Test of Injectivity via Interval Analysis

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Abstract. This paper presents a new numerical algorithm based on interval analysis able to verify that a differentiable function $\mathbf{f} : [\mathbf{x}] \subset \mathbb{R}^n \to \mathbb{R}^p$ is an injection. The efficiency of the method is demonstrated by illustrative examples. These examples have been treated by a C++ solver which is made available.

1 Introduction

The purpose of this paper is to present a new method based on guaranteed numerical computation [4, 5, 3, 1] able to verify that a function $\mathbf{f} : [\mathbf{x}] \subset \mathbb{R}^n \to \mathbb{R}^p$ satisfies

$$\forall \mathbf{x}_1 \in \mathcal{A}, \forall \mathbf{x}_2 \in \mathcal{A}, \mathbf{x}_1 \neq \mathbf{x}_2 \Rightarrow \mathbf{f}(\mathbf{x}_1) \neq \mathbf{f}(\mathbf{x}_2). \tag{1}$$

To our knowledge, it does not exist any numerical method able to perform this injectivity test. Moreover, the complexity of the algebraic manipulations involved often makes formal calculus too expensive, especially when the function $\mathbf{f}$ is not polynomial. A solver called ITVIA (Injectivity Test Via Interval Analysis) implemented in C++ is made available at `http://www.istia.univ-angers.fr/~lagrange/`. Note that many problems could be formulated as the injectivity verification of a specific function. For example, concerning the identification of parametric models, the problem of proving structural identifiability of parametric system amounts to checking the injectivity of the model structure [2, 6].

2 Injectivity Test

The paper provides a method to test function for injectivity. It exploits the following theorem which give a sufficient condition of injectivity.

Theorem 1. *Let $\mathbf{f} : [\mathbf{x}] \subset \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function and $\nabla \mathbf{f}$ its Jacobian matrix. The function $\mathbf{f}$ is injective over $[\mathbf{x}]$ if the CSP*

$$\begin{cases} \mathbf{y} = \mathbf{f}(\mathbf{x}_1), \mathbf{x}_1 \in [\mathbf{x}] \\ \mathbf{y} = \mathbf{f}(\mathbf{x}_2), \mathbf{x}_2 \in [\mathbf{x}] \\ \mathbf{x}_1 \neq \mathbf{x}_2 \\ \mathbf{M} \in [\nabla \mathbf{f}([\{\mathbf{x}_1, \mathbf{x}_2\}])] \\ \det_g(\mathbf{M}) = 0 \end{cases} \tag{2}$$

has no solution. Note that $[\{\mathbf{x}_1, \mathbf{x}_2\}]$ is the smallest box which contains $\mathbf{x}_1$ and $\mathbf{x}_2$ and $\det_g(\mathbf{M})$ is the generalized determinant of $\mathbf{M}$.

3 Examples

In this section, two examples are provided in order to illustrate the efficiency of the solver ITVIA. We are going to test two functions $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ for injectivity.

3.1 M-function

Consider the function $\mathbf{f}$, depicted in Figure 1, defined by

$$\mathbf{f} : \begin{cases} \quad \mathbb{R}^2 \quad \to \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2(x_1 \cos(x_1) + \sin(x_1)) \\ x_1 \sin(x_1) + x_2 \end{pmatrix} \end{cases} \tag{3}$$

and test its injectivity over the box $[\mathbf{x}] = \left([-4, 4], [0, \frac{4}{10}]\right)^T$.

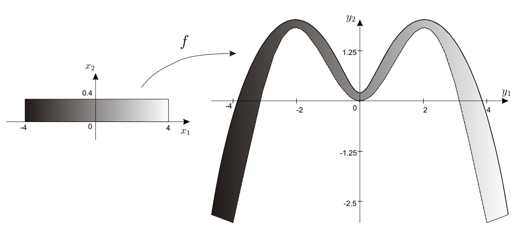

Fig. 1. Graph of the function $\mathbf{f}$ defined in (3).

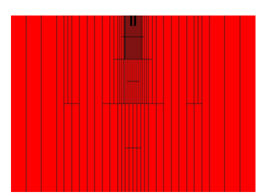

Fig. 2. Bisection of $[\mathbf{x}]$ obtained by ITVIA. In gray, the domain where the function $\mathbf{f}$ defined in (3) has been proved injective and, in white, the indeterminate domain (empty in this example).

After less than 0.1 sec on a Pentium $1.7GHz$, ITVIA proved that $\mathbf{f}$ is injective over $[\mathbf{x}]$ (see Figure 2).

3.2 Ribbon function

Consider the ribbon function $\mathbf{f}$ (depicted in Figure 3) defined by

$$\mathbf{f} : \begin{cases} \mathbb{R}^2 & \rightarrow \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1}{2} + (1 - x_2)\cos(x_1) \\ (1 - x_2)\sin(x_1) \end{pmatrix} \end{cases} \tag{4}$$

and get interest with its injectivity over the box $[\mathbf{x}] = ([-1, 4], [0, \frac{1}{10}])^T$. Since the ribbon overlapping (see Figure 3), one can see that $\mathbf{f}$ is not injective over $[\mathbf{x}]$.

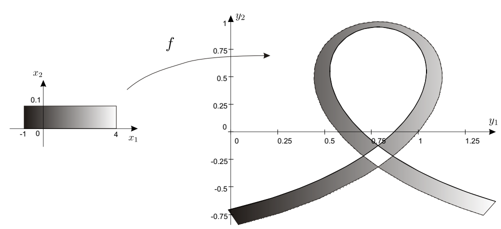

Fig. 3. Graph of the function $\mathbf{f}$ defined in (4). In the white zone, the function overlaps.

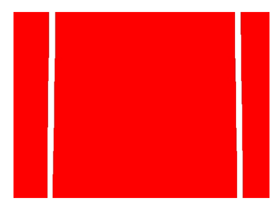

Fig. 4. Partition of the box $[\mathbf{x}]$ obtained by ITVIA. In white domain, the function has not been proved injective.

The solver ITVIA returns the solution presented in Figure 4. The function $\mathbf{f}$ has not been proved injective on the white domain. Indeed, this domain corresponds to the non injective zone of $[\mathbf{x}]$ where all points are mapped (by $\mathbf{f}$) in the overlapping zone of the ribbon.

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