } i = 2 \\ 1, & \text{if } i = 3 \end{cases} \tag{31}$$

(this is not the unique pair of dominances possible to be induced by $\mathbf{p}_{\mathtt{dl}}[0]$, and were chosen at random) and with those it is possible to obtain

$$\mathbf{p}_{\mathtt{pr}}[0] = (-6\ 0\ \ \perp\ \ -5\ \ \perp)^T. \tag{32}$$

The new objective function value is 1. Equation (17) has no solution and the algorithm halts. It took a single iteration of Algorithm 4.2 to do so.

If Step 3 in Algorithm 4.2 is avoided, the algorithm takes 4 steps to converge, beginning from objective function value equal to 4 and decreasing one unit by iteration.

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