CONTROL AND ROBUSTNESS ANALYSIS FOR (MAX, +)-LINEAR SYSTEMS

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Abstract: This paper deals with controller synthesis for (max,+) linear system. It aims at comparing the performances and the robustness of two control strategies introduced in (Cottenceau *et al.*, 2001) and (Maia *et al.*, 2003) respectively. In both strategies, the influences of the possible mismatches between the system and its model are analyzed. This work shows that the control strategy using simultaneously a precompensator and a feedback controller (introduced in (Maia *et al.*, 2003)) gives better performances.

Keywords: Discrete event dynamic systems, Timed Petri nets, $(\max, +)$ algebra, Timed event Graphs, dioid, idempotent semiring, Control Systems.

1. INTRODUCTION

Timed Event Graphs (TEG) constitute a subclass of timed Petri nets in which each place has exactly one upstream and one downstream transition. It is well known that the timed/event behavior of a TEG, under earliest functioning rule [1], can be expressed by linear relations over some dioids, namely idempotent semiring (Baccelli *et al.*, 1992). Strong analogies then appear between the classical linear system theory and the $(\max, +)$-linear system theory. In particular, the concept of *control* is well defined in the context of TEG study. It refers to the firing-control of the TEG input transitions in order to reach the desired performance. In the literature, an *optimal control* for TEG exists and is proposed in (Cohen *et al.*, 1989). For a given reference input, this open-loop structure control yields the latest input firing date in order to obtain the output before the desired date. One possible approach for the control of TEG is the model reference technique in which a given model describes the desired performance and the design goal is achieved through the calculation of a precompensator or a feedback controller (Cottenceau *et al.*, 2001; Luders and Santos-Mendes, 2002). The control strategies based on feedback control, although favoring stability, are limited in the sense that the reference model must satisfy certain restrictive conditions. Lately, a new technique for the design of controllers in which a precompensator and a feedback controllers are calculated simultaneously was introduced by (Maia *et al.*, 2003). This paper aims at comparing the performances and robustness

[1] *i.e.* a transition is fired as soon as it is enabled

of the above mentioned control methods. More precisely, we will compare performances regarding the just-in-time criterion and we will compare robustness, regarding possible mismatches between the system and its model. The paper is organized as follows. Section 2 introduces some algebraic tools concerning the dioid and residuation theories. Section 3 is devoted to recall some elements of DES representation over particular dioids and this section presents three control strategies. Section 4 is dedicated to the analysis of the performances and the robustness of these control strategies.

2. ALGEBRAIC PRELIMINARIES

A *dioid* $\mathcal{D}$ is an idempotent semiring, that is an algebraic structure with two internal operations denoted by $\oplus$ and $\otimes$. The neutral elements of $\oplus$ and $\otimes$ are represented by ε and e respectively. In a dioid, a partial order relation is defined by $a \succeq b$ iff $a = a \oplus b$ and $x \wedge y$ denotes the greatest lower bound between x and y. A dioid $\mathcal{D}$ is said to be complete if it is closed for infinite $\oplus$-sums and if $\otimes$ distributes over infinite $\oplus$-sums. Most of the time the symbol $\otimes$ will be omitted as in conventional algebra.

Theorem 1. ((Baccelli *et al.*, 1992), th. 4.75). The implicit equation $x = ax \oplus b$ defined over a complete dioid $\mathcal{D}$, admits $x = a^*b$ as least solution, where $a^* = \bigoplus_{i\in\mathbb{N}} a^i$ (Kleene star operator). It will be sometimes represented by the following mapping : $\mathcal{K} : \mathcal{D} \to \mathcal{D}, x \mapsto \bigoplus_{i\in\mathbb{N}} x^i$.

TEG control problems (Cohen *et al.*, 1989), stated in a just-in-time context, usually involves the inversion of isotone mappings [2], that is, one must find x such that $f(x) = y$ (where f is isotone). Residuation Theory (Blyth and Janowitz, 1972) deals with such problems stated in partially ordered sets.

Definition 2. (Residual and residuated mapping). A mapping $f : \mathcal{D} \to \mathcal{E}$ between two ordered sets is *residuated* if it is isotone, and if, for all $y \in \mathcal{E}$, the subset $\{x \in \mathcal{D} \mid f(x) \preceq y\}$ admits a maximal element, denoted $f^\sharp(y)$. The isotone mapping $f^\sharp : \mathcal{E} \to \mathcal{D}$ is called the *residual* of f. The residual $f^\sharp$ is the only isotone mapping satisfying the following properties :

$$f \circ f^\sharp \preceq \mathsf{Id} \text{ and } f^\sharp \circ f \succeq \mathsf{Id}, \tag{1}$$

where Id is the identity mapping respectively on $\mathcal{D}$ and $\mathcal{E}$.

Lemma 3. ((Cohen, 1998)).

- If $f : \mathcal{D} \to \mathcal{E}$ and $g : \mathcal{E} \to \mathcal{F}$ are residuated mappings, then $f \circ g$ is also residuated and $(f \circ g)^\sharp = g^\sharp \circ f^\sharp$.
- If f is a residuated mapping from $\mathcal{D} \to \mathcal{E}$, then $f \circ f^\sharp \circ f = f$.

The mappings $L_a : x \mapsto a \otimes x$ and $R_a : x \mapsto x \otimes a$ defined over a complete dioid $\mathcal{D}$ are both residuated ((Baccelli *et al.*, 1992), p. 181). Their residuals are isotone mappings denoted respectively by $L_a^\sharp(x) = a \backslash\!\!\circ\, x$ and $R_a^\sharp(x) = x \,\circ\!\!/ a$. Some useful dioid formulæ involving these residuals are given below.

$$a(a \backslash\!\!\circ\, x) \preceq x \text{ and } (x \,\circ\!\!/ a)a \preceq x \tag{2}$$
$$a(a \backslash\!\!\circ\,(ax)) = ax \tag{3}$$
$$a \backslash\!\!\circ\, a = (a \backslash\!\!\circ\, a)^* \tag{4}$$
$$(a^*)^2 = a^* \tag{5}$$
$$x \backslash\!\!\circ\,(a^*x) = (a^*x) \backslash\!\!\circ\,(a^*x) \tag{6}$$

Definition 4. (Restricted mapping). Let $f : \mathcal{D} \to \mathcal{E}$ be a mapping and $B \subset \mathcal{E}$ with $f(\mathcal{D}) \subset B$. We will denote ${}_{B|}f : \mathcal{D} \to B$ the mapping defined by $f = i_B \circ {}_{B|}f$, where $i_B : B \to \mathcal{E}, x \mapsto x$ is the canonical injection.

Definition 5. (Closure mapping). An isotone mapping $f : \mathcal{D} \to \mathcal{D}$ defined on an ordered set $\mathcal{D}$ is a closure mapping if $f \succeq \mathsf{Id}$ and $f \circ f = f$.

Remark 6. According to (5), the Kleene star operator is a closure mapping since $a^* \succeq a$ and $(a^*)^* = a^*$.

Theorem 7. ((Cottenceau *et al.*, 2001)). Let $f : \mathcal{D} \to \mathcal{D}$ be a closure mapping. Then, ${}_{\mathsf{Im}f|}f$ is a residuated mapping whose residual is the canonical injection $i_{\mathsf{Im}f} : \mathsf{Im}f \to \mathcal{D}, x \mapsto x$.

Example 8. Mapping ${}_{\mathsf{Im}\mathcal{K}|}\mathcal{K} : \mathcal{D} \to \mathsf{Im}\mathcal{K}$ is a residuated mapping whose residual is $\left({}_{\mathsf{Im}\mathcal{K}|}\mathcal{K}\right)^\sharp = i_{\mathsf{Im}\mathcal{K}}$. This means that $x = a^*$ is the greatest solution to inequality $x^* \preceq a^*$. Actually, this greatest solution achieves equality.

Theorem 9. ((Gaubert, 1992)). Let $f : \mathcal{D} \to \mathcal{D}$ be a residuated closure mapping, we have $f = f^\sharp \circ f$ and $f = f \circ f^\sharp$.

3. CONTROL METHOD

Firstly, let us consider the following $(\max, +)$-linear system

$$x(k) = Ax(k-1) \oplus Bu(k), \quad y(k) = Cx(k), \tag{7}$$

where $x(k) \in \overline{\mathbb{Z}}_{\max}^{n\times 1}, u(k) \in \overline{\mathbb{Z}}_{\max}^{p\times 1}$ and $y(k) \in \overline{\mathbb{Z}}_{\max}^{m\times 1}$ are respectively the state, input and output

[2] f is an isotone mapping if it preserves order, that is, $a \preceq b \Longrightarrow f(a) \preceq f(b)$.

vectors of the system. The matrices A, B and C are of proper sizes and have entries ranging over $\overline{\mathbb{Z}}_{\max}$. We know from (Baccelli *et al.*, 1992) that (7) represents the behavior of a class of discrete event systems called Timed Event Graphs (TEG). In the case of a TEG, x (resp. u and y) is a vector associated to the internal (resp. input and output) transitions, and $x_i(k)$ represents the kth firing dates of the internal transitions which are labelled x_i. Following the conventional approach, it is possible to define the transformation $x(\gamma) = \bigoplus_{k\in\mathbb{Z}} x(k)\gamma^k$ where γ is a backward shift operator in event domain (that is $y(\gamma) = \gamma x(\gamma) \Leftrightarrow \{y(k)\} = \{x(k-1), \forall k\}$, see (Baccelli *et al.*, 1992), p. 228). This transformation is analogous to the $\mathcal{Z}$-transform used in discrete-time classical control theory and the formal series $x(\gamma)$ is a synthetic representation of the trajectory $x(k)$. The set of the formal series in γ is a dioid denoted by $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$. By using γ-transform, we obtain the following representation of (7) :

$$X(\gamma) = A\gamma X(\gamma) \oplus BU(\gamma), \quad Y(\gamma) = CX(\gamma),$$

where $U(\gamma), X(\gamma)$ and $Y(\gamma)$ are the γ-transform of u, x and y respectively. The implicit equation for the vector X, namely $X = A\gamma X \oplus BU$ which is solved (thanks to theorem 1) by $X = (A\gamma)^* BU$. Finally, we obtain the input-output representation (transfer matrix)

$$Y = HU \quad \text{with} \quad H(\gamma) = C(A\gamma)^* B. \tag{8}$$

Herein, three control strategies for the systems are presented, and their performances are compared in section 4. They are based on the Just-in-Time criterion and on the model reference approach (Cottenceau *et al.*, 2001). They can be described as follows : let $H \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{m\times p}$ be the transfer matrix of the plant, given by (8), and $G_{ref} \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{m\times p}$ be the reference model, *i.e.*, the desired transfer matrix for the controlled system, what are the controllers leading to the greatest controlled system lower than the reference model.

The precompensation problem is depicted Fig. 1.(a). It is an open-loop strategy. The relation between the input $V \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^p$, and the output Y is denoted G_c and the relation between V and U is denoted G_{uv}. They are given by

$$Y = G_c V = HPV \text{ and } U = G_{uv} V = PV.$$

The aim is to compute the greatest precompensator P such that $G_c \preceq G_{ref}$. The residual of mapping L_H gives the optimal solution, denoted by P_{op} [3],

$$P_{op} = H \mathbin{\backslash\!\!\circ} G_{ref}. \tag{9}$$

This means that, for a given external input [4] $V \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^p$, the control input, given by $U = P_{op}V$, will be maximal. In fact, for any P such that $HP \preceq G_{ref}$, $P \preceq P_{op}$, therefore the isotony property assures that $U = PV \preceq P_{op}V$.

a) The output feedback control strategy is depicted Fig. 1.(b). It is obviously more robust than the open-loop strategy. For a given feedback controller F_1, the closed-loop transfer relation, denoted G_{1_c}, between Y and V, and the transfer relation between U and V, denoted $G_{1_{uv}}$, are given by

$$Y = G_{1_c} V = H(F_1 H)^* V \text{ and } U = G_{1_{uv}} V = (F_1 H)^* V.$$

The problem is then to compute the greatest controller F_1 such that $G_{1_c} \preceq G_{ref}$.

This problem can be solved via residuation theory if some restrictions are imposed on the reference model. The two following results are due to (Cottenceau *et al.*, 2001).

Proposition 10. Let $H \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{m\times p}$ be the transfer function of a TEG. Let $M_H : \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{p\times m} \to \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{m\times p}, X \mapsto H(XH)^*$ be a mapping. This mapping represents the influence of an output feedback x on the closed-loop transfer dynamics. Consider $G \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{m\times p}$, $D \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{m\times m}$ and $N \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{p\times p}$. Let us consider the following sets :

$$\begin{aligned} \mathcal{G}_1 &= \{G \mid \exists D \text{ such that } G = D^* H\}, \\ \mathcal{G}_2 &= \{G \mid \exists N \text{ such that } G = HN^*\}. \end{aligned}$$

The mapping ${}_{\mathcal{G}_1|}M_H$ and ${}_{\mathcal{G}_2|}M_H$ are both residuated. Their residuals are such that $\left({}_{\mathcal{G}_1|}M_H\right)^\sharp(X) = \left({}_{\mathcal{G}_2|}M_H\right)^\sharp(X) = H \mathbin{\backslash\!\!\circ} X \mathbin{\circ\!\!/} H$.

Proposition 11. If $G_{ref} \in \mathcal{G}_1 \cup \mathcal{G}_2$, there exists a greatest realizable output feedback $F_{1_{op}}$ such that $M_H(F_{1_{op}}) \preceq G_{ref}$. This greatest controller is given by

$$F_{1_{op}} = H \mathbin{\backslash\!\!\circ} G_{ref} \mathbin{\circ\!\!/} H. \tag{10}$$

b) The model-reference control scheme proposed in the following is a generalization of the two strategies described above, that is, it uses both a precompensator and feedback controller (Maia *et al.*, 2003). Fig. 1.(c) illustrates the approach.

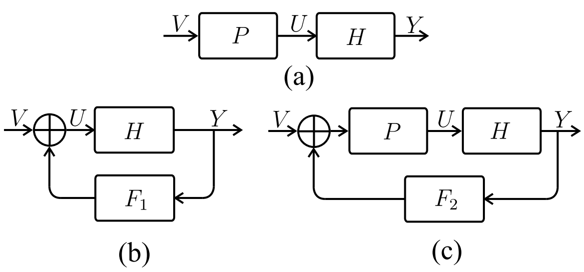

Fig. 1. Control structures.

By using theorem 1, one can obtain the closed-loop equations which relate U, V and Y:

[3] In a manufacturing context, control $U = P_{op}V$ will delay as far as possible the input of raw material while ensuring that output of manufacturing part Y be lower that $G_{ref}V$

[4] In a manufacturing context, V represents the available catering of raw material and U represents the allowance of the raw material into the system.

$$Y = G_{2_c} V = HP(F_2HP)^* V, \quad (11)$$

$$U = G_{2_{uv}} V = P(F_2HP)^* V. \quad (12)$$

The problem can be stated as follows. *Given a reference model G_{ref}, what are the controller matrices P and F_2 which assure the greatest transfer function between U and V, i.e. $G_{2_{uv}}$, such that $G_{2_c} \preceq G_{ref}$?* Again, considering the Just-in-Time context, one seeks the controllers which satisfy the reference specification $G_{2_c} \preceq G_{ref}$ while delaying as much as possible the input trajectory (*e.g.* the entrance of products to be processed). Formally, the problem can be stated as follows:

$$\bigoplus_{P,\ F_2} G_{2_{uv}}(P, F_2) \quad (13)$$

$$s.t.\ G_{2_c} = HP(F_2HP)^* \preceq G_{ref}.$$

It is clear that $P = [\varepsilon]_{p\times p}$ is always a subsolution to the problem independently of the choice of F_2, meaning that the subsolution set is not empty. Furthermore, it is easy to notice that the strategies using exclusively a precompensator (by setting $F_2 = [\varepsilon]_{p\times m}$) or exclusively a feedback controller (by setting $P = I_{p\times p}$, where $I_{p\times p}$ is the identity matrix in dioid) are particular cases of this problem.

Proposition 12. ((Maia *et al.*, 2003)). For the proposed control scheme shown in Fig. 1.(c), the three following inequalities are equivalent:

$$\begin{array}{ll} HP(F_2HP)^* & \preceq G_{ref} \\ P(F_2HP)^* & \preceq H \backslash\!\!\circ G_{ref} \\ HP(F_2HP)^* & \preceq H(H \backslash\!\!\circ G_{ref}). \end{array}$$

Lemma 13. ((Maia *et al.*, 2003)). A solution to problem 13 must satisfy $P \preceq G_{2_{uv}} \preceq H \backslash\!\!\circ G_{ref}$.

Proposition 14. ((Maia *et al.*, 2003)). A solution to the optimization problem proposed in (13) is given by :

$$P_{op} = H \backslash\!\!\circ G_{ref}. \quad (14)$$

$$F_{2_{op}} = (HP_{op}) \backslash\!\!\circ (HP_{op}) \circ\!\!/ (HP_{op}). \quad (15)$$

PROOF.

From lemma 13, $G_{2_{uv}}$ is maximum (it is equal to the upper bound) if $P = H \backslash\!\!\circ G_{ref}$ and $F_2 = \varepsilon$. Then, the greatest F_2 for this value of P is given by the greatest subsolution of inequality $P_{op}(F_2HP_{op})^* \preceq H \backslash\!\!\circ G_{ref}$, which in turn (by proposition 12) is equivalent to $HP_{op}(F_2HP_{op})^* \preceq H(H \backslash\!\!\circ G_{ref}) = HP_{op}$. Moreover, from the residuation definition this inequality is equivalent to $(F_2HP_{op})^* \preceq (HP_{op}) \backslash\!\!\circ (HP_{op})$. Equation (4) yields $((HP_{op}) \backslash\!\!\circ (HP_{op}))^* = (HP_{op}) \backslash\!\!\circ (HP_{op})$ then, thanks to corollary 8, $F_2HP_{op} \preceq (HP_{op}) \backslash\!\!\circ (HP_{op})$. Finally, by solving this last inequality one obtains $F_{2_{op}} = (HP_{op}) \backslash\!\!\circ (HP_{op}) \circ\!\!/ (HP_{op})$.

4. PERFORMANCES AND ROBUSTNESS ANALYSIS OF CONTROL METHODS

We will compare below the performances and the robustness of the control strategy given by proposition 11 and the one given by proposition 14. First, we must observe that unlike the first strategy, the second one does not restrict the reference model choice. Nevertheless, in order to compare performances of these strategies, we assume below that the controller $F_{1_{op}}$ exists and then that the reference model is such that $G_{ref} \in \mathcal{G}_1 \cup \mathcal{G}_2$ (see proposition 11).

4.1 Performance Comparison

Proposition 15. The control strategy given in proposition 14 leads to the same performances than the one obtained with the open-loop strategy, *i.e.*, the greatest closed-loop transfer functions $G_{2_{uv}}$ and G_{2_c} are equal to their upper bounds, that is, P_{op} and HP_{op} respectively. Formally, this means that

$$\begin{array}{lll} G_{2_c} & = & HP_{op}(F_{2_{op}}HP_{op})^* = HP_{op} \\ G_{2_{uv}} & = & P_{op}(F_{2_{op}}HP_{op})^* = P_{op}. \end{array}$$

PROOF. This proposition follows directly from proposition 14, lemma 13 and from the observation that $G_{2_c} = HG_{2_{uv}}$ (see (11) and (12)).

Proposition 16. Let $G_{ref} \in \mathcal{G}_1 \cup \mathcal{G}_2$ be a reference model. The transfer relation between U and V are such that

$$G_{1_{uv}} = (F_{1_{op}}H)^* \preceq G_{2_{uv}} = P_{op}(F_{2_{op}}HP_{op})^* = P_{op}.$$

PROOF. We suppose that $G_{ref} \in \mathcal{G}_1$, that is $G_{ref} = D^*H$. Then, we have

$$G_{1_{uv}} = (F_{1_{op}}H)^* = ((H \backslash\!\!\circ (D^*H) \circ\!\!/ H)\, H)^* \preceq (H \backslash\!\!\circ (D^*H))^*$$

which follows from (2). But we also have that

$$G_{1_{uv}} \preceq (H \backslash\!\!\circ (D^*H))^* = H \backslash\!\!\circ (D^*H) = P_{op} = G_{2_{uv}}$$

by making use of (6) and (4).

Proposition 17. Let $G_{ref} \in \mathcal{G}_1 \cup \mathcal{G}_2$ be a reference model. The controlled system transfer $G_{1_c} = H(F_{1_{op}}H)^* \preceq G_{2_c} = HP_{op}(F_{2_{op}}HP_{op})^* = HP_{op}$.

PROOF. From proposition 16, we have $G_{1_{uv}} = (F_{1_{op}}H)^* \preceq G_{2_{uv}} = P_{op}(F_{2_{op}}HP_{op})^* = P_{op}$ and by isotony of product we obtain $G_{1_c} = HG_{1_{uv}} = H(F_{1_{op}}H)^* \preceq G_{2_c} = HG_{2_{uv}} = HP_{op}(F_{2_{op}}HP_{op})^* = HP_{op}$.

4.1.1. Summary These results mean that the pair of controllers $(P_{op}, F_{2_{op}})$:

- allows to obtain the same performances than the one obtained with the open-loop control ;

- generates a control law greater than the one obtained by $F_{1_{op}}$;
- leads to a controlled system transfer closer to the reference model G_{ref} than the one obtained by the controller $F_{1_{op}}$.

4.2 Robustness analysis

a) In this section the aim is to analyse the robustness of the closed-loop control methods introduced previously.

First, we study the robustness of the controller given by proposition 11 (Fig. 1.(b)). We are looking for an upper bound denoted by $H_{1_{sup}}$ to the set of systems which preserves the optimal closed-loop control objective, that is,

$$H_{1_{sup}} = \sup \left\{ X | X(F_{1_{op}}X)^* = H(F_{1_{op}}H)^* \right\}.$$

Then we characterize the set of systems which preserves the input output behavior. It means that the system can evolve in this set without alter the input-output performances of the closed-loop system.

Lemma 18. Let $Q_A : \mathcal{D} \rightarrow \mathcal{D}, X \mapsto X(AX)^*$ be a mapping defined over a complete dioid. Then ${}_{\mathsf{Im}Q_A|}Q_A$ is a residuated mapping and the residual is $({}_{\mathsf{Im}Q_A|}Q_A)^\sharp = i_{\mathsf{Im}Q_A}$, where $i_{\mathsf{Im}Q_A}$ is the canonical injection.

PROOF.

The mapping Q_A is a closure mapping, indeed $Q_A \circ Q_A(X) = X(AX)^*(AX(AX)^*)^* = X(AX)^*\left((AX)^+\right)^* = X(AX)^*(AX)^* = X(AX)^*$. Then proposition 7 gives the result.

Proposition 19. The system $H_{1_{sup}} = H(F_{1_{op}}H)^*$ is the greatest system which does not alter the closed-loop transfer relation, *i.e.*, $H_{1_{sup}}(F_{1_{op}}H_{1_{sup}})^* = H(F_{1_{op}}H)^*$.

PROOF.

According to lemma 18, we seek the greatest X such that $Q_{F_{1_{op}}}(X) \preceq H(F_{1_{op}}H)^*$. Lemma 18 yields $({}_{\mathsf{Im}Q_{F_{1_{op}}}|}Q_{F_{1_{op}}})^\sharp = i_{\mathsf{Im}Q_{F_{1_{op}}}}$, and since $H(F_{1_{op}}H)^* \in \mathsf{Im}Q_{F_{1_{op}}}$, we have directly $H_{1_{sup}} = H(F_{1_{op}}H)^*$. Furthermore according to theorem 9, we have $Q_{F_{1_{op}}} = Q_{F_{1_{op}}} \circ Q^\sharp_{F_{1_{op}}}$, which leads to equality $H_{1_{sup}}(F_{1_{op}}H_{1_{sup}})^* = H(F_{1_{op}}H)^* = H_{1_{sup}}$.

Corollary 20. Whatever be the system behavior X such that $H \preceq X \preceq H_{1_{sup}}$ the closed-loop transfer relation is equal to $H(F_{1_{op}}H)^*$, *i.e.*, the input-output performances are not altered.

PROOF. Let X be a transfer relation such that $H \preceq X \preceq H_{1_{sup}}$. Since the product and star operators are isotone, we have $H(F_{1_{op}}H)^* \preceq X(F_{1_{op}}X)^* \preceq H_{1_{sup}}(F_{1_{op}}H_{1_{sup}})^*$, and proposition 19 leads to equality $H(F_{1_{op}}H)^* = X(F_{1_{op}}X)^* = H_{1_{sup}}(F_{1_{op}}H_{1_{sup}})^*$.

b) We are now interested in the robustness analysis of the control method which equations are given in proposition 14. We are looking for an upper bound, denoted $H_{2_{sup}}$, to the set of systems which preserves the optimal closed-loop control objective, that is

$$H_{2_{sup}} = \sup \left\{ X \mid XP_{op}(F_{2_{op}}XP_{op})^* = HP_{op}(F_{2_{op}}HP_{op})^* \right\}. \tag{16}$$

Proposition 21. The system $H_{2_{sup}} = HP_{op}(F_{2_{op}}HP_{op})^* \phi P_{op}$ is the greatest which satisfies (16), *i.e.*, $H_{2_{sup}}P_{op}(F_{2_{op}}H_{2_{sup}}P_{op})^* = HP_{op}(F_{2_{op}}HP_{op})^*$.

PROOF.

According to definition of the mappings $Q_{F_{2_{op}}}$ and $R_{P_{op}}$, we first seek the greatest X such that $XP_{op}(F_{2_{op}}XP_{op})^* \preceq HP_{op}(F_{2_{op}}HP_{op})^*$, that is, $(Q_{F_{2_{op}}} \circ R_{P_{op}})(X) \preceq (Q_{F_{2_{op}}} \circ R_{P_{op}})(H)$. Since $(Q_{F_{2_{op}}} \circ R_{P_{op}})(H) \in \mathsf{Im}Q_{F_{2_{op}}}$ and as ${}_{\mathsf{Im}Q_{F_{2_{op}}}|}Q_{F_{2_{op}}}$ is a residuated mapping (see lemma 18), $X \preceq ({}_{\mathsf{Im}Q_{F_{2_{op}}}|}Q_{F_{2_{op}}} \circ R_{P_{op}})^\sharp \circ (Q_{F_{2_{op}}} \circ R_{P_{op}})(H)$, and thanks to lemma 3, it follows $X \preceq R^\sharp_{P_{op}} \circ ({}_{\mathsf{Im}Q_{F_{2_{op}}}|}Q_{F_{2_{op}}})^\sharp \circ (Q_{F_{2_{op}}} \circ R_{P_{op}})(H)$.

By recalling that $({}_{\mathsf{Im}Q_{F_{2_{op}}}|}Q_{F_{2_{op}}})^\sharp = i_{\mathsf{Im}Q_{F_{2_{op}}}}$, we have $X \preceq R^\sharp_{P_{op}} \circ (Q_{F_{2_{op}}} \circ R_{P_{op}})(H) = HP_{op}(F_{2_{op}}HP_{op})^* \phi P_{op}$.

Now, we will show that this upper bound is solution of (16). From proposition 15 it follows $HP_{op}\left(F_{2_{op}}HP_{op}\right)^* \phi P_{op} = HP_{op} \phi P_{op}$, *i.e.*, $R^\sharp_{P_{op}} \circ Q_{F_{2_{op}}} \circ R_{P_{op}}(H) = R^\sharp_{P_{op}} \circ R_{P_{op}}(H)$. Then, from lemma 3 it follows $Q_{F_{2_{op}}} \circ R_{P_{op}} \circ R^\sharp_{P_{op}} \circ R_{P_{op}}(H) = Q_{F_{2_{op}}} \circ R_{P_{op}}(H)$, which yields to $H_{2_{sup}}P_{op}(F_{2_{op}}H_{2_{sup}}P_{op})^* = HP_{op}(F_{2_{op}}HP_{op})^*$.

Corollary 22. Whatever be the system behavior X such that $H \preceq X \preceq H_{2_{sup}}$ the closed-loop transfer relation is equal to $HP_{op}(F_{2_{op}}HP_{op})^*$, *i.e.*, the input-output performances are not altered.

PROOF.

Let X be a transfer relation such that $H \preceq X \preceq H_{2_{sup}}$. Since the product and star operators are isotone, we have $HP_{op}(F_{2_{op}}HP_{op})^* \preceq XP_{op}(F_{2_{op}}XP_{op})^* \preceq H_{2_{sup}}P_{op}(F_{2_{op}}H_{2_{sup}}P_{op})^*$, and proposition

21 leads to equality $HP_{op}(F_{2_{op}}HP_{op})^* = XP_{op}(F_{2_{op}}XP_{op})^* = H_{2_{sup}}P_{op}(F_{2_{op}}H_{2_{sup}}P_{op})^*$.

4.3 Robustness evaluation

In the previous section, the upper bound of the system set which achieve the control objective is given for the both closed-loop control strategy. In order to compare these bounds we assume below that the optimal controller $F_{1_{opt}}$ exists, *i.e.* $G_{ref} \in \mathcal{G}_1 \cup \mathcal{G}_2$. Nevertheless, we recall that this restriction is not useful to ensure the existence of $F_{2_{op}}$.

Lemma 23. Consider a reference model $G_{ref} \in \mathcal{G}_1 \cup \mathcal{G}_2$. Let $F_{1_{op}}$ be the greatest controller such that $H(F_{1_{op}}H)^* \preceq G_{ref}$ and P_{op} the greatest precompensator such that $HP_{op} \preceq G_{ref}$. Then

$$H(F_{1_{op}}H)^* \preceq HP_{op} \preceq G_{ref}.$$

PROOF.

Since L_H is a residuated mapping (see definition 2), we have the following equivalences $H(F_{1_{op}}H)^* \preceq G_{ref} \iff (F_{1_{op}}H)^* \preceq H \backslash G_{ref}$. Furthermore, by isotony of $\otimes$, we obtain $H(F_{1_{op}}H)^* \preceq H(H \backslash G_{ref}) = HP_{op} \preceq G_{ref}$ in which the latter inequality follows from (14) and (2).

Lemma 24. If $G_{ref} \in \mathcal{G}_1 \cup \mathcal{G}_2$ the upper bound $H_{2_{sup}}$ is equal to HP_{op}, that is

$$H_{2_{sup}} = HP_{op}(F_2HP_{op})^* / P_{op} = HP_{op} / P_{op} = HP_{op}.$$

PROOF. First assume that $G_{ref} \in \mathcal{G}_1$, *i.e.*, it exists D such that $G_{ref} = D^*H$. Then, thanks to (6),(3) and (4), it follows that $P_{op} = H \backslash (D^*H) = (D^*H) \backslash (D^*H) = ((D^*H) \backslash (D^*H))^* = P_{op}^*$. Then, we have $HP_{op} / P_{op} = HP_{op}^* / P_{op}^* = R^{\sharp}_{P_{op}^*} \circ R_{P_{op}^*}$. By recalling that $R_{P_{op}^*}$ is a closure mapping (*i.e.*, $R_{P_{op}^*} \circ R_{P_{op}^*}(x) = xP_{op}^*P_{op}^* = R_{P_{op}^*}(x)$ and $R_{P_{op}^*} \succeq \mathsf{Id}$) and by using proposition 9 we have $HP_{op} / P_{op} = R_{P_{op}^*}(H) = HP_{op}^* = HP_{op}$. The proof, if $G_{ref} \in \mathcal{G}_2$ can be given in a similar way.

Proposition 25. If $G_{ref} \in \mathcal{G}_1 \cup \mathcal{G}_2$ then

$$H_{1_{sup}} \preceq H_{2_{sup}}.$$

This means that the pair of controllers $(P_{op}, F_{2_{op}})$ is more robust with regard to the system variations.

PROOF. Thanks to lemma 24, we have

$$H_{2_{sup}} = HP_{op} = HP_{op}^* = H\left(H \backslash G_{ref}\right)^*.$$

From (2), we have $H \backslash G_{ref} \succeq ((H \backslash G_{ref}) / H)H$, then by isotony of the laws $\otimes$ and $\oplus$ it follows that

$$H_{2_{sup}} = H\left(H \backslash G_{ref}\right)^* \succeq H\left((H \backslash G_{ref} / H)H\right)^* = H_{1_{sup}}.$$

5. CONCLUSION

This paper compares the robustness and the performances of two control strategies for $(\max, +)$-linear systems. More precisely, we show that the control proposed by (Maia *et al.*, 2003) gives a greatest control and ensures a greatest insensitivity to the mismatch between the system and the model used for the controller synthesis. The next step for the control proposed by (Maia *et al.*, 2003) aims at designing robust feedback controller when the system includes some parametric uncertainties which can be described by intervals (Lhommeau *et al.*, 2003).

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