CONTROL AND ROBUSTNESS ANALYSIS FOR
(MAX, +)-LINEAR SYSTEMS

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Abstract: This paper deals with controller synthesis for (max, +) linear system. It
aims at comparing the performances and the robustness of two control strategies
introduced in (Cottenceau et al., 2001) and (Maia et al., 2003) respectively. In
both strategies, the influences of the possible mismatches between the system
and its model are analyzed. This work shows that the control strategy using
simultaneously a precompensator and a feedback controller (introduced in (Maia
et al., 2003)) gives better performances.

Keywords: Discrete event dynamic systems, Timed Petri nets, (max, +) algebra,
Timed event Graphs, dioid, idempotent semiring, Control Systems.

1. INTRODUCTION

Timed Event Graphs (TEG) constitute a subclass
of timed Petri nets in which each place has ex-
actly one upstream and one downstream transi-
tion. It is well known that the timed/event be-
havior of a TEG, under earliest functioning rule 1,
can be expressed by linear relations over some
dioids, namely idempotent semiring (Baccelli et
al., 1992). Strong analogies then appear be-
tween the classical linear system theory and the
(max, +)-linear system theory. In particular, the
concept of control is well defined in the context
of TEG study. It refers to the firing-control of
the TEG input transitions in order to reach the
desired performance. In the literature, an optimal
control for TEG exists and is proposed in (Cohen
et al., 1989). For a given reference input, this
open-loop structure control yields the latest input
firing date in order to obtain the output before
the desired date. One possible approach for the
control of TEG is the model reference technique
in which a given model describes the desired per-
formance and the design goal is achieved through
the calculation of a precompensator or a feed-
back controller (Cottenceau et al., 2001; Luders
and Santos-Mendes, 2002). The control strategies
based on feedback control, although favoring sta-
bility, are limited in the sense that the reference
model must satisfy certain restrictive conditions.
Lately, a new technique for the design of con-
trollers in which a precompensator and a feedback
controllers are calculated simultaneously was in-
troduced by (Maia et al., 2003). This paper aims
at comparing the performances and robustness

1 i.e. a transition is fired as soon as it is enabled
of the above mentioned control methods. More precisely, we will compare performances regarding the just-in-time criterion and we will compare robustness, regarding possible mismatches between the system and its model. The paper is organized as follows. Section 2 introduces some algebraic tools concerning the dioid and residuation theories. Section 3 is devoted to recall some elements of DES representation over particular dioids and this section presents three control strategies. Section 4 is dedicated to the analysis of the performances and the robustness of these control strategies.

2. ALGEBRAIC PRELIMINARIES

A dioid $D$ is an idempotent semiring, that is an algebraic structure with two internal operations denoted by $\oplus$ and $\otimes$. The neutral elements of $\oplus$ and $\otimes$ are represented by $e$ and $1$ respectively. In a dioid, a partial order relation is defined by $a \geq b$ iff $a = a \oplus b$ and $x \leq y$ denotes the greatest lower bound between $x$ and $y$. A dioid $D$ is said to be complete if it is closed for infinite $\oplus$-sums and if $\oplus$ distributes over infinite $\otimes$-sums. Most of the time the symbol $\ominus$ will be omitted as in conventional algebra.

**Theorem 1.** (Baccelli et al., 1992), th. 4.75). The implicit equation $x = ax \oplus b$ defined over a complete dioid $D$, admits $x = a^* b$ as least solution, where $a^* = \bigoplus_{i \in \mathbb{N}} a^i$ (Kleene star operator). It will be sometimes represented by the following mapping : $K : D \rightarrow D, x \mapsto \bigoplus_{i \in \mathbb{N}} x^i$.

TEG control problems (Cohen et al., 1989), stated in a just-in-time context, usually involves the inversion of isotope mappings\(^2\), that is, one must find $x$ such that $f(x) = y$ (where $f$ is isotope). Residuation Theory (Blyth and Janowitz, 1972) deals with such problems stated in partially ordered sets.

**Definition 2.** (Residual and residuated mapping). A mapping $f : D \rightarrow E$ between two ordered sets is **residuated** if it is isotope, and if, for all $y \in E$, the subset $\{x \in D \mid f(x) \leq y\}$ admits a maximal element, denoted $f^\sharp(y)$. The isotope mapping $f^\sharp : E \rightarrow D$ is called the **residual** of $f$. The residual $f^\sharp$ is the only isotope mapping satisfying the following properties:

$$f \circ f^\sharp \leq \text{Id} \quad \text{and} \quad f^\sharp \circ f \geq \text{Id},$$

where $\text{Id}$ is the identity mapping respectively on $D$ and $E$.

**Lemma 3.** (Cohen, 1998).

\(^2\) $f$ is an isotope mapping if it preserves order, that is, $a \leq b \implies f(a) \leq f(b)$.

- If $f : D \rightarrow E$ and $g : E \rightarrow F$ are residuated mappings, then $f \circ g$ is also residuated and $(f \circ g)^\sharp = g^\sharp \circ f^\sharp$.
- If $f$ is a residuated mapping from $D \rightarrow E$, then $f^\sharp \circ f = f$.

The mappings $L_a : x \mapsto a \otimes x$ and $R_a : x \mapsto x \otimes a$ defined over a complete dioid $D$ are both residuated ((Baccelli et al., 1992), p. 181). Their residu-als are isotope mappings denoted respectively by $L_a^\sharp(x) = a \otimes x$ and $R_a^\sharp(x) = x \otimes a$. Some useful dioid formulæ involving these residuals are given below.

$$a(a^\sharp x) \leq x \quad \text{and} \quad (x^\sharp a) a \leq x$$

$$a(a^\sharp x) = ax$$

$$a^\sharp a = (a^\sharp a)^*$$

$$(a^\sharp)^2 = a^*$$

$$x^\sharp (a^* x) = (a^* x)^\sharp (a^* x)$$

**Definition 4.** (Restricted mapping). Let $f : D \rightarrow E$ be a mapping and $B \subseteq E$ with $f(D) \subseteq B$. We will denote $B \circ f : D \rightarrow B$ the mapping defined by $f = i_B \circ_{B1} f$, where $i_B : B \rightarrow E, x \mapsto x$ is the canonical injection.

**Definition 5.** (Closure mapping). An isotope mapping $f : D \rightarrow D$ defined on an ordered set $D$ is a closure mapping if $f \geq \text{Id}$ and $f \circ f = f$.

**Remark 6.** According to (5), the Kleene star operator is a closure mapping since $a^* \geq a$ and $(a^*)^* = a^*$.

**Theorem 7.** (Cottenceau et al., 2001). Let $f : D \rightarrow D$ be a closure mapping. Then, $\text{im}_f \circ f$ is a residuated mapping whose residual is the canonical injection $\text{im}_f : \text{im}f \rightarrow D, x \mapsto x$.

**Example 8.** Mapping $\text{im}_K : D \rightarrow \text{im}K$ is a residuated mapping whose residual is $(\text{im}_K)^\sharp = \text{im}_K \circ f$. This means that $x = a^*$ is the greatest solution to inequality $x^\sharp \leq a^*$. Actually, this greatest solution achieves equality.

**Theorem 9.** (Gaubert, 1992). Let $f : D \rightarrow D$ be a residuated closure mapping, we have $f = f^\sharp \circ f$ and $f = f \circ f^\sharp$.

3. CONTROL METHOD

Firstly, let us consider the following (max,+)-linear system

$$x(k) = Ax(k - 1) \oplus Bu(k), \quad y(k) = Cx(k),$$

where $x(k) \in \mathbb{Z}_\text{max}^{n \times 1}, u(k) \in \mathbb{Z}_\text{max}^{p \times 1}$ and $y(k) \in \mathbb{Z}_\text{max}^{m \times 1}$ are respectively the state, input and output.
vectors of the system. The matrices $A, B$ and $C$ are of proper sizes and have entries ranging over $\mathbb{Z}_{\text{max}}$. We know from (Baccelli et al., 1992) that (7) represents the behavior of a class of discrete event systems called Timed Event Graphs (TEG). In the case of a TEG, $x$ (resp. $u$ and $y$) is a vector associated to the internal (resp. input and output) transitions, and $x_i(k)$ represents the $k$th firing dates of the internal transitions which are labelled $x$. Following the conventional approach, it is possible to define the transformation $x(\gamma) = \bigoplus_{i \in G} x(k)\gamma_i^k$ where $\gamma$ is a backward shift operator in event domain (that is $y(\gamma) = y(k) - \{y(k-1), \forall k\}$, see (Baccelli et al., 1992), p. 228). This transformation is analogous to the $\mathcal{Z}$-transform used in discrete-time classical control theory and the formal series $x(\gamma)$ is a synthetic representation of the trajectory $x(k)$.

The set of the formal series in $\gamma$ is a dioid denoted by $\mathbb{Z}_{\text{max}}[\gamma]$. By using $\gamma$-transform, we obtain the following representation of (7):

$$X(\gamma) = A \gamma X(\gamma) \oplus BU(\gamma), \quad Y(\gamma) = CX(\gamma),$$

where $U(\gamma), X(\gamma)$ and $Y(\gamma)$ are the $\gamma$-transform of $u, x$ and $y$ respectively. The implicit equation for the vector $X$, namely $X = A \gamma X \oplus BU$ which is solved (thanks to theorem 1) by $X = (A \gamma)^{-1} BU$. Finally, we obtain the input-output representation (transfer matrix)

$$Y = H U \quad \text{with} \quad H(\gamma) = C (A \gamma)^{-1} B.$$  \hspace{1cm} (8)

Herein, three control strategies for the systems are presented, and their performances are compared in section 4. They are based on the Just-in-Time criterion and on the model reference approach (Cottenceau et al., 2001). They can be described as follows: let $H \in \mathbb{Z}_{\text{max}}[\gamma]^{m \times p}$ be the transfer function of a TEG. Let $M_H : \mathbb{Z}_{\text{max}}[\gamma]^{p \times m} \rightarrow \mathbb{Z}_{\text{max}}[\gamma]^{m \times p}$, $X \mapsto H(XH)^*$ be a mapping. This mapping represents the influence of an output feedback $x$ on the closed-loop transfer dynamics. Consider $G \in \mathbb{Z}_{\text{max}}[\gamma]^{m \times p}$, $D \in \mathbb{Z}_{\text{max}}[\gamma]^{m \times n}$ and $N \in \mathbb{Z}_{\text{max}}[\gamma]^{p \times p}$. Let us consider the following sets:

$$G_1 = \{G \mid D \text{ such that } G = D^* H\}, \quad G_2 = \{G \mid 3N \text{ such that } G = H N^*\}.$$  

The mapping $g_{11}M_H$ and $g_{21}M_H$ are both residuated. Their residuals are such that $(g_{11}M_H)^2(X) = (g_{21}M_H)^2(X) = H X H^*.$

Proposition 11. If $G_{\text{ref}} \in G_1 \cup G_2$, there exists a greatest realizable output feedback $F_{\text{op}}$ such that $M_H(F_{\text{op}}) \leq G_{\text{ref}}$. This greatest controller is given by

$$F_{\text{op}} = H \gamma G_{\text{ref}} H^*.$$  \hspace{1cm} (10)

(b) The model-reference control scheme proposed in the following is a generalization of the two strategies described above, that is, it uses both a precompensator and feedback controller (Maia et al., 2003). Fig. 1.(c) illustrates the approach.

```
P ------ H ------ Y
|       |       |
Y ------ U ------ P
|       |       |  (a)
F_1    H    F_2
|       |       |
F_1    H    F_2
|       |       |
Y ------ U ------ P
|       |       |  (b)
Y ------ U ------ P
|       |       |  (c)
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Fig. 1. Control structures.

By using theorem 1, one can obtain the closed-loop equations which relate $U$, $V$ and $Y$:

$$HP \preceq G_{\text{ref}}, \quad P \preceq P_{\text{op}}, \quad \text{therefore the isotony property assures that } U = PV \preceq P_{\text{op}}V.$$
\[ Y = G_2 V = HP(F_2 HP)^* V, \quad (11) \]
\[ U = G_{2a2} V = P(F_2 HP)^* V. \quad (12) \]

The problem can be stated as follows. Given a reference model \( G_{ref} \), what are the controller matrices \( P \) and \( F_2 \) which assure the greatest transfer function between \( U \) and \( V \), i.e., \( G_{2a2} \), such that \( G_2 \leq G_{ref} \)? Again, considering the Just-in-Time context, one seeks the controllers which satisfy the reference specification \( G_2 \leq G_{ref} \) while delaying as much as possible the input trajectory (e.g., the entrance of products to be processed). Formally, the problem can be stated as follows:

\[ \bigoplus_{P, F_2} G_{2a2}(P, F_2) \quad (13) \]

s.t. \( G_2 = HP(F_2 HP)^* \leq G_{ref}. \)

It is clear that \( P = [\varepsilon]_{pxp} \) is always a subsolution to the problem independently of the choice of \( F_2 \), meaning that the subsolution set is not empty. Furthermore, it is easy to notice that the strategies using exclusively a precompensator (by setting \( F_2 = [\varepsilon]_{pxn} \)) or exclusively a feedback controller (by setting \( P = I_{pxp} \), where \( I_{pxp} \) is the identity matrix in dioid) are particular cases of this problem.

**Proposition 12.** (Maia et al., 2003). For the proposed control scheme shown in Fig. 1(c), the three following inequalities are equivalent:

\[ \begin{align*}
HP(F_2 HP)^* \leq G_{ref} \\
P(F_2 HP)^* \leq H \setminus G_{ref} \\
HP(F_2 HP)^* \leq H(H \setminus G_{ref}).
\end{align*} \]

**Lemma 13.** (Maia et al., 2003). A solution to problem 13 must satisfy \( P \leq G_{2a2} \leq H \setminus G_{ref}. \)

**Proposition 14.** (Maia et al., 2003). A solution to the optimization problem proposed in (13) is given by:

\[ \begin{align*}
P_{op} &= H \setminus G_{ref}. \\
F_{2op} &= (HP_{op}) \setminus (H(HP_{op}) \setminus (HP_{op})). \quad (15)
\end{align*} \]

**PROOF.**

From lemma 13, \( G_{2a2} \) is maximum (it is equal to the upper bound) if \( P = H \setminus G_{ref} \) and \( F_2 = \varepsilon \). Then, the greatest \( F_2 \) for this value of \( P \) is given by the greatest subsolution of inequality \( P_{op}(F_2 HP_{op})^* \leq H \setminus G_{ref} \), which in turn (by proposition 12) is equivalent to \( HP_{op}(F_2 HP_{op})^* \leq H(H \setminus G_{ref}) = HP_{op} \). Moreover, from the residuation definition this inequality is equivalent to \( (F_2 HP_{op})^* \leq (HP_{op}) \setminus (H(HP_{op}) \setminus (HP_{op})). \) Equation (4) yields \( (HP_{op}) \setminus (H(HP_{op}) \setminus (HP_{op})) \), then, to corollary 8, \( F_2 HP_{op} \leq (HP_{op}) \setminus (H(HP_{op}) \setminus (HP_{op})). \)

Finally, by solving this last inequality one obtains \( F_{2op} = (HP_{op}) \setminus (H(HP_{op}) \setminus (HP_{op})). \)

**4. PERFORMANCES AND ROBUSTNESS ANALYSIS OF CONTROL METHODS**

We will compare below the performances and the robustness of the control strategy given by proposition 11 and the one given by proposition 14. First, we must observe that unlike the first strategy, the second one does not restrict the reference model choice. Nevertheless, in order to compare performances of these strategies, we assume below that the controller \( F_{1op} \) exists and then that the reference model is such that \( G_{ref} \in G_1 \cup G_2 \) (see proposition 11).

**4.1 Performance Comparison**

**Proposition 15.** The control strategy given in proposition 14 leads to the same performances than the one obtained with the open-loop strategy, i.e., the greatest closed-loop transfer functions \( G_{2a2} \) and \( G_2 \) are equal to their upper bounds, that is, \( P_{op} \) and \( HP_{op} \) respectively. Formally, this means that

\[ \begin{align*}
G_{2c} &= H P_{op}(F_2 HP_{op})^* = HP_{op} \\
G_{2a2} &= P_{op}(F_2 HP_{op})^* = P_{op}.
\end{align*} \]

**PROOF.** This proposition follows directly from proposition 14, lemma 13 and from the observation that \( G_{2c} = HG_{2a2} \) (see (11) and (12)).

**Proposition 16.** Let \( G_{ref} \in G_1 \cup G_2 \) be a reference model. The transfer relation between \( U \) and \( V \) are such that

\[ G_{1a2} = (F_{1op} H)^* \leq G_{2a2} = P_{op}(F_2 HP_{op})^* = P_{op}. \]

**PROOF.** We suppose that \( G_{ref} \in G_1 \), that is, \( G_{ref} = D^* H \). Then, we have

\[ G_{1a2} = (F_{1op} H)^* = (H \setminus (D^* H)) \setminus (H \setminus (D^* H)) \]

which follows from (2). But we also have that

\[ G_{1a2} \leq (H \setminus (D^* H))^* = H \setminus (D^* H) = P_{op} = G_{2a2} \]

by making use of (6) and (4).

**Proposition 17.** Let \( G_{ref} \in G_1 \cup G_2 \) be a reference model. The controlled system transfer \( G_{1c} = H(F_{1op} H)^* \leq G_{2c} = HP_{op}(F_2 HP_{op})^* = HP_{op}. \)

**PROOF.** From proposition 16, we have \( G_{1a2} = (F_{1op} H)^* \leq G_{2a2} = P_{op}(F_2 HP_{op})^* = P_{op} \) and by isotony of product we obtain \( G_{1c} = HG_{1a2} = H(F_{1op} H)^* \leq G_{2c} = HG_{2a2} = HP_{op}(F_2 HP_{op})^* = HP_{op} \).

**4.1.1 Summary**

These results mean that the pair of controllers \( (P_{op}, F_{2op}) \) :  

- allows to obtain the same performances than the one obtained with the open-loop control;
4.2 Robustness analysis

a) In this section the aim is to analyse the robustness of the closed-loop control methods introduced previously.

First, we study the robustness of the controller given by proposition 11 (Fig. 1). We are looking for an upper bound denoted by $H_{1\text{sup}}$ to the set of systems which preserves the optimal closed-loop control objective, that is,$H_{1\text{sup}} = \sup \left\{ X | X(F_{1\text{op}}X)^* = H(F_{1\text{op}}H)^* \right\}.$

Then we characterize the set of systems which preserves the input output behavior. It means that the system can evolve in this set without altering the input–output performances of the closed-loop system.

**Lemma 18.** Let $Q_A : \mathcal{D} \rightarrow \mathcal{D}, X \mapsto X(A) X^*$ be a mapping defined over a complete dioid. Then $\text{Im} Q_A, Q_A$ is a residuated mapping and the residual is $(\text{Im} Q_A) Q_A = \text{Im} Q_A$, where $\text{Im} Q_A$ is the canonical injection.

**Proof.** The mapping $Q_A$ is a closure mapping, indeed $Q_A = (AX)^* (AX)^* = X (AX)^* = X (AX)^* = (AX)^* (AX)^* = X (AX)^* (AX)^*$. Then proposition 7 gives the result.

**Proposition 19.** The system $H_{1\text{sup}} = H(F_{1\text{op}}H)^*$ is the greatest system which does not alter the closed-loop transfer relation, i.e., $H_{1\text{sup}}(F_{1\text{op}} H_{1\text{sup}})^* = H(F_{1\text{op}}H)^*.$

**Proof.** According to lemma 18, we seek the greatest $X$ such that $F_{1\text{op}} X \preceq H(F_{1\text{op}} H)^*.$ Lemma 18 yields $(\text{Im} Q_{F_{1\text{op}}})(Q_{F_{1\text{op}}})^* = \text{Im} Q_{F_{1\text{op}}}$, and since $H(F_{1\text{op}} H)^* \in \text{Im} Q_{F_{1\text{op}}}$, we have directly $H_{1\text{sup}} = H(F_{1\text{op}} H)^*.$ Furthermore according to theorem 9, we have $Q_{F_{1\text{op}}} = Q_{F_{1\text{op}}} Q_{F_{1\text{op}}}^*$, which leads to equality $H_{1\text{sup}}(F_{1\text{op}} H_{1\text{sup}})^* = H(F_{1\text{op}} H)^* = H_{1\text{sup}}.$

**Corollary 20.** Whatever be the system behavior $X$ such that $H \preceq X \preceq H_{1\text{sup}}$ the closed-loop transfer relation is equal to $H(F_{1\text{op}} H)^*$, i.e., the input–output performances are not altered.

b) We are now interested in the robustness analysis of the control method which equations are given in proposition 14. We are looking for an upper bound, denoted $H_{2\text{sup}}$ to the set of systems which preserves the optimal closed-loop control objective, that is

$$H_{2\text{sup}} = \sup \left\{ X | X P_{op}(F_{2\text{op}} X P_{op})^* = H P_{op}(F_{2\text{op}} H P_{op})^* \right\}.$$  \hspace{1cm} (16)

**Proposition 21.** The system $H_{2\text{sup}} = H P_{op}(F_{2\text{op}} H P_{op})^*$ is the greatest which satisfies (16), i.e., $H_{2\text{sup}} P_{op}(F_{2\text{op}} H_{2\text{sup}} P_{op})^* = H P_{op}(F_{2\text{op}} H P_{op})^*.$

**Proof.** According to definition of the mappings $Q_{F_{2\text{sup}}}$ and $R_{P_{op}}$, we seek the greatest $X$ such that $X P_{op}(F_{2\text{op}} X P_{op})^* \preceq H P_{op}(F_{2\text{op}} H P_{op})^*,$ that is, $(Q_{F_{2\text{sup}}} R_{P_{op}})(X) \preceq (Q_{F_{2\text{sup}}} R_{P_{op}})(H).$ Since $(Q_{F_{2\text{sup}}} R_{P_{op}})(H) \in \text{Im} Q_{F_{2\text{sup}}}$ and as $\text{Im} Q_{F_{2\text{sup}}}$ is a residuated mapping (see lemma 18), $X \preceq (\text{Im} Q_{F_{2\text{sup}}})(Q_{F_{2\text{sup}}} R_{P_{op}})^* (Q_{F_{2\text{sup}}} R_{P_{op}})(H), \text{ and thanks to lemma 3, it follows } X \preceq R_{P_{op}}^* (Q_{F_{2\text{sup}}} R_{P_{op}})(H).$

By recalling that $(\text{Im} Q_{F_{2\text{sup}}})(Q_{F_{2\text{sup}}})^* = \text{Im} Q_{F_{2\text{sup}}},$ we have $X \preceq R_{P_{op}}^* (Q_{F_{2\text{sup}}} R_{P_{op}})(H) = H P_{op}(F_{2\text{op}} H P_{op})^*$.

Now, we will show that this upper bound is solution of (16). From proposition 15 it follows $H P_{op}(F_{2\text{op}} H P_{op})^* \preceq P_{op} = H P_{op} P_{op},$ i.e., $R_{P_{op}}^* Q_{F_{2\text{sup}}} R_{P_{op}}(H) = R_{P_{op}}^* P_{op} R_{P_{op}}(H).$ Then, from lemma 3 it follows $Q_{F_{2\text{sup}}} R_{P_{op}}^* R_{P_{op}}(H) = Q_{F_{2\text{sup}}} R_{P_{op}}(H),$ which yields to $H_{2\text{sup}} P_{op}(F_{2\text{op}} H_{2\text{sup}} P_{op})^* = H P_{op}(F_{2\text{op}} H P_{op})^*.$

**Corollary 22.** Whatever be the system behavior $X$ such that $H \preceq X \preceq H_{2\text{sup}}$ the closed-loop transfer relation is equal to $H P_{op}(F_{2\text{op}} H P_{op})^*,$ i.e., the input–output performances are not altered.

**Proof.** Let $X$ be a transfer relation such that $H \preceq X \preceq H_{2\text{sup}}.$ Since the product and star operators are isotone, we have $H P_{op}(F_{2\text{op}} H P_{op})^* \preceq X P_{op}(F_{2\text{op}} X P_{op})^* \preceq H_{2\text{sup}} P_{op}(F_{2\text{op}} H_{2\text{sup}} P_{op})^*$, and proposition
21 leads to equality $H P_{op}(F_{2_{op}} H P_{op})^* = X P_{op}(F_{2_{op}} X P_{op})^* = H_{2_{sup}} P_{op}(F_{2_{op}} H_{2_{sup}} P_{op})^*$.

4.3 Robustness evaluation

In the previous section, the upper bound of the system set which achieve the control objective is given for both closed-loop control strategy. In order to compare these bounds we assume below that the optimal controller $F_{1_{opt}}$ exists, i.e. $G_{ref} \in \mathcal{G}_1 \cup \mathcal{G}_2$. Nevertheless, we recall that this restriction is not useful to ensure the existence of $F_{2_{op}}$.

Lemma 23. Consider a reference model $G_{ref} \in \mathcal{G}_1 \cup \mathcal{G}_2$. Let $F_{1_{op}}$ be the greatest controller such that $H(F_{1_{op}} H)^* \leq G_{ref}$ and $P_{op}$ the greatest precompensator such that $H P_{op} \leq G_{ref}$. Then

$$H(F_{1_{op}} H)^* \leq H P_{op} \leq G_{ref}.$$  

**PROOF.**

Since $L_H$ is a residuated mapping (see definition 2), we have the following equivalences $H(F_{1_{op}} H)^* \leq G_{ref} \iff (F_{1_{op}} H)^* \leq H \lambda G_{ref}$.

Furthermore, by isotony of $\otimes$, we obtain $H(F_{1_{op}} H)^* \leq H(H \lambda G_{ref}) = H P_{op} \leq G_{ref}$ in which the latter inequality follows from (14) and (2).

Lemma 24. If $G_{ref} \in \mathcal{G}_1 \cup \mathcal{G}_2$ the upper bound $H_{2_{sup}}$ is equal to $H P_{op}$, that is

$$H_{2_{sup}} = H P_{op}(F_{2_{op}} H P_{op})^* P_{op} = H P_{op} P_{op} = H P_{op}.$$  

**PROOF.**

First assume that $G_{ref} \in \mathcal{G}_2$, i.e., it exists $D$ such that $G_{ref} = D^* H$. Then, thanks to (6),(3) and (4), it follows that $P_{op} = H \lambda (D^* H) = (D^* H) \lambda (D^* H) = ((D^* H) \lambda (D^* H))^* = P_{op}^*$. Then, we have $H P_{op} P_{op} = H P_{op}^* P_{op} = R_{P_{op}}^* \circ R_{P_{op}}$.

By recalling that $R_{P_{op}}$ is a closure mapping (i.e., $R_{P_{op}} \circ R_{P_{op}}(x) = x P_{op} P_{op} = R_{P_{op}}(x)$) and $R_{P_{op}} \geq \mathbb{I}$ and by using proposition 9 we have $H P_{op}^* P_{op} = R_{P_{op}}(H) = H P_{op} = H P_{op}$. The proof, if $G_{ref} \in \mathcal{G}_2$ can be given in a similar way.

Proposition 25. If $G_{ref} \in \mathcal{G}_1 \cup \mathcal{G}_2$ then

$$H_{1_{sup}} \leq H_{2_{sup}}.$$  

This means that the pair of controllers $(P_{op}, F_{2_{op}})$ is more robust with regard to the system variations.

**PROOF.**

Thanks to lemma 24, we have

$$H_{2_{sup}} = H P_{op} = H P_{op}^* = H (H \lambda G_{ref})^*.$$  

From (2), we have $H \lambda G_{ref} \geq ((H \lambda G_{ref}) \lambda H) H$, then by isotony of the laws $\otimes$ and $\oplus$ it follows that

$$H_{2_{sup}} = H (H \lambda G_{ref})^* \geq H ((H \lambda G_{ref}) \lambda H) H = H_{1_{sup}}.$$  

5. CONCLUSION

This paper compares the robustness and the performances of two control strategies for (max, +)-linear systems. More precisely, we show that the control proposed by (Maia et al., 2003) gives a greatest control and ensures a greatest insensitivity to the mismatch between the system and the model used for the controller synthesis. The next step for the control proposed by (Maia et al., 2003) aims at designing robust feedback controller when the system includes some parametric uncertainties which can be described by intervals (Lhommeau et al., 2003).

REFERENCES


