A precompensator synthesis for P-temporal event graphs

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1 introduction

Positive systems have the peculiar property that any nonnegative input and nonnegative initial state generates a nonnegative state trajectory and nonnegative output trajectory for all times. Positivity of the variables often emerges as the immediate consequence of the nature of the phenomenon itself ([1]).

Among those large class of systems, this paper focuses on the positive systems in which the state trajectories are non decreasing and for which the phenomena are mainly delays and synchronization. A relevant model of these systems involves the operators 'max' and '+' (or by duality, the operators 'min' and '+'). The nice and relatively old idea was to consider these systems in a specific algebraic structure, namely idempotent semiring, in which these models become linear and then can be studied in an analogous manner to the classical linear systems. The reader is invited to consult [2][3] for an exhaustive introduction to this system theory, and ([4],[5]and [6]) for a presentation to the controllers synthesis. This paper attempts to extend the theory of this kind of systems by considering systems subject to nonlinear constraints in the peculiar algebraic setting. More precisely the state trajectories must satisfy the following constraint : $X = \underline{A}X \oplus BU \preceq \overline{A} \odot x$ where $\underline{A}$ describes the internal dynamic of the system and $\overline{A}$ the constraints which must be respected.The main contribution of this paper demonstrates that the residuation theory makes possible to characterizing the greatest element in $Im\underline{A}^* \cap Im\overline{A}_*$. This result is used to synthesize a precompensator P, such that control $u = Pv$ ensures to achieve a model matching while satisfying the constraints.

2 Algebraic preliminaries

Definition 1. *A dioid $\mathscr{D}$ is a set endowed with two internal operations denoted by $\oplus$ (addition) and $\otimes$ (multiplication), both associative and both having neutral elements denoted by ε and e respectively, such that $\oplus$ is also commutative and idempotent (i.e. $a \oplus a = a$). The $\otimes$ operation is distributive with respect to $\oplus$, and ε is absorbing for* *the product (i.e. $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$, $\forall a$). Dioids can be endowed with a natural order : $a \succeq b$ iff $a = a \oplus b$. A dioid is complete if every subset $\mathscr{A} \subseteq \mathscr{D}$ admits a least upper bound equal to $\bigoplus_{x \in \mathscr{A}} x$ and if $\otimes$ distributes over finite and infinite sums. The greatest element of a Dioid is denoted $\top = \bigoplus_{x \in \mathscr{D}} x$. A complete dioid have a complete lattice structure, and then $a \succeq b \Leftrightarrow b = a \wedge b$.*

Theorem 1. *([2],[6])*
Over a complete dioid $\mathscr{D}$, the implicit equation $x = ax \oplus b$ admits $x = a^*b$ as least solution, where $a^* = \oplus_{i \in \mathbb{N}} a^i$ with $a^0 = e$. Furthermore, if $x, y \in \mathscr{D}$, we have $x(yx)^* = (xy)^*x$ and $x^* \otimes x^* = x^*$.

Remark 1. (Matrix dioid) We can extend the notion of scalar dioid to matrix dioid by considering the following two internal operations;

Let $A, B \in \mathscr{D}^{n \times p}$ and $C \in \mathscr{D}^{p \times q}$ $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$ and $(A \otimes C)_{ij} = \bigoplus_{k=1}^{k=p} A_{ik} \otimes C_{kj}$

Definition 2. *(Isotone mapping) A mapping Π from an ordered set $\mathscr{D}$ into an ordered set $\mathscr{C}$ such that $\forall$ a, $b \in \mathscr{D}$, $a \preceq b \Rightarrow \Pi(a) \preceq \Pi(b)$.*

Lemma 1. *([2]) Let Π be a mapping from a dioid $\mathscr{D}$ into another dioid $\mathscr{C}$. The following statements are equivalent:*
1. the mapping Π is isotone;
2. if lower bounds exist in $\mathscr{D}$ and $\mathscr{C}$, Π is a $\wedge$-submorphism, that is, $\forall$ a, $b \in \mathscr{D}$, $\Pi(a \wedge b) \preceq \Pi(a) \wedge \Pi(b)$.

Lemma 2. *([2]) If a admits a left inverse b and a right inverse c, then*
• $b = c$ and this unique inverse is denoted a^{-1};
• moreover, $\forall x, y$, $a(x \wedge y) = ax \wedge ay$.
The same holds true for right multiplication by a, and also for right and left multiplication by a^{-1}.

Definition 3. *([7],[2]) A multiplicative lattice-ordered group $\mathscr{G}$, means that, in addition to being a group and a lattice, the multiplication is isotone, and*
• the multiplication is necessarily distributive with respect to both the upper and the lower bounds ($\mathscr{G}$ is called a reticulated group),
• moreover, the lattice is distributive (that is, upper and lower bounds are distributive with respect to one another).

Theorem 2. *([2]) Let $\mathscr{G}$ be a multiplicative lattice-ordered group, and $a, b \in \mathscr{G}$. Since each element of $\mathscr{G}$ admits an inverse, one has the remarkable formulae:*

$$(a \wedge b)^{-1} = a^{-1} \oplus b^{-1},\ (a \oplus b)^{-1} = a^{-1} \wedge b^{-1},\ a \wedge b = a(a \oplus b)^{-1} b,$$

Proposition 1. *Let $\mathscr{G}$ be a multiplicative lattice-ordered group, a and b in $\mathscr{G}$ and $x \in \mathscr{D}$ with $\mathscr{D}$ a dioid and $\mathscr{G} \subseteq \mathscr{D}$ then $ax \wedge bx = (a \wedge b)x$.*

Proof. First, using lemma (1) we have $(a \wedge b)x \preceq ax \wedge bx$. Second, since a and b are in $\mathscr{G}$ from theorem (2), we have $(a \wedge b)^{-1} = a^{-1} \oplus b^{-1}$, so:

$$\begin{aligned} ax \wedge bx &= (a \wedge b)(a^{-1} \oplus b^{-1})(ax \wedge bx) \\ &= (a \wedge b)[(x \wedge a^{-1}bx) \oplus (b^{-1}ax \wedge x)] \preceq (a \wedge b)[x \oplus x](a \wedge b)x \end{aligned}$$

this leads to $ax \wedge bx = (a \wedge b)x$.

Definition 4. *In the sequel, we will endow the dioid $\mathscr{D}$ with the product $a \odot b = a \otimes b$ with the following convention $\varepsilon \odot \top = \top$ (we recall that $\varepsilon \otimes \top = \varepsilon$). And over matrix dioid, $(A \odot B)_{ij} = \bigwedge_{k=1}^{n} A_{ik} \odot B_{kj}$ with $A \in \mathscr{D}^{p \times n}$ and $B \in \mathscr{D}^{n \times q}$. In the sequel, $e^{\odot} \in \mathscr{D}^{n \times n}$ is the identity matrix of the law $\odot$, i.e, $e^{\odot}_{ii} = e$, and $e^{\odot}_{ij} = \top$ if $i \neq j$.*

Definition 5. *In the next, we will consider the dual star operator which is given by: $g_* = \bigwedge_{i \in N} g^{\odot i}$ with $g^{\odot i} = \underbrace{g \odot \cdots \odot g}_{i\ times}$ and $g^{\odot 0} = e^{\odot}$.*

Proposition 2. *Let $\mathscr{G}$ be a reticulated group and $A, B \in \mathscr{D}^{p \times n}$ be two matrices with each entry in $\mathscr{G}$ and $x \in \mathscr{D}^{n \times q}$ then we have $(A \wedge B) \odot x = A \odot x \wedge B \odot x$*

Proof. Let $A, B \in \mathscr{D}^{p \times n}$ be two matrices with each entry in $\mathscr{G}$ then

$$\begin{aligned} \big((A \wedge B) \odot x\big)_{ij} &= \bigwedge_{k=1}^{k=n} (A \wedge B)_{ik} \odot x_{kj} = \bigwedge_{k=1}^{k=n} (A_{ik} \wedge B_{ik}) \odot x_{kj} \\ &= \bigwedge_{k=1}^{k=n} (A_{ik} \odot x_{kj}) \wedge (B_{ik} \odot x_{kj}) \text{ thanks to proposition (1)} \\ &= \Big(\bigwedge_{k=1}^{k=n} (A_{ik} \odot x_{kj})\Big) \wedge \Big(\bigwedge_{k=1}^{k=n} (B_{ik} \odot x_{kj})\Big) = (A \odot x \wedge B \odot x)_{ij}. \end{aligned}$$

Residuation theory allows a kind of pseudo-inversion of mapping defined over lattices, it plays a central role in the control of systems. For (max,+) linear systems we refer the reader to [6] and [5] for an introduction.

Definition 6. *Let f be a mapping from a complete dioid $\mathscr{D}$ to a complete dioid $\mathscr{C}$, f is lower semi-continuous (l.s.c), respectively, upper semi-continuous (u.s.c), if for all subsets (finite or infinite) X of $\mathscr{D}$ $f(\bigoplus_{x \in X} x) = \bigoplus_{x \in X} f(x)$, respectively $f(\bigwedge_{x \in X} x) = \bigwedge_{x \in X} f(x)$.*

Definition 7. *An isotone mapping $f : \mathscr{D} \to \mathscr{C}$, where $\mathscr{D}$ and $\mathscr{C}$ are ordered sets, is a residuated mapping if for all $y \in \mathscr{C}$, the least upper bound of the subset $\{x | f(x) \preceq y\}$ exists and belongs to it. It is denoted by $f^{\#}(y)$, and $f^{\#}$ is called the residual of f.*
An isotone mapping $g : \mathscr{D} \to \mathscr{C}$ is a dual residuated mapping if for all $y \in \mathscr{C}$, the greatest lower bound of the subset $\{x | g(x) \succeq y\}$ exists and belongs to it. It is then denoted by $g^{\flat}(y)$, and $g^{\flat}$ is called the dual residual of g.

Theorem 3. *([2]) Let f, g be isotone mappings from : $\mathscr{D}$ to $\mathscr{C}$, where $\mathscr{D}$ and $\mathscr{C}$ are ordered sets, the following equivalences holds true:*
f is a residuated $\Leftrightarrow f \circ f^{\#} \preceq Id_{\mathscr{C}}$ and $f^{\#} \circ f \succeq Id_{\mathscr{D}} \Leftrightarrow f$ is l.s.c and $f(\varepsilon) = \varepsilon$.
g is dual residuated $\Leftrightarrow g \circ g^{\flat} \succeq Id_{\mathscr{C}}$ and $g^{\flat} \circ g \preceq Id_{\mathscr{D}}, \Leftrightarrow g$ is u.s.c and $g(\top) = \top$.

Example 1. ([2]) The mapping $L_a : \mathscr{D} \to \mathscr{D}$, $x \mapsto a \otimes x$ is isotone and l.s.c (i.e $L_a(\bigoplus_{x\in X} x) = \bigoplus_{x\in X} L_a(x)$), then it is residuated. The residual is denoted $L_a^{\#}(x) = a \mathbin{\backslash\!\!\circ} x$ in (max, +) literature. We recall that $\varepsilon \mathbin{\backslash\!\!\circ} x = \top$, $\top \mathbin{\backslash\!\!\circ} x = \varepsilon$ and $\top \mathbin{\backslash\!\!\circ} \top = \top$.

Proposition 3. *([2, §4.4.2]) If $\Pi : \mathscr{D} \to \mathscr{C}$ and $\Phi : \mathscr{C} \to \mathscr{B}$ are dually residuated mappings, then $\Phi \circ \Pi$ is also dually residuated and $(\Phi \circ \Pi)^\flat = \Pi^\flat \circ \Phi^\flat$. $\Pi \wedge \Phi$ is also dually residuated and $(\Pi \wedge \Phi)^\flat = \Pi^\flat \oplus \Phi^\flat$.*

Proposition 4. *If each entry of A admits an inverse then the mapping $\Gamma_A : x \mapsto A \odot x$ is u.s.c, with x an element of $\mathscr{D}^{n\times q}$, that is: $\Gamma_A(\bigwedge_{x\in X} x) = \bigwedge_{x\in X} \Gamma_A(x)$*

Proof. $\Gamma_A(\bigwedge_{x\in X} x) = A \odot (\bigwedge_{x\in X} x)$
$\Rightarrow (\Gamma_A(\bigwedge_{x\in X} x))_{ij} = \bigwedge_{k=1}^{n} A_{ik} \odot (\bigwedge_{x\in X} x_{kj})$ thanks to proposition 1;
$(\Gamma_A(\bigwedge_{x\in X} x))_{ij} = \bigwedge_{k=1}^{n} \bigwedge_{x\in X} (A_{ik} \odot x_{kj}) = \bigwedge_{x\in X} \bigwedge_{k=1}^{n} (A_{ik} \odot x_{kj}) = \bigwedge_{x\in X} (\Gamma_A(x))_{ij}$
then $\Gamma_A(\bigwedge_{x\in X} x) = \bigwedge_{x\in X} \Gamma_A(x)$

Corollary 1. *Let $A \in \mathscr{D}^{n\times n}$, $X \in \mathscr{D}^{n\times q}$ be two matrices, if each entry of A admits an inverse then the mapping $\Gamma_A : x \mapsto A \odot x$ is dually residuated and its dual residual is given by $\Gamma_A^\flat : x \mapsto A \mathbin{\backslash\!\!\bullet} x$ with $(A \mathbin{\backslash\!\!\bullet} x)_{ij} = \bigoplus_{l=1}^{l=n} A_{li} \mathbin{\backslash\!\!\bullet} x_{lj} = \bigoplus_{l=1}^{l=n} A_{li}^{-1} \odot x_{lj}$ and by respecting the following rules $\top \mathbin{\backslash\!\!\bullet} x = \varepsilon$, $\varepsilon \mathbin{\backslash\!\!\bullet} x = \top$ and $\varepsilon \mathbin{\backslash\!\!\bullet} \varepsilon = \varepsilon$ (i.e. $\varepsilon^{-1} \odot \varepsilon = \varepsilon$). It is important to note that $a \succeq b \Rightarrow a \mathbin{\backslash\!\!\bullet} x \preceq b \mathbin{\backslash\!\!\bullet} x$. Furthermore, if b admits an inverse $b \mathbin{\backslash\!\!\bullet} (a \otimes c) = (b \mathbin{\backslash\!\!\bullet} a) \otimes c$ (i.e. $b^{-1} \odot (a \otimes c) = (b^{-1} \odot a) \otimes c$).*

Proposition 5. *Let $\mathscr{G}$ be a reticulated group, $A, B \in \mathscr{D}^{p\times n}$ two matrices with each entry in $\mathscr{G}$ and $x \in \mathscr{D}^{n\times q}$ then we have $(A \wedge B) \mathbin{\backslash\!\!\bullet} x = A \mathbin{\backslash\!\!\bullet} x \oplus B \mathbin{\backslash\!\!\bullet} x$.*

Proof. The result is a direct application of proposition (3).

Theorem 4. *([2, §4.5]) Let $A \in \mathscr{D}^{n\times n}$, the following equivalences holds true:*

$$x = A^* \otimes x \Leftrightarrow x \succeq A \otimes x \Leftrightarrow A \mathbin{\backslash\!\!\circ} x \succeq x \Leftrightarrow A^* \mathbin{\backslash\!\!\circ} x = x.$$

Corollary 2. *The greatest solution of $Ax \preceq x$ and $x \preceq B$ is $A^* \mathbin{\backslash\!\!\circ} B$.*

Proposition 6. *Let $G \in \mathscr{D}^{n\times n}$ with each entry in a reticulated group then the following equivalences holds true:*

$$x \preceq G \odot x \Leftrightarrow G \mathbin{\backslash\!\!\bullet} x \preceq x \Leftrightarrow G_* \mathbin{\backslash\!\!\bullet} x = x \Leftrightarrow G_* \odot x = x.$$

Proof. First, we prove that: $x \preceq G \odot x \Rightarrow G \mathbin{\backslash\!\!\bullet} x \preceq x$.
If $x \preceq G \odot x$ then $G \mathbin{\backslash\!\!\bullet} x \preceq G \mathbin{\backslash\!\!\bullet} (G \odot x)$ since $(G \mathbin{\backslash\!\!\bullet} .)$ is isotone, furthermore theorem (3) yields $G \mathbin{\backslash\!\!\bullet} (G \odot x) \preceq x$, then $G \mathbin{\backslash\!\!\bullet} x \preceq x$.
Second, we prove that: $G \mathbin{\backslash\!\!\bullet} x \preceq x \Rightarrow G_* \mathbin{\backslash\!\!\bullet} x = x$.
If $x \succeq G \mathbin{\backslash\!\!\bullet} x \Rightarrow x \succeq (e^{\odot} \mathbin{\backslash\!\!\bullet} x) \oplus (G \mathbin{\backslash\!\!\bullet} x) \oplus (G^{\odot 2} \mathbin{\backslash\!\!\bullet} x) \oplus \cdots$ and thanks to proposition (5)

$\Rightarrow x \succeq (e^{\odot} \wedge G \wedge G^{\odot 2} \wedge \cdots) \bullet\!\!\backslash x = G_* \bullet\!\!\backslash x \succeq e^{\odot} \bullet\!\!\backslash x = x$ so $x = G_* \bullet\!\!\backslash x$.
Third, we prove that $x = G_* \bullet\!\!\backslash x \Rightarrow x = G_* \odot x$
$x = G_* \bullet\!\!\backslash x \Rightarrow G_* \odot x = G_* \odot (G_* \bullet\!\!\backslash x) \succeq x$ (thanks to theorem 3), but $G_* \odot x \preceq e^{\odot} \odot x = x$, then $G_* \odot x = x$.
Fourth, we prove that $G_* \odot x = x \Rightarrow x \preceq G \odot x$.
Thanks to proposition 4, $G_* \odot x = (x \wedge G \odot x \wedge G^{\odot 2} \odot x \wedge \cdots) \preceq G \odot x$.

Proposition 7. *Let $A \in \mathscr{D}^{n\times p}$, $X \in \mathscr{D}^{p\times q}$ and $B \in \mathscr{D}^{n\times n}$ be three matrices. If each element B_{ij} admits an inverse then we have $B \bullet\!\!\backslash (A \otimes X) = (B \bullet\!\!\backslash A) \otimes X$*

Proof.
$$\begin{aligned}(B \bullet\!\!\backslash (A \otimes X))_{ij} &= \bigoplus_{l=1}^{l=n} B_{li} \bullet\!\!\backslash (A \otimes X)_{lj} = \bigoplus_{l=1}^{l=n} B_{li} \bullet\!\!\backslash (\bigoplus_{k=1}^{k=p} A_{lk} \otimes X_{kj}) \\ &= \bigoplus_{l=1}^{l=n}\bigoplus_{k=1}^{k=p} B_{li} \bullet\!\!\backslash (A_{lk} \otimes X_{kj}) \text{ since } \Gamma_B^{\flat} \text{ is l.s.c} \\ &= \bigoplus_{k=1}^{k=p}\bigoplus_{l=1}^{l=n} (B_{li} \bullet\!\!\backslash A_{lk}) \otimes X_{kj} \text{ since } B_{li} \text{ admits an inverse} \\ &= \bigoplus_{k=1}^{k=p} (B \bullet\!\!\backslash A)_{ik} \otimes X_{kj} = ((B \bullet\!\!\backslash A) \otimes X)_{ij}.\end{aligned}$$

Proposition 8. *Let us consider a dioid $\mathscr{D}$, a reticulated group $\mathscr{G} \subseteq \mathscr{D}$ and two matrices $A, G \in \mathscr{D}^{n\times n}$ and each entry of G in $\mathscr{G}$. The greatest x such that $A \otimes x \preceq x \preceq G \odot x$ and $x \preceq B$ is given by $\hat{x} = ((G_* \bullet\!\!\backslash A^*)^*) \circ\!\!\backslash B$*

Proof. First, we prove that: $A \otimes x \preceq x \preceq G \odot x$ and $x \preceq B \Rightarrow x \preceq \hat{x}$.
Second we prove that $\hat{x}$ satisfy the following properties
(i) $\hat{x} \preceq \quad B$
(ii) $\hat{x} = A^* \otimes \hat{x}$
(iii) $\hat{x} = G_* \odot \hat{x}$
By considering propositions 6 and theorem 4, $A \otimes x \preceq x \preceq G \odot x$ implies that $x = G_* \odot x = G_* \bullet\!\!\backslash x = A^* \otimes x$, which means that $x \in ImG_* \cap ImA^*$. Then, x must be such that $x = G_* \bullet\!\!\backslash (A^* x)$. The assumption about entries of G and proposition 7 leads to $x = (G_* \bullet\!\!\backslash A^*) \otimes x \Rightarrow x \preceq (G_* \bullet\!\!\backslash A^*) \circ\!\!\backslash x$, which is equivalent to $x = ((G_* \bullet\!\!\backslash A^*)^*) \circ\!\!\backslash x$ (see theorem 4).
Then $A \otimes x \preceq x \preceq G \odot x$ and $x \preceq B \Leftrightarrow x = ((G_* \bullet\!\!\backslash A^*)^*) \circ\!\!\backslash x$ and $x \preceq B$
$\Leftrightarrow x \preceq \hat{x} = ((G_* \bullet\!\!\backslash A^*)^*) \circ\!\!\backslash B$. According to theorem 4 $\hat{x}$ is such that

$$(G_* \bullet\!\!\backslash A^*) \otimes \hat{x} \preceq \hat{x} \preceq (G_* \bullet\!\!\backslash A^*) \circ\!\!\backslash \hat{x} \tag{1}$$

Now it suffices to prove that (i), (ii) and (iii) are respected
First we prove that $\hat{x} \in ImA^*$, according to theorem 4, this is equivalent to $\hat{x} = A^* \otimes \hat{x} = A^* \circ\!\!\backslash \hat{x}$; $\hat{x} \preceq (G_* \bullet\!\!\backslash A^*) \circ\!\!\backslash \hat{x} \Rightarrow$
$$\begin{aligned}A^* \circ\!\!\backslash \hat{x} \preceq A^* \circ\!\!\backslash ((G_* \bullet\!\!\backslash A^*) \circ\!\!\backslash \hat{x}) &= ((G_* \bullet\!\!\backslash A^*) \otimes A^*) \circ\!\!\backslash \hat{x} \\ &= (G_* \bullet\!\!\backslash (A^* \otimes A^*)) \circ\!\!\backslash \hat{x} \text{ (see proposition 7)} \\ &= (G_* \bullet\!\!\backslash A^*) \circ\!\!\backslash \hat{x} \succeq \hat{x} \text{ since } (G_* \bullet\!\!\backslash A^*) \succeq e \text{ (see theorem 1).}\end{aligned}$$
Furthermore, $\hat{x} \succeq A^* \circ\!\!\backslash \hat{x}$ (since $A^* \succeq e$), then $\hat{x} = A^* \circ\!\!\backslash \hat{x}$, i.e, $\hat{x} \in ImA^*$.
Second, we prove that $\hat{x} \in ImG_*$, i.e, $\hat{x} = G_* \odot \hat{x} = G_* \bullet\!\!\backslash \hat{x}$ (see proposition 6), from

equation (1), $\hat{x} \succeq (G_* \backslash A^*) \otimes \hat{x} = G_* \backslash (A^* \otimes \hat{x}) = G_* \backslash \hat{x}$ since $\hat{x} = A^* \otimes \hat{x}$.
On the other side $G_* \preceq e^{\odot}$ then $\hat{x} \preceq G_* \backslash \hat{x}$ so $\hat{x} = G_* \backslash \hat{x} = G_* \odot \hat{x}$.
Third, since $(G_* \backslash A^*)^* \succeq e$, $\hat{x} \preceq B$.
To summarize, $\hat{x}$ is the greatest solution of $A \otimes x \preceq x \preceq G \odot x$ and $x \preceq B$.

3 Application to P-temporal event graphs

3.1 P-temporal event graphs

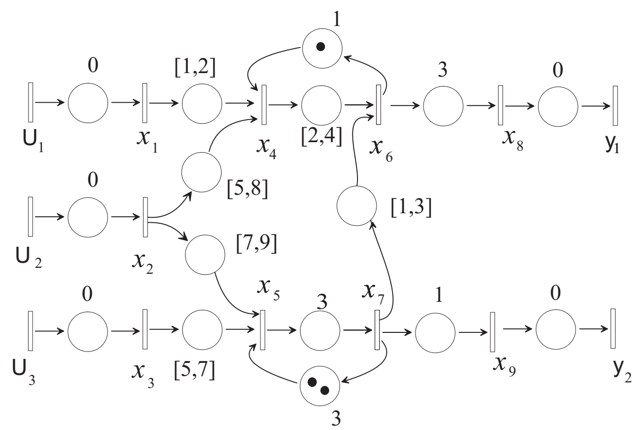

Fig. 1. A P-temporal event graph

The P-temporal Petri net model defined in [8], enables to model manufacturing systems whose activities times are included between a minimum and a maximum value, that is the sojourn time associated to each place P_i is included in an interval $[s_{\min_i}, s_{\max_i}]$ with $0 \leq s_{\min_i} \leq s_{\max_i}$. Before the duration $s_{\min_i}$, the token in p_i is in the non-available state. After $s_{\min_i}$ and before $s_{\max_i}$, the token in p_i is in the available state for the firing of a transition. After $s_{\max_i}$, the token in p_i violates the constraints.

A transition is fired as soon as there is an available token in each upstream place. The behavior of a transition may be described as a sequence of firing dates. The variable $x(k)$ is a "dater" and it represents the $k+1^{th}$ firing date of the transition labeled x. For each increasing sequence $x(k)$, it is possible to define the transformation $x(\gamma) = \bigoplus_{k \in \mathbb{Z}} x(k) \otimes \gamma^k$, where γ is a backward shift operator in event domain (that is $y(\gamma) = \gamma x(\gamma) \Leftrightarrow y(\gamma) = x(k-1)$, see [2], p. 228). This transformation is analogous to the Z-transform used in discrete-time classical control theory and the formal series $x(\gamma)$ is a synthetic representation of the trajectory $\{x(k)\}_{k \in \mathbb{Z}}$. The set of formal series in γ is denoted $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ and constitutes a dioid. In general, the behavior of a p-TEG (i.e, the firing sequence of each transition) can be represented by linear relations over this dioid:

$$\begin{cases} x(\gamma) = \underline{A}x(\gamma) \oplus \underline{B}u(\gamma) = \underline{A}^*\underline{B}u(\gamma), \\ y(\gamma) = \underline{C}x(\gamma) = \underline{C}\underline{A}^*\underline{B}u(\gamma), \end{cases} \tag{2}$$

but trajectories must respect non-linear constraints which are given by:

$$\begin{cases} x(\gamma) \preceq \overline{A} \odot x(\gamma) \wedge \overline{B} \odot u(\gamma), \\ y(\gamma) \preceq \overline{C} \odot x(\gamma). \end{cases}, \tag{3}$$

in which $\odot$ is given in definition (4), and $\overline{A}$ represents the constraints relations between internal transitions, $\overline{B}$ represents the constraints relations between internal transitions and input transitions and $\overline{C}$ represents the constraints relations between output transitions and internal transitions. Entries of matrices $\overline{A}, \overline{B}$ and $\overline{C}$ are assumed to be in the reticulated group $\overline{\mathbb{Z}}_{\max} \subset \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$.

Example 2. Figure 1 gives an example of P-temporal timed event graph. We suppose that there is no constraint on the input and output transition (this condition doesn't affect the generality of results).

The behavior of the P-temporal event graph can be represented by

$$\underline{A} \otimes X(\gamma) \preceq X(\gamma) \preceq \overline{A} \odot X(\gamma).$$

$$\underline{A} = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 1 & 5 & \varepsilon & \varepsilon & \varepsilon & 1\gamma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 7 & 5 & \varepsilon & \varepsilon & \varepsilon & 3\gamma^2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon \end{pmatrix}, \overline{A} = \begin{pmatrix} \top & \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top & \top \\ 2 & 8 & \top & \top & \top & \top & \top & \top & \top \\ \top & 9 & 7 & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & 4 & \top & \top & 3 & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top & \top \end{pmatrix}, B = \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, C = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix}.$$

Entry $\underline{A}(5,7) = 3\gamma^2$, corresponds to the place between transition x_7 and x_5, and means that there are two tokens in the place and that the minimum sojourn time is 3 time units. $\overline{A}(6,4) = 4$ means that the tokens in the place between transitions x_4 and x_6 must not stay more than 4 time units. The entries equal to $\top = +\infty$, mean that there is no constraint on the sojourn time. It must be noted that each entry admits an inverse (with the convention $\top^{-1} = \varepsilon$). This assumption is essential to solve the control problem.

3.2 Optimal control of p-temporal event graphs

The control method proposed herein is based on the Just-in- Time strategy and on the model reference approach (see [6],[9],[10]). Let $H \in (\mathbb{Z}_{\max}[\![\gamma]\!])^{p\times q}$ be the transfer matrix of the plant and $G_{ref} \in (\mathbb{Z}_{\max}[\![\gamma]\!])^{p\times q}$ be the reference model, i.e., the desired transfer matrix for the controlled system. The precompensation problem for TEG is solved by finding the greatest precompensator P such that $HP \preceq G_{ref}$. The optimal

solution, denoted by P_o, is given by $P_o = H \circ\!\!\!\!\backslash\, G_{ref}$.
This means that, for a given external input $v \in (\mathbb{Z}_{\max}[\![\gamma]\!])^q$, the input variable, given by $u = P \otimes v$, will be maximal and ensures that $\forall\, v\; HP_o v \preceq G_{ref} v$.

For P-TEG, the problem is to compute the greatest P such that $HP = C\underline{A}^* BP \preceq G_{ref}$ and $\underline{A}^* BP \preceq \overline{A} \odot \underline{A}^* BP$. By considering $X = \underline{A}^* BP$, this problem may be written:

$$X \preceq \underline{A}^* B((C\underline{A}^* B) \circ\!\!\!\!\backslash\, G_{ref}) = X_0 \text{ and } \underline{A} \otimes X \preceq X \preceq \overline{A} \odot X,$$

which admits the following optimal solution: $X_{opt} = ((\overline{A}_* \bullet\!\!\!\!\backslash\, \underline{A}^*)^*) \circ\!\!\!\!\backslash\, X_0 = \overline{\underline{A}}^* \circ\!\!\!\!\backslash\, X_0$ (see proposition 8). Therefore, the precompensator must be such that $P \preceq P_{opt} = \underline{A}^* B \circ\!\!\!\!\backslash\, X_{opt}$. The last question arising is to know if P_{opt} respect the constraints. It suffices that $\underline{A}^* BP_{opt} = \overline{\underline{A}}^* \otimes \underline{A}^* BP_{opt}$. By lack of places, this problem is not adressed here.

Example 3. Let us consider the p-temporal given in figure (1) and a model reference given by: $G_{ref} = \begin{pmatrix} 14(4\gamma)^* & 21(4\gamma)^* & 19(4\gamma)^* \\ 11(4\gamma)^* & 18(4\gamma)^* & 16(4\gamma)^* \end{pmatrix}$ the optimal precompensator is given by: $P_{opt} = \begin{pmatrix} 7(4\gamma)^* & 14(4\gamma)^* & 12(4\gamma)^* \\ (4\gamma)^* & 7(4\gamma)^* & 5(4\gamma)^* \\ 2(4\gamma)^* & 9(4\gamma)^* & 7(4\gamma)^* \end{pmatrix}$.

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